LAZY GROUPOIDS

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ABSTRACT. A binary operation $f(x, y)$ is said to be lazy if every operation that can be obtained from f by composition is equivalent to $f(x, y)$, $f(x, x)$ or x. We describe lazy operations by identities (i.e., we determine all varieties of lazy groupoids), and we also characterize lazy groupoids up to isomorphism.

1. INTRODUCTION

Given a (not necessarily associative) binary operation $f(x, y) = xy$, we can form many other operations by composing f by itself, such as $(xy)z$, $((xy)(zu))(u(yv))$, $x_1(x_2(x_3 \cdots (x_{n-1}x_n)))$, etc. These composite operations can have arbitrarily many variables, but sometimes it happens that they do not depend on all of their variables. Consider, for example, a *rectangular band*, i.e., a semigroup satisfying the identities $xx \approx x$ (idempotence) and $xyz \approx xz$. These identities imply $x_1 x_2 \cdots x_n \approx x_1 x_n$ for all $n \in \mathbb{N}$, thus every product can be reduced to a product of at most two variables. It is natural to say that the multiplication of a rectangular band is *lazy*, since it only generates the operations $f(x, y)$ and $f(x, x)$ (up to renaming variables), and we can get these from f by simply identifying variables, hence composition is "unproductive" in this case.

Motivated by this example, we shall say that a binary operation f on a set A is *lazy*, if the only operations that can be obtained from f by composition are $f(x, y)$ and $f(x, x)$. We will give a more precise definition of laziness for operations of arbitrary arities in the Section 2. The main goal of this paper is to describe all lazy binary operations and the corresponding groupoids $(A; f)$. In Section 3 we will characterize lazy groupoids by identities: we will prove that they fall into 15 varieties (Theorem 3.5). One of these varieties is the semigroup variety defined by $(xy)z \approx x(yz) \approx xz$, which contains rectangular bands as a subvariety. We will determine all subvarieties of the 15 maximal lazy groupoid varieties in Section 4 (Theorem 4.2).

In Section 5 we give a more concrete description of lazy groupoids: we characterize them up to isomorphism by explicitly constructing their multiplication tables. This description is similar in spirit to the well known construction of rectangular bands as groupoids of the form $(A_1 \times A_2; \cdot)$, where the multiplication is defined by $(a_1, a_2) \cdot (b_1, b_2) = (a_1, b_2)$.

Lazy operations were originally defined in [5] in connection with essentially minimal clones. The 15 varieties of lazy groupoids were described already in the conference paper [6] (but the proof of Theorem 3.5 was only sketched there), and then an application to essentially minimal clones was given. Thus, the present paper can be regarded as an extended version of [6], the new contributions being the determination of all (sub)varieties of lazy groupoids and the explicit description of lazy groupoids up to isomorphism.

2. PRELIMINARIES

An *n-ary operation* on a nonempty set A is a map $f: A^n \to A$. We denote the set of n-ary operations on A by $\mathcal{O}_A^{(n)}$, and $\mathcal{O}_A$ stands for the set of all finitary operations

on A. We say that the i-th variable of $f \in \mathcal{O}_A^{(n)}$ is *essential* (in other words, f depends on its i-th variable) if there exist $a_1, \ldots, a_i, a_i', \ldots, a_n \in A$ such that

$$f(a_1, \ldots, a_i, \ldots, a_n) \neq f(a_1, \ldots, a_i', \ldots, a_n).$$

For $1 \leq i \leq n \in \mathbb{N}$ we define the i-th n-ary *projection* $e_i^{(n)} \in \mathcal{O}_A^{(n)}$ by $e_i^{(n)}(x_1, \ldots, x_n) = x_i$. The set of all projections on A is denoted by $\mathcal{J}_A$. Observe that $e_1^{(1)} = \mathrm{id}$ is the identity function on A.

A *clone* is a set $\mathcal{C} \subseteq \mathcal{O}_A$ of operations that is closed under composition and contains every projection. The clone generated by a given operation f is the clone $[f]$ containing all operations that can be obtained from f and the projections by composition. Equivalently, $[f]$ is the clone of term functions of the algebra $\mathbb{A} = (A; f)$. If f is a binary operation, then we will use the notation $f(x, y) = x \cdot y = xy$, and then the algebra $\mathbb{A} = (A; f) = (A; \cdot)$ is called a *groupoid*. For the sake of simplicity, let us say that the groupoid $\mathbb{A}$ is essentially binary (essentially at most unary) if f depends on both of its variables (f depends on at most one variable).

For $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$, we say that g is an *identification minor* (or simply a *minor*) of f (notation: $g \preceq f$), if there exists a map $\sigma \colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ such that

$$g(x_1, \ldots, x_m) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

This means that g can be obtained from f by identifying variables, permuting variables and introducing inessential variables. The relation $\preceq$ gives rise to a quasiorder on $\mathcal{O}_A$. The corresponding equivalence relation is defined by $f \equiv g \iff f \preceq g$ and $g \preceq f$, and it is clear that $f \equiv g$ if and only if they differ only in inessential variables and/or in the order of their variables. Note that for any $f \in \mathcal{O}_A$, we have $f \equiv \mathrm{id}$ if and only if $f \in \mathcal{J}_A$. We use the notation $\downarrow f$ for the principal ideal (downset) generated by f in the subfunction quasiorder: $\downarrow f := \{g \in \mathcal{O}_A \colon g \preceq f\}$. Note that the set $\downarrow f$ contains only one unary operation, namely $f(x, \ldots, x)$. For more information on the minor quasiorder and its principal ideals, see [4].

Clearly, $\mathcal{J}_A \cup \downarrow f \subseteq [f]$ holds for every operation f. If $[f] = \mathcal{J}_A \cup \downarrow f$, then we say that f is a *lazy operation* and $[f]$ is a *lazy clone*. Thus f is a lazy operation if it does not generate any other operations but its identification minors and projections.

Example 2.1. A unary operation $f \in \mathcal{O}_A^{(1)}$ is lazy if and only if the algebra $(A; f)$ satisfies $f^2(x) \approx x$ or $f^2(x) \approx f(x)$ (where $f^2(x)$ stands for $f(f(x))$). Indeed, if f is lazy, then $f^2 \in \mathcal{J}_A \cup \downarrow f$, and the latter set contains only two operations up to equivalence, namely id and f. Thus we have $f^2 \equiv \mathrm{id}$ (hence $f^2(x) \approx x$) or $f^2 \equiv f$ (hence $f^2(x) \approx f(x)$). Conversely, each one of the given identities implies that $[f] = \{\mathrm{id}, f\}$, and thus f is lazy.

Example 2.2. If f is idempotent, i.e., $f(x, \ldots, x) \approx x$, then $\mathcal{J}_A \subseteq \downarrow f$, hence in this case laziness is equivalent to $[f] = \downarrow f$. Lazy idempotent operations can be constructed as follows. Let $A_1, \ldots, A_n$ be nonempty sets, and let us define an n-ary operation f on $A_1 \times \cdots \times A_n$ by

$$f\big((a_1^1, \ldots, a_n^1), \ldots, (a_1^n, \ldots, a_n^n)\big) = (a_1^1, \ldots, a_n^n)$$

for all $a_i^j \in A_i$ $(i, j = 1, 2, \ldots, n)$. Note that the algebra $(A_1 \times \cdots \times A_n; f)$ is the direct product of the algebras $\big(A_i; e_i^{(n)}\big)$ $(i = 1, 2, \ldots, n)$. These algebras were called n-dimensional diagonal algebras in [7] and n-ary rectangular bands in [6] (cf. the construction of binary rectangular bands in Section 1). It was shown in [7, 6] that an idempotent operation $f \in \mathcal{O}_A^{(n)}$ is lazy if and only if $(A; f)$ is (isomorphic to) an n-ary rectangular band.

Remark 2.3. For $f \in \mathcal{O}_A^{(n)}$ and $k \in \{1, 2, \ldots, n\}$, let $f_k \in \mathcal{O}_A^{(2n-1)}$ denote the function obtained from f by substituting f for its k-th variable; more precisely,

$$f_k(x_1, \ldots, x_{2n-1}) := f(x_1, \ldots, x_{k-1}, f(x_k, \ldots, x_{k+n-1}), x_{k+n}, \ldots, x_{2n-1}).$$

Clearly, if the operation $f \in \mathcal{O}_A^{(n)}$ is lazy then $f_k \preceq f$ or $f_k \in \mathcal{J}_A$ for every $k \in \{1, 2, \ldots, n\}$. This is a simple necessary condition for laziness, and it will serve as a starting point for our investigation of lazy binary operations. Note, however, that this condition is not sufficient for laziness, as it is shown by the following example. Let us consider the binary operation $f(x, y) = xy$ on the set $A = \{0, 1, 2\}$ that is defined by the multiplication table below.

	0	1	2
0	1	1	0
1	1	1	0
2	0	0	0

One can verify that this groupoid satisfies the identities $(xy)z \approx z^2$ and $x(yz) \approx x^2$, which means that the functions $f_1(x, y) = (xy)z$ and $f_2(x, y) = x(yz)$ are both minors of f. However, f is not a lazy operation, because $g(x, y, z, u) := (xy)(zu)$ is constant 1, and $g \notin \mathcal{J}_A \cup \downarrow f$.

