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DeMorgan systems with an infinitely many negations in the strict monotone operator case

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ABSTRACT

We give a new representation theorem of negation based on the generator function of the strict operator. We study a certain class of strict monotone operators which build the DeMorgan class with infinitely many negations. We show that the necessary and sufficient condition for this operator class is $f_c(x)f_d(x) = 1$, where $f_c(x)$ and $f_d(x)$ are the generator functions of the strict t-norm and strict t-conorm.

1. Introduction and some elementary considerations

When studying continuous-valued logical operators, one of the most important questions is how to choose an element in a consistent way (i.e. the DeMorgan law is valid). One of the most important Boolean identities is the DeMorgan class. Several authors have studied this identity. The first article appeared in 1983 [8]. Esteva in 1984 [11] made a study on some representable DeMorgan algebras. One of the most important studies was done by Garcia and Valverde [13], where they focused on isomorphism between DeMorgan triplets. In this article the authors deal with the main types of fuzzy t-norm and t-conorms, i.e. with the min and max operators, the nilpotent operator and with strict monotonously increasing operators. The main equivalence classes of DeMorgan triplets are extensively studied. The next important step was done by Gehrke, Walker and Walker [14]. They used an algebraic approach that is general and very extensive. We have to mention the book of Nguyen and Walker [24]. Here we can find a summary of the results on the existence of DeMorgan triplets.

In this article we shall focus on the DeMorgan systems which correspond to infinitely many negations. These types of operators are important because the fix point of the negation (see Eq. (4) later) can be varied, this value can be interpreted as a decision level and this kind of logic is very flexible. Such logic is very important. Cintula, Klement, Mesiar and Pap focus on fuzzy logic with an additional involutive negation operator [7], but in our case we have infinitely many.

This general characterisation makes it possible for us to construct a new type of operator system.

In the introductory part we give an elementary discussion for readers not familiar with this topic. In the Section 2 we extend the operators with weights and then we describe the relation between strict t-norm, strict t-conorm generator function and negation. This result is a reformulation of the known results. We show that the involutive properties of the negation (given $f_c(x)$ and $f_d(x)$) ensure that $k(x)$ is a function (see Fig. 1). We give the general form of the negation by using $k(x)$. We show that all involutive negation operators can be represented in this form and we will give some examples. The main result

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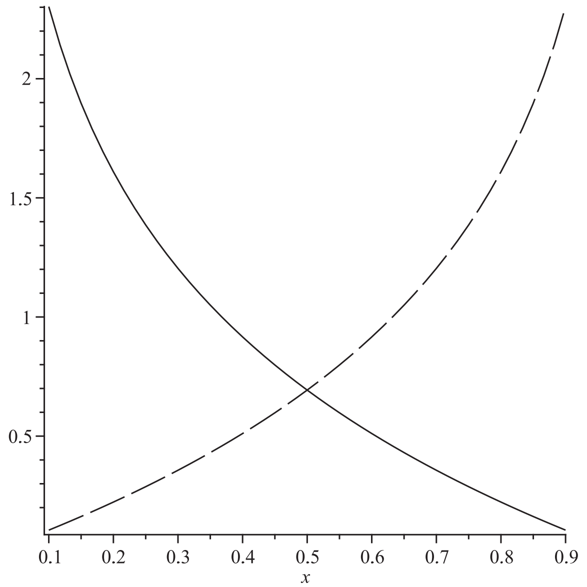

Fig. 1. The shape of the generator function of the strict t-norm and t-conorm. Here it $f_c(x) = -\ln(x)$ and $f_d(x) = -\ln(1-x)$ (dash).

of the article can be found in Section 3. We show that a DeMorgan triplet is valid with an infinitely many negations if and only if $f_c(x)f_d(x) = 1$ (i.e. $k(x) = \frac{1}{x}$). We call such a system a pliant system. In Section 4 we characterize the pliant operators.

From an application point of view, the strict monotonously increasing operators are useful. They have many applications. This is the reason why in this article we will focus on strictly monotonously increasing operators.

1.1. Triangular norms and conorms

Here we summarize the necessary notations and some previous results which will be used in the sequel.

For the basic properties of triangular norms (t-norms for short) and triangular conorms (t-conorms) Klement, Mesiar and Pap [18–20] refer to [17–26]. By definition, a t-norm $c(x,y)$ and a t-conorm $d(x,y)$ turn $[0,1]$ into an abelian, fully ordered semigroup with neutral element 1 and 0, respectively. In this paper Klement, Mesiar and Pap [18–20] shall restrict their selves to continuous t-norms and t-conorms. Let us only recall that a continuous t-norm $c(x,y)$ is Archimedean if it satisfies $c(x,x) < x$ for all $x \in]0,1[$. A continuous Archimedean t-norm is called strict if $0 \leqslant x < y \leqslant 1$ and $0 < z \leqslant 1$ implies $c(x,y) < c(y,z)$. Non-strict continuous Archimedean t-norms are called nilpotent. The basic result can be found in the book of Aczél [1].

From [21,23] Klement, Mesiar and Pap [18–20] state that a t-norm $c(x,y)$ is continuous Archimedean if and only if it has a continuous additive generator, i.e.,there is a continuous, strictly decreasing function t: $[0,1] \to [0,\infty]$ satisfying $t(1) = 0$ such that for all $(x,y) \in [0,1]^2$

$$c(x,y) = t^{(-1)}(t(x) + t(y)), \tag{1}$$

where the pseudo-inverse $t^{(-1)}$: $[0,\infty] \to [0,1]$ of t in this special context is given by $t^{(-1)}(x) = t^{-1}(\min(t(x), t(0)))$. Observe that the additive generator of a continuous Archimedean t-norm is unique up to a positive multiplicative constant. The case $t(0) = \infty$ occurs if and only if $c(x,y)$ is strict (in which case the pseudo-inverse in Eq. (1) is an ordinary inverse).

