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Coherent and incoherent superposition of transition matrix elements of the squeezing operator

Sándor Varró[*]

Wigner Research Centre for Physics, Eötvös Loránd Research Network, Budapest, Hungary
ELI-ALPS Attosecond Light Pulse Source, ELI-HU Non-Profit Ltd., Szeged, Hungary
* Author to whom any correspondence should be addressed.

E-mail: varro.sandor@wigner.hu

Abstract

We discuss the general matrix elements of the squeezing operator between number eigenstates of a harmonic oscillator (which may also represent a quantized mode of the electromagnetic radiation). These matrix elements have first been used by Popov and Perelomov (1969 *Zh. Eksp. Teor. Fiz.* **56** 1375–90) long ago, in their thorough analysis of the parametric excitation of harmonic oscillators. They expressed the matrix elements in terms of transcendental functions, the associated Legendre functions. In the present paper we will show that these matrix elements can also be derived in a different form, expressed by the classical Gegenbauer polynomials. This new expression makes it possible to determine coherent and incoherent superpositions of these matrix elements in closed analytic forms. As an application, we describe multiphoton transitions in the system 'charged particle + electromagnetic radiation', induced by a (strong) coherent field or by a black-body radiation component (with a Planck–Bose photon number distribution). The exact results are compared with the semi-classical ones. We will show that in case of interaction with a thermal field, the semi-classical result (with a Gaussian stochastic field amplitude) yields an acceptable approximation only in the Rayleigh–Jeans limit, however, in the Wien limit it completely fails.

1. Introduction

The 'non-classical states of light', in particular, 'squeezed light' have been a subject of intensive theoretical and experimental research for many decades (Stoler 1970, 1971, Yuen 1976, Walls 1979, Loudon and Knight 1987, Teich and Saleh 1989, Dodonov 2002, Dodonov and Man'ko 2003, Andersen *et al* 2016), and have become standard subjects in any textbooks on quantum optics (Scully and Zubairy 1997, Loudon 2000, Schleich 2001). The phenomenon 'squeezing of the amplitudes' (of a mechanical oscillator, or of a component of electromagnetic radiation) often appears in parametric processes (Husimi 1953, Louisell *et al* 1961, Mollow and Glauber 1967, Mollow 1967, Popov and Perelomov 1969, Malkin *et al* 1970, Milburn and Walls 1983, Kiss *et al* 1994, Dodonov 2003, Fedorov *et al* 2008, Straupe *et al* 2011). Concerning the oscillatory behaviour of the photon number distribution in highly-squeezed coherent states, and Wigner phase-space considerations, see Wheeler (1985), Schleich and Wheeler (1987), Schleich *et al* (1988), Schleich *et al* (1989), Schleich (2001). The squeezing operator is the generator of the Bogoljubov transformation, which has played an important role in the theory of superfluidity (Bogoljubov 1947, Dodonov 2007). The quantum dynamics of charged particles in time-varying external (electro)magnetic fields also shows interesting phenomena, related to non-classical states of linear oscillators (Malkin *et al* 1970, Varró 1984, Varró and Ehlotzky 1985, 1987, Dodonov 2018, Dodonov and Horovits 2021). In the context of Landau states, concerning coherent states and non-classicality, see the detailed discussions by Fakhri, Mojaveri and Gomshi Nobary (2010), Dehghani *et al* (2012), Dehghani and Mojaveri (2013) and Mojaveri and Dehghani (2015). Recently the interest in entangled states is more pronounced then in nonclassical pure states. Anyway, undoubtely, the 'classic non-classical states' are the squeezed coherent

states, which have been the ones most extensively studied in the last decades. It is lesser known by the quantum optics community that such states of the radiation field naturally appear as exact stationary solutions in the simplest (and most fundamental) system of quantum electrodynamics, namely in the interaction of a free electron with a quantized radiation field (see Berson 1969, Fedorov and Kazakov 1973, Bergou and Varró 1981a, 1981b, Gazazyan and Sherman 1989, Varró 2008, 2010, 2021a). The exact stationary states of the system of a free electron and a quantized mode of the radiation field (whose interaction is represented by the minimal coupling) are eigenstates of the total energy and momentum of the system. In general, the photon part of these states are squeezed displaced number states (or, in other words, n-photon-added squeezed coherent states). For instance, in the nonrelativistic limit and in dipole approximation, the exact solutions are simple product states $|\mathbf{p}\rangle$ SD $|n\rangle$, where $|\mathbf{p}\rangle$ is a momentum eigenstate of the electron, and $|n\rangle$ is a Fock state of the quantized mode. The parameters of the squeezing operator S and of the displacement operator D depend on the (continuous) momentum parameter $\mathbf{p}$ (see e.g. Bergou and Varró 1981a). These states incorporate the interaction of the electron and a quantized mode exactly, and, moreover, they form a complete orthonormal set in the product space of the system, so they offer a natural basis for various non-perturbative calculations (for the treatment of processes like induced Bremsstrahlung, strong-field ionization or high-harmonic generation). Accordingly, matrix elements of the type $S_{mn} = \langle m|S|n\rangle$ automatically appear in the analysis. This is a simple indication for that, the 'squeezing transitions' also play an important role in the theory of high-order interactions of a free electron in a quantized radiation field. In the theory of interactions of strong laser radiation with electrons the exact solutions of the Schrödinger or Dirac equations are of outstanding importance in treating the multiphoton processes non-perturbatively (though the laser field is mostly taken as an external field). If there is a need to treat the influence of the photon statistics on the transition probabilities, one has to go beyond the external field approximation, of course, and use the quantum description of the strong radiation field. This approach is certainly not of academic interest, since in most of the recents experiments on multiphoton processes the inducing radiation stems from parametric amplifiers (see e.g. Kühn *et al* 2017).

Our motivation for the present study has been to calculate the coherent and incoherent superposition of multiphoton transition amplitudes, which we have derived in the frame of non-relativistic quantum electrodynamics, for considering high-order harmonic generation (Varró 2021a). In the non-perturbative treatment of such processes one necessarily encounters matrix elements between photon number eigenstates of the displacement and squeezing operators (which are standard objects in quantum optics). It is very remarkable that, though the photon number distributions of squeezed coherent states (Hermite polynomials), or displaced number states (Laguerre polynomials) have long been known, the matrix elements $S_{mn} = \langle m|S|n\rangle$ of the squeezing operator alone have not been published in a classical polynomial form. More precisely, S_{mn} has been expressed by Bargmann (1947) in his paper on the irreducible representation of the Lorentz group, in terms of hypergeometric functions (see also Sannikov 1965). In the context of parametric processes, Popov and Perelomov (1969) and Malkin *et al* (1970) derived a form of S_{mn}, expressed in terms of associated Legendre functions. Later S_{mn} has been related also to Jacobi polynomials (see, in particular, the very thorough recent studies of Wünsche 2017a, 2017b), however a compact, classical polynomial expression has not been derived (see also Tanabe 1973, Rashid 1975, Satyanarayana 1985 and Mendaš and Popović 1995). Recently we have derived a compact classical polynomial expression, in term of Gegenbauer polynomials (Varró 2021b), and the larger part of the present paper will be devoted to the derivation and the main properties of the matrix element S_{mn} of the squeezing operator.

In section 2 we summarize the main steps of the derivation of the Gegenbauer polynomial expression of the matrix elements of the squeezing operator between photon number eigenstates. In order to have the paper self-contained, we have included an appendix A which contains all the necessary details of this derivation. We shall also present some numerical illustrations to display the main features of the photon number distribution in a squeezed number state. In section 3 we apply the new formula for the matrix elements for calculating the coherent and incoheren superpositions of multiphoton squeezing transitions.

