On a certain class of aggregative operators

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Abstract

Here our starting point is a study of connection with Dombi aggregative operators, uninorms, strict t-norms and t-conorms. We present a new representation theorem of strong negations that explicitly contains the neutral value. Then the relationships for aggregative operators and strong negations are verified as well as those for the t-norm and t-conorm using the Pan operator concept. We introduce the multiplicative pliant concept and give the necessary and sufficient conditions for it. We study a certain class of weighted aggregative operators (representable uninorms) which build a self-DeMorgan class wih infinitely many negations. We provide the necessary and sufficient conditions for these operators.

Keywords: aggregative operators, uninorms, pliant systems, Dombi operators

1 Introduction

The term *uninorm* was first introduced by Yager and Rybalov [37] in 1996. Uninorms are generalization of t-norms and t-conorms by relaxing the constraint on the identity element from $\{0, 1\}$ to the unit interval. Since then many articles have focused on uninorms, both from a theoretical [7, 17, 21, 22, 24, 28] and a practical point of view [36]. The paper of Fodor, Yager and Rybalov [12] is important since it defined a new subclass of uninorms called representable uninorms. This characterization is similar to the representation theorem of strict t-norms and t-conorms, in the sense that both originate from the solution of the associativity functional equation given by Aczél [1].

The aggregative operators were introduced in the paper [9] by selecting a set of minimal concepts which must be fulfilled by an evaluation like operator.

As mentioned in [12], there is a close relationship between Dombi's aggregative operators and uninorms.

We shall distinguish between logical operators (strict, continuous t-norms and t-conorms) and aggregative operators, where the former means strict, continuous operators.

The first goal is to show the close correspondence between strong negations, aggregative and logical operators. The second goal is to introduce and characterize multiplicative pliant operator systems.

The reader may recall that the field of uninorm, t-norm, t-conorm and its application were discussed in recent Information Science issues. For example see articles [2, 4, 6, 14–16, 20, 23, 25, 29, 35, 38].

This paper is organized as follows. First we give some basic definitions. We emphasis the role of the neutral value in Section 2. Section 3 describes the correspondence between strong negations, aggregative and logical operators. Lastly, we present and give a characterization the so-called multiplicative pliant systems using examples in Section 4. We study a certain class of weighted aggregative operators (representable uninorms) which build a self-DeMorgan class wih infinitely many negations. We provide the necessary and sufficient conditions for these operators.

1.1 Basic definition and known results

In 1982 Dombi [9] defined the aggregative operator in the following way:

Definition 1. *An aggregative operator is a function* $a : [0,1]^2 \to [0,1]$ *with the properties:*

1. *Continuous on* $[0,1]^2 \backslash \{(0,1),(1,0)\}$

2. $a(x,y) < a(x,y')$ *if* $y < y', x \neq 0, x \neq 1$
   $a(x,y) < a(x',y)$ *if* $x < x', y \neq 0, y \neq 1$

3. $a(0,0) = 0$ *and* $a(1,1) = 1$ *(boundary conditions)*

4. $a(x,a(y,z)) = a(a(x,y),z)$ *(associativity)*

5. *There exists a strong negation* η *such that* $a(x,y) = \eta(a(\eta(x),\eta(y)))$ *(self DeMorgan identity) if* $\{x,y\} \neq \{0,1\}$ *or* $\{x,y\} \neq \{1,0\}$

6. $a(1,0) = a(0,1) = 0$ *or* $a(1,0) = a(0,1) = 1$

We note that the original definition of aggregative operators has the condition of correct cluster formation instead of the self DeMorgan identity (see [9]), which later proved to be equivalent.

For the sake of completeness, strong negation will be defined by the following:

Definition 2. $\eta(x)$ *is strong negation iff* $\eta : [0,1] \to [0,1]$ *satisfies the following conditions:*

1. $\eta(x)$ *is continuous*

2. $\eta(0) = 1$, $\eta(1) = 0$ *(boundary conditions)*

3. $\eta(x) < \eta(y)$ *for* $x > y$ *(monotonicity)*

4. $\eta(\eta(x)) = x$ *(involution)*

The definition of uninorms, originally given by Yager and Rybalov [37] in 1996, is the following:

Definition 3. *A uninorm* U *is a mapping* $U : [0,1]^2 \to [0,1]$ *having the following properties:*

- $U(x,y) = U(y,x)$ *(commutativity)*

- $U(x_1,y_1) \geq U(x_2,y_2)$ *if* $x_1 \geq x_2$ *and* $y_1 \geq y_2$ *(monotonicity)*

- $U(x,U(y,z)) = U(U(x,y),z)$ *(associativity)*

- $\exists \nu_* \in [0,1] \, \forall x \in [0,1] \, U(x,\nu_*) = x$ *(neutral element)*

A uninorm is a generalization of t-norms and t-conorms. By adjusting its neutral element, a uninorm is a t-norm if $\nu_* = 1$ and a t-conorm if $\nu_* = 0$. The following representation theorem of strict, continuous on $[0,1] \times [0,1] \setminus (\{0,1\},\{1,0\})$ uninorms (or *representable uninorms*) was given by Fodor et al. [12] (see also Klement et al. [18]).