In this section, besides the min/max and the drastic operators, we shall be concerned with strict t-norms, that is

$$c(x,y) < c(x',y) \quad \text{if } x < x' \quad x,y \in (0,1]$$

and t-conorms, that is

$$d(x,y) < d(x',y) \quad \text{if } x < x' \quad x,y \in [0,1).$$

Later on we shall look for the general form of $c(x,y)$ and $d(x,y)$. We assume that the following conditions are satisfied:

1. Continuity:

$$c : [0,1] \times [0,1] \to [0,1] \quad d : [0,1] \times [0,1] \to [0,1].$$

2. Strict monotonous increasing:

$$c(x,y) < c(x,y') \text{ if } \quad y < y' \quad x \neq 0 \quad d(x,y) < d(x,y') \text{ if } \quad y < y' \quad x \neq 0.$$

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3. Compatibility with two-valued logic:

$$c(0,0)=0 \; c(1,1)=1 \quad d(0,0)=0 \quad d(1,1)=1$$
$$c(0,1)=0 \; c(1,0)=0 \quad d(0,1)=1 \quad d(1,0)=1.$$

4. Associativity:

$$c(x,c(y,z))=c(c(x,y),z) \quad d(x,d(y,z))=d(d(x,y),z).$$

5. Archimedean:

$$c(x,x)<x, \quad x\in(0,1) \quad d(x,x)>x, \quad x\in(0,1).$$

So

$$c(x,y)=f_c^{-1}(f_c(x)+f_c(y)). \tag{2}$$

Similarly, the strict t-conorm on $(0,1]\times(0,\infty]$ has the form:

$$d(x,y)=f_d^{-1}(f_d(x)+f_d(y)). \tag{3}$$

Here $f_c(x)$: $[0,1]\to[0,\infty]$ ($f_d(x)$: $[0,1]\to[0,\infty]$) are continuous and strictly increasing (decreasing) monotone functions and they are the generator functions of the strict t-norms and strict t-conorms (see Fig. 1).

In our case.

- we do not use the pseudo inverse and ordinal sum to construct a general t-norm and t-conorm.
- we do not use the commutativity axiom of the t-norm and t-conorm because it is always valid for the strict t-norm.
- we do not use the boundary condition of the t-norm and t-conorm, just the compatibility condition with binary logic. (The boundary condition can be proved by using associativity.) [17]
- we will call the elements of pliant logic conjunctive, disjunctive and negation operators.

Those familiar with fuzzy logic theory will find that the terminology used here is slightly different from that used in standard texts [17,5,2,4,22,15].

I would like to emphasise that only three changes have been made in pliant logic, namely.

- conjunctive operator = strict monotonously increasing t-norms,
- disjunctive operator = strict monotonously increasing t-conorms,
- negation = strong negation.

Consistent many-valued (fuzzy) operators have to satisfy of certain Boole identities. The most important one is the DeMorgan law. Esteva [11] and Dombi [8] were the first two researchers who carefully analysed the DeMorgan identity. It corresponds to the conjunction, disjunction and negation operators.

In order to analyse the DeMorgan identity we first need a good definition of negation. Strong negations are order reversing automorphisms of the unit interval. Because here we deal only with strong negations we shall refer to them as *negation*.

The usual requirements for such a negation (η) are the following.

Definition 1. We say that $\eta(x)$ is a negation if η: $[0,1]\to[0,1]$ satisfies the following conditions:

C1:	η: $[0,1]\to[0,1]$ is continuous	(Continuity)
C2:	$\eta(0)=1$, $\eta(1)=0$	(Boundary conditions)
C3:	$\eta(x)<\eta(y)$ for $x>y$	(Monotonicity)
C4:	$\eta(\eta(x))=x$	(Involution)

From C1 and C3 it follows that there exists a fix point $\nu_*\in[0,1]$ of the negation where

$$\eta(\nu_*)=\nu_*. \tag{4}$$

Since this value and its negated form are the same, it may be termed a neutral value. Furthermore, since the negation of values smaller than the neutral value gives values larger than the neutral value, and vice versa, the neutral value naturally divides the evaluation interval into two parts. The values larger than ν_* may be interpreted as the positive or acceptable evaluation range, and those smaller than ν_* as the negative evaluation range; ν_* is thus a threshold value, and can be interpreted as an expectation value.

Threshold logic has many applications. In this concept there is a previously fixed ν_0 value, which is the threshold.

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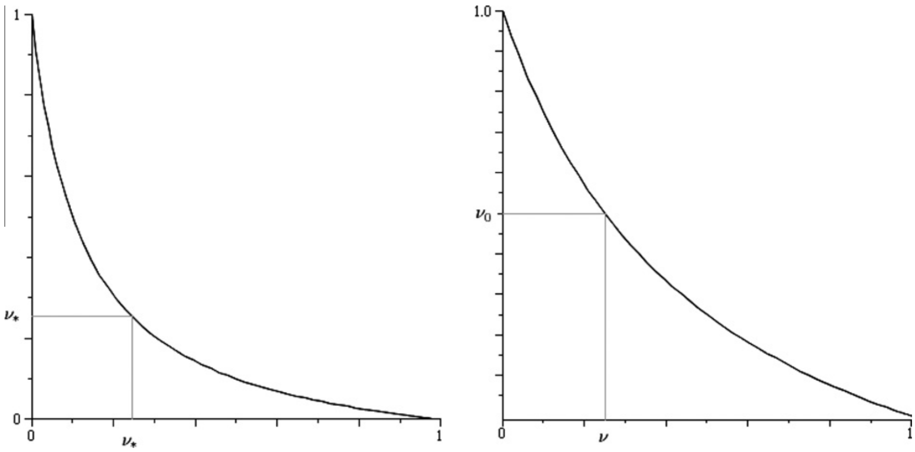

Fig. 2. The two interpretation of the parameter of the negation.