2. The matrix elements of the squeezing operator between photon number eigenstates

In the present section we summarize the basic steps of the calculation of the matrix elements between the number eigenstates of the squeezing operator, $S_{m,n} = \langle m|S(\xi)|n\rangle$, where

$$S(\xi) = \exp[\tfrac{1}{2}\xi(a^+)^2 - \tfrac{1}{2}\xi^*(a)^2], \qquad S(\xi) = \exp(\xi K_+ - \xi^* K_-), \qquad \xi = |\xi|e^{i\phi} \qquad (1)$$

with $K_+ = \tfrac{1}{2}(a^+)^2$ and $K_- = \tfrac{1}{2}(a)^2$. Here a and a^+ are the photon absorption and emission operators, respectively. (In a more general context, they are the quantum amplitudes associated to the decrease and

increase by 1 of the excitation index of the system, like a mechanical linear oscillator.) The a and a^+ satisfy the commutation relation $[a, a^+] = aa^+ - a^+ a = I$, where I is the identity operator of the complex Hilbert space $\mathscr{H}$, which models the system. The effects of a and a^+ on the Fock states $|n\rangle$ (which are the photon number eigenstates, satisfying the eigenvalue equation $a^+ a |n\rangle = n |n\rangle$) are expressed by equations $a |n\rangle = \sqrt{n} |n-1\rangle$ and $a^+ |n\rangle = \sqrt{n+1} |n+1\rangle$. For the ground state $a |0\rangle = \vec{0}$, where $\vec{0}$ is the zero vector of H. The Fock states $|n\rangle$ form a complete orthonormal set ($\langle m | n \rangle = \delta_{m,n}$) in H. By introducing $K_0 = \frac{1}{4}(aa^+ + a^+ a)$, we have the closed set of generators of a special Lie algebra of the $SU(1,1)$ group, having the commutation relations $[K_0, K_\pm] = \pm K_\pm$, and $[K_-, K_+] = 2K_0$. In appendix A we summarize some basic properties of this group, on the basis of which we derive the normally ordered form of $S(\xi)$. The details of the derivations are presented in appendix A, in the present section we just outline the basic steps, leading to the final result.

According to equation (A.5), the normal form of $S(\xi)$ becomes

$$S(\xi) = \exp[\tfrac{1}{2}\zeta(a^+)^2] \exp[-\eta\tfrac{1}{2}(a^+ a + \tfrac{1}{2})] \exp[-\tfrac{1}{2}\zeta^*(a)^2], \qquad \eta = 2\log\cosh|\xi|, \qquad \zeta = e^{i\phi}\tanh|\xi|. \tag{2}$$

By consecutively applying the relation $a^2 |n\rangle = \sqrt{n(n-1)} |n-2\rangle$, we have the finite sum

$$\exp(-\tfrac{1}{2}\zeta^* a^2)|n\rangle = \sum_{k=0}^{[n/2]} \frac{(-\zeta^*/2)^k}{k!} \sqrt{n(n-1)(n-2)\ldots(n-2k+1)} \, |n-2k\rangle, \tag{3}$$

as the right-hand factor in the scalar product $S_{m,n} = \langle m|S(\xi)|n\rangle$. Similarly, the left-hand factor becomes also a finite sum, running up to $[m/2]$. The operator in the middle of the normal form in equation (2) results in a multiplication for any $|k\rangle$ by a number, namely, $\exp[-\eta\tfrac{1}{2}(a^+ a + \tfrac{1}{2})]|k\rangle = \exp[-\eta\tfrac{1}{2}(k+\tfrac{1}{2})]|k\rangle$. Owing to the orthogonality of the Fock states, the scalar product, as a double sum, reduces to a single sum, as is shown in equations (A.7) and (A.8) for $m \geqslant n$ and $m \leqslant n$, respectively. Only those matrix elements are non-vanishing in which m and n have the same parity, so $\tfrac{1}{2}(m-n)$ must be an integer. The finite sums in equations (A.7) and (A.8) are expressed by hypergeometric functions in equations (A.9) and (A.10). Here we display the result in the case $m \geqslant n$,

$$\langle m|S(\xi)|n\rangle = \exp[-\tfrac{1}{2}(n+\tfrac{1}{2})\eta](\zeta/2)^{(m-n)/2}\sqrt{\frac{m!}{n!}}\frac{1}{\Gamma(\alpha+1)}F\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{m-n}{2}+1; z\right) \quad (m \geqslant n), \tag{4}$$

where $z = -\sinh^2|\xi|$ and $\exp(\eta/2) = \cosh|\xi|$. A similar expression comes out in the other case $m \leqslant n$, as is shown in equation (A.10). This formula has been first published by Bargmann (1947), who used the analytic function representation in the Fock–Bargmann space. It was rederived by Popov and Perelomov (1969), by performing the scalar products in the L^2 representation, and using generating functions.

The crucial step in our derivation is that now we apply formula (8.962.1) of Gradshteyn and Ryzhik (2000), to connect the hypergeometric functions with the Jacobi polynomials $P_{n/2}^{(\alpha,-1/2)}(x)$, and $P_{(n-1)/2}^{(\alpha,+1/2)}(x)$, where $\alpha = \tfrac{1}{2}(m-n) \geqslant 0$, and $n/2$ and $(n-1)/2$ are integers (corresponding to even or odd n, respectively). This is shown in equations (A.11) and (A.12) in the appendix A. The problem with these expressions is that they contain different Jacobi polynomials with different parameters, depending on the parity. This means that the functional forms of the matrix elements are not the same for even–even and odd–odd transitions. Fortunately, we can get rid off this severe asymmetry by using the formulas (22.5.22) and (22.5.21) of Abramowitz and Stegun (1965) in equations (A.11) and (A.12), to be applied for the even–even and odd–odd transitions, respectively. These formulas, equations (A.13a) and (A.13b), connect the Jacobi polynomials of the type $P_k^{(\alpha,\pm 1/2)}$ with the Gegenbauer polynomials $C_n^{\alpha+\frac{1}{2}}$. In this way we receive a *unifying formula (in the sub-case $m \geqslant n$), valid for both even and odd indices*, as is expressed in equation (A.14). A similar unifying formula (which is also valid for both even and odd paritys) results in the other sub-case $m \leqslant n$ as is shown in equation (A.15). Some additional details of the calculation are explained in appendix A.

On the basis of these considerations, the final result can be summarized in the form

$$\langle m|S(\xi)|n\rangle = e^{-\eta/4}(2e^{i\phi}\tanh|\xi|)^\alpha\sqrt{\frac{n!}{m!}}\frac{\Gamma(\alpha+\tfrac{1}{2})}{\sqrt{\pi}}C_n^{\alpha+\frac{1}{2}}\left(\frac{1}{\cosh|\xi|}\right), \qquad \alpha = \frac{1}{2}(m-n), \quad (m \geqslant n), \tag{5}$$

$$\langle m|S(\xi)|n\rangle = e^{-\eta/4}(-2e^{-i\phi}\tanh|\xi|)^\lambda\sqrt{\frac{m!}{n!}}\frac{\Gamma(\lambda+\tfrac{1}{2})}{\sqrt{\pi}}C_m^{\lambda+\frac{1}{2}}\left(\frac{1}{\cosh|\xi|}\right), \qquad \lambda = \frac{1}{2}(n-m), \quad (m \leqslant n). \tag{6}$$

The formulas in equations (5) and (6) are valid for both even–even and odd–odd transitions, so they are valid in the whole Hilbert space $H = H_{1/4} \oplus H_{3/4}$. The order ν of the Gegenbauer polynomials $C_n^\nu(x)$ in equations (5) and (6) satisfy the condition $\nu > -1/2$, and this means that the matrix elements are expressed by the classical orthogonal polynomials.