3. CHARACTERIZING LAZY GROUPOIDS BY IDENTITIES

Let $f(x, y) = x \cdot y = xy$ denote a binary operation on an arbitrary nonempty set A, and let $\mathbb{A}$ denote the groupoid $(A; f) = (A; \cdot)$. The dual of $\mathbb{A}$ is the groupoid $\mathbb{A}^d = (A; g)$, where $g(x, y) = yx$, and the dual of a groupoid variety V is $V^d = \left\{ \mathbb{A}^d \colon \mathbb{A} \in V \right\}$. Clearly, a groupoid is lazy if and only if its dual is lazy.

If f depends only on at most one variable, then we have either $f(x, y) = g(x)$ or $f(x, y) = g(y)$ for some unary operation g. According to Example 2.1, f is lazy if and only if g satisfies either $g(g(x)) \approx g(x)$ or $g(g(x)) \approx x$. This yields the following description of essentially at most unary lazy groupoids.

Lemma 3.1. *If $\mathbb{A}$ is a lazy essentially at most unary groupoid, then $\mathbb{A}$ belongs to one of the following four varieties:*

$$U \colon xy \approx xz, \ (xy)z \approx xy; \qquad U^d \colon xy \approx zy, \ x(yz) \approx yz;$$
$$\widetilde{U} \colon xy \approx xz, \ (xy)z \approx x; \qquad \widetilde{U}^d \colon xy \approx zy, \ x(yz) \approx z.$$

Proof. The fact that f does not depend on its second variable can be expressed by the identity $xy \approx xz$. Using the unary operation g as above, $g(g(x)) \approx g(x)$ translates to $(xy)z \approx xy$ and $g(g(x)) \approx x$ translates to $(xy)z \approx x$, yielding the varieties U and $\widetilde{U}$. If f does not depend on its first variable, then we obtain the dual varieties U^d and $\widetilde{U}^d$. $\square$

In the sequel, we will assume that f depends on both of its variables. We have $f_1(x, y) = (xy)z$ and $f_2(x, y) = x(yz)$; see Remark 2.3. If f is a lazy operation then $f_1, f_2 \in \mathcal{J}_A \cup \downarrow f$, hence $\mathbb{A}$ satisfies the identities $(xy)z \approx t_1$ and $x(yz) \approx t_2$ for some choice of the terms $t_1, t_2 \in \left\{ x, y, z, x^2, y^2, z^2, xy, yx, yz, zy, xz, zx \right\}$. This gives us 144 possibilities; we will prove that only 20 of these are possible. Examining these cases, we will find that essentially binary lazy groupoids belong to 13 varieties, each being defined by two identities.

Lemma 3.2. *If the binary operation $f(x, y) = xy$ satisfies the identity $(xy)z \approx t_1$ for some $t_1 \in \{x, y, z, zy, zx, yx\}$, or it satisfies $x(yz) \approx t_2$ for some $t_2 \in \{z, y, x, yx, zx, zy\}$, then f is essentially at most unary.*

Proof. The identity $(xy)z \approx t_1(x, y, z)$ implies

$$t_1(xy, z, u) \approx ((xy)z)u \approx t_1(x, y, z) \cdot u.$$

If $t_1 = x$, then we obtain $xy \approx xu$, which shows that xy does not depend on y. If $t_1 = zy$, then we get $uz \approx (zy)u$, which yields $uz \approx uy$ after applying $(zy)u \approx t_1(z, y, u)$, hence f does not depend on its second variable.

Let us now consider the case $t_1 = yx$. Then we have

$$z(xy) \approx t_1(xy, z, u) \approx ((xy)z)u \approx t_1(x, y, z) \cdot u \approx (yx)u \approx t_1(y, x, u) \approx xy,$$

which immediately implies $(vw)(xy) \approx xy$. On the other hand, $(vw)(xy) \approx t_1(v, w, xy) \approx wv$. Thus $xy \approx wv$, i.e., f is a constant operation.

Similar arguments work for the remaining three cases; we summarize them in Table 1 of Appendix A. The identities $x(yz) \approx t_2$ are the duals of the above ones. $\quad\square$

Now we are left with 36 pairs (t_1, t_2); these possibilities are summarized in Table 2. We will prove in the next two lemmas that the entries marked by '$-$' contradict the assumption that f is essentially binary, while the other cases give rise to 7 varieties $L_1, \ldots, L_7$ of lazy groupoids together with their duals (note that L_7 is selfdual).

Lemma 3.3. *Let $\mathbb{A}$ be an essentially binary groupoid. If $\mathbb{A}$ is lazy, then it belongs to one of the 13 varieties $L_1, \ldots, L_7, L_1^d, \ldots, L_6^d$, which are defined by the following identities:*

$$L_1 : \quad (xy)z \approx x^2, \quad x(yz) \approx x^2;$$
$$L_2 : \quad (xy)z \approx x^2, \quad x(yz) \approx xy;$$
$$L_3 : \quad (xy)z \approx xy, \quad x(yz) \approx x^2;$$
$$L_4 : \quad (xy)z \approx xz, \quad x(yz) \approx x^2;$$
$$L_5 : \quad (xy)z \approx xy, \quad x(yz) \approx xy;$$
$$L_6 : \quad (xy)z \approx xz, \quad x(yz) \approx xy;$$
$$L_7 : \quad (xy)z \approx xz, \quad x(yz) \approx xz.$$

Proof. We can derive the following three identities from $(xy)z \approx t_1$ and $x(yz) \approx t_2$:

(3.1a) $$t_1(x, y, zu) \approx (xy)(zu) \approx t_2(xy, z, u);$$

(3.1b) $$t_1(x, yz, u) \approx (x(yz))u \approx t_2(x, y, z) \cdot u;$$

(3.1c) $$x \cdot t_1(y, z, u) \approx x((yz)u) \approx t_2(x, yz, u).$$

In all the 16 cases marked by '$-$' in Table 2, at least one of the above three identities contradicts the essentiality of the operation f. We work out the details only for $t_1 \approx y^2, t_2 \approx xy$ (here we will need the identity (3.1c)); the other cases are similar or even simpler (see Table 3):

$$xz^2 \approx x \cdot t_1(y, z, u) \approx x((yz)u)$$
$$\approx t_2(x, yz, u) \approx x(yz) \approx xy.$$

Now it only remains to verify the entries marked by $L_1 \cap L_1^d$ in Table 2. These can be handled with the help of the identities (3.1); again, we provide details only for one case, namely for $t_1 \approx y^2, t_2 \approx x^2$, and refer to Table 3 for the remaining cases. Note that the variety $L_1 \cap L_1^d$ is axiomatized by the identities $(xy)z \approx x(yz) \approx x^2 \approx z^2$. This means that a groupoid $\mathbb{A}$ belongs to $L_1 \cap L_1^d$ if and only if $\mathbb{A}$ is a semigroup and there is a constant $c \in A$ such that $xyz \approx x^2 \approx c$. It is clear that such semigroups satisfy $(xy)z \approx y^2$ and $x(yz) \approx x^2$. Conversely, assume now that $(xy)z \approx y^2$ and $x(yz) \approx x^2$ hold in a groupoid $\mathbb{A}$. Let us write out (3.1b):

$$(yz)^2 \approx t_1(x, yz, u) \approx (x(yz))u \approx t_2(x, y, z) \cdot u \approx x^2 u.$$

We can conclude that $(yz)^2$ depends neither on y nor on z, hence there is a constant $c \in A$ such that $(yz)^2 \approx c$. Now let us use (3.1a):

$$z^2 \approx t_1(y, z, vu) \approx (yz)(vu) \approx t_2(yz, v, u) \approx (yz)^2.$$

This implies that z^2 is constant c, hence $\mathbb{A}$ satisfies $(xy)z \approx x(yz) \approx x^2 \approx c$; therefore, $\mathbb{A} \in L_1 \cap L_1^d$.

The entry marked by $L_1 \cap L_1^d(!)$ in Table 2 is special in the sense that the identities $(xy)z \approx z^2, x(yz) \approx x^2$ do not guarantee laziness (see the example in Remark 2.3). Here (3.1a) yields $(zu)^2 \approx (xy)^2$, i.e., $(xy)^2$ is constant. Since the only constant in $\mathcal{J}_A \cup \downarrow f$ is the diagonal operation $f(x) = x^2$, laziness implies that x^2 must be constant. Then we have $(xy)z \approx x(yz) \approx x^2 \approx z^2$, hence $\mathbb{A} \in L_1 \cap L_1^d$. $\quad\square$

In order to complete the description of lazy groupoids, we still need to verify that every groupoid in the varieties $L_1, \ldots, L_7, L_1^d, \ldots, L_6^d$ defined in Lemma 3.3 is indeed lazy. In the following, whenever we use one of the two defining identities for any one of our varieties, we write "$\overset{1.}{\approx}$" or "$\overset{2.}{\approx}$" to indicate whether we have used the first or the second identity (as listed in Lemma 3.3).