Theorem 1. *Let* $U : [0,1] \to [0,1]$ *be a function and* $\nu_* \in]0,1[$. *The following are equivalent:*

1. U *is a uninorm with neutral element* ν_* *which is strictly monotone on* $]0,1[^2$ *and continuous on* $[0,1]^2 \backslash \{(0,1),(1,0)\}$.

2. *There exists a strictly increasing bijection $g_u : [0, 1] \to [-\infty, \infty]$ with $g_u(\nu_*) = 0$ such that for all $(x, y) \in [0, 1]^2$ we have*

$$U(x, y) = g_u^{-1}\left(g_u(x) + g_u(y)\right),\tag{1}$$

where, in the case of a conjunctive uninorm U, we use the convention $\infty + (-\infty) = -\infty$, while, in the disjunctive case, we use $\infty + (-\infty) = \infty$.

If Eq.(1) holds, the function g_u is uniquely determined by U up to a positive multiplicative constant, and it is called an additive generator of the uninorm U.

Strong negation plays an important role. Besides Trillas' representation theorem, we introduced another form of negation [9]. In the following theorem which is well-known, we show that this representation is universal.

Theorem 2. *Let $\eta : [0, 1] \to [0, 1]$ be a continuous function, then the following are equivalent:*

1. *η is a strong negation.*

2. *There exists a continuous and strictly monotone function $g : [0, 1] \to [-\infty, \infty]$ with $g(\nu_*) = 0$, $\nu_* \in]0, 1[$ such that for all $x \in [0, 1]$*

$$\eta(x) = g^{-1}(-g(x)).\tag{2}$$

If Eq.(2) holds, then ν_* is called the neutral value of the strong negation, i.e. for which $\eta(\nu_*) = \nu_*$.

   Sketch of the proof. The representation theorem of Trillas [33] states that all strong negations can be written as
$$\eta(x) = \varphi^{-1}\left(1 - \varphi(x)\right),\tag{3}$$
where φ is an automorphism of the unit interval. Let

$$g(x) = \ln\left(\frac{1}{\varphi(x)} - 1\right).\tag{4}$$

It is easy to see that the Theorem 2 is valid for this function.

∎

2   The neutral value

Theorem 2 tells us that all strong negations have the form $g^{-1}(-g(x))$ for a suitable g generator function. In this formula the neutral value of the strong negation is implicitly present the generator function. The following representation theorem of strong negations explicitly contains the neutral value.

Theorem 3 (Additive form of strong negations). *Let $\eta : [0, 1] \to [0, 1]$ be a continuous function, then the following are equivalent:*

1. *η is a strong negation with neutral value ν_*.*

2. *There exists a continuous and strictly monotone function $g : [0, 1] \to [-\infty, \infty]$ and $\nu_* \in]0, 1[$ such that for all $x \in [0, 1]$*

$$\eta(x) = g^{-1}(2g(\nu_*) - g(x)).\tag{5}$$

Proof. A similar proof can be found in Dombi [9].

Suppose $\eta(x) = g^{-1}(2g(\nu_*) - g(x))$, then it is not hard to check that η is a strong negation with neutral value ν_*.

Suppose η_* is a strong negation. By Theorem 2 there exists a generator function g_* for which $\eta_*(x) = g_*^{-1}(-g_*(x))$ and $\nu_* = g_*(0)$. Let $g : [0,1] \to [-\infty, \infty]$ be a continuous, strictly monotone function such that $g_*(x) = g(x) - g(\nu_*)$, i.e. $g_*^{-1}(x) = g^{-1}(x + g(\nu_*))$. Then

$$\eta_*(x) = g_*^{-1}(-g_*(x)) = g^{-1}(2g(\nu_*) - g(x)). \tag{6}$$

∎

Similar to strong negations, the representation theorem of aggregative operators (Theorem 1) does not explicitly contain the neutral value of the aggregative operator. With the help of the following lemma, a new representation can be given.

Lemma 1 (Dombi [9]). *If g is the additive generator function of an aggregative operator a, then the function displaced by $d \in \mathbb{R}$, $g_*(x) = g(x) + d$ is also a generator function of an aggregative operator with neutral value $\nu_* = g^{-1}(-d)$.*

Proof. It follows from $g_*(x)$ that $g_*^{-1}(x) = g^{-1}(x - d)$. Then

$$\begin{aligned} a_*(x,y) &= g_*^{-1}\left(g_*(x) + g_*(y)\right) = \\ &= g^{-1}\left(g(x) + g(y) + d\right). \end{aligned} \tag{7}$$