In the simplest special case, when $m = n$, the Gegenbauer polynomials reduce to the Legendre polynomials, according to the relation $C_n^{\frac{1}{2}}(x) = P_n(x)$ (see e.g. Gradshteyn and Ryzhik (2000)), i.e.

$$\langle n|S(\xi)|n\rangle = \sqrt{x}P_n(x), \quad x = 1/\cosh r, \; r = |\xi| \; (m = n). \tag{7}$$

In the general case the new formulas for the matrix elements $S_{mn} = \langle m|S(\xi)|n\rangle$, equations (5) and (6), can be brought to another equivalent form, expressed in terms of the associated Legendre functions $P_\mu^\nu(x)$. By using the formula (11.4) (10) of Erdélyi (1953), the Gegenbauer polynomials can be interrelated with the associated Legendre functions,

$$\frac{1}{\sqrt{\pi}}\Gamma\left(\mu + \frac{1}{2}\right)C_{\nu-\mu}^{\mu+\frac{1}{2}}(x) = (-1)^\mu 2^{-\mu}(1-x^2)^{-\frac{1}{2}\mu}P_\nu^\mu(x)$$

and we have

$$\langle m|S(\xi)|n\rangle = (-1)^k\sqrt{\frac{n!}{m!}}\sqrt{x}P_l^k(x)e^{ik\phi}, \qquad k = \frac{1}{2}(m-n), \quad (m \geqslant n), \tag{8}$$

$$\langle m|S(\xi)|n\rangle = \sqrt{\frac{m!}{n!}}\sqrt{x}P_l^{k'}(x)e^{-ik'\phi}, \qquad k' = \frac{1}{2}(n-m), \quad (n \geqslant m), \tag{9}$$

where $x = 1/\cosh|\xi|$ and $l = \frac{1}{2}(m+n)$. The transition probability then has the compact form

$$w_{m,n} = |\langle m|S(\xi)|n\rangle|^2 = \frac{n_<!}{n_>!}x|P_{(m+n)/2}^{|m-n|/2}(x)|^2, \tag{10}$$

where $n_< = \min(m,n)$, $n_> = \max(m,n)$, $x = 1/\cosh r$ (see Popov and Perelomov (1969), Malkin *et al* (1970)). The result in equation (10) directly shows the symmetry of the transition probabilities with respect to the interchange of n and m. Besides, the matrix elements $S_{mn} = \langle m|S(\xi)|n\rangle$ are formally related to the representation of spatial rotations (see e.g. Aronson *et al* (1974), Witschel (1975)). By introducing the non-normalized spherical harmonics $X_l^k(\vartheta, \phi)$,

$$X_l^k(\vartheta, \phi) = \sqrt{\frac{4\pi}{(2l+1)}}Y_l^k(\vartheta, \phi) = (-1)^k\left[\frac{(l-k)!}{(l+k)!}\right]^{1/2}P_l^k(\cos\vartheta)e^{ik\phi}, \tag{11}$$

Equations (8) and (9) can be summarized in a compact formula for the general matrix elements of the squeezing operator. On the basis of equations (8), (9) and (11), we have
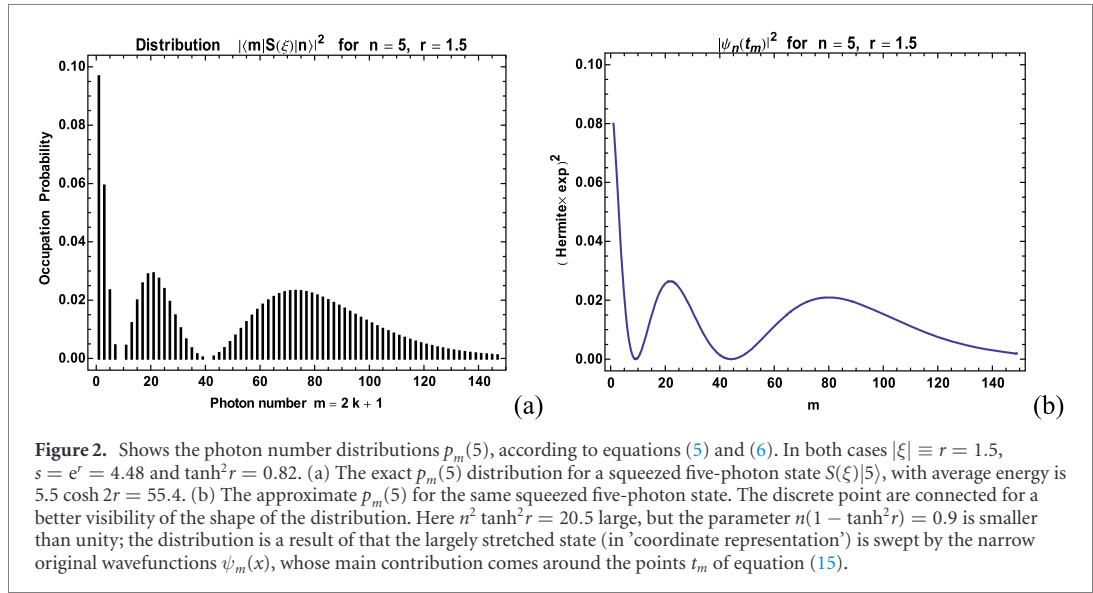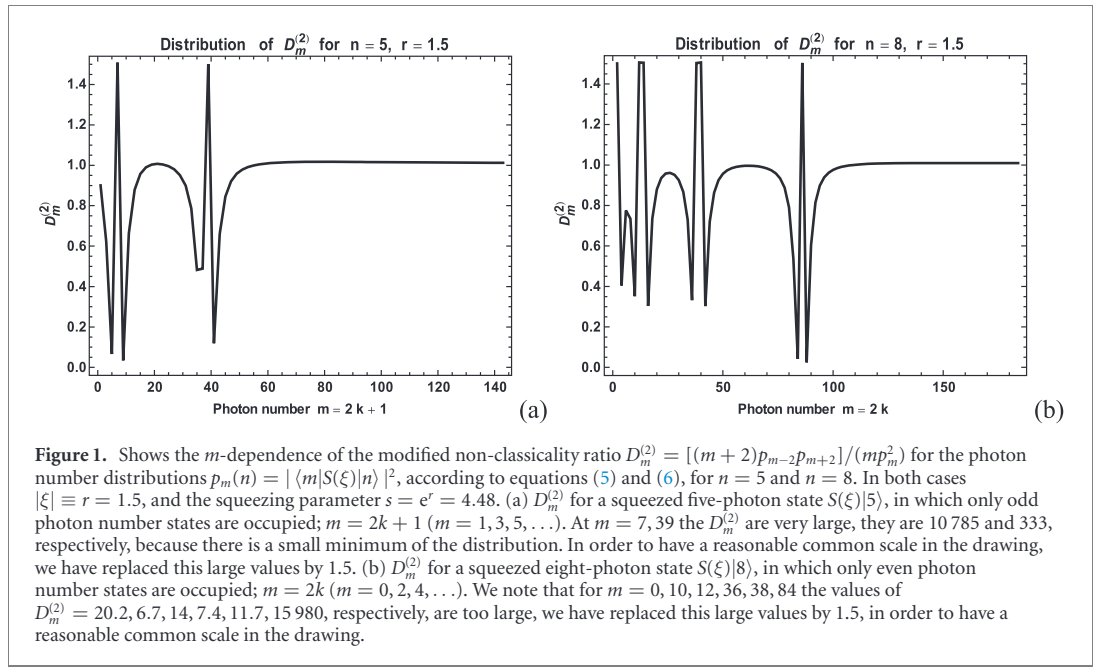
$$\langle m|S(\xi)|n\rangle = \sqrt{\cos\theta}X_l^k(\theta, \phi), \qquad \cos\theta = 1/\cosh r, \quad l = \frac{1}{2}(m+n), k = \frac{1}{2}(m-n), \; (m,n \geqslant 0), \tag{12}$$

