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The construction of multidimensional membership functions and its application to feasibility problems

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Abstract

We present a novel idea of the membership function using logical expressions over inequalities. This is achieved by introducing the multidimensional membership function (distending function) using inequalities by including more variables instead of just one. In this paper, we concentrate on the application of this new approach to different kinds of feasibility problems. This new concept serves as a good tool for describing various kinds of feasibility regions. Another result here is that we present algorithms that can handle linear, nonlinear, convex and non-convex feasibility problems. Usually, in practice only the conjunction operator is used in the case of feasibility problems. A novelty of our result is that we drop this restriction, and based on our construction we will describe regions where other types of operators can also be used. We shall introduce the fundamentals of this new concept and we will show how it can be used in practical applications. Later, this approach may be applied to neural networks as well.
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Keywords: Membership function; Multidimensional fuzzy sets; Soft inequality systems; Fuzzy operators; Linear and nonlinear inequalities; Feasibility regions

1. Introduction

Membership functions play a crucial role in fuzzy theory. There are several definitions of membership functions and there is no precise definition for the notion of these functions. All the existing membership functions are one dimensional. In [1], Bilgic and Türksen provided a comprehensive overview of the most relevant interpretations. For the construction of membership functions, Dombi [2] offered an axiomatic point of view. Furthermore, in [3] we can also find a detailed description of different membership functions. In [4], Sotoudeh-Anvari extensively reviewed the literature on fuzzy methods. There we can read about the most frequently used 21 types of fuzzy sets, such as intuitionistic [5,6], interval-valued [7], and complex fuzzy sets [8], etc.

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In each area of fuzzy logic the membership functions are constructed by separating them into right and left hand sides. If we consider only the left hand side, we create a soft inequality, which behaves like a threshold, and it has the value of $\frac{1}{2}$. The most commonly used functions are the trapezoidal or triangular membership functions. These are described by four parameters a, b, c and d. Note that the degree of membership increases between a and b, flattens between b and c with a degree of 1, then decreases between c and d. Triangular functions are a particular case of trapezoidal functions where $b = c$. However, it is difficult to interpret the semantic meaning of these without any simplification and a crucial drawback of these functions is their lack of differentiability. A solution to the problem of differentiability is to use sigmoid functions. They play significant role in applications such as artificial neural networks, optimization, and economics. Furthermore, they are defined on the whole $\mathbb{R}$. In the case of the trapezoid methods we have to analyze assumptions and this makes it more complicated. In the case of sigmoid functions we use two parameters, λ which gives the sharpness and a, which defines the threshold.

In science and engineering, feasibility problems arise in several practical problems. For example, the task of neural network is to determine the feasibility region, which defines the output of the neural networks. In [9], the application of convex feasibility problem to adaptive image denoising is analysed. Usually, these regions are different (non-convex, nonlinear, disjunct sets). We formulate an approach with which we can construct these regions. The concept provided here can assist the interpretation of neural networks, where the nodes are logical operators. We focus on the description of the feasibility regions.

Finding the solution(s) of a system of linear or nonlinear inequalities plays an important role in optimization as well. Frequently, optimization problems are CPU intensive and require a lot of memory space. If we have limited time and memory, it is enough to find an approximate solution or a feasible point. However, in most cases only linear and convex feasibility problems are analyzed in the literature. Here, we introduce a new approach for finding a feasible point for all types of feasibility problems, including those for nonlinear and non-convex cases.

It was once common to convert constrained problems into unconstrained subproblems by using penalty and barrier methods. From the literature, we know that Frisch [10,11] proposed the use of logarithmic barrier functions in linear optimization. Later on, Fiacco and McCormick [12] developed the sequential unconstrained minimization technique. Since then, the barrier functions have been extensively studied (see [13,14]). Our new approach is similar to the one that uses the logarithmic barrier method. However, the logarithmic barrier function is not defined for points that lie outside the feasibility region. In our approach, we use a sigmoid-type function defined on the whole region.

In 1948, Von Neumann proposed an interior algorithm for finding a feasible solution to a linear program. This was presented by Dantzig in [15]. In 2015, Chubanov [16] gave a polynomial algorithm for linear feasibility problems. The problem is formulated as a system of linear equations and non-negativity constraints. Chubanov proved that the method either proves in polynomial time that the original system is infeasible or it finds a solution in the relative interior of the feasible set. Later on, Roos [17] presented a method which speeds up implementations of Chubanov's original algorithm. The existing results for solving linear feasibility problems are related to projections. There are other algorithms that invoke surrogate constraints. In his paper, Dudek [18] combined these two approaches in his method for solving these types of problems. Furthermore, S.D. Ventura et al. [19] presented a heuristic for solving nonlinear feasibility problems. In [20], Hu et al. gave a unified framework of subgradient methods for solving quasi-convex feasibility problems. Andreani et al. [21] proposed a projected-gradient underdetermined Newton-like algorithm for computing a feasible solution of a mathematical programming problem with complementarity constraints. Bozóki et al. studied the monostatic property of convex polyhedra [22]. Polynomial optimization (see [23]) has become a fundamental computational problem in algebraic geometry and it has a wide range of applications in different fields. Also, Egon Balas introduced disjunctive programming as an extension of linear programming with disjunctive constraints (see [24,25]). Kis and Horváth [26] proposed a systematic way of constructing ideal, nonextended mixed integer programming formulations for disjunctive constraints. Recently, Artacho et al. [27] introduced a Douglas-Rachford type algorithm for solving convex and nonconvex feasibility problems. The convergence of their algorithm is given in the convex setting, but their approach turned out to be a successful heuristic for solving different combinatorial problems. In [28], Shehu and Gibali presented a new inertial relaxed method for solving split feasibility problems. In all the above-mentioned algorithms particular types of feasibility problems are discussed.

The aim of this manuscript is to introduce the multidimensional membership function (distending function) using inequalities. This new membership function describes soft inequalities. With this concept, different fuzzy sets can be obtained using this approach. However, a comparison of this approach with other fuzzy sets presented in [4] is not the topic of this study. Here, we focus on the applications of this novel method to different types of feasibility problems, such as linear, nonlinear, convex and non-convex feasibility problems. The key is to apply continuous-valued logic, which has become increasingly important in the theory and application of fuzzy logic. Several representations of continuous-valued logical operators and so-called membership functions have been introduced here. The strictly monotone logical operators play a key role in this area. Zadeh introduced the min and max operators [29], which are not strict monotone. However, in practice the monotone operators have proved to be more efficient. Later on, academic investigations were conducted to find some good operators (see Hamacher [30], Dombi [31]). Here, we will focus on the Dombi operator system [32,33]. Utilizing it, we introduce multidimensional membership functions and we present a method that can be used to handle feasibility regions. We shall add an extra dimension represented by the new type of membership function and we will employ continuous-valued operators to describe and handle the various stated problems. We will also present an algorithm for finding feasible points in various types of feasibility problems. The novelty of this method is that it can handle feasibility problems like this as well, where the constraints are given using the disjunction or other types of operators, not just the conjunction operator as in the case of the approaches that appear in the literature. Moreover, it provides a quite new universal approach for handling all types of feasibility problems by using continuous-valued logic. We will present several small examples to illustrate how the algorithm works. Furthermore, in practical applications, several feasibility problems arise, where only 5-10 variables appear. Here, we will propose a quite new method to handle these types of problems. Note that the aim of this paper is to show how this new technology works. There is room for improving this technology and handling degenerated problems. Furthermore, the method can also be tested on large-sized problems. This is part of our future research plans. Later on, we would also like to link this approach to fuzzy relations. However, in the case of fuzzy relations the operations are in general discrete, while in our case they are continuous.