Substituting $x = y = \nu_*$ and using the fact that $g(\nu_*) = -d$,

$$\begin{aligned} \nu_* &= a_*(\nu_*, \nu_*) = \\ &= g^{-1}\left(g(\nu_*) + g(\nu_*) + d\right) = g^{-1}(-d). \end{aligned} \tag{8}$$

∎

Theorem 4 (Dombi [9]). *Let $a : [0,1]^n \to [0,1]$ be a function and let a be an aggregative n-valued operator with additive generator g. The neutral value of the aggregative operator is ν_* if and only if $\mathbf{x} \in [0,1]^n, \forall x$. It has the following form:*

$$a(\boldsymbol{x}) = g^{-1}\left(\sum_{i=1}^{n} g(x_i) - (n-1)g(\nu_*)\right), \text{ or} \tag{9}$$

$$a_{\nu_*}(\boldsymbol{x}) = g^{-1}\left(g(\nu_*) + \sum_{i=1}^{n}(g(x_i) - g(\nu_*))\right).$$

Proof. The proposition follows from Theorem 1, Lemma 1 and the associativity of the aggregative operator.

∎

According to this, one can construct an aggregative operator from any given generator function that has the desired neutral value.

There are infinitely many possible neutral values, and with each different neutral value, a different aggregative operator can be given.

Definition 4. *Let a_f and a_g be aggregative operators, with the additive generator functions f and g, respectively. The functions a_f and a_g belong to the same family if $f(x) = g(x) + d$, for all $x \in [0,1]$ and a suitable $d \in \mathbb{R}$. Note that a_f and a_g do not necessarily have the same neutral value.*

The following theorem shows that there is a one-to-one correspondence between aggregative operators and strong negations. (see [9] as well)

Theorem 5. *Let a be an aggregative operator with generator function g and neutral value ν_*. The only strong negation satisfies the self DeMorgan identity is*

$$\eta(x) = g^{-1}\left(2g(\nu_*) - g(x)\right). \tag{10}$$

Proof. *(Existence)* By Theorem 4,

$$
\begin{aligned}
\eta(a(\eta(x), \eta(y))) &= g^{-1}(g(x) + g(y) - g(\nu_*)) = \\
&= g^{-1}(2g(\nu_*) - 2g(\nu_*) + g(x) - 2g(\nu_*) + g(y) + g(\nu_*)) = \\
&= a(x, y).
\end{aligned} \tag{11}
$$

(Unicity) The functions for which

$$
\begin{aligned}
a_*(\mathbf{x}) &= g^{-1}\left(\sum_{i=1}^{n} g(x_i) - (n-1)g(\nu_*)\right) = \\
&= g_*^{-1}\left(\sum_{i=1}^{n} g_*(x_i)\right)
\end{aligned} \tag{12}
$$

fulfils the self DeMorgan identity are

$$\eta_*(x) = g_*^{-1}(cg_*(x)), \tag{13}$$

where $c \in \mathbb{R}$. The involution of η_* means that

$$
\begin{aligned}
\eta_*(\eta_*(x)) &= g_*^{-1}(cg_*(g_*^{-1}(cg_*(x)))) = \\
&= g_*^{-1}(c^2 g_*(x)) = x,
\end{aligned} \tag{14}
$$

which can only be true if $c = \pm 1$. For $c = 1$ $\eta_*(x)$ is the identity function, and for $c = -1$ $\eta_*(x)$ is strong negation. Using the substitutions $g_*(x) = g(x) - g(\nu)$ and $g_*^{-1}(x) = g^{-1}(x + g(\nu))$, we get

$$\eta_*(x) = g^{-1}\left(2g(\nu_*) - g(x)\right). \tag{15}$$

∎

Definition 5. *Let a be an aggregative operator with the additive generator function g and neutral value ν_*. Let us call the strong negation $\eta(x) = g^{-1}\left(2g(\nu_*) - g(x)\right)$ the corresponding strong negation of the aggregative operator.*

By Theorem 5 every aggregative operator has exactly one corresponding strong negation, which is a strong negation that fulfils the self DeMorgan identity. Conversely, every strong negation has exactly one corresponding aggregative operator.

Definition 6. *The neutral element ν_* of the aggregative operator has the property*

$$a(x, \nu_*) = x. \tag{16}$$

strict t-norm	the corresponding aggregative operator
Dombi	$\dfrac{1}{1+\frac{1-x}{x}\frac{1-y}{y}}$
Frank	$\log_s\left(1+(s-1)\exp\left[-\ln\frac{s^x-1}{s-1}\ln\frac{s^y-1}{s-1}\right]\right)$
Hamacher	$\alpha\left(\alpha-1+\exp\left[\ln\left(\frac{\alpha+(1-\alpha)x}{x}\right)\ln\left(\frac{\alpha+(1-\alpha)y}{y}\right)\right]\right)^{-1}$
Aczél-Alsina	$\exp\left[-\ln(x)\ln(y)\right]$

strict t-conorm	the corresponding aggregative operator
Dombi	$\dfrac{1}{1+\frac{1-x}{x}\frac{1-y}{y}}$
Frank	$1-\log_s\left(1+(s-1)\exp\left[-\ln\frac{s^{1-x}-1}{s-1}\ln\frac{s^{1-y}-1}{s-1}\right]\right)$
Hamacher	$\dfrac{\exp\left[\ln\left(\frac{1+\beta x}{1-x}\right)\ln\left(\frac{1+\beta y}{1-y}\right)\right]-1}{\exp\left[\ln\left(\frac{1+\beta x}{1-x}\right)\ln\left(\frac{1+\beta y}{1-y}\right)\right]-\beta}$
Aczél-Alsina	$1-\exp\left[-\ln(1-x)\ln(1-y)\right]$