Lemma 3.4. *If a groupoid $\mathbb{A}$ belongs to one of the 13 varieties $L_1, \ldots, L_7, L_1^d, \ldots, L_6^d$, then $\mathbb{A}$ is lazy.*

Proof. Assume that $\mathbb{A}$ is a groupoid in L_4 (the proof for the other varieties is very similar; see Table 4). We prove by term induction that every term of $\mathbb{A}$ is equivalent to x or xy (allowing that x and y are the same variable). Let t be a term that contains at least two multiplications (i.e., at least three, not necessarily distinct variables). Then $t = s_1 \cdot s_2$, where the terms s_1 and s_2 are shorter than t, hence, by the induction hypothesis, they are equivalent to a variable or to a product of two variables. Therefore, we have the following three possibilities with some (not necessarily distinct) variables x, y, z, u:

$$s_1 \approx xy, \qquad s_2 \approx z \qquad \implies s_1 s_2 \approx (xy)z \overset{1.}{\approx} xz;$$

$$s_1 \approx x, \qquad s_2 \approx yz \qquad \implies s_1 s_2 \approx x(yz) \overset{2.}{\approx} x^2;$$

$$s_1 \approx xy, \qquad s_2 \approx zu \qquad \implies s_1 s_2 \approx (xy)(zu) \overset{1.}{\approx} x(zu) \overset{2.}{\approx} x^2.$$

Thus, every term over L_4 is indeed equivalent to a variable or a product of two variables, showing that every member of L_4 is a lazy groupoid. $\square$

Theorem 3.5. *A groupoid $\mathbb{A}$ is lazy if and only if it belongs to one of the 15 varieties $L_1, \ldots, L_7, L_1^d, \ldots, L_6^d, \widetilde{U}, \widetilde{U}^d$.*

Proof. For essentially binary groupoids the "only if" part is covered by Lemma 3.3, while the "if" part follows from Lemma 3.4 and its dual. For essentially unary groupoids we can use Lemma 3.1. We do not need to list U and U^d, since $U \subseteq L_i$ for every i (cf. Figure 1). $\square$

4. VARIETIES OF LAZY GROUPOIDS

It is easy to verify that the proper subvarieties of U, $\widetilde{U}$ and their duals are the varieties LZ (left zero semigroups), RZ (right zero semigroups), Z (zero semigroups) and T (trivial semigroups), as shown in Figure 1:

$$LZ\colon x \approx xy; \qquad RZ\colon x \approx yx; \qquad Z\colon xy \approx zu; \qquad T\colon x \approx y.$$

Having determined all varieties of essentially at most unary lazy groupoids, let us now deal with subvarieties of $L_1, \ldots, L_7$. We get such subvarieties by adding some extra identities to the defining two identities of L_i ($i = 1, \ldots, 7$). Let V be the intersection of L_i and the variety defined by the identity $p \approx q$ (we assume $p \neq q$). Laziness of L_i implies that p and q are both equivalent to a product of two (not necessarily distinct) variables over V. If one of p and q involves two different variables, say $p = xy$, but q does not involve both x and y, then $p \approx q$ implies that f is essentially at most unary, hence V is a subvariety of U, U^d, $\widetilde{U}$ or $\widetilde{U}^d$. If x and y both occur in q, then we get $xy \approx yx$ (commutativity). The remaining cases, when at most one variable occurs on both sides, are the following: $x^2 \approx y^2$ (the main diagonal of the multiplication table is constant), $x^2 \approx y$ (satisfied only by trivial groupoids), $x^2 \approx x$ (idempotence), $x \approx y$ (satisfied only by trivial groupoids). Thus, in order to determine all non-unary subvarieties of L_i, we need to compute the intersection of L_i by one or more of the following three varieties:

$$I\colon x^2 \approx x; \qquad D\colon x^2 \approx y^2; \qquad C\colon xy \approx yx.$$

Lemma 4.1. *If a groupoid $\mathbb{A}$ belongs to one of the 21 varieties $L_i \cap V$ ($i = 1, \ldots, 7$, $V = I, D, C$), then $\mathbb{A}$ is essentially at most unary, with the exception of the 3 varieties*
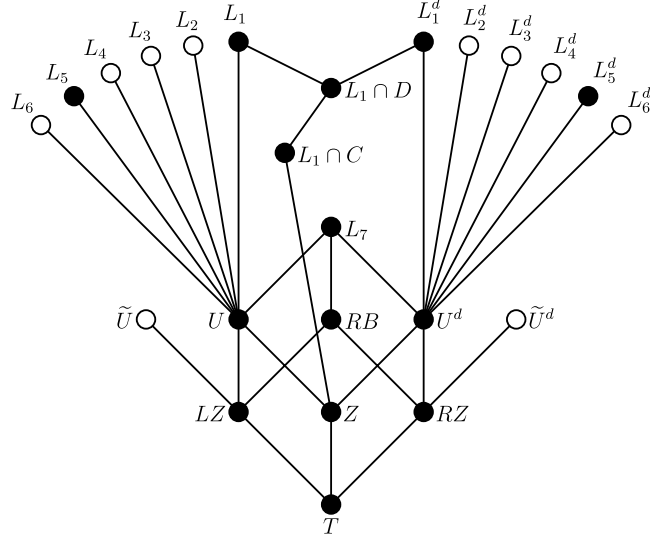
FIGURE 1. The semilattice of lazy groupoid varieties

$L_1 \cap D$, $L_1 \cap C$ and $L_7 \cap I = RB$ *(here, and in the sequel, RB denotes the variety of rectangular bands).*

Proof. We consider only L_3; the other cases can be seen in Table 5. Recall that L_3 is defined by $(xy)z \approx xy$ and $x(yz) \approx x^2$. As before, we will use $\overset{1.}{\approx}$ and $\overset{2.}{\approx}$ when we use one of these two identities, and we write $\overset{I}{\approx}$, $\overset{D}{\approx}$ and $\overset{C}{\approx}$ to indicate that we use the defining identity of I, D and C, respectively. In the variety $L_3 \cap I$ we have $xy \overset{I}{\approx} (xx)y \overset{1.}{\approx} xx$, showing that xy does not depend on y. In $L_3 \cap D$ we have $xy \overset{1.}{\approx} (xy)(zu) \overset{2.}{\approx} (xy)^2 \overset{D}{\approx} x^2$, which means again that xy does not depend on y. Finally, in $L_3 \cap C$ we can deduce $xy \overset{1.}{\approx} (xy)z \overset{C}{\approx} z(xy) \overset{2.}{\approx} z^2$, thus xy depends neither on x nor on y. □

Theorem 4.2. *There are 24 varieties of lazy groupoids, and they form the meet semilattice shown in Figure 1 (semigroup varieties are indicated by filled circles).*

Proof. By Theorem 3.5 every lazy groupoid belongs to $L_1, \ldots, L_7, U, \widetilde{U}$ or to the duals of these varieties. We have already determined the subvarieties of U and $\widetilde{U}$, and Lemma 4.1 describes all subvarieties of $L_2, \ldots, L_7$ (and their duals). Lemma 4.1 also implies that for L_1 and its dual we need to consider $L_1 \cap D$, $L_1 \cap C$, $L_1^d \cap D$, $L_1^d \cap C$ and any intersections of these. It is clear that $L_1 \cap C$ is selfdual, i.e., $L_1 \cap C = L_1^d \cap C$. The variety $L_1 \cap D$ is defined by $(xy)z \approx x(yz) \approx x^2 \approx z^2$, hence it it also selfdual: $L_1 \cap D = L_1^d \cap D$. Therefore, the only variety that could be possibly missing from Figure 1 is $L_1 \cap D \cap C$. However, this coincides with $L_1 \cap C$, as $L_1 \cap C \subseteq L_1 \cap D$. Indeed, we can derive $x^2 \approx y^2$ from commutativity and the two defining identities of L_1:

$$x^2 \overset{1.}{\approx} (xz)y \overset{C}{\approx} y(xz) \overset{2.}{\approx} y^2.$$

To prove that the 24 varieties in Figure 1 are all distinct, we give the operation tables of their two-generated free algebras in Appendix B (we include only one member of each pair of dual varieties, and we omit T). It remains to prove the containments indicated in Figure 1. All of these are straightforward to verify, with the exception of $L_1 \cap C \subseteq L_1 \cap D$, which we have already proved. □

5. Characterizing lazy groupoids up to isomorphism

We will give a concrete description of groupoids in the varieties $L_1, \ldots, L_7$ in this section (we do not write out the details for the dual varieties $L_1^d, \ldots, L_6^d$, and we also ignore the trivial unary cases $\widetilde{U}$ and $\widetilde{U}^d$).

Let us start with the variety L_7, which is the semigroup variety defined by $xyz \approx xz$. If $\mathbb{A} \in L_7$, then the set E of the idempotent elements in $\mathbb{A}$ forms a rectangular band. We will prove below in Theorem 5.1 that $\mathbb{A}$ is an *inflation* of the rectangular band $\mathbb{E} = (E; \cdot)$. This means that each idempotent $e \in E$ has a "neighborhood" N_e containing e such that $\{N_e : e \in E\}$ is a partition of A (in particular, the only idempotent in N_e is e), and for each $a \in N_e$, $b \in N_f$ we have $ab = ef$. In other words, $\mathbb{A}$ can be constructed from the rectangular band $\mathbb{E}$ by "inflating" each element $e \in E$ to a set N_e. Together with the well-known characterization of rectangular bands as direct products of a left zero semigroup and a right zero semigroup, this provides an explicit description of the members of L_7. This result appeared in [1] (and perhaps it has been known even earlier), but we include the proof for completeness.