So another possible characterization of negation, is when we assign a so-called decision value ν for a given ν_0, i.e. one can specify a point (ν, ν_0) that the curve must intersect. This tells us something about how strong the negation operator is.

$$\eta(\nu) = \nu_0. \tag{5}$$

For the interpretation of the two concepts, see Fig. 2.

If $\eta(x)$ has a fix point ν_*, we use the notation $\eta_{\nu_*}(x)$ and if the decision value is ν, then we use the notation $\eta_\nu(x)$. If $\eta(x)$ is used without a suffix for then the parameter has no importance in the proof. Later on we will characterize the negation by the ν_*, ν_0 and νparameters.

In the following, we will examine the relations between f_c, f_d and η, to see whether c, d and η satisfy the DeMorgan law.

The ν_* value has several terminologies. It is called the fixpoint eigenvalue, or equilibrium point. In the article of De Baets and Fodor [3] the negations are induced by the uninorm. Here ν_* is called the neutral element. In multicriteria decision making 'expectation value' has a meaning and the pliant logic can be applied in this area.

2. DeMorgan law and general form of negation

We will use the generalized operator based on strict t-norms and strict t-conorms introduced by the authors. Calvo [6] and Yager [28].

Definition 2. Generalized operators based on strict t-norms and t-conorms which are

$$c(\mathbf{w}, \mathbf{x}) = c(w_1, x_1; w_2, x_2; \ldots; w_n, x_n) = f_c^{-1}\left(\sum_{i=1}^{n} w_i f_c(x_i)\right), \tag{6}$$

$$d(\mathbf{w}, \mathbf{x}) = d(w_1, x_1; w_2, x_2; \ldots w_n, x_n) = f_d^{-1}\left(\sum_{i=1}^{n} w_i f_d(x_i)\right), \tag{7}$$

where $w_i \geqslant 0$.

If $w_i = 1$ we get the t-norm and t-conorm. If $w_i = \frac{1}{n}$, then we get mean operators. If $\sum_{i=1}^{n} w_i = 1$, then we get weighted operators.

Definition 3. The DeMorgan law holds for the generalized operator based strict t-norms and strict t-conorms and for negation if and only if the following equation holds.

$$c(w_1, \eta(x_1); w_2, \eta(x_2); \ldots; w_n, \eta(x_n)) = \eta(d(w_1, x_1; w_2, x_2; \ldots; w_n, x_n)). \tag{8}$$

We call Eq. (8) later on generalized DeMorgan law.

Theorem 1 (*DeMorgan Law*). *If $\eta(x)$ and $f_d(x)$ are given, then $c(x, y)$, $d(x, y)$ and $\eta(x)$ form a DeMorgan triplet iff*

$$f_c^{-1}(x) = \eta(f_d^{-1}(ax)), \tag{9}$$

where $a \neq 0$.

Eq. (9) can be written in other form, see later (Eq. (17)).

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Remark 1. It is well-known that based on Eq. (9) from a generator function of a strict t-norm (or strict t-conorm) operator using negation we can get a generator function os the strict t-conorm (or strict t-norm), see [4].

Proof. We exploit the fact that η is an automorphism on the unit interval. In the proof we shall use the solution of the Cauchy functional equality, i.e.

$$h(x+y) = h(x) + h(y), \tag{10}$$

where h is continuous. The solution of this equation is

$$h(x) = ax, \quad a \neq 0. \tag{11}$$

A description of the DeMorgan law using the generator function is

$$f_c^{-1}\left(\sum_{i=1}^{n} w_i f_c(\eta(x_i))\right) = \eta\left(f_d^{-1}\left(\sum_{i=1}^{n} w_i f_d(x_i)\right)\right). \tag{12}$$

Substituting $x_i^* = f_d(x_i)$ and $x_i = f_d^{-1}(x_i^*)$, into this equation, we get

$$f_c^{-1}\left(\sum_{i=1}^{n} w_i f_c(\eta(f_d^{-1}(x_i^*)))\right) = \eta\left(f_d^{-1}\left(\sum_{i=1}^{n} w_i x_i^*\right)\right). \tag{13}$$

Let us apply f_c on both sides of the equation and introduce the following notation:

$$h(x_i^*) = f_c(\eta(f_d^{-1}(x_i^*))). \tag{14}$$

Then we have

$$\sum_{i=1}^{n} w_i h(x_i^*) = h\left(\sum_{i=1}^{n} w_i x_i^*\right). \tag{15}$$

This is the Cauchy equation. Using Eq. (11), we get

$$ax^* = f_c(\eta(f_d^{-1}(x^*))). \tag{16}$$

Substituting x: = ax^* and applying f_c^{-1} on both sides, we get the desired result. □

Remark 2. On the basis of Theorem 1 and given $f_c(x)$ and $\eta(x)$, $f_d(x)$ can be determined, so that c,d and η is a DeMorgan triple. Similar to the above-mentioned consideration, with a given $f_d(x)$ and $\eta(x)$, $f_c(x)$ can be determined.

2.1. Form of negations

Here the following question naturally arises. If f_c and f_d are given, what kind of condition ensures that η is a negation (i.e. fulfils C1–C4)? From Theorem 1 we know that the necessary and sufficient condition of the DeMorgan Law is Eq. (9). Substituting the $x := f_d^{-1}(ax)$, we have

$$\eta(x) = f_c^{-1}\left(\frac{1}{a} f_d(x)\right), \quad a \neq 0. \tag{17}$$

Let us give a parametric form of negation.