Table 1: The corresponding aggregative operators of the principal strict t-norm or t-conorm

strict t-norm	the corresponding negation
Dombi	$\dfrac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\frac{x}{1-x}}$
Frank	$\log_s\left(1+(s-1)\exp\left[\ln^2\frac{s_*^\nu-1}{s-1}\left(\ln\frac{s^x-1}{s-1}\right)^{-1}\right]\right)$
Hamacher	$\dfrac{1}{1+\frac{1}{\alpha}\left(\exp\left[\ln^2\left(1+\alpha\frac{1-\nu_*}{\nu_*}\right)\left(\ln\left(1+\alpha\frac{1-x}{x}\right)\right)^{-1}\right]-1\right)}$
Aczél-Alsina	$\exp\left[\frac{\ln^2\nu_*}{\ln x}\right]$

strict t-conorm	the corresponding negation
Dombi	$\dfrac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\frac{x}{1-x}}$
Frank	$1-\log_s\left(1+(s-1)\exp\left[\ln^2\frac{s^{1-\nu_*}-1}{s-1}\left(\ln\frac{s^{1-x}-1}{s-1}\right)^{-1}\right]\right)$
Hamacher	$\dfrac{1}{1+\beta\left(\exp\left[\ln^2\left(1+\beta\frac{\nu_*}{1-\nu_*}\right)\left(\ln\left(1+\beta\frac{x}{1-x}\right)\right)^{-1}\right]-1\right)^{-1}}$
Aczél-Alsina	$1-\exp\left[\frac{\ln^2(1-\nu_*)}{\ln(1-x)}\right]$

Table 2: The corresponding negations of the principal strict t-norm or t-conorm

3 On additive and multiplicative representations of the operators

Let

$$c(x,y) = f_c^{-1}\left(f_c(x) + f_c(y)\right) \qquad d(x,y) = f_d^{-1}\left(f_d(x) + f_d(y)\right),$$

where f_c and f_d are the generator functions of the operators. The shape of these function can be seen in Figure 1.
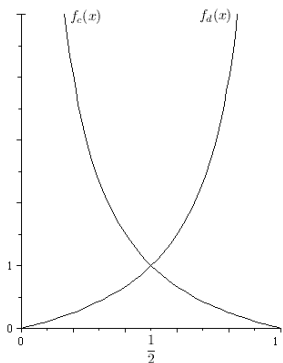

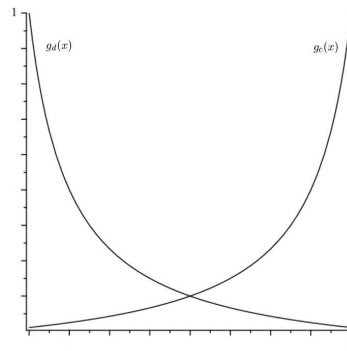Figure 1: The generator function of the conjunctive and disjunctive operators (additive representation)

Figure 2: The generator function of the conjunctive and disjunctive operators (multiplicative representation)

Let

$$g_c(x) = e^{-f_c(x)} \qquad g_d(x) = e^{-f_d(x)} \tag{17}$$

Then

$$f_c(x) = -\ln(g_c(x)) \qquad f_d(x) = -\ln(g_d(x)). \tag{18}$$

So

$$c(x,y) = f_c^{-1}\left(-\ln(g_c(x)) - \ln(g_c(y))\right) = g_c^{-1}\left(e^{-(-\ln(g_c(x)) - \ln(g_c(y)))}\right)$$
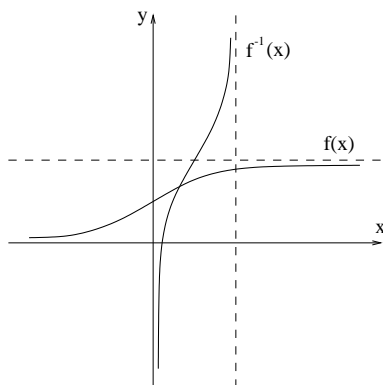