Theorem 5.1. *A groupoid $\mathbb{A}$ belongs to the variety L_7 if and only if there is a subset $E \subseteq A$ and a partition $\{N_e : e \in E\}$, such that $\mathbb{E} = (E; \cdot)$ is a rectangular band, and*

(5.1)   *for all $e, f \in E$ and $a \in N_e$, $b \in N_f$, we have $a^2 = e$ and $ab = ef$.*

Proof. Suppose first that $\mathbb{A} \in L_7$, let $E = \{e \in A : e^2 = e\}$ and let $N_e = \{a \in A : a^2 = e\}$. Clearly, the sets N_e $(e \in E)$ are pairwise disjoint and nonempty (since $e \in N_e$); moreover, for all $a \in A$ we have $(a^2)^2 = (aa)(aa) \overset{1.}{=} a(aa) \overset{2.}{=} a^2$, hence $e := a^2$ is idempotent and $a \in N_e$. This shows that $\{N_e : e \in E\}$ is indeed a partition of A. The set E is closed under multiplication, as $(ef)^2 = (ef)(ef) \overset{1.}{=} e(ef) \overset{2.}{=} ef$ for all $e, f \in E$. Therefore $(E; \cdot)$ is a subsemigroup, and it satisfies the identities $xyz \approx xz$ and $x^2 \approx x$, hence it is a rectangular band. It only remains to prove $ab = ef$ for all $a \in N_e$, $b \in N_f$:

$$ab \overset{1.}{=} (aa)b \overset{2.}{=} (aa)(bb) = a^2 b^2 = ef.$$

Now assume that $E \subseteq A$ such that $\mathbb{E} = (E; \cdot)$ is a rectangular band, and $\{N_e : e \in E\}$ is a partition of A such that (5.1) holds. In order to verify that $\mathbb{A}$ satisfies the identity $(xy)z \approx xz$, let us consider arbitrary elements $a, b, c \in A$. Then $a \in N_e$, $b \in N_f$ and $c \in N_g$ for some $e, f, g \in E$. From the second equality of (5.1) it follows that $ab = ef$ and $ac = eg$. Since E is closed under multiplication, $h := ef$ belongs to E, and $h^2 = h$, as $\mathbb{E}$ is a band. The first equality of (5.1) implies that $h \in N_h$, and then we have $hc = hg$ by the second equality of (5.1). Putting everything together, we obtain $(ab)c = ac$:

$$(ab)c = (ef)c = hc = hg = (ef)g \overset{RB}{=} eg = ac,$$

where in the equality marked by RB we used the assumption that $\mathbb{E}$ is a rectangular band. We have proved that $\mathbb{A}$ satisfies the identity $(xy)z \approx xz$, and the identity $x(yz) \approx xz$ can be verified in a similar way, proving that $\mathbb{A} \in L_7$. $\qquad \square$

For the varieties $L_1, \ldots, L_6$, we will give similar characterizations in the following six theorems. Groupoids belonging to L_6 were described in [3] as unions of constant semigroups, which is essentially the same as Theorem 5.9 below, but our proof is different. Semigroups satisfying $xyz \approx xy$ were investigated and characterized in [2]; our Theorem 5.8 gives a different (and perhaps simpler) description. The characterizations will be given in terms of a partition $\{N_e : e \in E\}$ similarly to Theorem 5.1, but we will also need to specify a subset $S_e \subseteq N_e$ for all $e \in E$. Let us fix the notation for later reference:

Notation 5.2. Let $E \subseteq A$, and let $\{N_e : e \in E\}$ be a partition of A; moreover, for every $e \in E$, let $S_e \subseteq N_e$ such that $e \in S_e$.

Remark 5.3. Most of the time (with the exception of L_5), the setup will be the same as in Theorem 5.1: E will be the set of idempotents, $N_e = \{a \in A : a^2 = e\}$ and S_e

will be the set of *factorizable* elements in N_e, i.e., $S_e = N_e \cap \{ab : a, b \in A\}$. To see that $\{N_e : e \in E\}$ is indeed a partition of A, it suffices to note that $(xx)(xx) \approx xx$ holds in $L_1, \ldots, L_6$ (by the same argument as in the second sentence of the proof of Theorem 5.1). This implies that the square of any element is idempotent, therefore the sets N_e cover A.

Theorem 5.4. *A groupoid $\mathbb{A}$ belongs to the variety L_1 if and only if there is a partition $\{N_e : e \in E\}$ as in Notation 5.2, such that*

(5.2)   *for all $e, f \in E$ and $a \in N_e$, $b \in N_f$, we have $a^2 = e$ and*

$$
\begin{aligned}
ab &= e, & &\text{if } a \in S_e, & &b \in S_f; \\
ab &= e, & &\text{if } a \in S_e, & &b \in N_f \setminus S_f; \\
ab &= e, & &\text{if } a \in N_e \setminus S_e, & &b \in S_f; \\
ab &\in S_e, & &\text{if } a \in N_e \setminus S_e, & &b \in N_f \setminus S_f.
\end{aligned}
$$

Proof. First assume that $\mathbb{A} \in L_1$ and recall that L_1 is defined by $(xy)z \approx x^2$ and $x(yz) \approx x^2$. Let us consider the partition described in Remark 5.3; then the first equality of (5.2) is automatically satisfied. For the other four statements, let $a \in N_e$ and $b \in N_f$. We have $(ab)^2 = (ab)(ab) \overset{1.}{=} a^2 = e$, thus $ab \in N_e$. Therefore $ab \in S_e$, as ab is obviously factorizable, and this verifies the last statement of (5.2). If $b \in S_f$, i.e., $b = b_1 b_2$ for some $b_1, b_2 \in A$, then $ab = a(b_1 b_2) \overset{2.}{=} a^2 = e$. Similarly, if $a = a_1 a_2 \in S_e$, then $ab = (a_1 a_2)b \overset{1.}{=} a_1^2 \overset{1.}{=} (a_1 a_2)^2 = a^2 = e$. This proves that (5.2) is satisfied.

Conversely, suppose that $\{N_e : e \in E\}$ is a partition of A as in Notation 5.2, such that (5.2) holds. Let us compute $(ab)c$ and $a(bc)$ for $a \in N_e$, $b \in N_f$, $c \in N_g$. From (5.2) we see that $ab \in S_e$ and $bc \in S_f$, and then, again from (5.2), we obtain

$$(ab)c = e = a^2, \qquad a(bc) = e = a^2.$$

This shows that $\mathbb{A}$ satisfies $(xy)z \approx x(yz) \approx x^2$, hence $\mathbb{A} \in L_1$. $\square$

Theorem 5.4 allows us to construct the multiplication table of any groupoid in L_1 as follows. Fix an arbitrary nonempty set A, and choose a partition $\{N_e : e \in E\}$ of A as in Notation 5.2. Then define a multiplication on A such that (5.2) is satisfied (if $a \in N_e \setminus S_e$ and $b \in N_f \setminus S_f$, then we can choose ab to be any element of S_e). This gives a groupoid in L_1, and every member of L_1 can be obtained this way. A part of such an operation table can be seen in Appendix C. Only two blocks of the partition are displayed; the elements of S_e are denoted by $e = s_0, s_1, \ldots$ and the elements of $N_e \setminus S_e$ are denoted by $a_1, a_2, \ldots$ (and similarly for S_f and $N_f \setminus S_f$). However, this is only for notational convenience: these sets can have arbitrary cardinalities (not necessarily countable). Appendix C contains similar tables for the varieties $L_2, \ldots, L_6$, illustrating the following five theorems.

Theorem 5.5. *A groupoid $\mathbb{A}$ belongs to the variety L_2 if and only if there is a partition $\{N_e : e \in E\}$ as in Notation 5.2, such that*

(5.3)   *for all $e, f \in E$ and $a \in N_e$, $b \in N_f$, we have $a^2 = e$ and*

$$
\begin{aligned}
ab = af &= e, & &\text{if } a \in S_e, & &b \in S_f; \\
ab = af &= e, & &\text{if } a \in S_e, & &b \in N_f \setminus S_f; \\
ab = af &\in S_e, & &\text{if } a \in N_e \setminus S_e, & &b \in S_f; \\
ab = af &\in S_e, & &\text{if } a \in N_e \setminus S_e, & &b \in N_f \setminus S_f.
\end{aligned}
$$

Proof. First assume that $\mathbb{A} \in L_2$ and recall that L_2 is defined by $(xy)z \approx x^2$ and $x(yz) \approx xy$. Let us consider the partition described in Remark 5.3. Let $a \in N_e$ and $b \in N_f$. We have $(ab)^2 = (ab)(ab) \overset{1.}{=} a^2 = e$, hence $ab \in S_e$. Moreover, $ab \overset{2.}{=} a(bb) = af$, and if $a = a_1 a_2 \in S_e$, then $ab = (a_1 a_2)b \overset{1.}{=} a_1^2 \overset{1.}{=} (a_1 a_2)^2 = a^2 = e$. This proves that (5.3) is satisfied.