The paper is organized as follows. In Section 2, we will present a technique that can model inequalities using sigmoid functions. In Section 3, we will show how feasibility regions can be represented with the help of sigmoid functions. We will present examples containing linear and nonlinear inequalities as well. In Section 4, we will define logical expressions over inequalities and we will present our notion of a sigmoid-type distending function. Doing so, we will describe a new approach that can handle feasibility regions characterized by any type of logical expressions over linear and nonlinear inequalities. We will give several examples that have convex and non-convex feasibility regions. In Section 5, we will present our new algorithm for linear feasibility problems. In Section 6, we will introduce an algorithm that handles nonlinear feasibility problems. We will show how it works by providing several illustrative examples. Then, in Section 7, we will make some concluding remarks and offer some suggestions for future study.

2. Modeling inequalities by using sigmoid functions

Membership functions play a key role in fuzzy logic theory. In many cases these can be determined by using sigmoid functions. Let us now consider the following sigmoid function:

$$\sigma^{(\lambda')}(x) = \frac{1}{1+e^{-\lambda' x}}. \tag{1}$$

Remark 2.1. Note that we can use the interpretation of the sigmoid function as a soft inequality. We will get additional information, i.e. the value of σ will characterize how far we are from the origin and the 'distance' depends on λ'.

We have

$$\sigma^{(\lambda')}(x) > \frac{1}{2}, \text{ if } x > 0,$$
$$\sigma^{(\lambda')}(x) = \frac{1}{2}, \text{ if } x = 0,$$
$$\sigma^{(\lambda')}(x) < \frac{1}{2}, \text{ if } x < 0,$$

which is a soft inequality. If we consider the sigmoid function in (1), then

$$\lim_{\lambda'\to\infty} \sigma^{(\lambda')}(x) = \begin{cases} 1, & \text{if } x > 0, \\ \frac{1}{2}, & \text{if } x = 0, \\ 0, & \text{if } x < 0, \end{cases}$$

which is a hard inequality. We can generalize this approach by considering functions like $g(\mathbf{x})$, where $g : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable and $\mathbf{x} = [x_1, \dots, x_n]^T$. Then, we have the following:

$$\sigma^{(\lambda')}(g(\mathbf{x})) > \frac{1}{2}, \text{ if } g(\mathbf{x}) > 0,$$
$$\sigma^{(\lambda')}(g(\mathbf{x})) = \frac{1}{2}, \text{ if } g(\mathbf{x}) = 0,$$
$$\sigma^{(\lambda')}(g(\mathbf{x})) < \frac{1}{2}, \text{ if } g(\mathbf{x}) < 0,$$

which is a soft inequality.

If $\sigma^{(\lambda')}(x) \geq \frac{1}{2}$ for some $\lambda' > 0$, then we can say that the inequality $x \geq 0$ is satisfied. If $\sigma^{(\lambda')}(x) < \frac{1}{2}$ for some $\lambda' > 0$, then $x < 0$ is satisfied. Note that $\sigma(x)$ tells us the degree of the truth that the inequality states. The farther we are from 0, the more likely it is that $x \geq 0$ holds. If we add an extra dimension we can represent inequalities using sigmoid functions. We will use this approach in order to find the feasibility region of different inequalities and inequality systems. In this paper, the expressions appearing in the inequalities are always nonnegative.

Our requirement is to have $\sigma^{(\lambda')}(x)_{x=0} = \frac{1}{2}$ and $(\sigma^{(\lambda')}(x))'_{x=0} = \lambda'$. We can use a new parametrical form of the sigmoid function such that these conditions hold. If we replace λ' by 4λ, then we get

$$\sigma^{(\lambda)}(x) = \frac{1}{1 + e^{-4\lambda x}}, \tag{2}$$

where $\lambda \in (0, \infty)$. Then, if we use expression (2), then we get $(\sigma^{(\lambda)}(x))'_{x=0} = \lambda$.

An important property of the sigmoid function given in (2) is that it is continuously differentiable and analytic. As it is well known, this sigmoid function has applications in many different areas. It is frequently used in logistic regression, and neural networks.

Below, we will present an approach for determining feasibility regions using sigmoid functions.

3. Representing feasibility regions described by sigmoid functions

If instead of x we consider implicit functions, then we can represent different feasibility regions. We shall use the following form of the sigmoid function:

$$\mathcal{D}^{(\lambda)}(g(\mathbf{x})) = \sigma^{(\lambda)}(g(\mathbf{x})) = \frac{1}{1 + e^{-4\lambda g(\mathbf{x})}}. \tag{3}$$

In the following example, we will model the interior of an ellipse.

Example 3.1. Suppose we have the following nonlinear inequality:

$$g_1(x, y) = 1 - \left(\frac{x}{2}\right)^2 - \left(\frac{y}{5}\right)^2 \geq 0. \tag{4}$$

The interior of the ellipse described by the inequality given in (4) using the sigmoid function with $\lambda = 1$ is shown in Fig. 1.

We shall always consider the $g(x) \geq 0$ forms of the inequalities. If we multiply both sides of $g(x) \geq 0$ by a non-negative constant, we will get the same inequality. However, the sigmoid function over these two forms of inequalities will be different. Moreover, it is not just the multiplication by a constant that can cause problems. We will give another example.

Example 3.2. The following two inequalities describe the same circle:

$$g_1(x, y) = 1 - (x^2 + y^2) \geq 0 \tag{5}$$

and

$$g_2(x, y) = 1 - \sqrt{x^2 + y^2} \geq 0. \tag{6}$$

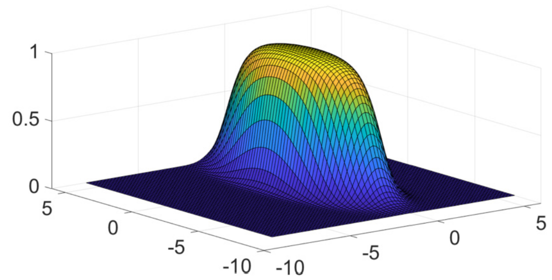

Fig. 1. Interior of ellipse described by sigmoid function.