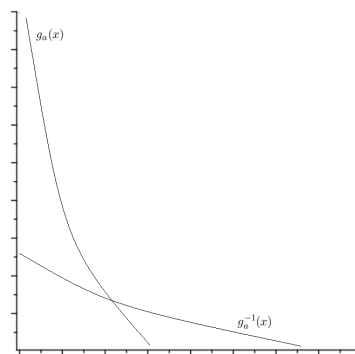Figure 3: The generator function of the aggregative operator in additive representation case

Figure 4: The generator function of the aggregative operator in multiplicative representation case

3.1 The multiplicative form of the aggregative operator

We will use the transformation defined in (17) and (18) to get the multiplicative operator

$$a_{\nu_*}(x) = f^{-1}\left(f(\nu_*)\prod_{i=1}^{n}\frac{f(x_i)}{f(\nu_*)}\right) = f^{-1}\left(f^{1-n}(\nu_*)\prod_{i=1}^{n}f(x_i)\right) \tag{19}$$

In the Dombi operator case, we get

$$a_{\nu_*}(\mathbf{x}) = \cfrac{1}{1 + \left(\frac{\nu_*}{1-qnu_*}\right)^{n-1}} \prod_{i=1}^{n} \frac{1-x_i}{x_i} = \cfrac{(1-\nu_*)^{n-1}\prod_{i=1}^{n} x_i}{(1-\nu_*)^{n-1}\prod_{i=1}^{n} x_i + \nu_*^{n-1}\prod_{i=1}^{n}(1-x_i)} \tag{20}$$

$$a_{\nu_*}(\mathbf{x}) = \cfrac{1}{1 + \frac{1-\nu_*}{\nu_*}\prod_{i=1}^{n}\left(\frac{1-x_i}{x_i}\frac{\nu_*}{1-\nu_*}\right)} \tag{21}$$

If $\nu_* = \frac{1}{2}$, then we get

$$a_{\frac{1}{2}}(\mathbf{x}) = \cfrac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i + \prod_{i=1}^{n}(1-x_i)}. \tag{22}$$
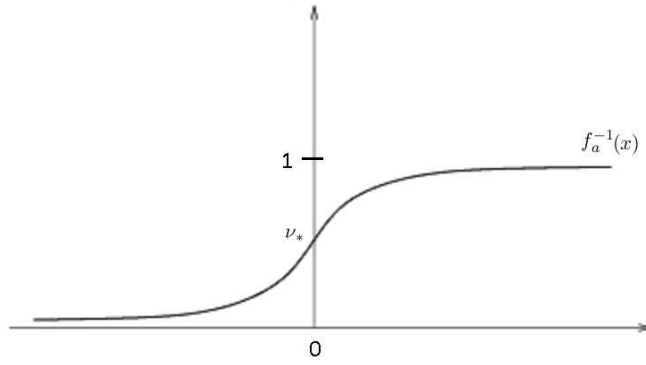

Figure 5: ν_* is the neutral element of the aggregative operator

(22) is called $3\,\Pi$ operator, i.e. it consists of three product operators [9].

3.2 Determining the ν_* value in the aggregative operator case

Given $x_1, x_2, \ldots, x_n$ and also z, its aggregative value, we can express z in terms of the x_i variables like so:

$$z = f^{-1}\left(f^{1-n}(\nu_*)\prod_{i=1}^{n} f(x_i)\right). \tag{23}$$

For ν_*, we have

$$\nu_* = f^{-1}\left(\left(\cfrac{f(z)}{\prod_{i=1}^{n} f(x_i)}\right)^{\frac{1}{1-n}}\right) = f^{-1}\left(\left(\cfrac{\prod_{i=1}^{n} f(x_i)}{f(z)}\right)^{\frac{1}{n-1}}\right) \tag{24}$$

In the Dombi operator case

$$\nu_* = \cfrac{1}{1 + \left(\frac{z}{1-z}\prod_{i=1}^{n}\frac{1-x_i}{x_i}\right)^{\frac{1}{n-1}}} \tag{25}$$

8

4 Aggregative operator and negation

4.1 The negation of the aggregative operator

The basic identity of the aggregative operator is the self-De Morgan identity:

$$\eta(a_{\nu_*}(x,y)) = a_{\nu_*}(\eta(x), \eta(y)). \tag{26}$$

Let $y = \eta(x)$. Then

$$\eta(a_{\nu_*}(x, \eta(x))) = a_{\nu_*}(\eta(x), x).$$

Because $a_{\nu_*}(x,y) = a_{\nu_*}(y,x)$, we get

$$a_{\nu_*}(x, \eta(x)) = \nu_*.$$

Using the multiplicative form of the aggregative operator

$$f^{-1}\left(\frac{f(x)f(\eta(x))}{f(\nu_*)}\right) = \nu_*. \tag{27}$$

Expressing $\eta(x)$, we have

$$\eta_{\nu_*}(x) = f^{-1}\left(\frac{f^2(\nu_*)}{f(x)}\right). \tag{28}$$

4.2 The Self-De Morgan identity with two different negation operators

Let us suppose that the following identity holds:

$$a_\nu\left(\eta_{\nu_1}(x), \eta_{\nu_1}(y)\right) = \eta_{\nu_2}(a_\nu(x,y)), \tag{29}$$

in analogy to (26), where

$$\eta_\nu(x) = f^{-1}\left(\frac{f^2(\nu)}{f(x)}\right). \tag{30}$$

Theorem 6. *(29) is valid if and only if*

$$\nu = \eta_{\nu_1}(\nu_2). \tag{31}$$

    Proof. (29) has the form:

$$f^{-1}\left(\frac{\frac{f^2(\nu_1)}{f(x)}\frac{f^2(\nu_1)}{f(y)}}{f(\nu)}\right) = f^{-1}\left(\frac{f^2(\nu_2)}{\frac{f(x)f(y)}{f(\nu)}}\right),$$

or, in terms of ν, we have

$$\nu = f^{-1}\left(\frac{f^2(\nu_1)}{f(\nu_2)}\right) = \eta_{\nu_1}(\nu_2).$$

4.3 Infinitely many negation when the self-De Morgan identity holds

The general form of the weighted operator in the additive representation case is

$$a(\mathbf{w}, \mathbf{x}) = f^{-1}\left(\sum_{i=1}^n w_i f(x_i)\right). \tag{32}$$

    We will derive the weighted aggregative operators when ν_* is given.
    First, we use the construction

$$f_1(x) = f_a(x) - f_a(\nu_*) \qquad f_1^{-1}(x) = f_a(x + f(\nu_*)),$$

where $\nu_* \in (0, 1)$.

$$a_{\nu_*}(\mathbf{w}, \mathbf{x}) = f_1^{-1}\left(\sum_{i=1}^{n} w_i f_1(x_i)\right) =$$

$$= f_a^{-1}\left(\sum_{i=1}^{n} w_i(f_a(x_i) - f_a(\nu_*)) + f_a(\nu_*)\right) \qquad (33)$$

$$= f_a^{-1}\left(\sum_{i=1}^{n} w_i f_a(x_i) + f(\nu_*)\left(1 - \sum_{i=1}^{n} w_i\right)\right)$$

To get the multiplicative form of the aggregative operator, we will use (18)

$$a_{\nu_*}(\mathbf{w}, \mathbf{x}) = f^{-1}\left(f(\nu_*)\prod_{i=1}^{n}\left(\frac{f(x)}{f(\nu_*)}\right)^{w_i}\right) = f^{-1}\left(f^{1-\sum_{i=1}^{n} w_i}(\nu_*)\prod_{i=1}^{n} f^{w_i}(x_i)\right) \qquad (34)$$

From (4.3) if $\sum_{i=1}^{n} w_i = 1$, then $a_{\nu_*}(\mathbf{w}, \mathbf{x})$ is independent of ν_* and

$$a(\mathbf{w}, \mathbf{x}) = f^{-1}\left(\prod_{i=1}^{n} f^{w_i}(x_i)\right). \qquad (35)$$

In the Dombi operator case:

$$a_{\nu_*}(\mathbf{w}, \mathbf{x}) = \frac{1}{1 + \frac{1-\nu_*}{\nu_*}\prod_{i=1}^{n}\left(\frac{1-x_i}{x_i}\frac{\nu_*}{1-\nu_*}\right)} \qquad (36)$$

$$a_{\nu_*}(\mathbf{w}, \mathbf{x}) = \frac{\nu_*(1-\nu_*)^{\sum_{i=1}^{n} w_i}\prod_{i=1}^{n} x_i^{w_i}}{\nu_*(1-\nu_*)^{\sum_{i=1}^{n} w_i}\prod_{i=1}^{n} x_i^{w_i} + (1-\nu_*)\nu_*^{\prod_{i=1}^{n} w_i}\prod_{i=1}^{n}(1-x_i)^{w_i}} \qquad (37)$$

If $\sum_{i=1}^{n} w_i = 1$, then

$$a(\mathbf{w}, \mathbf{x}) = \frac{\prod_{i=1}^{n} x_i^{w_i}}{\prod_{i=1}^{n} x_i^{w_i} + \prod_{i=1}^{n}(1-x_i)_i^{w}}. \qquad (38)$$

For the next theorem we have to proof the following lemma:

Lemma 2. *The general solutions of the functional equation*

$$\prod_{i=1}^{n} F^{a_i}(x_i) = F(\prod_{i=1}^{n} x_i^{a_i}) \qquad a_i \neq 0 \qquad (39)$$

is

$$F(x) = \alpha x^{\beta} \quad if \quad \sum_{i=1}^{n} a_i = 1 \qquad (40)$$

$$F(x) = x^{\beta} \quad if \quad \sum_{i=1}^{n} a_i \neq 1 \qquad (41)$$

Proof. Let for all i $x_j = (\prod_{i=1}^{n} \hat{x}_j{}^{a_j})^{\frac{1}{\sum_{j=1}^{n} a_j}} = \prod_{j=1}^{n} \hat{x}_j{}^{p_j}$ $\sum_{j=1}^{n} p_j = 1$ (39)   has the form

$$\prod_{i=1}^{n} F^{a_i}\left((\prod_{j=1}^{n} x_j^{a_j})^{\frac{1}{\sum_{j=1}^{n} a_j}} \right) = F\left(\prod_{i=1}^{n} \left(\prod_{j=1}^{n} x_j^{a_j} \right)^{\frac{1}{\sum_{j=1}^{n} a_j}} \right)^{a_i},$$

$$F^{\sum_{i=1}^{n} a_i}\left(\prod_{j=1}^{n} \hat{x}_j{}^{p_j} \right) = F\left(\prod_{j=1}^{n} \hat{x}_j{}^{a_j} \right)$$

(42)

If $\sum_{i=1}^{n} a_i = 0$ then $F(x) = 1$. Let us suppose that $\sum_{i=1}^{n} a_i \neq 0$ and let us substitute $\hat{x}_j$ by x.

Using (39) and (42), we get

$$F^{\sum_{i=1}^{n} a_i}\left(\prod_{i=1}^{n} x_i^{p_i} \right) = F\left(\prod_{i=1}^{n} x_i^{a_i} \right) = F^{a_i}\left(\prod_{i=1}^{n}(x_i) \right)$$

(43)

So

$$F\left(\prod_{i=1}^{n} x_i^{p_i} \right) = \prod_{i=1}^{n} F^{p_i}(x_i)$$

(44)

Let $x_i = u_i^{\frac{1}{p_i}}, x_j = 1$ $j \neq i$ and $F(1) = \gamma$, then we get

$$F\left(\prod_{i=1}^{n} x_i^{p_i} \right) = F(u_i) = F^{p_i}(u_i^{\frac{1}{p_i}}) \prod_{j=1,i\neq j}^{n} F^{p_j}(1) = F^{p_i}(u_i^{\frac{1}{p_i}})\gamma^{1-p_i}$$

so:

$$F^{p_i}\left(u_i^{\frac{1}{p_i}} \right) = \frac{F(u_i)}{\gamma^{1-p_i}}$$

(45)

Let $x_i = u_i^{\frac{1}{p_i}}$ $i = 1, \ldots n$. Then we get from (44)

$$F\left(\prod_{i=1}^{n} u_i \right) = \prod_{i=1}^{n} F^{p_i}\left(u_i^{\frac{1}{p_i}} \right)$$

(46)

From (45), (46)   we can infer:

$$F\left(\prod_{i=1}^{n} u_i \right) = \frac{1}{\gamma^{n-1}} \prod_{i=1}^{n} F(u_i)$$

(47)

From (45),(46) $\gamma \neq 0$.
Let $G(u) = \frac{1}{\gamma}F(u)$. Then (47) has the form

$$G\left(\prod_{i=1}^{n} u_i \right) = \prod_{i=1}^{n} G(u_i)$$

(48)

The solution of (48) is [1].

$$G(u) = u^{\beta}, \quad \text{and therefore} \quad F(u) = \alpha u^{\beta},$$

(49)

where $\alpha = \frac{1}{\gamma}$.

Now let us check this solution. Substituting solution (49) into (39), we find that

$$\prod_{i=1}^{n}\left(\alpha x_i^{\beta}\right)^{a_i} = \alpha \left(\prod_{i=1}^{n} x_i^{a_i}\right)^{\beta}$$

$$\alpha^{\sum_{i=1}^{n} a_i} = \alpha$$

From this either $\alpha = 1$ or $\sum_{i=1}^{n} a_i = 1$.

∎

Definition 7. *The self-De Morgan identity is*

$$a(w_1, \eta(x_1); \ldots; w_n, \eta(x_n)) = \eta(a(\boldsymbol{w}, \boldsymbol{x})).$$

Theorem 7. *The self-De Morgan identity (7) is valid if and only if*

a) $\displaystyle\sum_{i=1}^{n} w_i \neq 1$, *then* $\eta(x) = f^{-1}\left(\dfrac{1}{f(x)}\right)$

b) $\displaystyle\sum_{i=1}^{n} w_i = 1$, *then* $\eta(x) = f^{-1}\left(\dfrac{\alpha}{f(x)}\right) = f^{-1}\left(f(\nu_0)\dfrac{f(\nu)}{f(x)}\right) = f^{-1}\left(\dfrac{f^2(\nu_*)}{f(x)}\right)$

Remark 1. *In case b) infinitely many negations fulfills the self-DeMorgan identity.*

Sufficiency:

If $\displaystyle\sum_{i=1}^{n} w_i = 1$, then for $a(\mathbf{w}, \mathbf{x})$ the self-De Morgan identity hold for all ν_*, i.e. $a(x, y)$ is independent of the ν_* value.