Conversely, suppose that $\{N_e \colon e \in E\}$ is a partition of A as in Notation 5.2, such that (5.3) holds. Let us compute $(ab)c$ and $a(bc)$ for $a \in N_e$, $b \in N_f$, $c \in N_g$. From (5.3) we see that $ab \in S_e$ and $bc \in S_f$, and then, again from (5.3), we obtain

$$(ab)c = e = a^2, \qquad a(bc) = af = ab.$$

This shows that $\mathbb{A}$ satisfies $(xy)z \approx x^2$ and $x(yz) \approx xy$, hence $\mathbb{A} \in L_2$. $\qquad \square$

Theorem 5.6. *A groupoid $\mathbb{A}$ belongs to the variety L_3 if and only if there is a partition $\{N_e : e \in E\}$ as in Notation 5.2, such that*

(5.4) *for all $e, f \in E$ and $a \in N_e$, $b \in N_f$, we have $a^2 = e$ and*

$$
\begin{aligned}
ab &= e, && \text{if } a \in S_e, && b \in S_f; \\
ab &= e, && \text{if } a \in S_e, && b \in N_f \setminus S_f; \\
ab &= e, && \text{if } a \in N_e \setminus S_e, && b \in S_f; \\
ab &\in E, && \text{if } a \in N_e \setminus S_e, && b \in N_f \setminus S_f.
\end{aligned}
$$

Proof. First assume that $\mathbb{A} \in L_3$ and recall that L_3 is defined by $(xy)z \approx xy$ and $x(yz) \approx x^2$. Let us consider the partition described in Remark 5.3. Let $a \in N_e$ and $b \in N_f$; then $(ab)(ab) \overset{1.}{=} ab$, which means that every product is idempotent (this already proves the last statement of (5.4)). In particular, if $a = a_1 a_2 \in S_e$, then $a^2 = a$. However, the construction of the partition implies that $a^2 = e$, thus we can conclude that $a = e$. (Note that this means that $S_e = \{e\}$.) Now we can write ab as $ab = (a_1 a_2)b \overset{1.}{=} a_1 a_2 = a = e$. If $b = b_1 b_2 \in S_f$, then $ab = a(b_1 b_2) \overset{2.}{=} a^2 = e$, hence (5.4) holds.

Conversely, suppose that $\{N_e \colon e \in E\}$ is a partition of A as in Notation 5.2, such that (5.4) is satisfied. Let us compute $(ab)c$ and $a(bc)$ for $a \in N_e$, $b \in N_f$, $c \in N_g$. From (5.4) we see that $ab =: e'$ and $bc =: f'$ belong to E, and the construction of the partition implies that $e' \in S_{e'}$ and $f' \in S_{f'}$ (cf. Notation 5.2). Now we can calculate using (5.4) as follows:

$$(ab)c = e'c = e' = ab, \qquad a(bc) = af' = e = a^2.$$

This proves that $\mathbb{A}$ satisfies $(xy)z \approx xy$ and $x(yz) \approx x^2$, hence $\mathbb{A} \in L_3$. $\qquad \square$

In the first half of the proof we observed that if the partition is chosen as in Remark 5.3, then each S_e is a singleton. However, if we choose an arbitrary partition as in Notation 5.2 and we define the multiplication according to (5.4), then the resulting groupoid will be in L_3. This is not a contradiction: given such an operation table (like the one in Appendix C), we can redefine the set S_e so that $S_e = \{e\}$ (i.e., we "move" all elements of S_e to $N_e \setminus S_e$ except for the element e) for each $e \in E$, without changing the multiplication table. Then the groupoid will still satisfy (5.4) for these new sets.

Theorem 5.7. *A groupoid $\mathbb{A}$ belongs to the variety L_4 if and only if there is a partition $\{N_e : e \in E\}$ as in Notation 5.2, such that*

(5.5) *for all $e, f \in E$ and $a \in N_e$, $b \in N_f$, we have $a^2 = e$ and*

$$
\begin{aligned}
ab &= eb = e, && \text{if } a \in S_e, && b \in S_f; \\
ab &= eb \in S_e, && \text{if } a \in S_e, && b \in N_f \setminus S_f; \\
ab &= eb = e, && \text{if } a \in N_e \setminus S_e, && b \in S_f; \\
ab &= eb \in S_e, && \text{if } a \in N_e \setminus S_e, && b \in N_f \setminus S_f.
\end{aligned}
$$

Proof. First assume that $\mathbb{A} \in L_4$ and recall that L_4 is defined by $(xy)z \approx xz$ and $x(yz) \approx x^2$. Let us consider the partition described in Remark 5.3. If $a \in N_e$ and $b \in N_f$, then $(ab)^2 = (ab)(ab) \overset{1.}{=} a(ab) \overset{2.}{=} a^2 = e$, which means that $ab \in S_e$. We also have $ab \overset{1.}{=} (aa)b = eb$; furthermore, if $b = b_1 b_2 \in S_f$, then $eb = e(b_1 b_2) \overset{2.}{=} e^2 = e$. This proves that (5.5) holds.

Conversely, suppose that $\{N_e : e \in E\}$ is a partition of A as in Notation 5.2, such that (5.5) is satisfied. Let us compute $(ab)c$ and $a(bc)$ for $a \in N_e$, $b \in N_f$, $c \in N_g$. From (5.5) we see that $ab \in S_e$ and $bc \in S_f$. Therefore, using (5.5) again, we obtain

$$(ab)c = ec = ac, \qquad a(bc) = e = a^2.$$

This shows that $\mathbb{A}$ satisfies $(xy)z \approx xz$ and $x(yz) \approx x^2$, hence $\mathbb{A} \in L_4$. $\qquad\square$

Theorem 5.8. *A groupoid $\mathbb{A}$ belongs to the variety L_5 if and only if there is a partition $\{N_e : e \in E\}$ as in Notation 5.2, such that*

(5.6)   *for all $e, f \in E$ and $a \in N_e$, $b \in N_f$, we have*

$$
\begin{aligned}
ab = af = a, \quad &\text{if } a \in S_e, && b \in S_f; \\
ab = af = a, \quad &\text{if } a \in S_e, && b \in N_f \setminus S_f; \\
ab = af \in S_e, \quad &\text{if } a \in N_e \setminus S_e, && b \in S_f; \\
ab = af \in S_e, \quad &\text{if } a \in N_e \setminus S_e, && b \in N_f \setminus S_f.
\end{aligned}
$$

Proof. First assume that $\mathbb{A} \in L_5$ and recall that L_5 is defined by $(xy)z \approx xy$ and $x(yz) \approx xy$. This time our partition will be different from that of Remark 5.3. We introduce an equivalence relation on A: let us write $a \sim b$ if the right multiplications by a and b coincide, i.e., if $ca = cb$ for all $c \in A$ (in other words, the columns of a and b in the multiplication table are the same). Since $ca \overset{2.}{=} c(aa)$ for every $c \in A$, we have $a \sim a^2$. As noted in Remark 5.3, a^2 is idempotent: $(aa)(aa) \overset{1.}{=} aa$. Thus every equivalence class with respect to $\sim$ contains at least one idempotent element. Therefore, we can choose a complete set of representatives E consisting of idempotent elements. Let N_e denote the equivalence class of $e \in E$, and let S_e be the set of factorizable elements in N_e, as before.

If $a \in N_e$ and $b \in N_f$, then $ab = af$, as $b \sim f$. Moreover, $a \sim e$ implies that for every $c \in A$, we have $ce = ca \overset{2.}{=} c(af)$, hence $e \sim af$, which means that $af \in S_e$. If $a = a_1 a_2 \in S_e$, then $ab = af = (a_1 a_2)f \overset{1.}{=} a_1 a_2 = a$. This proves that (5.6) holds.

Conversely, suppose that $\{N_e : e \in E\}$ is a partition of A as in Notation 5.2, such that (5.6) is satisfied. Let us compute $(ab)c$ and $a(bc)$ for $a \in N_e$, $b \in N_f$, $c \in N_g$. From (5.6) we see that $ab \in S_e$ and $bc \in S_f$. Therefore, using (5.6) again, we obtain

$$(ab)c = ab, \qquad a(bc) = af = ab.$$

This shows that $\mathbb{A}$ satisfies $(xy)z \approx x(yz) \approx xy$, hence $\mathbb{A} \in L_5$. $\qquad\square$

Theorem 5.9. *A groupoid $\mathbb{A}$ belongs to the variety L_6 if and only if there is a partition $\{N_e : e \in E\}$ as in Notation 5.2, such that*

(5.7)   *for all $e, f \in E$ and $a \in N_e$, $b \in N_f$, we have $ab = ef \in S_e$.*

Proof. First assume that $\mathbb{A} \in L_6$ and recall that L_6 is defined by $(xy)z \approx xz$ and $x(yz) \approx xy$. We use again the partition described in Remark 5.3. If $a \in N_e$ and $b \in N_f$, then $(ab)^2 = (ab)(ab) \overset{1.}{=} a(ab) \overset{2.}{=} a^2 = e$, therefore $ab \in S_e$. On the other hand, we can express ab as $ab \overset{1.}{=} (aa)b \overset{2.}{=} (aa)(bb) = ef$, which proves that (5.7) holds.