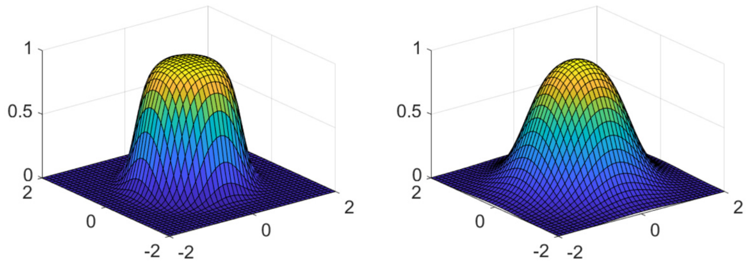

Fig. 2. Sigmoid function of circles described in (5) and (6).

The sigmoid functions of these circles with $\lambda = 1$ have been plotted in Fig. 2. Here, we can see that $\sigma(g_1(x, y)) \neq \sigma(g_2(x, y))$.

Hence, it should be remarked that using different equivalent forms of inequalities, we will get different surfaces. And if we multiply both sides of an inequality by a nonnegative number, we will change the sharpness of the sigmoid function.

Our goal is to standardize these inequalities in the following sense: when at a distance from a boundary point, our intention is to get the same value for its sigmoid function. We will do this by normalizing the inequalities in those cases where it is possible.

Definition 3.1. Consider the sigmoid function given in (3), where $g(\mathbf{x}) = g(x_1, x_2, \dots, x_n) \geq 0$, describes an inequality. We use the standard definition of the distance of a point from a surface described by $g(\mathbf{x}) = 0$, hence we deal with the shortest distance. Let $P_1(y_1, \dots, y_n)$ and $P_2(\bar{y}_1, \dots, \bar{y}_n)$ be two points at the same distance from the boundary point, where $g(y_1, \dots, y_n) \cdot g(y_1, \dots, y_n) \geq 0$. Then, we say that the inequality is normalized if $\mathcal{D}^{(\lambda)}(g(y_1, \dots, y_n)) = \mathcal{D}^{(\lambda)}(g(\bar{y}_1, \dots, \bar{y}_n))$.

Proposition 3.1. *In the case of an n-dimensional sphere we can normalize the inequality, namely:*

$$g(\mathbf{x}) = 1 - \sum_{i=1}^{n} \left(\frac{x_i - x_{i0}}{r} \right)^2 \geq 0, \tag{7}$$

where $(x_{10}, \dots, x_{n0})$ is the center and r is the radius.

Proof. *If we substitute (7) into (3) we get*

$$\mathcal{D}^{(\lambda)}(g(\mathbf{x})) = \frac{1}{1 + e^{-4\lambda\left(1 - \sum_{i=1}^{n}\left(\frac{x_i - x_{i0}}{r}\right)^2\right)}}.$$

Let $Q_1(x_1, \dots, x_n)$ and $Q_2(\bar{x}_1, \dots, \bar{x}_n)$ be two points at distance $k \cdot r$ from the center of the circle, where $k \in \mathbb{N}$. This means that $\sum_{i=1}^{n} \left(\frac{x_i - x_{i0}}{r}\right)^2 = \sum_{i=1}^{n} \left(\frac{\bar{x}_i - x_{i0}}{r}\right)^2$. Hence, we have $\mathcal{D}^{(\lambda)}(g(x_1, \dots, x_n)) = \mathcal{D}^{(\lambda)}(g(\bar{x}_1, \dots, \bar{x}_n))$, which means that we have the same value of the sigmoid function at distance $k \cdot r$ from the center, where $k \in \mathbb{N}$.

Remark 3.1. Note that if the center is the origin, then $\mathcal{D}^{(\lambda)}(g(\mathbf{x}))|_{\mathbf{x}=r\,\mathbf{e}} = \frac{1}{2}$, where $\mathbf{e}$ is the n-dimensional all-one vector and $\frac{\partial \mathcal{D}^{(\lambda)}(g(\mathbf{x}))}{\partial x_i}|_{\mathbf{x}=r\,\mathbf{e}} = \frac{2\lambda}{r}$.

Proposition 3.2. *Consider the following system of linear inequalities:*

$$A\mathbf{x} + \mathbf{b} \geq \mathbf{0}, \tag{8}$$

where $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{0}$ is the m-dimensional zero vector. We can normalize the inequality given in (8) in the following way:

$$l_j(\mathbf{x}) = \sum_{i=1}^{n} \frac{|a_{ji}x_i + b_j|}{\left(\sum_{i=1}^{n}(a_{ji})^2\right)^{\frac{1}{2}}}, \tag{9}$$

where $j = 1, \ldots, m$.

Proof. *Note that if we substitute the coordinates of a point $R_1(z_1, \ldots, z_n)$ into (9), then we get exactly the distance of the point R_1 from the line $A\mathbf{x} + \mathbf{b} = \mathbf{0}$. Hence, we can see that if we consider two points $R_1(z_1, \ldots, z_n)$ and $R_2(\bar{z}_1, \ldots, \bar{z}_n)$ at a distance d from the line, we get from (3) that $\mathcal{D}^{(\lambda)}(l_j(z_1, \ldots, z_n)) = \mathcal{D}^{(\lambda)}(l_j(\bar{z}_1, \ldots, \bar{z}_n))$.*

Remark 3.2. There are cases where we cannot perform normalization, such as in the case of an ellipse. In general, with normalization the value of λ will depend on the given boundary point. In the cases presented in Propositions 3.1 and 3.2, λ does not depend on the boundary points.

4. Construction of multidimensional membership function

In fuzzy theory, several operators have been invented for union and intersection. These operators may be viewed as continuous-valued operators. For this logic we need a negation operator as well. One of the most interesting classes is the Dombi operator system. The properties of this system have been described in several articles.

In the case of $\mathbf{x} = [x_1, \ldots, x_n]^T$, the conjunction operator is

$$c(\mathbf{x}) = \frac{1}{1 + \sum_{i=1}^{n} \frac{1-x_i}{x_i}}, \tag{10}$$

and the disjunction operator is

$$d(\mathbf{x}) = \frac{1}{1 + \left(\sum_{i=1}^{n} \frac{x_i}{1-x_i}\right)^{-1}}. \tag{11}$$

Furthermore, the following negation operator belongs to this operator system (see [34]):

$$\eta_\nu(x) = \frac{1}{1 + \frac{x}{1-x}\frac{1-\nu}{\nu}}. \tag{12}$$

Remark 4.1. The conjunction and disjunction operators form the DeMorgan identity with respect to this negation, independent of the value of ν. It should also be mentioned that $\eta_{\frac{1}{2}}(x) = 1 - x$ and $\eta_{\frac{1}{2}}\left(\frac{1}{2}\right) = \frac{1}{2}$.