Theorem 2. *If $f_c(x)$ and $f_d(x)$ are given, then $c(x,y)$, $d(x,y)$ and $\eta(x)$ form a DeMorgan triplet iff*

$$\eta_{\nu_*}(x) = f_d^{-1}\left(\frac{f_d(\nu_*)}{f_c(\nu_*)} f_c(x)\right), \tag{18}$$

$$\eta_{\nu_*}(x) = f_c^{-1}\left(\frac{f_c(\nu_*)}{f_d(\nu_*)} f_d(x)\right). \tag{19}$$

Proof. Based on Eq. (17)

$$\nu_* = \eta(\nu_*) = f_d^{-1}(af_c(\nu_*)). \tag{20}$$

Expressing a and using Eq. (17), we get Eq. (18). On the basis of the involution of the negation we get Eq. (19). Because $\eta(\nu) = \nu_0$ from Eq. (20) we get $a = \frac{f_d(\nu)}{f_c(\nu_0)}$.

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$$\eta_v(x) = f_d^{-1}\left(\frac{f_d(v)}{f_c(v_0)} f_c(x)\right). \tag{21}$$

In a similar way we can prove that

$$\eta_v(x) = f_c^{-1}\left(\frac{f_c(v_0)}{f_d(v)} f_d(x)\right), \tag{22}$$

where v and v_0 are defined by Eq. (5). □

Negation is not always necessary involutive if it has the form (18) or (19), i.e. if

$$f_c(x) = \ln(x), \quad f_d(x) = \ln(1-x), \quad i.e.: \quad f_c^{-1}(x) = e^x, \quad f_d^{-1}(x) = 1 - e^x$$

then

$$\eta(x) = f_d^{-1}(Kf_c(x)) = 1 - e^{K\ln x} = 1 - x^K, \tag{23}$$

$$\eta(x) = f_c^{-1}(Kf_d(x)) = e^{K\ln(1-x)} = (1-x)^K. \tag{24}$$

If $K \neq 1$, then Eqs. (23) and (24) are not involutive.

This negation satisfies (C1–C3). The next important question is whether they obey the involution condition C4: $\eta(x) = \eta^{-1}(x)$.

Theorem 3. *Let η be given by (17). Then $\eta(x)$ is involutive if and only if*

$$f_c(x) = \frac{1}{a} k(f_d(x)) \text{ or } k(x) = af_c(f_d^{-1}(x)), \quad a \neq 0, \tag{25}$$

where k: $[0,\infty] \to [0,\infty]$ is a strictly decreasing, continuous function and

$$k^{-1}(x) = k(x). \tag{26}$$

Proof. ⇒Assume that $f_c(x) = \frac{1}{a} k(f_d(x))$, where $a \neq 0$. Then

$$f_c^{-1} = f_d^{-1}(k(ax)) \tag{27}$$

and substituting (27) into Eq. (17) we get:

$$\eta(x) = f_d^{-1}(k(f_d(x))) \tag{28}$$

and so $\eta(x)$ is involutive.

$\eta(0)$ and $\eta(1)$ is defined as a limit Eq. (28), i.e. $\eta(0) = 1$ and $\eta(1) = 0$.

⇐ Involutions mean that: $\eta(x) = \eta^{-1}(x)$, so from Eq. (17) we have

$$f_c^{-1}\left(\frac{1}{a} f_d(x)\right) = f_d^{-1}(af_c(x)). \tag{29}$$

Let define $h(x)$ is the following way

$$h(x) = f_c(f_d^{-1}(x)), \tag{30}$$

h is continuous strictly decreasing function, h: $[0,\infty] \to [0,\infty]$. So $h(x)$ exists. From Eq. (30) we get

$$f_c(x) = h(f_d(x)). \tag{31}$$

From this

$$f_c^{-1}(x) = f_d^{-1}(h^{-1}(x)). \tag{32}$$

Substituting Eqs. (31) and (32) into (29),

$$f_d^{-1}\left(h^{-1}\left(\frac{1}{a} f_d(x)\right)\right) = f_d^{-1}(ah(f_d(x))). \tag{33}$$

Let us apply f_d and use the notation $x = f_d(x)$. Then

$$h^{-1}\left(\frac{1}{a} x\right) = ah(x). \tag{34}$$

Choosing $k(x)$ in the following way:

$$k(x) = ah(x). \tag{35}$$

Then using Eq. (34), we can verify that

$$k^{-1}(x) = k(x), \tag{36}$$

so

$$f_c(x) = h(f_d(x)) = \frac{1}{a}k(f_d(x)). \qquad \square$$

In Fig. 3 we show the shape of $k(x)$ function.

We can obtain a new representation theorem for the negation using Theorem 3.

Theorem 4 (*General form of the negation*). *We have that $c(\mathbf{w},\mathbf{x})$, $d(\mathbf{w},\mathbf{x})$ and $\eta(x)$ is a DeMorgan triple if and only if*

$$\eta(x) = f^{-1}(k(f(x))), \tag{37}$$

where $f(x) = f_c(x)$ or $f(x) = f_d(x)$ and $k(x)$ is a strictly decreasing function with the property

$$k(x) = k^{-1}(x), \tag{38}$$

where $k: [0,\infty] \to [0,\infty]$.

Proof. Because of Eqs. (31) and (25), $f_c(x) = h(f_d(x)) = \frac{1}{a}k(f_d(x))$ and, on the basis of Eq. (17), $\eta(x) = f_d^{-1}(af_c(x))$, so

$$\eta(x) = f_d^{-1}(k(f_d(x))). \tag{39}$$

Note that $\eta(x)$ can be expressed in terms of $f_c(x)$ too, using that $k(f_d(x)) = a\, f_c(x)$ and $f_d^{-1}(x) = f_c^{-1}(\frac{1}{a}k^{-1}(x)) = f_c^{-1}(\frac{1}{a}k(x))$. Hence from Eq. (39) we get

$$\eta(x) = f_c^{-1}(k(f_c(x))). \tag{40}$$

Also, it is clear that Eq. (40) follows from Eqs. (25) and (27).