Conversely, suppose that $\{N_e : e \in E\}$ is a partition of A as in Notation 5.2, such that (5.7) is satisfied. Let us compute $(ab)c$ and $a(bc)$ for $a \in N_e$, $b \in N_f$, $c \in N_g$. From (5.7) we see that $ab \in S_e$ and $bc \in S_f$. Therefore, using (5.7) again, we obtain

$$(ab)c = eg = ac, \qquad a(bc) = ef = ab.$$

This shows that $\mathbb{A}$ satisfies $(xy)z \approx xz$ and $x(yz) \approx xy$, hence $\mathbb{A} \in L_6$. $\qquad\square$

APPENDIX A. DETAILED CASE ANALYSES FOR THE PROOFS OF SOME LEMMAS

TABLE 1. Summary of the proof of Lemma 3.2

t_1		
x	$xy \approx t_1(xy, z, u) \approx t_1(x, y, z)u \approx xu$	xu does not depend on u
y	$z \approx t_1(xy, z, u) \approx t_1(x, y, z)u \approx yu$	yu does not depend on y, u
z	$u \approx t_1(xy, z, u) \approx t_1(x, y, z)u \approx zu$	zu does not depend on z
zy	$uz \approx t_1(xy, z, u) \approx t_1(x, y, z)u$ $\approx (zy)u \approx t_1(z, y, u) \approx uy$	uy does not depend on y
zx	$u(xy) \approx t_1(xy, z, u) \approx t_1(x, y, z)u$ $\approx (zx)u \approx t_1(z, x, u) \approx uz$	uz does not depend on z
yx	$wv \approx t_1(v, w, xy) \approx (vw)(xy)$ $\approx t_1(xy, vw, u) \approx t_1(x, y, vw)u$ $\approx (yx)u \approx t_1(y, x, u) \approx xy$	xy does not depend on x, y

TABLE 2. The 36 cases of Lemma 3.3

		t_2					
		x^2	y^2	z^2	xy	yz	xz
t_1	x^2	L_1	$L_1 \cap L_1^d$	$L_1 \cap L_1^d$	L_2	$-$	$-$
	y^2	$L_1 \cap L_1^d$	$L_1 \cap L_1^d$	$L_1 \cap L_1^d$	$-$	$-$	$-$
	z^2	$L_1 \cap L_1^d(!)$	$L_1 \cap L_1^d$	L_1^d	$-$	L_3^d	L_4^d
	xy	L_3	$-$	$-$	L_5	$-$	$-$
	yz	$-$	$-$	L_2^d	$-$	L_5^d	L_6^d
	xz	L_4	$-$	$-$	L_6	$-$	$L_7 = L_7^d$

TABLE 3. Summary of the proof of Lemma 3.3

t_1	t_2		
x^2	yz	(a)	$zu \approx t_2(xy, z, u) \approx t_1(x, y, zu) \approx x^2$
y^2	yz	(a)	$zu \approx t_2(xy, z, u) \approx t_1(x, y, zu) \approx y^2$
xy	y^2	(a)	$xy \approx t_1(x, y, zu) \approx t_2(xy, z, u) \approx z^2$
xy	z^2	(a)	$xy \approx t_1(x, y, zu) \approx t_2(xy, z, u) \approx u^2$
xy	yz	(a)	$xy \approx t_1(x, y, zu) \approx t_2(xy, z, u) \approx zu$
yz	x^2	(b)	$zu \approx t_1(y, z, u) \approx (yz)u \approx t_1(x, yz, u) \approx t_2(x, y, z)u \approx x^2u$
yz	y^2	(b)	$zu \approx t_1(y, z, u) \approx (yz)u \approx t_1(x, yz, u) \approx t_2(x, y, z)u \approx y^2u$
yz	xy	(b)	$zu \approx t_1(y, z, u) \approx (yz)u \approx t_1(x, yz, u) \approx t_2(x, y, z)u \approx (xy)u$
xz	y^2	(b)	$xu \approx t_1(x, yz, u) \approx t_2(x, y, z)u \approx y^2u$
xz	z^2	(b)	$xu \approx t_1(x, yz, u) \approx t_2(x, y, z)u \approx z^2u$
xz	yz	(b)	$xu \approx t_1(x, yz, u) \approx t_2(x, y, z)u \approx (yz)u$
y^2	xy	(c)	$xy \approx t_2(x, y, z) \approx x(yz) \approx t_2(x, yz, u) \approx xt_1(y, z, u) \approx xz^2$
z^2	xy	(c)	$xy \approx t_2(x, y, z) \approx x(yz) \approx t_2(x, yz, u) \approx xt_1(y, z, u) \approx xu^2$
x^2	xz	(c)	$xu \approx t_2(x, yz, u) \approx xt_1(y, z, u) \approx xy^2$
y^2	xz	(c)	$xu \approx t_2(x, yz, u) \approx xt_1(y, z, u) \approx xz^2$
xy	xz	(c)	$xu \approx t_2(x, yz, u) \approx xt_1(y, z, u) \approx x(yz)$
x^2	y^2	(a)	$x^2 \approx t_1(x, y, zu) \approx t_2(xy, z, u) \approx z^2$
x^2	z^2	(a)	$x^2 \approx t_1(x, y, zu) \approx t_2(xy, z, u) \approx u^2$
y^2	y^2	(a)	$y^2 \approx t_1(x, y, zu) \approx t_2(xy, z, u) \approx z^2$
y^2	z^2	(a)	$y^2 \approx t_1(x, y, zu) \approx t_2(xy, z, u) \approx u^2$
y^2	x^2	(a)	$z^2 \approx t_1(y, z, vu) \approx t_2(yz, v, u) \approx (yz)^2$
		(b)	$\approx t_1(x, yz, u) \approx t_2(x, y, z)u \approx x^2u$
z^2	y^2	(a)	$y^2 \approx t_2(xu, y, z) \approx t_1(x, u, yz) \approx (yz)^2$
		(c)	$\approx t_2(x, yz, u) \approx xt_1(y, z, u) \approx xu^2$

TABLE 4. Summary of the proof of Lemma 3.4

$L_1:$ $s_1 \approx xy,\ s_2 \approx z \implies s_1s_2 \approx (xy)z \overset{1.}{\approx} x^2$

$\qquad s_1 \approx x,\quad s_2 \approx yz \implies s_1s_2 \approx x(yz) \overset{2.}{\approx} x^2$

$\qquad s_1 \approx xy,\ s_2 \approx zu \implies s_1s_2 \approx (xy)(zu) \overset{1.}{\approx} x^2$

$L_2:$ $s_1 \approx xy,\ s_2 \approx z \implies s_1s_2 \approx (xy)z \overset{1.}{\approx} x^2$

$\qquad s_1 \approx x,\quad s_2 \approx yz \implies s_1s_2 \approx x(yz) \overset{2.}{\approx} xy$

$\qquad s_1 \approx xy,\ s_2 \approx zu \implies s_1s_2 \approx (xy)(zu) \overset{1.}{\approx} x^2$

$L_3:$ $s_1 \approx xy,\ s_2 \approx z \implies s_1s_2 \approx (xy)z \overset{1.}{\approx} xy$

$\qquad s_1 \approx x,\quad s_2 \approx yz \implies s_1s_2 \approx x(yz) \overset{2.}{\approx} x^2$

$\qquad s_1 \approx xy,\ s_2 \approx zu \implies s_1s_2 \approx (xy)(zu) \overset{1.}{\approx} xy$

$L_4:$ $s_1 \approx xy,\ s_2 \approx z \implies s_1s_2 \approx (xy)z \overset{1.}{\approx} xz$

$\qquad s_1 \approx x,\quad s_2 \approx yz \implies s_1s_2 \approx x(yz) \overset{2.}{\approx} x^2$

$\qquad s_1 \approx xy,\ s_2 \approx zu \implies s_1s_2 \approx (xy)(zu) \overset{1.}{\approx} x(zu) \overset{2.}{\approx} x^2$

$L_5:$ $s_1 \approx xy,\ s_2 \approx z \implies s_1s_2 \approx (xy)z \overset{1.}{\approx} xy$

$\qquad s_1 \approx x,\quad s_2 \approx yz \implies s_1s_2 \approx x(yz) \overset{2.}{\approx} xy$

$\qquad s_1 \approx xy,\ s_2 \approx zu \implies s_1s_2 \approx (xy)(zu) \overset{1.}{\approx} xy$

$L_6:$ $s_1 \approx xy,\ s_2 \approx z \implies s_1s_2 \approx (xy)z \overset{1.}{\approx} xz$

$\qquad s_1 \approx x,\quad s_2 \approx yz \implies s_1s_2 \approx x(yz) \overset{2.}{\approx} xy$

$\qquad s_1 \approx xy,\ s_2 \approx zu \implies s_1s_2 \approx (xy)(zu) \overset{1.}{\approx} x(zu) \overset{2.}{\approx} xz$

$L_7:$ $s_1 \approx xy,\ s_2 \approx z \implies s_1s_2 \approx (xy)z \overset{1.}{\approx} xz$

$\qquad s_1 \approx x,\quad s_2 \approx yz \implies s_1s_2 \approx x(yz) \overset{2.}{\approx} xz$

$\qquad s_1 \approx xy,\ s_2 \approx zu \implies s_1s_2 \approx (xy)(zu) \overset{1.}{\approx} x(zu) \overset{2.}{\approx} xu$