where $0 < \theta < \pi/2$, $l = 0, 1, 2, \ldots$, $k = -l, -l+1, \ldots, l-1, l$. In the special case $n = 0$, the order equals to $k = l = m/2$, and we encounter with the well-known special formula for spherical harmonics,

$$X_l^l(\theta, \phi) = (-1)^l\sqrt{\frac{(2l)!}{2^{2l}(l!)^2}}(\sin\theta)^l e^{il\phi}, \quad l = m/2, \; \sin\theta = \tanh r. \tag{13}$$

By multiplying this with $\sqrt{\cos\theta}$ we receive the photon number distribution amplitude of a *squeezed vacuum state* $S(\xi)|0\rangle$. We note that, in contrast to our Gegenbauer polynomial expressions, equations (5) and (6), in the other equivalent formulas, given by equations (8), (9) and (12), the degree $l = \frac{1}{2}(m+n)$ of the functions is a combination of the initial and final indices, n and m, respectively. This circumstance would severely complicate the calculation for building up the coherent and incoherent superpositions, to be discussed in section 3.

In the rest of the present section we illustrate some main features of the photon number distribution of a squeezed number state, determined by the Gegenbauer polynomials in equations (5) and (6). In the photon number distributions $p_m(n) = |\langle m|S|n\rangle|^2$ the m are either even or odd, depending on the parity of n. This means that the squeezed number states $S|n\rangle$ are 'quite non-classical', like the even and odd coherent states (Dodonov and Man'ko (2003)). Klyshko's original non-classicality criterion $D_k(1) = [(k+1)p_{k-1}p_{k+1}]/(kp_k^2) < 1$ (see Klyshko (1996), Lee (1997) and Peřina et al (2019)) does not give a directly usable result in the present case, because if, say, $p_k \neq 0$, then $p_{k\pm1} = 0$, thus $D_k(1) = 0$ for all such k indices. In order to get rid of this difficulty, we can introduce the modified combination

Figure 1. Shows the m-dependence of the modified non-classicality ratio $D_m^{(2)} = [(m+2)p_{m-2}p_{m+2}]/(mp_m^2)$ for the photon number distributions $p_m(n) = |\langle m|S(\xi)|n\rangle|^2$, according to equations (5) and (6), for $n = 5$ and $n = 8$. In both cases $|\xi| \equiv r = 1.5$, and the squeezing parameter $s = e^r = 4.48$. (a) $D_m^{(2)}$ for a squeezed five-photon state $S(\xi)|5\rangle$, in which only odd photon number states are occupied; $m = 2k + 1$ ($m = 1, 3, 5, \ldots$). At $m = 7, 39$ the $D_m^{(2)}$ are very large, they are 10 785 and 333, respectively, because there is a small minimum of the distribution. In order to have a reasonable common scale in the drawing, we have replaced this large values by 1.5. (b) $D_m^{(2)}$ for a squeezed eight-photon state $S(\xi)|8\rangle$, in which only even photon number states are occupied; $m = 2k$ ($m = 0, 2, 4, \ldots$). We note that for $m = 0, 10, 12, 36, 38, 84$ the values of $D_m^{(2)} = 20.2, 6.7, 14, 7.4, 11.7, 15\,980$, respectively, are too large, we have replaced this large values by 1.5, in order to have a reasonable common scale in the drawing.

Figure 2. Shows the photon number distributions $p_m(5)$, according to equations (5) and (6). In both cases $|\xi| \equiv r = 1.5$, $s = e^r = 4.48$ and $\tanh^2 r = 0.82$. (a) The exact $p_m(5)$ distribution for a squeezed five-photon state $S(\xi)|5\rangle$, with average energy is $5.5 \cosh 2r = 55.4$. (b) The approximate $p_m(5)$ for the same squeezed five-photon state. The discrete point are connected for a better visibility of the shape of the distribution. Here $n^2 \tanh^2 r = 20.5$ large, but the parameter $n(1 - \tanh^2 r) = 0.9$ is smaller than unity; the distribution is a result of that the largely stretched state (in 'coordinate representation') is swept by the narrow original wavefunctions $\psi_m(x)$, whose main contribution comes around the points t_m of equation (15).

$D_k^{(2)} = [(k+2)p_{k-2}p_{k+2}]/(kp_k^2)$, which is nontrivial, and seems to give an extra information on the non-classicality. In figures 1(a) and (b) we have plotted the k-dependence of $D_k^{(2)}$ for $p_k(5)$ and $p_k(8)$. The appearance of the alternating concave and convex regions is not a surprise, because the distribution is oscillatory.

In figure 2 the photon number distribution $p_m(5)$ for a squeezed five-photon state $S(\xi)|5\rangle$ have been plotted, for $|\xi| \equiv r = 1.5$. In this case $n/\cosh^2 r < 1$, and it is justified to use the asymptotic formula 8.936 in Gradshteyn and Ryzhik (2000),

$$\lim_{\nu \to \infty} \nu^{-n/2} C_n^{\nu/2}[t(2/\nu)^{1/2}] = (1/2^{n/2}n!)H_n(t), \tag{14}$$

which yields $p_m(n) \approx |\psi_n(t_m)|^2/t_m$, where $\psi_n(x)$ is the usual Hermite function for the oscillator eigenfunction in L^2 representation, so

$$\langle m|S(\xi)|n\rangle \approx \frac{e^{i\phi(m-n)/2}}{(\cosh r)\sqrt{t_m}}\psi_n(t_m), \qquad \psi_n(x) = \frac{1}{\sqrt{\pi^{1/2}2^n n!}}H_n(x)\exp(-\frac{1}{2}x^2), \qquad t_m = \sqrt{\frac{m}{2\cosh^2 r}}, \tag{15}$$
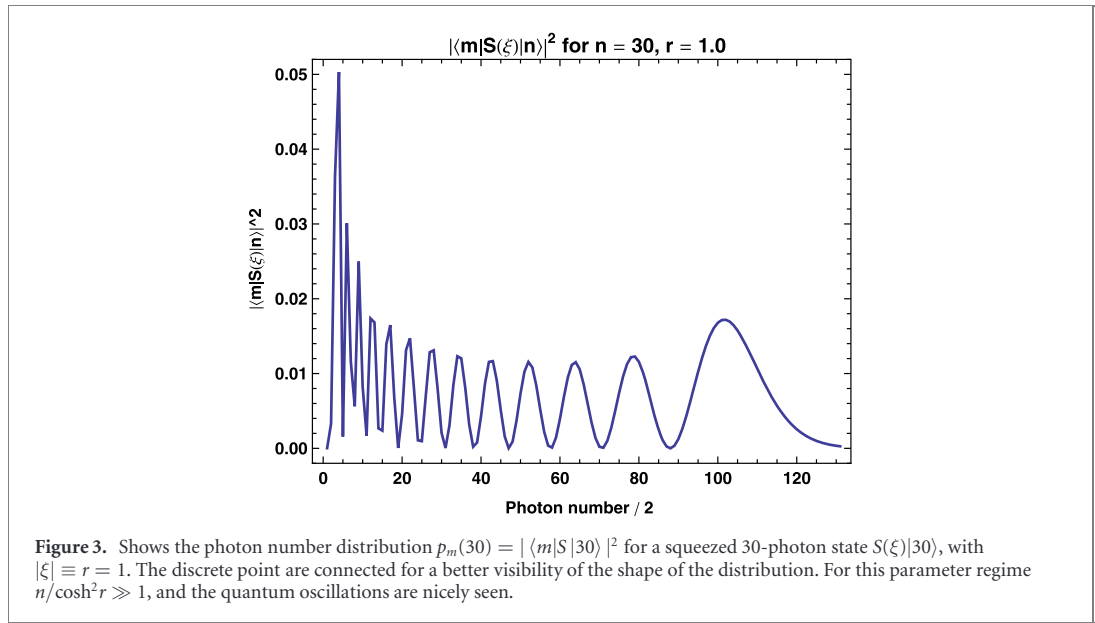
Figure 3. Shows the photon number distribution $p_m(30) = |\langle m|S|30\rangle|^2$ for a squeezed 30-photon state $S(\xi)|30\rangle$, with $|\xi| \equiv r = 1$. The discrete point are connected for a better visibility of the shape of the distribution. For this parameter regime $n/\cosh^2 r \gg 1$, and the quantum oscillations are nicely seen.