Moreover, let $x_i := \sigma(g_i(\mathbf{x}))$. If we substitute these into (10), we get

$$\begin{aligned} c(\sigma(g_1(\mathbf{x})), \ldots, \sigma(g_n(\mathbf{x}))) &= \frac{1}{1 + \sum_{i=1}^{n}\left(\frac{1-\frac{1}{1+e^{-4\lambda g_i(\mathbf{x})}}}{\frac{1}{1+e^{-4\lambda g_i(\mathbf{x})}}}\right)} \\ &= \frac{1}{1 + \sum_{i=1}^{n} e^{-4\lambda g_i(\mathbf{x})}}, \end{aligned} \tag{13}$$

where $\lambda > 0$. If we substitute it into (11), we get in a similar way

$$d(\sigma(g_1(\mathbf{x})), \ldots, \sigma(g_n(\mathbf{x}))) = \frac{1}{1 + \left(\sum_{i=1}^{n} e^{4\lambda g_i(\mathbf{x})}\right)^{-1}},$$

Furthermore, if we substitute $\sigma(g(\mathbf{x}))$ into (12) with $\nu = \frac{1}{2}$, then we obtain

$$\eta^{(\lambda)}(\sigma(g(\mathbf{x}))) = \frac{1}{1 + e^{4\lambda g(\mathbf{x})}}. \tag{14}$$

Using these operators we can define logical expressions over inequalities. It should be mentioned that usually when we have several inequalities, then we consider the conjunction operator. However, with our approach, we can also include other operators as well.

In this way, we can define the multidimensional membership function in terms of a system of inequalities and we can represent several feasibility regions. It should be added that throughout the paper we will call the multidimensional membership function the distending function. These functions are responsible for an inflated region.

Definition 4.1. Let $\mathcal{L}$ be a logical expression based on the Dombi operator which consists of logical operators like a conjunction, disjunction, negation and we can use brackets as well. Each variable represents the truth value of an inequality. If we substitute the soft inequality into $\mathcal{L}$, then we get the distending function (multidimensional membership function) over the logical expression and we will denote it by $\mathcal{D}_{\mathcal{L}}^{(\lambda)}$. That is, $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) = \mathcal{L}(\sigma^{(\lambda)}(g_1(\mathbf{x})), \ldots, \sigma^{(\lambda)}(g_m(\mathbf{x})))$.

Remark 4.2.

- The sigmoid functions used in Section 3 are univariate distending functions.
- If $\lambda \to \infty$, then $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) = 1$.
- If $\mathcal{D}_{\mathcal{L}}(x_0) = 1$, then x_0 is a feasible point of the region determined by the inequalities.

In the following example we model the symmetric difference with logical expressions over inequalities.

Example 4.1. Let

$$\begin{aligned} g_1(x, y) &= 1 - x^2 - y^2 \geq 0, \\ g_2(x, y) &= \left(\frac{x}{\sqrt{6}}\right)^2 + \left(\frac{y}{\sqrt{6}}\right)^2 - 1 \geq 0, \\ g_3(x, y) &= x^2 + y^2 - 1 \geq 0, \\ g_4(x, y) &= 1 - \left(\frac{x}{\sqrt{6}}\right)^2 - \left(\frac{y}{\sqrt{6}}\right)^2 \geq 0. \end{aligned} \tag{15}$$

Now, we wish to compute the distending function of the region determined by

$$\mathcal{L}: (g_1(x, y) \geq 0 \text{ “and” } g_2(x, y) \geq 0) \text{ “or” } (g_3(x, y) \geq 0 \text{ “and” } g_4(x, y) \geq 0). \tag{16}$$

The distending function of the region determined by these inequalities with $\lambda = 1$ has been plotted in Fig. 3.

We will now present two different methods that show how we can model a heart using sigmoid functions.

Example 4.2. We define an equation which describes a cardioid:

$$\mathcal{L}: g(x, y) = 1 - x^2 - \left(\frac{5y}{4} - \sqrt{|x|}\right)^2 \geq 0. \tag{17}$$

Then, the distending function of the cardioid is

$$\mathcal{D}_{\mathcal{L}}^{(\lambda)}(x) = \frac{1}{1 + e^{-4\lambda\left(1 - x^2 - \left(\frac{5y}{4} - \sqrt{|x|}\right)^2\right)}}. \tag{18}$$

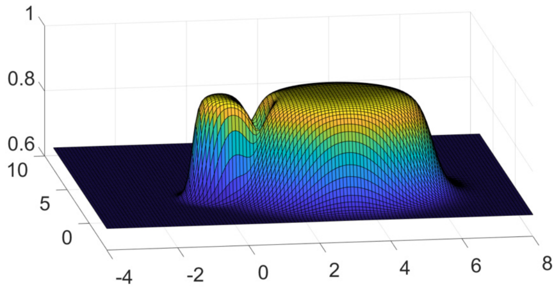

Fig. 3. The distending function of region determined by the inequalities stated in (16).

In order to describe the cardioid in another way, we will define the following inequalities describing half spaces and interiors of circles:

$$g_1(x, y) = -x^2 + x - y^2 + 2y - 1 \geq 0,$$
$$g_2(x, y) = -x^2 - x - y^2 + 2y - 1 \geq 0,$$
$$g_3(x, y) = 1 - y \geq 0,$$
$$g_4(x, y) = 2.5x + y + 1 \geq 0,$$
$$g_5(x, y) = -2.5x + y + 1 \geq 0.$$

The union of the circles is:

$$\mathcal{L}_1 : g_1(x, y) \geq 0 \text{ “or” } g_2(x, y) \geq 0. \tag{19}$$

The triangle is described in the following way:

$$\mathcal{L}_2 : g_3(x, y) \geq 0 \text{ “and” } g_4(x, y) \geq 0 \text{ “and” } g_5(x, y) \geq 0. \tag{20}$$

Using (19) and (20), the cardioid can also be described in the following way:

$$\mathcal{L} : \mathcal{L}_1 \text{ “or” } \mathcal{L}_2. \tag{21}$$

The distending function of the cardioid got by using the disjunction and conjunction operators:

$$\mathcal{D}_{\mathcal{L}}^{(\lambda)}(x, y) = \frac{1}{1 + \left(e^{4\lambda \mathcal{D}_{\mathcal{L}_1}^{(\lambda)}(x,y)} + e^{4\lambda \mathcal{D}_{\mathcal{L}_2}^{(\lambda)}(x,y)}\right)^{-1}},$$

where

$$\mathcal{D}_{\mathcal{L}_1}^{(\lambda)}(x, y) = \frac{1}{1 + j(x, y, \lambda)}$$

with

$$j(x, y, \lambda) = \left(e^{4\lambda(-x^2+x-y^2+2y-1)} + e^{4\lambda(-x^2-x-y^2+2y-1)}\right)^{-1}$$

and

$$\mathcal{D}_{\mathcal{L}_2}^{(\lambda)}(x, y) = \frac{1}{1 + h(x, y, \lambda)}$$

with

$$h(x, y, \lambda) = e^{-4\lambda(1-y)} + e^{-4\lambda(2.5x+y+1)} + e^{-4\lambda(-2.5x+y+1)}.$$

The distending function of the cardioids in these two cases has been plotted in Fig. 4.