TABLE 5. Summary of the proof of Lemma 4.1

$L_1 \cap I$	$xy \overset{I}{\approx} (xx)y \overset{1.}{\approx} x^2$	xy does not depend on y
$L_2 \cap I$	$xy \overset{I}{\approx} (xx)y \overset{1.}{\approx} x^2$	xy does not depend on y
$L_2 \cap D$	$xy \overset{2.}{\approx} x(yy) \overset{D}{\approx} x(xx)$	xy does not depend on y
$L_2 \cap C$	$xy \overset{2.}{\approx} x(yz) \overset{C}{\approx} (yz)x \overset{1.}{\approx} y^2$	xy does not depend on x
$L_3 \cap I$	$xy \overset{I}{\approx} (xx)y \overset{1.}{\approx} x^2$	xy does not depend on y
$L_3 \cap D$	$xy \overset{1.}{\approx} (xy)(zu) \overset{2.}{\approx} (xy)^2 \overset{D}{\approx} x^2$	xy does not depend on y
$L_3 \cap C$	$xy \overset{1.}{\approx} (xy)z \overset{C}{\approx} z(xy) \overset{2.}{\approx} z^2$	xy does not depend on y
$L_4 \cap I$	$xy \overset{I}{\approx} x(yy) \overset{2.}{\approx} x^2$	xy does not depend on y
$L_4 \cap D$	$xy \overset{1.}{\approx} (xx)y \overset{D}{\approx} (yy)y$	xy does not depend on x
$L_4 \cap C$	$xy \overset{1.}{\approx} (xz)y \overset{C}{\approx} y(xz) \overset{2.}{\approx} y^2$	xy does not depend on x
$L_5 \cap I$	$xy \overset{I}{\approx} (xx)y \overset{1.}{\approx} x^2$	xy does not depend on y
$L_5 \cap D$	$xy \overset{2.}{\approx} x(yy) \overset{D}{\approx} x(xx)$	xy does not depend on y
$L_5 \cap C$	$xy \overset{1.}{\approx} (xy)z \overset{C}{\approx} z(xy) \overset{2.}{\approx} zx$	xy does not depend on y
$L_6 \cap I$	$xy \overset{I}{\approx} (xy)(xy) \overset{1.}{\approx} x(xy) \overset{2.}{\approx} x^2$	xy does not depend on y
$L_6 \cap D$	$xy \overset{2.}{\approx} x(yy) \overset{D}{\approx} x(xx)$	xy does not depend on y
$L_6 \cap C$	$xy \overset{2.}{\approx} x(yz) \overset{C}{\approx} (zy)x \overset{1.}{\approx} zx$	xy does not depend on y
$L_7 \cap D$	$xy \overset{2.}{\approx} x(yy) \overset{D}{\approx} x(xx)$	xy does not depend on y
$L_7 \cap C$	$xy \overset{1.}{\approx} (xz)y \overset{C}{\approx} y(xz) \overset{2.}{\approx} yz$	xy does not depend on x

Appendix B. Two-generated free algebras in lazy groupoid varieties

L_1	x	y	x^2	y^2	xy	yx
x	x^2	xy	x^2	x^2	x^2	x^2
y	yx	y^2	y^2	y^2	y^2	y^2
x^2	x^2	x^2	x^2	x^2	x^2	x^2
y^2	y^2	y^2	y^2	y^2	y^2	y^2
xy	x^2	x^2	x^2	x^2	x^2	x^2
yx	y^2	y^2	y^2	y^2	y^2	y^2

L_2	x	y	x^2	y^2	xy	yx
x	x^2	xy	x^2	xy	x^2	xy
y	yx	y^2	yx	y^2	yx	y^2
x^2	x^2	x^2	x^2	x^2	x^2	x^2
y^2	y^2	y^2	y^2	y^2	y^2	y^2
xy	x^2	x^2	x^2	x^2	x^2	x^2
yx	y^2	y^2	y^2	y^2	y^2	y^2

$L_1 \cap D$	x	y	x^2	xy	yx
x	x^2	xy	x^2	x^2	x^2
y	yx	x^2	x^2	x^2	x^2
x^2	x^2	x^2	x^2	x^2	x^2
xy	x^2	x^2	x^2	x^2	x^2
yx	x^2	x^2	x^2	x^2	x^2

L_3	x	y	x^2	y^2	xy	yx
x	x^2	xy	x^2	x^2	x^2	x^2
y	yx	y^2	y^2	y^2	y^2	y^2
x^2	x^2	x^2	x^2	x^2	x^2	x^2
y^2	y^2	y^2	y^2	y^2	y^2	y^2
xy	xy	xy	xy	xy	xy	xy
yx	yx	yx	yx	yx	yx	yx

$L_1 \cap C$	x	y	x^2	xy
x	x^2	xy	x^2	x^2
y	xy	x^2	x^2	x^2
x^2	x^2	x^2	x^2	x^2
xy	x^2	x^2	x^2	x^2

L_4	x	y	x^2	y^2	xy	yx
x	x^2	xy	x^2	x^2	x^2	x^2
y	yx	y^2	y^2	y^2	y^2	y^2
x^2	x^2	xy	x^2	x^2	x^2	x^2
y^2	yx	y^2	y^2	y^2	y^2	y^2
xy	x^2	xy	x^2	x^2	x^2	x^2
yx	yx	y^2	y^2	y^2	y^2	y^2

L_5	x	y	x^2	y^2	xy	yx
x	x^2	xy	x^2	xy	x^2	xy
y	yx	y^2	yx	y^2	yx	y^2
x^2	x^2	x^2	x^2	x^2	x^2	x^2
y^2	y^2	y^2	y^2	y^2	y^2	y^2
xy	xy	xy	xy	xy	xy	xy
yx	yx	yx	yx	yx	yx	yx

U	x	y	x^2	y^2
x	x^2	x^2	x^2	x^2
y	y^2	y^2	y^2	y^2
x^2	x^2	x^2	x^2	x^2
y^2	y^2	y^2	y^2	y^2

L_6	x	y	x^2	y^2	xy	yx
x	x^2	xy	x^2	xy	x^2	xy
y	yx	y^2	yx	y^2	yx	y^2
x^2	x^2	xy	x^2	xy	x^2	xy
y^2	yx	y^2	yx	y^2	yx	y^2
xy	x^2	xy	x^2	xy	x^2	xy
yx	yx	y^2	yx	y^2	yx	y^2

$\widetilde{U}$	x	y	x^2	y^2
x	x^2	x^2	x^2	x^2
y	y^2	y^2	y^2	y^2
x^2	x	x	x	x
y^2	y	y	y	y

L_7	x	y	x^2	y^2	xy	yx
x	x^2	xy	x^2	xy	xy	x^2
y	yx	y^2	yx	y^2	y^2	yx
x^2	x^2	xy	x^2	xy	xy	x^2
y^2	yx	y^2	yx	y^2	y^2	yx
xy	x^2	xy	x^2	xy	xy	x^2
yx	yx	y^2	yx	y^2	y^2	yx

Z	x	y	xy
x	xy	xy	xy
y	xy	xy	xy
xy	xy	xy	xy

RB	x	y	xy	yx
x	x	xy	xy	x
y	yx	y	y	yx
xy	x	xy	xy	x
yx	yx	y	y	yx

LZ	x	y
x	x	x
y	y	y

Appendix C. Operation tables of lazy groupoids

L_1	e	s_1	s_2	$\dots$	a_1	a_2	$\dots$	f	t_1	t_2	$\dots$	b_1	b_2	$\dots$	$\dots$
$s_0 = e$	e	e	e	$\dots$	e	e	$\dots$	e	e	e	$\dots$	e	e	$\dots$	$\dots$
$s_1 \in S_e$	e	e	e	$\dots$	e	e	$\dots$	e	e	e	$\dots$	e	e	$\dots$	$\dots$
$s_2 \in S_e$	e	e	e	$\dots$	e	e	$\dots$	e	e	e	$\dots$	e	e	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$a_1 \notin S_e$	e	e	e	$\dots$	e	$\in S_e$	$\dots$	e	e	e	$\dots$	$\in S_e$	$\in S_e$	$\dots$	$\dots$
$a_2 \notin S_e$	e	e	e	$\dots$	$\in S_e$	e	$\dots$	e	e	e	$\dots$	$\in S_e$	$\in S_e$	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$t_0 = f$	f	f	f	$\dots$	f	f	$\dots$	f	f	f	$\dots$	f	f	$\dots$	$\dots$
$t_1 \in S_f$	f	f	f	$\dots$	f	f	$\dots$	f	f	f	$\dots$	f	f	$\dots$	$\dots$
$t_2 \in S_f$	f	f	f	$\dots$	f	f	$\dots$	f	f	f	$\dots$	f	f	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$b_1 \notin S_f$	f	f	f	$\dots$	$\in S_f$	$\in S_f$	$\dots$	f	f	f	$\dots$	f	$\in S_f$	$\dots$	$\dots$
$b_2 \notin S_f$	f	f	f	$\dots$	$\in S_f$	$\in S_f$	$\dots$	f	f	f	$\dots$	$\in S_f$	f	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		