Figure 2(b) has been plotted by using the approximate formula, equation (15), for a comparison. The agreement with figure 2(a) is quite good. It is very remarkable, that the approximation in equation (15), is derived from the exact expressions equations (5) and (6), which are the result of an exclusively abstract, algebraic procedure. So, we would even say, that we 'did not know at the beginning that wave functions in L^2 exist, at all'. Still, we find that a quite faithful 'coarse-grained picture' of the modulus square of the wave function has been created from the photon number distribution. We note that Popov and Perelomov (1969) have also found this approximation (see their note at the very end of their paper), however they have calculated the matrix element *a priori* in the L^2 coordinate representation.

In figure 3 the characteristic quantum oscillations are nicely seen in this parameter regime $n/\cosh^2 r \gg 1$. Such oscillations have first been explained by Popov and Perelomov (1969) by using the quasi-classical approximation.

3. Coherent and incoherent superposition of the matrix elements of squeezing transitions

In the present section we determine the coherent and incoherent superpositions of the matrix elements of the squeezing operator, on the basis of the newly found Gegenbauer polynomial expressions in equations (5) and (6). First we note that in the semi-classical description, when the photon field is considered as an external field $F_0 \sin \omega t$, for a free electron in a Volkov state (see e.g. Varró 2021a), the oscillating phase factor $\exp[-i(U_0/4\hbar\omega)\sin 2\omega t]$ appears, which comes from the A^2-interaction. U_0 is the ponderomotive energy of the electron which is proportional to the intensity of the (laser) field. So, in the semiclassical approximation this factor contributes to the double-photon emission and absorption processes by the Fourier components, governed by the Bessel functions,

$$\exp[-iz \sin 2\omega t] = \sum_{k=-\infty}^{\infty} J_k(z)e^{-i2k\omega \cdot t}. \tag{16}$$

The corresponding coherent superposition of the transition amplitudes can be calculated on the basis of the generating formula for the Gegenbauer polynomials (see Gradshteyn and Ryzhik 2000),

$$\Gamma(k + \frac{1}{2})\frac{2^k}{\sqrt{\pi}}\sum_{n=0}^{\infty} C_n^{k+\frac{1}{2}}(x)\frac{z^n}{\Gamma(k+1+n)} = e^{z \cos \theta}\frac{1}{(z \sin \theta)^k}J_k(z \sin \theta).$$

By applying this formula for the (fixed) $2k$-order emission processes, taking equation (5), we have

$$\sum_{n=0}^{\infty}\frac{(\beta^*)^{n+2k}}{\sqrt{(n+2k)!}}e^{-\frac{1}{2}|\beta|^2}\frac{\alpha^n}{\sqrt{n!}}e^{-\frac{1}{2}|\alpha|^2}\langle n + 2k|S(\xi)|n\rangle$$

$$= \exp(-\frac{1}{2}|\beta|^2 - \frac{1}{2}|\alpha|^2 + \beta^*\alpha \cos \theta)e^{-\eta/4}e^{ik\phi}\left(\frac{\beta^*}{\alpha}\right)^k J_k(\beta^*\alpha \tanh|\xi|). \tag{17}$$

In equation (17) α and β are parameters of the (optionally highly-excited) initial and final coherent states, $|\alpha\rangle$ and $|\beta\rangle$, which represent the (laser) field, and the argument of the Bessel function can be related to the average photon number, which is connected to the classical intensity. For a correct probability interpretation of this expression, the initial and final coherent states can be considered as von Neumann lattice coherent states (Neumann 1932), as is explained in our recent works in Gombkötő *et al* (2016, 2020), Földi *et al* (2021) and Varró (2021a). By comparing equation (17) and the Bessel function amplitudes in equation (16), we see that if $\alpha = \beta$, the semiclassical result reproduces the result obtained with the quantized description of the field excitations. However, in the quantal description there is always a chance for α and β being different, which expresses the effect of the back-reaction of the electron. So, according to the exact result, equation (17), if this back-reaction is not negligible, then the emission (absorption) probability distribution of the $2k$-order processes can be qualitatively different from the semi-classical one.

In the case of the interaction with a spectral component of black-body radiation, or with some chaotic radiation, in the semi-classical description we use an averaging with respect to the field amplitude. In case of a Gaussian distribution we need the average of $J_n^2(z)$ in equation (16). We have

$$\frac{1}{I_0}\int_0^\infty \mathrm{d}I e^{-I/I_0} J_k^2(\gamma I) = \frac{1}{\pi\gamma I_0} Q_{k-\frac{1}{2}}(1 + 1/2\gamma^2 I_0^2) \quad (k = 0, 1, 2, \ldots), \tag{18}$$

where I_0 is the average intensity of the field, and $Q_{k-\frac{1}{2}}(z)$ is a Legendre function of second kind (here it is a so-called toroidal function, see Gradshteyn and Ryzhik, 2000). For the absorption processes we receive the similar expression. In equation (18) the parameter γI_0, in case of a free electron, means $\gamma I_0 = U_0/2\hbar\omega$, where $U_0 = \mu^2 mc^2/4$ is the ponderomotive energy of the electron in a radiation field, $\mu^2 = 10^{-18} I_0 \lambda^2$ is the dimensionless intensity parameter (where I_0 measured in W cm^{-2}, and λ, the central wavelength, measured in microns).