In the following example we model the feasibility region described in two different cases, making use of conjunction and disjunction operators.

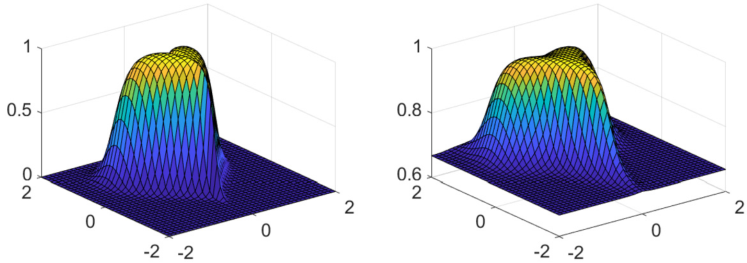

Fig. 4. The distending function of cardioids described by (18) and (21).

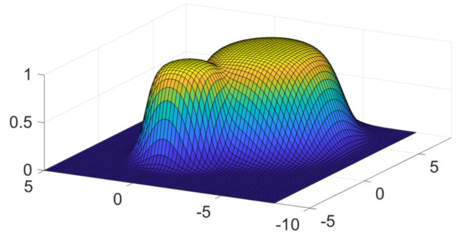

Fig. 5. The union of two circles with $\lambda = 1$.

Example 4.3. We have

$$g_1(x, y) = 1 - \left(\frac{x}{2}\right)^2 - \left(\frac{y}{2}\right)^2 \geq 0, \tag{22}$$
$$g_2(x, y) = 1 - \left(\frac{x-4}{\sqrt{13}}\right)^2 - \left(\frac{y+2}{\sqrt{13}}\right)^2 \geq 0.$$

We will look at the following case:

$$\mathcal{L}_2 : g_1(x, y) \geq 0 \text{ ``or'' } g_2(x, y) \geq 0. \tag{23}$$

The distending function of the region determined by the inequalities given in (23) with $\lambda = 1$ has been plotted in Fig. 5.

The following example describes a torus defined by two inequalities.

Example 4.4. Let

$$g_1(x, y) = 1 - \left(\frac{x}{4}\right)^2 - \left(\frac{y}{4}\right)^2 \geq 0 \tag{24}$$
$$g_2(x, y) = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 - 1 \geq 0$$

and the region determined by these inequalities:

$$\mathcal{L} : g_1(x, y) \geq 0 \text{ ``and'' } g_2(x, y) \geq 0. \tag{25}$$

The distending function of the region determined by this inequality with and $\lambda = 1$ has been plotted in Fig. 6.

In the following example we define a non-convex feasibility region.

Example 4.5. Consider the following nonlinear inequalities:

$$g_1(x, y) = -1 + \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 \geq 0,$$
$$g_2(x, y) = 1 - \left(\frac{x}{4}\right)^2 - \left(\frac{x}{4}\right)^2 \geq 0, \tag{26}$$

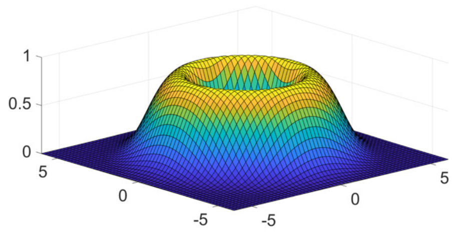

Fig. 6. The distending function of the region determined by the inequalities given in (24).

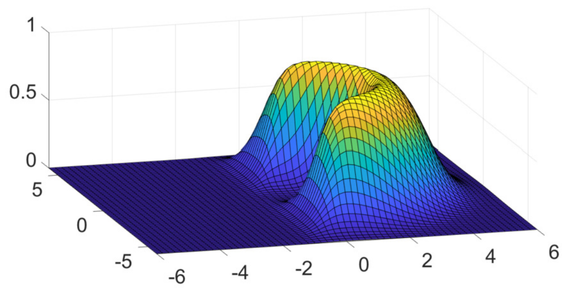

Fig. 7. The distending function of the region bounded by the inequalities stated in (26).

$$g_3(x, y) = x \geq 0.$$

We wish to determine the distending function of the region determined by

$$\mathcal{L} : g_1(x, y) \geq 0 \text{ “and” } g_2(x, y) \geq 0 \text{ “and” } g_3(x, y) \geq 0. \tag{27}$$

The distending function of the region determined by these inequalities with $\lambda = 1$ has been plotted in Fig. 7.

All above examples show how powerful a combination of logical expressions with sigmoid functions can be. The next example includes other nonlinear expressions.

Example 4.6. Let

$$\begin{aligned} g_1(x, y) &= 1 - \left(\frac{x}{\sqrt{80}}\right)^2 - \left(\frac{y}{\sqrt{80}}\right)^2 \geq 0, \\ g_2(x, y) &= y - e^x \geq 0, \\ g_3(x, y) &= \sin(x) - y \geq 0. \end{aligned} \tag{28}$$

We wish to compute the distending function of the region determined by

$$\mathcal{L} : g_1(x, y) \geq 0 \text{ “and” } g_2(x, y) \geq 0 \text{ “and” } g_3(x, y) \geq 0. \tag{29}$$

The distending function of the region determined by inequalities from (28) has been plotted in Fig. 8.

Next, we will analyze systems of linear inequalities and we will examine the distending function of different feasibility regions defined by these inequalities.