It is easy to check that for $\eta(x)$, conditions C1–C4 hold. $\square$

Corollary 1. *From Eq. (37) it is easy to see that*

$$k(x) = f(\eta(f^{-1}(x))), \tag{41}$$

i.e. if $f(x)$ and $\eta(x)$ is given, then $k(x)$ is determined by Eq. (41).

2.2. Representation theorem of negation

Another interesting question is whether Eq. (37) is a general representation form of the negation? The following theorem ensures that all negations can be written in this form.

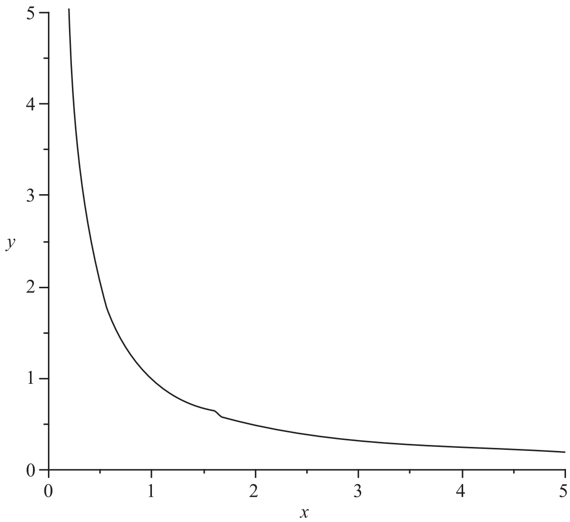

Fig. 3. The shape of $k(x)$ function

While Trillas' theorem [27] represents negations (from our point of view) for the nilpotent class of t-norms and t-conorms, our next result provides a representation theorem for the strict t-norms and t-conorms.

Let $k(x)=\frac{A}{x}$. In the next theorem we show that all negation can be expressed by using this $k(x)$.

Theorem 5 (*Representation theorem of negation*). *For any given* $\eta(x)$ *there exists an* $f(x)$ *such that*

$$\eta(x)=f^{-1}\left(\frac{A}{f(x)}\right), \tag{42}$$

where f is the generator function of some strict t-norm, or strict t-conorm and $A > 0$.

Remark 3. This theorem is similar to what Trillas' theorem states, i.e. for any given $\eta(x)$ there exists a $f(x)$ such that $\eta(x) = f^{-1}(1-f(x))$, where $f(x)$ is the generator function of some non-strict operator. In Theorem 5 the generator function is the generator function a strict monotonously increasing operator.

Proof. According to Trillas' result [27], for every $\eta(x)$ there exists an $f_*(x)$, such that

$$\eta(x)=f_*^{-1}(1-f_*(x)), \tag{43}$$

where f_*: $[0,1]\to[0,1]$ is a continuous, strictly increasing function.

First we give a particular solution of Eq. (42) in the case where $\eta(x) = 1-x$. We will show that if $\eta(x) = 1-x$ then there always exists an $f_c(x)(\text{or}f_d(x))$ which has the form of Eq. (42) because

$$1-x=f^{-1}\left(\frac{A}{f(x)}\right). \tag{44}$$

This can be written in the following form

$$f(x)f(1-x)=A. \tag{45}$$

If $f_c(x)$ has the form

$$f_c(x)=\sqrt{A}\frac{1-x}{x}, \tag{46}$$

then $f_c(x)$ is a generator function of a t-norm and this is solution of Eq. (45).

If $f_d(x)$ has the form

$$f_d(x)=\sqrt{A}\frac{x}{1-x}, \tag{47}$$

then $f_d(x)$ is a generator function of a t-conorm and this is solution of Eq. (45).

Let us denote the solution of Eq. (44) by $f_i(x)$ (when $\eta(x) = 1-x$),$i = c$ or d, so

$$1-x=f_i^{-1}\left(\frac{A}{f_i(x)}\right). \tag{48}$$

Based on Trillas' result, we get

$$\eta(x)=f_*^{-1}(1-f_*(x))=f_*^{-1}\left(f_i^{-1}\left(\frac{A}{f_i(f_*(x))}\right)\right). \tag{49}$$

Let us define $f(x) = f_i(f_*(x))$, this is also a generator function of a t-norm (or t-conorm), so Eq. (42) is valid for $f(x)$. □

Remark 4. A DeMorgan triple can be built by using one the generator function of just one operator and choosing a $k(x)$. That is,

$$\eta(x)=f_c^{-1}(k(f_c(x))), \tag{50}$$

$$c(\mathbf{w},\mathbf{x})=f_c^{-1}\sum_{i=1}^{n}(w_if(x_i)), \tag{51}$$

$$d(\mathbf{w},\mathbf{x})=f_c^{-1}\left(k\left(\sum_{i=1}^{n}w_ik(f_c(x_i))\right)\right), \tag{52}$$

form a DeMorgan triple, and

$$f_c(x)=k(f_d(x)), \tag{53}$$

so $k(x)$ can be understood as a kind of negation.

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2.3. Examples of DeMorgan systems

Using the above results, we can construct classical systems and also new operator systems.

- If $f_c(x) = -\ln(x)$ and $\eta(x) = 1 - x$, then

$$k(x) = f_c(\eta(f_c^{-1}(x))) = -\ln(1 - e^{-x}) \quad c(x,y) = xy, \quad d(x,y) = x + y - xy.$$

- If $f_d(x) = -\ln(1 - x)$ and $\eta(x) = 1 - x$, then

$$k(x) = f_d(\eta(f_d^{-1}(x))) = -\ln(1 - e^{-x}) \quad c(x,y) = xy, \quad d(x,y) = x + y - xy.$$

We give a new example of where $\eta(x)$ can be varied.
If $f_c(x) = -\ln(x)$ and $f_d(x) = -\frac{1}{\ln(x)}$, then $k(x) = \frac{1}{x}$.

$$c(x,y) = xy \quad d(x,y) = e^{\frac{1}{\frac{1}{\ln x} + \frac{1}{\ln y}}} \quad \eta(x) = e^{\frac{a}{\ln(x)}}$$

where $a > 0$.
Here $d(1,x) = d(x,1) = \lim_{y \to 1} d(x,y) = 1$ and $\eta(1) = \lim_{x \to 1} \eta(x) = 0$ and $\eta(0) = \lim_{x \to 0} y(x) = 1$.