We have also calculated the incoherent superpositions of the Gegenbauer amplitudes in equations (5) and (6), by using Planck–Bose weights,

$$p_n = (1 - b)b^n, \quad b = e^{-h\nu/k_B T} = \frac{\overline{n}}{1 + \overline{n}}, \qquad p_n = \frac{\overline{n}^n}{(1 + \overline{n})^{1+n}}, \qquad \overline{n} = \frac{1}{e^{h\nu/k_B T} - 1}, \tag{19}$$

where k_B is the Boltzmann constant, and T is the absolute temperature of the black-body radiation. For $2k$-order emission processes we have the result

$$\sum_{n=0}^\infty p_n |\langle n + 2k|S(\xi)|n\rangle|^2 = \frac{\cos\theta}{\pi\sqrt{\overline{n}(1 + \overline{n})}\sin^2\theta} b^{-k} Q_{k-\frac{1}{2}}(z), \qquad z = 1 + \frac{1}{2\overline{n}(1 + \overline{n})\sin^2\theta}, \qquad k = 0, 1, 2\ldots \tag{20}$$

For $2l$-order absorptions we have received the average probability

$$\sum_{n=2l}^\infty p_n |\langle n - 2l|S(\xi)|n\rangle|^2 = \frac{\cos\theta}{\pi\sqrt{\overline{n}(1 + \overline{n})}\sin^2\theta} b^{+l} Q_{l-\frac{1}{2}}(z) \qquad z = 1 + \frac{1}{2\overline{n}(1 + \overline{n})\sin^2\theta}, \qquad l = 0, 1, 2\ldots, \tag{21}$$

In equations (20) and (21) $\sin^2\theta = \tanh^2 r$. For electrons, interacting with a quantized mode of the radiation field the hyperbolic tangent of r is expressed by the relation (see equation (12) in Varró, 2021a),

$$\tanh^2 r = \frac{1 + \beta - \sqrt{1 + 2\beta}}{1 + \beta + \sqrt{1 + 2\beta}} = \frac{1}{4}\beta^2 + O(\beta^3), \qquad \beta = \frac{\omega_p^2}{2\omega^2}, \qquad \omega_p = \sqrt{\frac{4\pi e^2}{mL^3}}, \tag{22}$$

where L^3 is the quantization volume, and ω_p is the one-electron plasma frequency. This means that the denominator in parameter z, equation (21), is proportional with $\overline{n}(1 + \overline{n})/L^6$, which is essentially the square of the average photon density. The semi-classical result, equation (18), approximates quite well the exact result, equations (20) and (21), calculated on the basis of equations (5) and (6), if the prefactors $b^l = e^{-l \cdot h\nu/k_B T}$ are close to unity, i.e. if $h\nu/k_B T \ll 1$, which corresponds to the Rayleigh–Jeans limit of the Planck distribution. In the opposite case, in the Wien-limit, when $h\nu/k_B T \gg 1$, the two expressions are qualitatively different, because $\exp[k \cdot (h\nu/k_B T)]Q_{|k|-\frac{1}{2}}(z)$ $(k = 0, \pm 1, \pm 2, \ldots)$ contains the exponential factor.

4. Summary

In section 2 (and in the appendix A) we have proved that matrix elements of the squeezing operator (Bogoljubov transformation, 1947) can be expressed in terms of the classical Gegenbauer polynomials, equations (5) and (6). On the basis of this new formula, we have given few illustrative examples for the

photon number distribution of a squeezed number state, and briefly discussed the non-classicality of these states. This new expression makes it possible to determine coherent and incoherent superpositions of these matrix elements in closed analytic forms. In section 3 we have described multiphoton transitions in the system 'charged particle + electromagnetic radiation', induced by a (strong) coherent field or by a spectral component of black-body radiation. The exact results have been compared with the semi-classical ones (the latter are based on non-perturbative matrix elements with c-number radiation fields). It has been found that in the case of interaction with a thermal field, the semi-classical result (calculated with a *c*-number, Gaussian stochastic field amplitude) yields an acceptable approximation only in the Rayleigh–Jeans limit ($h\nu/k_B T \ll 1$). It has been explicitly shown that in the Wien limit ($h\nu/k_B T \gg 1$) the semi-classical formula for the multiphoton absorption and emission probabilities cannot even 'mimic' its quantum counterpart, because it has a different functional form.

Acknowledgments

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Algebraic derivation of the matrix elements of the squeezing operator between photon number eigenstates

Derivation of the normal form of the squeezing operator

In the present appendix A we give the details of the calculation of the matrix elements between the number eigenstates of the unitary squeezing operator, $S_{m,n} = \langle m|S(\xi)|n\rangle$, where

$$S(\xi) = \exp[\frac{1}{2}\xi(a^+)^2 - \frac{1}{2}\xi^*(a)^2], \qquad S(\xi) = \exp(\xi K_+ - \xi^* K_-), \qquad \xi = |\xi|e^{i\phi}, \qquad (A.1)$$

with $K_+ = \frac{1}{2}(a^+)^2$ and $K_- = \frac{1}{2}(a)^2$. By introducing $K_0 = \frac{1}{4}(aa^+ + a^+a)$, we have the closed set of generators of a special Lie algebra of the $SU(1,1)$ group, having the commutation relations $[K_0, K_\pm] = \pm K_\pm$, and $[K_-, K_+] = 2K_0$. The Casimir operator $\hat{C}_2 = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+)$ commutes with all the generators, hence it is proportional with the unit operator, $C_2 = \kappa(\kappa - 1)\hat{I}$, where κ is called the Bargmann index. In the special case under discussion, we have $C_2 = -(3/16)\hat{I}$, which yields $\kappa = 1/4$ or $\kappa = 3/4$. For $\kappa = 1/4$, the states $|n\rangle$ with even n form a basis of the unitary irreducible representation space of the group $SU(1,1)$, and the states $|n\rangle$ with odd n form a basis corresponding to $\kappa = 3/4$. In the 'even representation space' $K_0 |2k\rangle = (k + 1/4)|2k\rangle$ ($k = 0, 1, 2, \ldots$), and in the 'odd representation space' $K_0 |2l + 1\rangle = (l + 3/4)|2l + 1\rangle$ ($l = 0, 1, 2, \ldots$). Each equation can be written in the form $K_0 |\psi_\nu\rangle = \nu |\psi_\nu\rangle$, where $\nu - \kappa = 0, 1, 2, \ldots$. Of course, the complete Hilbert space is the direct sum of these sub-spaces, $H = H_{1/4} \oplus H_{3/4}$. In the present and in similar contexts, the properties of the $SU(1,1)$ group have been used by Popov and Perelomov (1969), Malkin *et al* (1970), Gilmore (1974), Perelomov (1977), Kelemen (1975), Witschel (1975), Fisher *et al* (1984), Schumaker and Caves (1985), Varró (2015), Wünsche (1999), (2003), (2017a), (2017b).

The various ordered expressions for the squeezing operator have long been known (see. e.g. Wünsche 2003). However, we wish to have the material of the present appendix possibly self-contained, so we include the derivation of the normal form of $S(\xi)$. Since neither the commutation relations, nor the parameters depend on the Bargmann index, the ordering (factorization) of $S(\xi)$ can be performed in any representation, of course. Perhaps the simplest way to derive the normal (and anti-normal) form of $S(\xi)$ is to use the 'spinor representation' of the generators (see e.g. Gilmore (1974), Fisher *et al* (1984) or Schumaker and Caves (1985)). It is an easy matter to check that the following 2×2 matrices

$$-\sigma_+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \frac{1}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy the same commutation rules as K_+ and K_- and K_0, respectively, thus $-\sigma_+$, σ_- and $\frac{1}{2}\sigma_3$ also form an $SU(1,1)$ algebra. The ordering can be performed by expanding the exponential expressions of the 2×2 matrix representants (which are also 2×2 matrices), and comparing the parameters of the original exponential expression with the parameters of the new exponential factors. From this we can determine the coefficients of $(-\sigma_+)$, σ_- and $\frac{1}{2}\sigma_3$ in the new exponential factors, and these coefficients must be identified with that of K_+ and K_- and K_0 in the normal (anti-normal) form. In the 'spinor representation' we have

$$S(\xi) = \exp(\xi K_+ - \xi^* K_-) \to \exp\left[-\begin{pmatrix} 0 & \xi \\ \xi^* & 0 \end{pmatrix}\right] = \begin{pmatrix} \cosh|\xi| & -e^{i\phi}\sinh|\xi| \\ -e^{-i\phi}\sinh|\xi| & \cosh|\xi| \end{pmatrix}, \quad \text{(A.2)}$$

$$\exp(\zeta K_+)\exp(-\eta K_0)\exp(\zeta' K_-) \to \begin{pmatrix} 1 & -\zeta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \zeta' & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\eta/2} - \zeta\zeta'e^{\eta/2} & -\zeta e^{\eta/2} \\ e^{\eta/2}\zeta' & e^{\eta/2} \end{pmatrix}. \quad \text{(A.3)}$$