L_2	e	s_1	s_2	$\dots$	a_1	a_2	$\dots$	f	t_1	t_2	$\dots$	b_1	b_2	$\dots$	$\dots$
$s_0 = e$	e	e	e	$\dots$	e	e	$\dots$	e	e	e	$\dots$	e	e	$\dots$	$\dots$
$s_1 \in S_e$	e	e	e	$\dots$	e	e	$\dots$	e	e	e	$\dots$	e	e	$\dots$	$\dots$
$s_2 \in S_e$	e	e	e	$\dots$	e	e	$\dots$	e	e	e	$\dots$	e	e	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$a_1 \notin S_e$	e	e	e	$\dots$	e	e	$\dots$	s_i	s_i	s_i	$\dots$	s_i	s_i	$\dots$	$\dots$
$a_2 \notin S_e$	e	e	e	$\dots$	e	e	$\dots$	s_j	s_j	s_j	$\dots$	s_j	s_j	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$t_0 = f$	f	f	f	$\dots$	f	f	$\dots$	f	f	f	$\dots$	f	f	$\dots$	$\dots$
$t_1 \in S_f$	f	f	f	$\dots$	f	f	$\dots$	f	f	f	$\dots$	f	f	$\dots$	$\dots$
$t_2 \in S_f$	f	f	f	$\dots$	f	f	$\dots$	f	f	f	$\dots$	f	f	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$b_1 \notin S_f$	t_k	t_k	t_k	$\dots$	t_k	t_k	$\dots$	f	f	f	$\dots$	f	f	$\dots$	$\dots$
$b_2 \notin S_f$	t_l	t_l	t_l	$\dots$	t_l	t_l	$\dots$	f	f	f	$\dots$	f	f	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		

L_3	e	s_1	s_2	$\ldots$	a_1	a_2	$\ldots$	f	t_1	t_2	$\ldots$	b_1	b_2	$\ldots$	$\ldots$
$s_0 = e$	e	e	e	$\ldots$	e	e	$\ldots$	e	e	e	$\ldots$	e	e	$\ldots$	$\ldots$
$s_1 \in S_e$	e	e	e	$\ldots$	e	e	$\ldots$	e	e	e	$\ldots$	e	e	$\ldots$	$\ldots$
$s_2 \in S_e$	e	e	e	$\ldots$	e	e	$\ldots$	e	e	e	$\ldots$	e	e	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$a_1 \notin S_e$	e	e	e	$\ldots$	e	$\in E$	$\ldots$	e	e	e	$\ldots$	$\in E$	$\in E$	$\ldots$	$\ldots$
$a_2 \notin S_e$	e	e	e	$\ldots$	$\in E$	e	$\ldots$	e	e	e	$\ldots$	$\in E$	$\in E$	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$t_0 = f$	f	f	f	$\ldots$	f	f	$\ldots$	f	f	f	$\ldots$	f	f	$\ldots$	$\ldots$
$t_1 \in S_f$	f	f	f	$\ldots$	f	f	$\ldots$	f	f	f	$\ldots$	f	f	$\ldots$	$\ldots$
$t_2 \in S_f$	f	f	f	$\ldots$	f	f	$\ldots$	f	f	f	$\ldots$	f	f	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$b_1 \notin S_f$	f	f	f	$\ldots$	$\in E$	$\in E$	$\ldots$	f	f	f	$\ldots$	f	$\in E$	$\ldots$	$\ldots$
$b_2 \notin S_f$	f	f	f	$\ldots$	$\in E$	$\in E$	$\ldots$	f	f	f	$\ldots$	$\in E$	f	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		

L_4	e	s_1	s_2	$\ldots$	a_1	a_2	$\ldots$	f	t_1	t_2	$\ldots$	b_1	b_2	$\ldots$	$\ldots$
$s_0 = e$	e	e	e	$\ldots$	e	e	$\ldots$	e	e	e	$\ldots$	s_i	s_j	$\ldots$	$\ldots$
$s_1 \in S_e$	e	e	e	$\ldots$	e	e	$\ldots$	e	e	e	$\ldots$	s_i	s_j	$\ldots$	$\ldots$
$s_2 \in S_e$	e	e	e	$\ldots$	e	e	$\ldots$	e	e	e	$\ldots$	s_i	s_j	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$a_1 \notin S_e$	e	e	e	$\ldots$	e	e	$\ldots$	e	e	e	$\ldots$	s_i	s_j	$\ldots$	$\ldots$
$a_2 \notin S_e$	e	e	e	$\ldots$	e	e	$\ldots$	e	e	e	$\ldots$	s_i	s_j	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$t_0 = f$	f	f	f	$\ldots$	t_k	t_l	$\ldots$	f	f	f	$\ldots$	f	f	$\ldots$	$\ldots$
$t_1 \in S_f$	f	f	f	$\ldots$	t_k	t_l	$\ldots$	f	f	f	$\ldots$	f	f	$\ldots$	$\ldots$
$t_2 \in S_f$	f	f	f	$\ldots$	t_k	t_l	$\ldots$	f	f	f	$\ldots$	f	f	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$b_1 \notin S_f$	f	f	f	$\ldots$	t_k	t_l	$\ldots$	f	f	f	$\ldots$	f	f	$\ldots$	$\ldots$
$b_2 \notin S_f$	f	f	f	$\ldots$	t_k	t_l	$\ldots$	f	f	f	$\ldots$	f	f	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		

L_5	e	s_1	s_2	$\ldots$	a_1	a_2	$\ldots$	f	t_1	t_2	$\ldots$	b_1	b_2	$\ldots$	$\ldots$
$s_0 = e$	e	e	e	$\ldots$	e	e	$\ldots$	e	e	e	$\ldots$	e	e	$\ldots$	$\ldots$
$s_1 \in S_e$	s_1	s_1	s_1	$\ldots$	s_1	s_1	$\ldots$	s_1	s_1	s_1	$\ldots$	s_1	s_1	$\ldots$	$\ldots$
$s_2 \in S_e$	s_2	s_2	s_2	$\ldots$	s_2	s_2	$\ldots$	s_2	s_2	s_2	$\ldots$	s_2	s_2	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$a_1 \notin S_e$	s_{i_1}	s_{i_1}	s_{i_1}	$\ldots$	s_{i_1}	s_{i_1}	$\ldots$	s_{j_1}	s_{j_1}	s_{j_1}	$\ldots$	s_{j_1}	s_{j_1}	$\ldots$	$\ldots$
$a_2 \notin S_e$	s_{i_2}	s_{i_2}	s_{i_2}	$\ldots$	s_{i_2}	s_{i_2}	$\ldots$	s_{j_2}	s_{j_2}	s_{j_2}	$\ldots$	s_{j_2}	s_{j_2}	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$t_0 = f$	f	f	f	$\ldots$	f	f	$\ldots$	f	f	f	$\ldots$	f	f	$\ldots$	$\ldots$
$t_1 \in S_f$	t_1	t_1	t_1	$\ldots$	t_1	t_1	$\ldots$	t_1	t_1	t_1	$\ldots$	t_1	t_1	$\ldots$	$\ldots$
$t_2 \in S_f$	t_2	t_2	t_2	$\ldots$	t_2	t_2	$\ldots$	t_2	t_2	t_2	$\ldots$	t_2	t_2	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$b_1 \notin S_f$	t_{k_1}	t_{k_1}	t_{k_1}	$\ldots$	t_{k_1}	t_{k_1}	$\ldots$	t_{l_1}	t_{l_1}	t_{l_1}	$\ldots$	t_{l_1}	t_{l_1}	$\ldots$	$\ldots$
$b_2 \notin S_f$	t_{k_2}	t_{k_2}	t_{k_2}	$\ldots$	t_{k_2}	t_{k_2}	$\ldots$	t_{l_2}	t_{l_2}	t_{l_2}	$\ldots$	t_{l_2}	t_{l_2}	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		

L_6	e	s_1	s_2	$\ldots$	a_1	a_2	$\ldots$	f	t_1	t_2	$\ldots$	b_1	b_2	$\ldots$	$\ldots$
$s_0 = e$	s_i	s_i	s_i	$\ldots$	s_i	s_i	$\ldots$	s_j	s_j	s_j	$\ldots$	s_j	s_j	$\ldots$	$\ldots$
$s_1 \in S_e$	s_i	s_i	s_i	$\ldots$	s_i	s_i	$\ldots$	s_j	s_j	s_j	$\ldots$	s_j	s_j	$\ldots$	$\ldots$
$s_2 \in S_e$	s_i	s_i	s_i	$\ldots$	s_i	s_i	$\ldots$	s_j	s_j	s_j	$\ldots$	s_j	s_j	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$a_1 \notin S_e$	s_i	s_i	s_i	$\ldots$	s_i	s_i	$\ldots$	s_j	s_j	s_j	$\ldots$	s_j	s_j	$\ldots$	$\ldots$
$a_2 \notin S_e$	s_i	s_i	s_i	$\ldots$	s_i	s_i	$\ldots$	s_j	s_j	s_j	$\ldots$	s_j	s_j	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$t_0 = f$	t_k	t_k	t_k	$\ldots$	t_k	t_k	$\ldots$	t_l	t_l	t_l	$\ldots$	t_l	t_l	$\ldots$	$\ldots$
$t_1 \in S_f$	t_k	t_k	t_k	$\ldots$	t_k	t_k	$\ldots$	t_l	t_l	t_l	$\ldots$	t_l	t_l	$\ldots$	$\ldots$
$t_2 \in S_f$	t_k	t_k	t_k	$\ldots$	t_k	t_k	$\ldots$	t_l	t_l	t_l	$\ldots$	t_l	t_l	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$b_1 \notin S_f$	t_k	t_k	t_k	$\ldots$	t_k	t_k	$\ldots$	t_l	t_l	t_l	$\ldots$	t_l	t_l	$\ldots$	$\ldots$
$b_2 \notin S_f$	t_k	t_k	t_k	$\ldots$	t_k	t_k	$\ldots$	t_l	t_l	t_l	$\ldots$	t_l	t_l	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		

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