5. Algorithm in case of linear feasibility problems

Now, we shall consider systems of linear inequalities. Let

$$A\mathbf{x} + \mathbf{b} \geq \mathbf{0}, \tag{30}$$

where $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{0}$ is the m-dimensional zero vector. We will standardize this inequality using $l_j(\mathbf{x})$ given in (9). Furthermore, consider $\mathbf{l}(\mathbf{x}) = [l_1(\mathbf{x}), \ldots, l_m(\mathbf{x})]^T$. With this vector, we will consider the set of inequalities having the form $\mathbf{l}(\mathbf{x}) \geq \mathbf{0}$. Furthermore, we will consider the following regions:

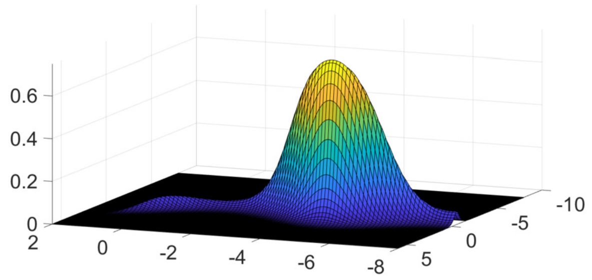

Fig. 8. The distending function of the region bounded by the inequalities stated in (28).

$$\mathcal{L} : l_1(\mathbf{x}) \geq 0 \text{ "and" } l_2(\mathbf{x}) \geq 0 \text{ "and" } \ldots \text{ "and" } l_m(\mathbf{x}) \geq 0, \tag{31}$$

hence we will only apply the conjunction operator in this case.

We can characterize linear inequalities by using the distending function. Using (3) and (10), we get the distending function of a polytope D determined by the inequalities $l_j(\mathbf{x})$:

$$\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) = \frac{1}{1 + \sum_{j=1}^{m} e^{-4\lambda l_j(\mathbf{x})}}, \tag{32}$$

where $\lambda \in (0, \infty)$.

Remark 5.1. If $\mathbf{x} \in \mathcal{L}$, then there exists a λ such that $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) > \frac{1}{2}$. Moreover, if $\mathbf{x} \notin \mathcal{L}$, then there exists a λ such that $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) < \frac{1}{2}$.

5.1. Communication between the optimization algorithm and objective function

We implemented an algorithm to handle the feasibility problem stated in (30). Our goal is to find a point where $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) > \frac{1}{2}$. For this, we will maximize the distending function $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x})$, for which we use some type of gradient method. We will utilize an optimization algorithm, the input being some parameters and the objective function. Here, we applied the Broyden-Fletcher-Goldfarb-Shanno method, which is used for solving unconstrained optimization problems (see [35]). In our case the objective function is not fixed, i.e. we have the free parameter λ that can be set for the optimization algorithm. Doing it this way is different from the classical algorithms where we set the objective function and the algorithm provides a feasible solution. In this case, after k iterations we have communication between the optimizer and the objective function, the value of λ which appears in the objective function, is recalculated. This type of interaction is a novelty of our approach. Note that in our case $k = 1$, hence we have a communication between the optimizer and objective function after each iteration.

This communication process has been plotted in Fig. 9.

If $\lambda \to \infty$, then the gradient will tend to zero and we cannot find the proper direction. But how can we choose the value of λ? This is explained in the following part.

Let $y_0 = \mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x})$. Moreover, let

$$Y_0 = \frac{1 - y_0}{y_0} = \sum_{j=1}^{n} e^{-4\lambda l_j(\mathbf{x})}, \tag{33}$$

where $l_j(\mathbf{x})$ given in (9). We wish to maximize $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x})$, which means that our goal is to minimize Y_0. Let

$$\lambda(j) = \tan\left(\frac{\pi}{2} \cdot \frac{j}{100}\right). \tag{34}$$

For each $j = 1, \ldots, 50$ we calculate the value of Y_0 given in (33) and we are looking for the smallest value. Thus, the calculation of λ can be expressed in the following way:

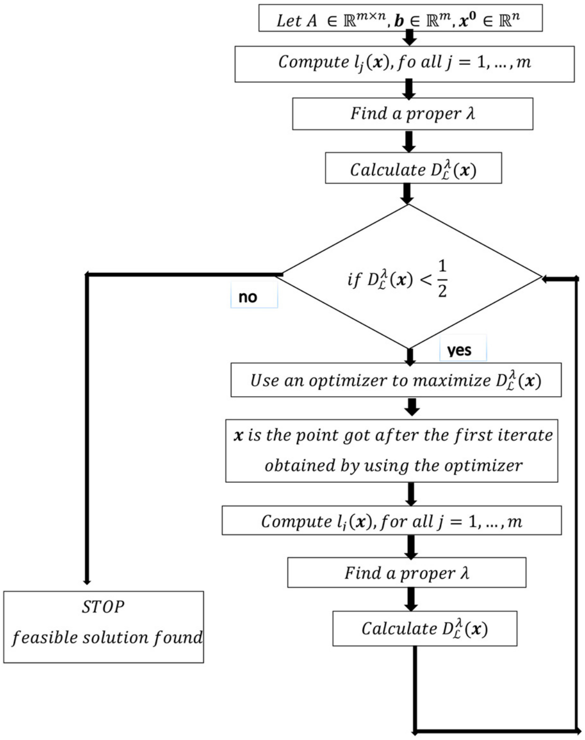

Fig. 9. Communication between optimizer and λ.

$$\lambda = arg\min\left\{\sum_{j=1}^{n} e^{-4\lambda l_j(\mathbf{x})}\right\}. \tag{35}$$

The implemented algorithm is presented in Algorithm 1.

Algorithm 1. Let $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{x}^{\mathbf{0}} \in \mathbb{R}^n$.
begin
$\mathbf{x} := \mathbf{x}^0$;
Compute $l_j(\mathbf{x}),\ j = 1, \ldots, m$ using (9);
Find a proper $\lambda > 0$ using (35);
Calculate $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x})$ using (32);
while $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) < \frac{1}{2}$ **do begin**
Use an optimizer (for instance, the BFGS algorithm) to maximize $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x})$;
Determine $\mathbf{x}$ as the point following subsequent to the first iterate got by using the optimizer;
Compute $l_j(\mathbf{x}),\ j = 1, \ldots, m$ using (9);
Find a proper $\lambda > 0$ using (35);
Calculate $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x})$;
end
end.

Remark 5.2. We can use other stopping criterion as well. We can calculate the value of the sigmoid function at a point with a proper value λ and with 2λ, as well. If $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) < \mathcal{D}_{\mathcal{L}}^{(2\lambda)}(\mathbf{x})$, then we have found a feasible point.

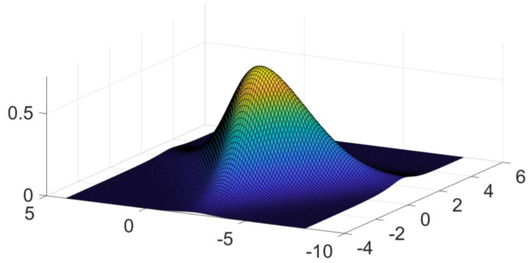

Fig. 10. The distending function of a triangle.