By equating the components of the right hand sides,

$$\cosh|\xi| = e^{\eta/2}, \ \text{i.e.}\,\eta = 2\log\cosh|\xi|, \ -e^{i\phi}\sinh|\xi| = -\zeta e^{\eta/2} = -\zeta\cosh|\xi|, \ \text{i.e.}\,\zeta = e^{i\phi}\tanh|\xi|,$$

$$-e^{-i\phi}\sinh|\xi| = \zeta'e^{\eta/2} = \zeta'\cosh|\xi|, \ \text{i.e.} \ \zeta' = -e^{-i\phi}\tanh|\xi| = -\zeta^*. \quad \text{(A.4)}$$

Thus, the normal form of $S(\xi)$ becomes

$$S(\xi) = \exp(\xi K_+ - \xi^* K_-) = \exp(\zeta K_+)\exp(-\eta K_0)\exp(-\zeta^* K_-). \quad \text{(A.5)}$$

On the basis of (A.4), the new parameters are determined to be $\eta = 2\log\cosh|\xi|$, $\zeta = e^{i\phi}\tanh|\xi|$. We note that the anti-normal form can also be obtained similarly,

$$\hat{S}(\xi) = \exp(\xi K_+ - \xi^* K_-) = \exp(-\zeta^* K_-)\exp(\eta K_0)\exp(\zeta K_+). \quad \text{(A.6)}$$

With the help of the normal form in (A.5) the calculation of the matrix element $\langle m|S(\xi)|n\rangle$ is now straightforward. The effect of $\exp(-\zeta^*\frac{1}{2}a^2)$ on $|n\rangle$ yields a finite sum, in which the highest power of $-\zeta^*$ is $[n/2]$, where $[x]$ denotes the integer part of x (i.e. the smallest integer, which is larger or equal to x). The factor in the middle of the expression in (A.5) is diagonal, i.e. $\exp(-\eta K_0)|k\rangle = \exp[-(\eta/2)(k+\frac{1}{2})]|k\rangle$ for any k. The factor $\langle m|\exp(\zeta K_+) = (\exp(\zeta^*\frac{1}{2}a^2)|m\rangle)^+$ on the left yields also a finite sum, in which the highest power of ζ is $[m/2]$. The scalar product of the two sums terminates at the smaller maximum summation index $\min([m/2],[n/2])$. Any term like $\langle m|(a^{+2})^l(a^2)^k|n\rangle$ is proportional to $\langle m|n - 2k + 2l\rangle = \delta_{m,n-2k+2l}$, thus, only those matrix elements are non-vanishing in which m and n have the same parity, and this means that their difference $m - n$ must be an even number, $m - n = 2\alpha$, where α is an integer. By taking all these considerations into account, a straightforward calculation leads to the explicit result,

$$\langle m|S(\xi)|n\rangle = \exp[-\tfrac{1}{2}(n+\tfrac{1}{2})\eta](\zeta/2)^{(m-n)/2}\sqrt{m!}\sqrt{n!}\sum_{k=0}^{[n/2]}\frac{(-|\zeta|^2 e^\eta/4)^k}{[\frac{1}{2}(m-n)+k]!k!(n-2k)!}, \quad (m \geqslant n), \quad \text{(A.7)}$$

$$\langle m|S(\xi)|n\rangle = \exp[-\tfrac{1}{2}(m+\tfrac{1}{2})\eta](-\zeta^*/2)^{(n-m)/2}\sqrt{m!}\sqrt{n!}\sum_{l=0}^{[m/2]}\frac{(-|\zeta|^2 e^\eta/4)^l}{[\frac{1}{2}(n-m)+l]!l!(m-2l)!} \quad (m \leqslant n). \quad \text{(A.8)}$$

These expressions (A.7) and (A.8) are equivalent to the ones derived long ago by Husimi (1953), who used two-variable generating functions of S_{mn}, and the L^2 representation (Hermite functions) for the oscillator basis states (see also Popov and Perelomov 1969). For a real ζ equation (A.7) reduces to the equation (27) in Kelemen's (1975) paper. See also Tanabe (1973), Rashid (1975), Satyanarayana (1985) and Mendaš and Popović (1995).

Explicit form of the matrix elements of the squeezing operator between photon number eigenstates, expressed in terms of Gegenbauer polynomials.

Now, our task is to express the finite sums (A.7) and (A.8) in terms of known functions, which makes it possible to perform the analytic calculations shown in the main text. Let us consider the sum on the

right-hand side of (A.7), which refers to the sub-case $m \geqslant n$. We observe that the first factorial in the denominator can be expressed as

$$\left[\frac{1}{2}(m-n)+k\right]! = \left[\frac{1}{2}(m-n)\right]!\left(\frac{1}{2}(m-n)+1\right)_k, \qquad (a)_k = (a)\cdot(a+1)\ldots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)},$$

where we have introduced Pochhammer's symbol $(a)_k$, which can also be expressed in terms of the gamma functions (Abramowitz and Stegun 1965). Moreover, from the explicit form of the factorials, one can show

$$\frac{n!}{(n-2k)!} = 2^{2k}\left(-\frac{n}{2}\right)_k\left(\frac{1-n}{2}\right)_k.$$

These manipulations allow us to bring into consideration the hypergeometric series $F(a,b;\ c;\ z)$, which may be used to represent orthogonal polynomials,

$$F(a,b;c;z) = 1 + \frac{a\cdot b}{c\cdot 1}z + \frac{a(a+1)b(b+1)}{c(c+1)\cdot 1.2}z^2 + \cdots = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k}z^k.$$

In our formula (A.7) this series terminates when the summation index takes on the value $k = [n/2]$ (and in (A.8) the sum terminates at $k = [m/2]$). The expressions (A.7) and (A.8) for the matrix elements can then be brought to the equivalent forms, expressed in terms of the hypergeometric functions,

$$\langle m|S(\xi)|n\rangle = \exp[-\frac{1}{2}(n+\frac{1}{2})\eta](\zeta/2)^{(m-n)/2}\sqrt{\frac{m!}{n!}}\frac{1}{\Gamma(\alpha+1)}F\left(-\frac{n}{2},\frac{1-n}{2};\frac{m-n}{2}+1;z\right) \quad (m\geqslant n),$$
(A.9)

$$\langle m|S(\xi)|n\rangle = \exp[-\frac{1}{2}(m+\frac{1}{2})\eta](-\zeta^*/2)^{(n-m)/2}\sqrt{\frac{n!}{m!}}\frac{1}{\Gamma(\lambda+1)}F\left(-\frac{m}{2},\frac{1-m}{2};\frac{n-m}{2}+1;z\right) \quad (m\leqslant n),$$
(A.10)

where $z = -|\zeta|^2 e^\eta = -\sinh^2|\xi|$, and $\alpha = \frac{1}{2}(m-n)$ in (A.9) and $\lambda = \frac{1}{2}(n-m) = |\alpha|$ in (A.10) are non-negative integers. It can be seen that equations (A.9) and (A.10) go over into each other if we exchange n and m, take the complex conjugate of the resulting expression, and multiply this by $(-1)^{(m-n)/2}$. Thus, it is enough to detail the forthcoming derivation only for one of these equations, e.g. (A.9). In addition we note that equations (A.9) and (A.10) are equivalent to equations (10.28a) and (10.28b) of Bargmann (1947) (see also Sciarrino and Toller 1967).