2.4. Parametric form of the negations

Lemma 1. *The parametric form of the negation is*

$$\eta(x) = f^{-1}\left(f(v_*)\frac{k(f(x))}{k(f(v_*))}\right), \tag{54}$$

$$\eta(x) = f^{-1}\left(f(v_0)\frac{k(f(x))}{k(f(v))}\right). \tag{55}$$

Proof. We will use the definitions of v_*, v and v_0 (i.e. $\eta(v_*) = v_*$ and $\eta(v) = v_0$ are valid). Using Theorem 2 and Eqs. (37) and (38) and $f_d(x) = k(f_c(x))$,

$$\eta(x) = f_d^{-1}\left(\frac{f_d(v_*)}{k(f_d(v_*))}k(f_d(x))\right),$$

$$\eta(x) = f_c^{-1}\left(\frac{f_c(v_*)}{k(f_c(v_*))}k(f_c(x))\right),$$

which is Eq. (54), i.e. here we can drop the index of *f*. Eq. (55) can be proved in a similar way. Negation does not depend on the type of operator used (i.e. strict t-conorm or strict t-norm). So we can drop the *c* and *d* indexes in Eqs. (54) and (55). ☐

3. Operators with an infinitely many negations

Now we will characterize the operator class (strict t-norm and strict t-conorm) for which various negations exist and build a DeMorgan class. The fixpoint v_* or the neutral value v can be regarded as decision threshold. Operators with various negations are useful because the threshold can be varied.

It is straightforward to see that the min and max operators belong to this class, as does the drastic operator. The next theorem characterizes those strict operator systems that have infinitely many negations and build a DeMorgan system. It is easy to see that $c(x,y) = xy$, $d(x,y) = x + y - xy$ and $\eta(x) = 1 - x$ build a DeMorgan system. There are no other negations for building a DeMorgan system, as we will see below.

Theorem 6. *$c(x,y)$ and $d(x,y)$ build a DeMorgan system for $\eta_{v_*}(x)$ where $\eta_{v_*}(v_*) = v_*$ for all $v_* \in (0,1)$ if and only if*

$$f_c(x)f_d(x) = 1. \tag{56}$$

Proof. Define

$$d_1(x,y) = \eta_{v_1}\left(f_c^{-1}\left(f_c(\eta_{v_1}(x)) + f_c(\eta_{v_1}(y))\right)\right) \tag{57}$$

and

$$d_2(x,y) = \eta_{v_2}\left(f_c^{-1}\left(f_c(\eta_{v_2}(x)) + f_c(\eta_{v_2}(y))\right)\right), \tag{58}$$

where $v_1 \neq v_2$.

Suppose that

$$d_1(x,y) = d_2(x,y). \tag{59}$$

This can be written as

$$f_c\left(t\left(f_c^{-1}(x'+y')\right)\right) = f_c\left(t\left(f_c^{-1}(x') + f\left(t\left(f_c^{-1}(y')\right)\right)\right)\right), \tag{60}$$

where $t(x) = \eta_2(\eta_1(x))$.

Since $t(x)$ is strictly monotonously increasing, $t(0) = 0$, $t(1) = 1$. We can assume that $t(x) \neq x$. If this is not the case, that $\eta_1(\eta_2(x)) = x$ therefore $\eta_1(x) = \eta_2^{-1}(x) = \eta_2(x)(\eta_2(x)$is involutive). This leads to contradiction because $v_1 \neq v_2$. Let

$$F_c(x) = f_c(t(f_c^{-1}(x))), \tag{61}$$

so Eq. (60) has the form

$$F_c(x+y) = F_c(x) + F_c(y). \tag{62}$$

The solution of the functional form of Eq. (62) is

$$F_c(x) = a_c x.$$

Using Eq. (61) we get

$$f_c(t(f_c^{-1}(x))) = a_c x,$$

thus

$$f_c(t(x)) = a_c f_c(x). \tag{63}$$

Similar considerations give:

$$f_d(t(x)) = a_d f_d(x). \tag{64}$$

Multiplying Eq. (63) by Eq. (64) and letting $f_c(x)f_d(x) = g(x)$, where g: $[0,1] \to [0,\infty]$, we have

$$g(t(x)) = a_c a_d g(x). \tag{65}$$

Because $t(x)$ is strict monotone, $t(x) \neq x$ and Eq. (65) is valid for all $x \in [0,1]$ and a_c, a_d are constant values, then the solution of Eq. (65) is $g(x) = \text{const}$ and $a_c = \frac{1}{a_d}$. So we get:

$$f_c(x)f_d(x) = \text{const}.$$

Because the generator function is determined up to a multiplicative constant we can get the result Eq. (57) of the theorem. □

4. Multiplicative Pliant systems

From Dombi's result [9] we know that if $f(x)$ is a generator function, then $f^\alpha(x)$ is a generator function. As we saw earlier $k(x)$ plays an important role in DeMorgan systems. Let us define the multiplicative pliant system by one of the simplest $k(x)$ functions.

Definition 4. If $k(x) = 1/x$, that is

$$f_c(x)f_d(x) = 1, \tag{66}$$

then we call the generated connectives a multiplicative pliant system.

If we have a generator function, then its power is also a generator function. Therefore in the pliant system we can use the power function of the generator function and define $f_c(x)$ by

$$f_c(x) = f^\alpha(x).$$

In Fig. 4 we can see a plot of the generator of pliant operator.