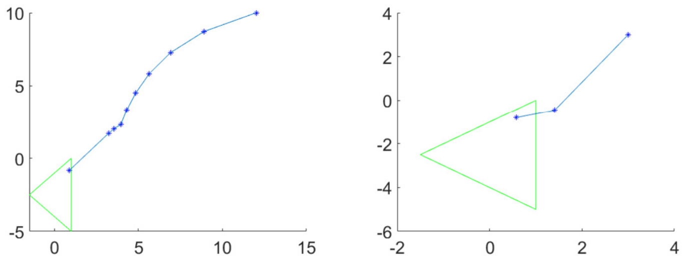

Fig. 11. Steps of the algorithm with communication between the optimizer and λ; $\mathbf{x}^0 = [12, 10]'$ and $\mathbf{x}^0 = [3, 3]'$.

Remark 5.3. The algorithm can also determine whether the problem is infeasible. We should take $-A\mathbf{x} - \mathbf{b} \geq \mathbf{0}$ instead of (30) and compute $l_j(\mathbf{x})$ and the corresponding value of the sigmoid function in this case. If this value is less than the original value of the sigmoid function, then we have infeasible problem.

Here, it is also worth investigating how the algorithm works with different types of optimizers. We will give an example to show how our algorithm works.

Example 5.1. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}.$$

Then, we have the following system of linear inequalities:

$$\begin{aligned} l_1(x, y) &= x + y + 4 \geq 0, \\ l_2(x, y) &= x - y - 1 \geq 0, \\ l_3(x, y) &= -x + 1 \geq 0. \end{aligned} \tag{36}$$

After normalizing the inequalities, the distending function of the polytope determined by these inequalities using $\lambda = 0.5$ is shown in Fig. 10.

We handled the feasibility problem given in (36) using Algorithm 1. Fig. 11 shows the steps of the algorithm with starting point $\mathbf{x}^0 = [12, 10]'$ and $\mathbf{x}^0 = [3, 3]'$, respectively.

In Fig. 12, we show the steps of the algorithm without the communication effects between λ and the optimizer. This means that we do not change the value of λ, and the algorithm for each iteration uses the original value of λ. We observe here that the algorithm cannot find a solution in every case, and the value of λ plays a key role in the progress of the algorithm.

We will now construct three circles enclosing this point. We will add these nonlinear inequalities to the system given in (36) and then we have the following new feasibility problem:

$$l_1(x, y) = x + y + 4 \geq 0,$$

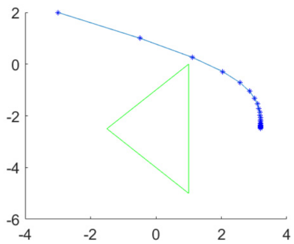

Fig. 12. Steps of the algorithm without communication between the optimizer and λ.

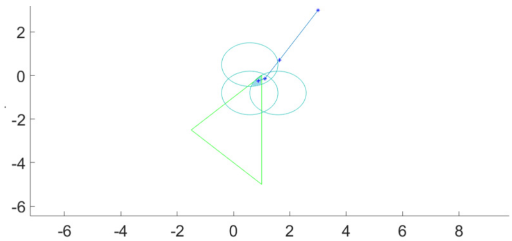

Fig. 13. Steps of the algorithm with $\mathbf{x}^0 = [3, 3]'$.

$$l_2(x, y) = x - y - 1 \geq 0,$$
$$l_3(x, y) = -x + 1 \geq 0,$$
$$g_1(x, y) = 0.022 - x^2 + 1.154x - y^2 - 1.606y \geq 0,$$
$$g_2(x, y) = -x^2 + 1.154x - y^2 + y + 0.417 \geq 0,$$
$$g_3(x, y) = -x^2 + 3.154x - y^2 - 1.607y - 2.133 \geq 0.$$

The region described by these inequalities can be defined by

$$\mathcal{L} : l_1(x, y) \geq 0 \text{ “and” } l_2(x, y) \geq 0 \text{ “and” } l_3(x, y) \geq 0$$
$$\text{“and” } g_1(x, y) \geq 0 \text{ “and” } g_2(x, y) \geq 0 \text{ “and” } g_3(x, y) \geq 0.$$

We handled the feasibility problem expressed by the above set of inequalities using Algorithm 1. Fig. 13 shows the steps of the algorithm. We notice that we get a feasible solution even in the case of a more complicated problem with a relatively small number of iterations.

In the following example we will show how the algorithm works on a larger-sized problem.

Example 5.2. Consider the following linear inequalities:

$$\begin{aligned}
x_1 \qquad\qquad + x_4 + x_5 + x_6 + x_7 &\geq 17 \\
x_1 + x_2 \qquad\qquad + x_5 + x_6 + x_7 &\geq 13 \\
x_1 + x_2 + x_3 \qquad\qquad + x_6 + x_7 &\geq 15 \\
x_1 + x_2 + x_3 + x_4 \qquad\qquad + x_7 &\geq 19 \\
x_1 + x_2 + x_3 + x_4 + x_5 \qquad\qquad &\geq 14 \\
x_2 + x_3 + x_4 + x_5 + x_6 \qquad &\geq 16 \\
x_3 + x_4 + x_5 + x_6 + x_7 &\geq 11 \\
x_i \geq 0 \qquad (i &= \overline{1, 7})
\end{aligned}$$

We applied Algorithm 1 with starting point $[4, 4, 2, 2, 5, 2, 4]'$. We got a feasible point after one iteration. Namely, $[10.5287, 10.8463, 9.0617, 9.3224, 10.1364, 7.5261, 10.3845]'$.

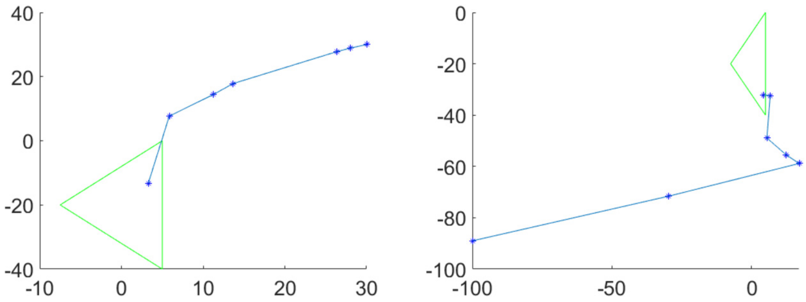

Fig. 14. Steps of the algorithm with communication between the optimizer and λ in the case of $\mathbf{x}^0 = [30, 30]'$ and $\mathbf{x}^0 = [-100, -89]'$.

5.2. *Normalization of variables*

In practice, the interval in which the variables are bounded is usually given. In this way we included in the algorithm the normalization of variables, by dividing each variable by the length of the interval.

We will consider Example 5.1 and we will divide the variable x by 5 and variable y by 8. Fig. 14 shows the steps of Algorithm 1 using the normalization of variables with $\mathbf{x}^0 = [30, 30]'$ and $\mathbf{x}^0 = [-100, -89]'$, a point far from the triangle.