Consider the transitions between *even* photon number states, i.e., assume that $n = 2k$ where $k = 0, 1, 2, \ldots$ (and $m = 2q$ with $q = 0, 1, 2, \ldots$). Then, on the right-hand side of equation (A.9) we have $F(-k, -k-\beta;\ \alpha+1;\ z)$, where $\beta = -1/2$ and $\alpha = \frac{1}{2}(m-n) = k-q$ is the same non-negative integer, as has been defined above. We apply the formula (8.962.1) of Gradshteyn and Ryzhik (2000), p 990, which gives the connection between the Jacobi polynomials and the hypergeometric functions,

$$P_k^{(\alpha,\beta)}(x) = \frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)}\left(\frac{x+1}{2}\right)^k F\left(-k,-k-\beta;\alpha+1;\frac{x-1}{x+1}\right), \qquad \frac{x-1}{x+1} = z, \qquad (*)$$

where $P_k^{(\alpha,\beta)}(x)$ is a Jacobi polynomial of degree $k = n/2$, and $\beta = -1/2$ in the present case. The argument of the Jacobi polynomial can simply be expressed as $x = (1+z)/(1-z)$, where $z = -\sinh^2|\xi|$. By taking this connection ($*$) into account in equation (A.9), we receive

$$\langle m|S(\xi)|n\rangle = e^{-\eta/4}(\zeta/2)^\alpha\frac{\sqrt{m!}}{\sqrt{n!}}\frac{(\frac{1}{2}n)!}{(12m)!}P_{n/2}^{(\alpha,-1/2)}(x), \qquad x = \frac{1+z}{1-z},$$
(A.11)

$(m \geqslant n$, even m and even $n \geqslant 0)$.

Consider now the transitions between *odd* photon number states, i.e., assume that $n = 2k+1$ where $k = 0, 1, 2, \ldots$ (and $m = 2q+1$ with $q = 0, 1, 2\ldots$). We take into account that the hypergeometric functions are symmetric in the first two parameters; $F(a,b;\ c;\ z) = F(b,a;\ c;\ z)$. Then, on the right-hand side of equation (A.9) we have $F(-k, -k-\beta';\ \alpha+1;\ z)$, but now $\beta' = +1/2$. As before, we again apply the same formula (8.962.1) of Gradshteyn and Ryzhik (2000), denoted by ($*$) above, for the connection of the Jacobi polynomials and the hypergeometric functions, and receive

$$\langle m|S(\xi)|n\rangle = e^{-\eta/4}(\zeta/2)^\alpha\frac{\sqrt{m!}}{\sqrt{n!}}\frac{[\frac{1}{2}(n-1)]!}{[12(m-1)]!}\sqrt{\frac{1+x}{2}}P_{(n-1)/2}^{(\alpha,+1/2)}(x), \qquad x = \frac{1+z}{1-z},$$
(A.12)

$(m \geqslant n$, odd m and odd $n \geqslant 1)$.

The argument of the Jacobi polynomial is $x = (1 + z)/(1 - z)$ (where $z = -\sinh^2|\xi|$), and the other parameters have been defined in (A.4); $\zeta = e^{i\phi}\tanh|\xi|$, and $e^{\eta/2} = \cosh|\xi|$.

Finally, we use the formulas (22.5.22) and (22.5.21) of Abramowitz and Stegun (1965) in equations (A.11) and (A.12), referring to the even–even and odd–odd transitions, respectively,

$$P_k^{(\alpha,-1/2)}(x) = \frac{(\frac{1}{2})_k}{(\alpha+1/2)_k}C_{2k}^{\alpha+\frac{1}{2}}\left(\sqrt{\frac{1+x}{2}}\right), \qquad P_k^{(\alpha,+1/2)}(x) = \frac{(\frac{1}{2})_{k+1}}{(\alpha+1/2)_{k+1}\sqrt{\frac{1+x}{2}}}C_{2k+1}^{\alpha+\frac{1}{2}}\left(\sqrt{\frac{1+x}{2}}\right),$$
(A.13a)

$$x = \frac{1+z}{1-z}, \quad \frac{1+x}{2} = \frac{1}{1-z}, \quad z = -\sinh^2|\xi|, \quad \sqrt{\frac{1+x}{2}} = \frac{1}{\cosh|\xi|}.$$
(A.13b)

Here $C_n^\nu(x)$ are the Gegenbauer polynomials (see e.g. Gradshteyn and Ryzhik (2000)). Inserting the first and the second equations of (A.13a) in equations (A.11) and (A.12), respectively, and taking (A.13b) into account, we receive a *unifying formula* (in the sub-case $m \geqslant n$), *valid for both even and odd indices*,

$$\langle m|S(\xi)|n\rangle = e^{-\eta/4}(2e^{i\phi}\tanh|\xi|)^\alpha\sqrt{\frac{n!}{m!}}\frac{\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}}C_n^{\alpha+\frac{1}{2}}\left(\frac{1}{\cosh|\xi|}\right), \qquad \alpha = \frac{1}{2}(m-n), \quad (m \geqslant n).$$
(A.14)

A similar unifying formula (which is also valid for both parities) results in the other sub-case $m \leqslant n$,

$$\langle m|S(\xi)|n\rangle = e^{-\eta/4}(-2e^{-i\phi}\tanh|\xi|)^\lambda\sqrt{\frac{m!}{n!}}\frac{\Gamma(\lambda+\frac{1}{2})}{\sqrt{\pi}}C_m^{\lambda+\frac{1}{2}}\left(\frac{1}{\cosh|\xi|}\right), \qquad \lambda = \frac{1}{2}(n-m), \quad (m \leqslant n).$$
(A.15)

In deriving equation (A.14) from equations (A.11) and (A.12), we have also taken into account the so-called doubling formula $\Gamma(2x) = \pi^{-1/2}2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2})$ of the gamma function (see e.g. Gradshteyn and Ryzhik 2000), and the explicit form of the parameter $\eta = 2\log\cosh|\xi|$. According to the functional relation $C_n^{\frac{1}{2}}(x) = P_n(x)$, if $m = n$, then, both in (A.14) and (A.15), the Gegenbauer polynomials reduce to the Legendre polynomial $P_n(x)$,

$$\langle n|S(\xi)|n\rangle = \sqrt{x}P_n(x), \qquad x = 1/\cosh r, r = |\xi|, \quad (m = n).$$
(A.16)

We also note that, in contrast to equations (A.11) and (A.12) being separately valid in the sub-spaces $\mathcal{H}_{1/4}$ and $\mathcal{H}_{3/4}$, respectively, the unifying formulas (A.14) (and also (A.15)) are valid in the whole Hilbert space $\mathcal{H} = \mathcal{H}_{1/4} \oplus \mathcal{H}_{3/4}$. On 'unifying formula' we mean that the functional form of the expressions are the same for both the $\mathcal{H}_{1/4}$ (even) and the $\mathcal{H}_{3/4}$ (odd) subspaces. This is not the case for equations (A.11) and (A.12), because they contain different Jacobi polynomials, $P_{n/2}^{(\alpha,-1/2)}(x)$ and $P_{(n-1)/2}^{(\alpha,+1/2)}(x)$, with different parameters, respectively.

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