Remark 5. A similar operator system was in fact presented by Roychowdhury [25]. Theorem 6 above gives the necessary and sufficient conditions for a such system.

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Theorem 7. *The general form of the multiplicative pliant system is*

$$o_\alpha(x,y) = f^{-1}\left((f^\alpha(x) + f^\alpha(y))^{1/\alpha}\right) \quad (67)$$

$$\eta_v(x) = f^{-1}\left(f(v_0)\frac{f(v)}{f(x)}\right) \quad \text{or} \quad (68)$$

$$\eta_{v_*}(x) = f^{-1}\left(\frac{f^2(v_*)}{f(x)}\right), \quad (69)$$

where f(x) is the generator function of the strict t-norm operator and f: [0, 1] → [0, ∞] continuous and strictly decreasing function. Depending on the value of α, the operator is

$$\begin{aligned} \alpha > 0 \quad & o_\alpha(x,y) = c(x,y), \\ \alpha < 0 \quad & o_\alpha(x,y) = d(x,y), \end{aligned} \quad (70)$$

$$\begin{aligned} &\lim_{\alpha\to\infty} o_\alpha(x,y) = \min(x,y), \\ &\lim_{\alpha\to-\infty} o_\alpha(x,y) = \max(x,y), \end{aligned} \quad (71)$$

$$\alpha = 0^+ \quad \lim_{\alpha\to 0^+} o_\alpha(x,y) = \begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{otherwise}, \end{cases} \quad (72)$$

$$\alpha = 0^- \quad \lim_{\alpha\to 0^-} o_\alpha(x,y) = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 1 & \text{otherwise}, \end{cases} \quad (73)$$

This operator called the drastic operator.

Proof. The $\alpha \geqslant 0$ case can be proved using the fact that $f_\alpha^{-1}(x) = f(x^{1/\alpha})$. From the involution of $\eta(x)$ we have

$$f_c(x) = \frac{1}{a}\frac{1}{f(x)}, \quad f_c^{-1}(x) = f_d^{-1}\left(\frac{a}{x}\right)$$

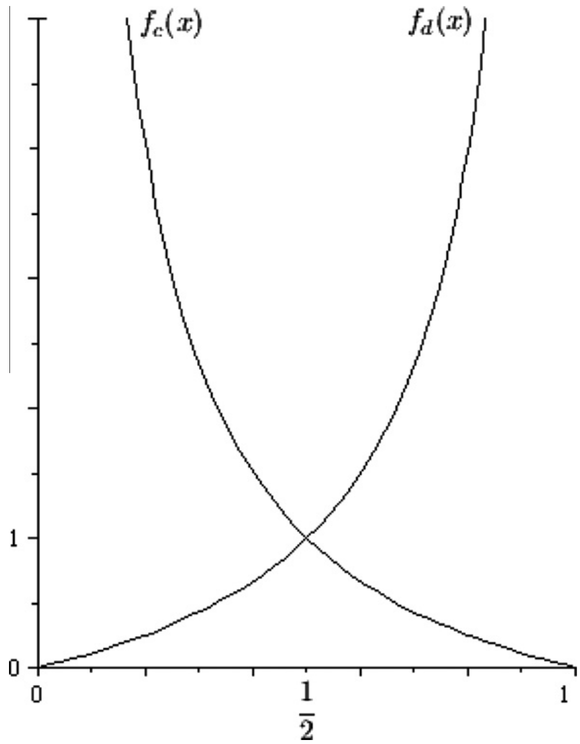

Fig. 4. The pliant generator function $f_c(x) = \frac{1}{f_d(x)}$. Here $f_c(x) = (\frac{1-x}{x})^2$ and $f_d(x) = (\frac{x}{1-x})^2$.

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and so

$$\begin{aligned} d(x,y) &= f_d^{-1}(f_d(x)+f_d(y)) \\ &= f_c^{-1}\left(\frac{1}{f_c(x)}+\frac{1}{f_c(y)}\right)^{-1} \\ &= f^{-1}(f^{-\alpha}(x)+f^{-\alpha}(y))^{-1/\alpha}. \end{aligned}$$

The form of negation comes from Eqs. (54) and (55).

If $\eta_{v_*}(x)$ has the same form as that in Eq. (54) and $k(x)=\frac{1}{x}$, we get:

$$\eta_{v_*}(x) = f^{-1}\left(\frac{f^2(v_*)}{f(x)}\right). \tag{74}$$

Let f_c be a generator function of a strict t-norm operator c, and $\alpha > 0$. Then, since $f_c(1) = 0$ implies $f_c^\alpha(1) = 0$ and since positive powers do not affect monotonicity, f_c^α is also a generator function of a strict t-norm operator denoted by c_α. Let $x < y$ then

$$\begin{aligned} c_\alpha(x,y) &= f_c^{-1}\left((f_c^\alpha(x)+f_c^\alpha(y))^{1/\alpha}\right) \\ &= f_c^{-1}\left(f_c(x)\left(1+\frac{f_c^\alpha(y)}{f_c^\alpha(x)}\right)^{1/\alpha}\right). \end{aligned} \tag{75}$$

Because $A = f_c^\alpha(y)/f_c^\alpha(x) < 1$ and

$$\lim_{\alpha\to\infty}(1+A^\alpha)^{1/\alpha} = 1 \quad 0 < A < 1,$$

it follows that

$$\lim_{\alpha\to\infty} c_\alpha(x,y) = x = \min(x,y).$$

For the drastic operator it is not hard to see that

$$c_\alpha(x,1) = 1, \quad d_\alpha(x,0) = x, \tag{76}$$

i.e. the boundary conditions are satisfied.

In the conjunction case we have to prove that

$$\lim_{x\to 0}(f^\alpha(x)+f^\alpha(y))^{\frac{1}{\alpha}} = \infty, \tag{77}$$

which is self-evident. □

Remark 6. In the multiplicative pliant system it is vital that negation be independent of the value and the sign of α. (In other words, it does not depend on whether the generator function belongs to the strict t-norm or strict t-conorm.)