6. The nonlinear feasibility problems case

We will now look at nonlinear expressions instead of the linear inequalities $l_j(\mathbf{x})$ stated in Section 5. In this case, the feasibility problem is the following:

$$g_j(\mathbf{x}) \geq \mathbf{0}, \quad j = 1, \ldots, m, \tag{37}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{0}$ is the m-dimensional zero vector and g_j are nonlinear expressions. Furthermore, consider

$$\mathbf{g}(\mathbf{x}) = [g_1(\mathbf{x}), \ldots, g_m(\mathbf{x})]. \tag{38}$$

Let

$$\mathcal{F} = \{\mathbf{x} \in \mathbb{R}^n, g_j(\mathbf{x}) \geq \mathbf{0}, j = 1, \ldots, m\} \tag{39}$$

be the feasible region determined by the nonlinear inequalities given in (37). We will also use the conjunction, disjunction operators and the logical expression $\mathcal{L}$ given in the previous sections.

Using (10), we have the following distending function:

$$\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) = \frac{1}{1 + \sum_{j=1}^{m} e^{-4\lambda g_j(\mathbf{x})}}, \tag{40}$$

where $\lambda \in (0, \infty)$. Moreover, if we apply (11), we have

$$\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) = \frac{1}{1 + \left(\sum_{j=1}^{m} e^{4\lambda g_j(\mathbf{x})}\right)^{-1}}. \tag{41}$$

Remark 6.1. If $\mathbf{x} \in \mathcal{L}$, then there exists a λ such that $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) > \frac{1}{2}$. Moreover, if $\mathbf{x} \notin \mathcal{L}$, then there exists a λ such that $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) < \frac{1}{2}$.

The algorithm used here is given in Algorithm 2. The idea is similar to the linear case; that is, we need to determine the value of λ in such a way as to ensure communication between the value of λ and the optimizer.

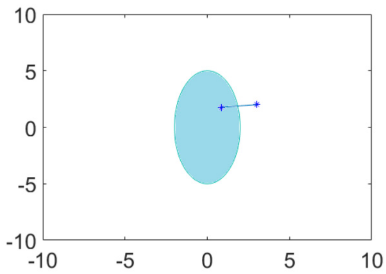

Fig. 15. Steps of the algorithm for the ellipsoid with $\mathbf{x}^0 = [3, 2]'$.

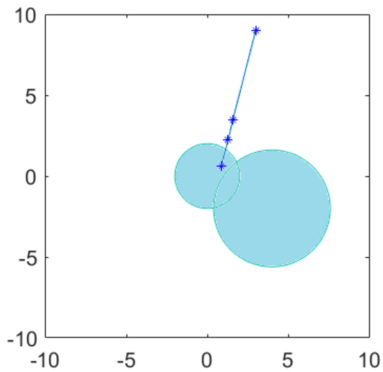

Fig. 16. Steps of the algorithm for the disjunction case with $\mathbf{x}^0 = [3, 9]'$.

Algorithm 2. Let $g_j(\mathbf{x}) \geq \mathbf{0}, \; j = 1, \ldots, m$, $\mathbf{x^0} \in \mathbb{R}^n$.
begin
$\mathbf{x} := \mathbf{x}^0$;
Find a proper $\lambda > 0$;
Calculate $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x})$;
while $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x}) < \frac{1}{2}$ **do begin**
Use an optimizer (for instance, the BFGS algorithm) to maximize $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x})$;
Determine $\mathbf{x}$ as the point after the first iterate got by using the optimizer;
Find a proper $\lambda > 0$ using (35);
Calculate $\mathcal{D}_{\mathcal{L}}^{(\lambda)}(\mathbf{x})$;
end
end.

Remark 6.2. It should be mentioned that in cases where the inequalities cannot be normalized, it is worth running the algorithm with different starting points. A refinement of this approach is the subject of future work.

We handled the feasibility problems given in (4) and (23) using Algorithm 2. Figs. 15 and 16 show the steps of the algorithm in both cases.

We also handled the feasibility problems described by (25) and (27) using Algorithm 2. In Figs. 17 and 18, we have plotted the steps of the algorithm in these two cases. Note that in case of problem (27) we combined Algorithms 1 and 2.

We considered the feasibility problem described in (29) and we applied Algorithm 2. Fig. 19 shows the steps of the algorithm.

Now, we will round off this paper with a short summary and make some brief remarks.

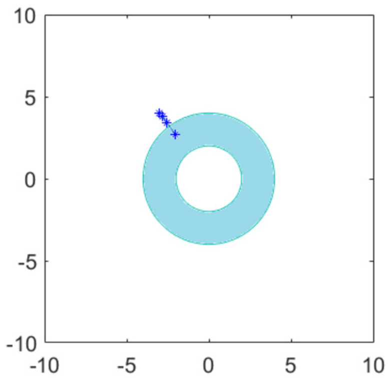

Fig. 17. Steps of the algorithm with $\mathbf{x}^0 = [-3, 4]'$.

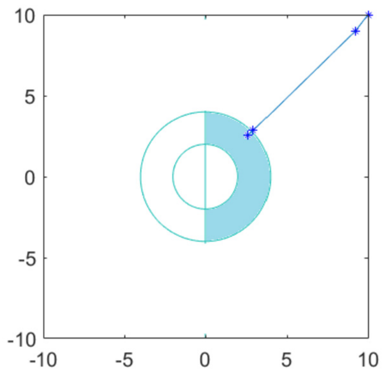

Fig. 18. Steps of the algorithm with $\mathbf{x}^0 = [10, 10]'$.

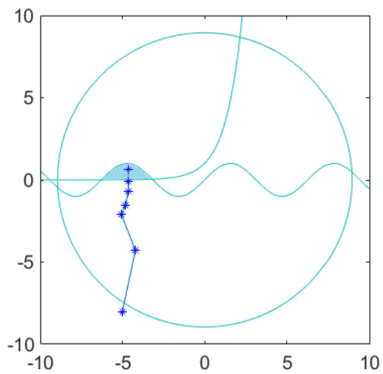

Fig. 19. Steps of the algorithm with $\mathbf{x}^0 = [-5, -8]'$.

7. Conclusions and future research

Here, we introduced the multidimensional membership function (distending function) using inequalities that include more variables instead of just one. We presented an application of this approach to feasibility problems, because in several optimization problems, the determination of a starting point plays an important role. However, this approach could lead to new applications in different fields, such as fuzzy approximations. In [36] we can read about such applications in the field of fuzzy control. We utilized this novel approach to represent feasibility regions described by any type of logical expressions over linear and nonlinear inequalities. We also introduced an algorithm to handle different types of feasibility problems. Several interesting questions arise on this topic that we are working on. For example, it would be nice to be able to choose a suitable optimizer that could be used in the algorithm. The determination of the step length could be explored and it would be nice to see how we could integrate this approach into a neural network framework.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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