Remark 7. The limit values of the pliant operators (min, max and drastic) also have the property that the DeMorgan triplet is valid for infinitely many negations.

Theorem 8. *If $g(x) = f^\alpha(x)$ is the generator function, negation does not change in the pliant system.*

Proof.

$$\begin{aligned} \eta_{v,v_0}(x) &= g^{-1}\left(g(v_0)\frac{g(v)}{g(x)}\right) = f^{-1}\left(\left(f^\alpha(v_0)\frac{f^\alpha(v)}{f^\alpha(x)}\right)^{1/\alpha}\right) \\ &= f^{-1}\left(f(v_0)\frac{f(v)}{f(x)}\right). \qquad \square \end{aligned}$$

Theorem 9. *Let $c(x,y) = f^{-1}(f(x)+f(y)), d(x,y) = f^{-1}\left(\frac{1}{\frac{1}{f(x)}+\frac{1}{f(y)}}\right)$ and $\eta(x) = f^{-1}\left(f(v_0)\frac{f(v)}{f(x)}\right)$, then c(x,y), d(x,y) and $\eta(x)$ form a DeMorgan triplet.*

Proof. The proof of this is straightforward.

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5. Summary

We can summarize the elements of the multiplicative pliant system (operators and their weighted form) like so:

$$c(\mathbf{x}) = f^{-1}\left(\sum_{i=1}^{n} f(x_i)\right) \quad c(\mathbf{w},\mathbf{w}) = f^{-1}\left(\sum_{i=1}^{n} w_i f(x_i)\right), \tag{78}$$

$$d(\mathbf{x}) = f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\frac{1}{f(x_i)}}\right) \quad d(\mathbf{w},\mathbf{x}) = f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\frac{w_i}{f(x_i)}}\right), \tag{79}$$

$$a(\mathbf{x}) = f^{-1}\left(\prod_{i=1}^{n} f(x_i)\right) \quad a(\mathbf{w},\mathbf{x}) = f^{-1}\left(\prod_{i=1}^{n} f^{w_i}(x_i)\right), \tag{80}$$

$$\eta(x) = f^{-1}\left(\frac{f^2(\nu_*)}{f(x)}\right), \tag{81}$$

where $f(x)$ is the generator function of the strict t-norm operator and in (80) $a(\mathbf{x})$ is the aggregative operator (i.e. representable uninorms, see [9,12,16]).

5.0.1. The operator system of Dombi

The Dombi operators form a pliant system and the operators are:

$$c(\mathbf{x}) = \frac{1}{1+\left(\sum_{i=1}^{n}\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{1/\alpha}} \quad c(\mathbf{w},\mathbf{x}) = \frac{1}{1+\left(\sum_{i=1}^{n} w_i\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{1/\alpha}}, \tag{82}$$

$$d(\mathbf{x}) = \frac{1}{1+\left(\sum_{i=1}^{n}\left(\frac{1-x_i}{x_i}\right)^{-\alpha}\right)^{-1/\alpha}} \quad d(\mathbf{w},\mathbf{x}) = \frac{1}{1+\left(\sum_{i=1}^{n} w_i\left(\frac{1-x_i}{x_i}\right)^{-\alpha}\right)^{-1/\alpha}}, \tag{83}$$

$$a(\mathbf{x}) = \frac{1}{1+\prod_{i=1}^{n}\frac{1-x_i}{x_i}} \quad a(\mathbf{w},\mathbf{x}) = \frac{1}{1+\prod_{i=1}^{n}\left(\frac{1-x_i}{x_i}\right)^{w_i}}, \tag{84}$$

$$\eta(x) = \frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\frac{x}{1-x}}, \tag{85}$$

where $\nu_* \in]0,1[$, with generator functions

$$f_c(x) = \left(\frac{1-x}{x}\right)^{\alpha} \quad f_d(x) = \left(\frac{1-x}{x}\right)^{-\alpha} \tag{86}$$

where $\alpha > 0$. The operators c, d and η fulfil the DeMorgan identity for all ν, a and η fulfil the self DeMorgan identity for all ν and the aggregative operator is distributive with the logical operators.

Eqs. (82)–(85) can be found in different articles by Dombi. Eqs. (82) and (83) can be found in [8], (84) in [9] and Eq. (85) can be found in [10]. These are all previous results by Dombi.

Eq. (84) is called 3π operator because it can be written in the following form:

$$a(\mathbf{x}) = \frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i + \prod_{i=1}^{n}(1-x_i)}. \tag{87}$$

The main results of this article can be summarized in the following way.

Given a strict t-norm c and a strict t-conorm d with generators f_c, f_d the paper determines the conditions for which a strong negation η exists such that c, d, η form a DeMorgan triple. To this end, a helper negation function k: $[0,\infty] \to [0,\infty]$ is required. For one particular $k(x) = \frac{1}{x}$ the conditions on f_c, f_d are given such that c, d, η form a DeMorgan triple, where η was obtained using k.

- We employ weighted operators. See Eqs. (6) and (7).
- We provide an involutive negation operator given $f_c(x)$ and $f_d(x)$. See in Eq. (25).
- We give the general form of the DeMorgan triplet using the $k(x)$ function. See Eqs. (50)–(52).
- We give the parametric form of the negation operator. See Eqs. (54) and (55).
- We show that the DeMorgan triplet has infinitely many negation operators if and only if $f_c(x)f_d(x) = 1$ (the main result) and such a system is called a pliant system. This condition is the same if the representable uninorm (aggregative operator) corresponds to the strict t-norms and strict t-conorms.

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- We give the general form of the pliant operators. See Eqs. (78)–(81).
- We show that consistent aggregation can be achieved.
- The special case of the pliant system is the Dombi operator class.

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