The self-De Morgan identity is

$$a(w, \eta_{\nu_*}(\mathbf{x})) = \eta_{\nu_*}(a(\mathbf{w}, \mathbf{x})). \tag{50}$$

Therefore

$$f^{-1}\left(\prod_{i=1}^{n} \frac{(f^{w_i}(\nu_*))^2}{\prod_{i=1}^{n} f^{w_i}(x_i)}\right) = f^{-1}\left(\frac{f^2(\nu_*)}{\prod_{i=1}^{n} f^{w_i}(x_i)}\right)$$

and because

$$\prod_{i=1}^{n} (f^{w_i}(\nu_*))^2 = \prod_{i=1}^{n} \left(f^2(\nu_*)\right)^{w_i} = \left(f^2(\nu_*)\right)^{\sum_{i=1}^{n} w_i} = f^2(\nu_*),$$

we get (50). Similar way we can show the case a), when $\sum_{i=1}^{n} w_i \neq 1$.

Necessity:

Using the multiplicative representation, we have

$$f^{-1}\left(\prod_{i=1}^{n} f^{w_i}(\eta(x_i))\right) = \eta\left(f^{-1}\left(\prod_{i=1}^{n} f^{w_i}(x_i)\right)\right). \tag{51}$$

Now let $F(x)$ denote $f(\eta(f^{-1}(x)))$, then (51) has the form

$$\prod_{i=1}^{n} F^{w_i}(x_i) = F\left(\prod_{i=1}^{n} x_i^{w_i}\right). \tag{52}$$

A general solution of this function equation is

$$F(x) = x^\beta \quad \text{if} \quad \sum_{i=1}^{n} w_i \neq 1$$

$$F(x) = \alpha x^\beta \quad \text{if} \quad \sum_{i=1}^{n} w_i = 1$$

$$\alpha, \beta \in R$$

See Lemma 2.

Using the definition of $F(x)$, we have

$$f(\eta(f^{-1}(x))) = x^\beta \quad \text{or}$$
$$f(\eta(f^{-1}(x))) = \alpha x^\beta$$

In terms of $\eta(x)$,

$$\eta(x) = f^{-1}(f^\beta(x)) \quad \text{or} \tag{53}$$

$$\eta(x) = f^{-1}(\alpha f^\beta(x)) \tag{54}$$

Because $\eta(x)$ is involutive using (53) we get

$$x = \eta(\eta(x)) = f^{-1}\left(f^\beta\left(f^{-1}\left(f^\beta(x)\right)\right)\right). \tag{55}$$

Now let $f(x) = e^{g(x)}$, then $f^{-1}(x) = g^{-1}(\ln(x))$, so $f^{-1}(f^\beta(x)) = g^{-1}(\beta g(x))$. Therefore

$$x = g^{-1}(\beta^2 g(x))$$

This means that $\beta = \pm 1$ and (53)

$$\eta(x) = f^{-1}\left(\frac{1}{f(x)}\right) \quad \text{or} \quad \eta(x) = x.$$

$\eta(x) = x$ is not a negation, so we get 7(a). After we will show that 7(b) is also true. Because $\eta(x)$ is involutive, using (54) we get

$$x = \eta(\eta(x)) = f^{-1}(\alpha f^\beta(f^{-1}(\alpha f^\beta(x)))) \tag{56}$$

Now let $f(x) = e^{g(x)}$, then $f^{-1}(x) = g^{-1}(\ln(x))$, so

$$f^{-1}(\alpha f^\beta(x)) = g^{-1}(\ln(\alpha e^{\beta g(x)})) = g^{-1}(\ln\alpha + \beta g(x)).$$

(56) can be written in the following way:

$$x = g^{-1}(\ln\alpha + \beta g(g^{-1}(\ln\alpha + \beta g(x)))) = g^{-1}((1+\beta)\ln\alpha + \beta^2 g(x)).$$

From this we get:

$$g(x)(1 - \beta^2) = (1+\beta))\ln\alpha.$$

If $\beta \neq \pm 1$ then $g(x)$ is a constant, so $\beta = -1$.
Then we get the result 7(b).

5   Strict t-norms, t-conorms and aggregative operators

From an application point of view, the strict monotonously increasing operators are useful. They have many applications. This is the reason why in this article we will focus on strictly monotonously increasing operators.

T-norms are commutative, associative and monotone operations on the real unit interval with 1 as the unit element. t-conorms are in some sense dual to t-norms. A t-conorm is a commutative, associative and monotone operation with 0 as the unit element [18]. In this section, besides the min/max and the drastic operators, we shall be concerned with strict t-norms and t-conorms, that is,

$$c(x,y) < c(x',y) \qquad \text{if} \quad x < x' \qquad x,y \in (0,1]$$
$$d(x,y) < d(x',y) \qquad \text{if} \quad x < x' \qquad x,y \in [0,1)$$

We will call the elements of pliant logic conjunctive, disjunctive and negation operators denote them by $c(x,y)$ and $d(x,y)$ respectively. Those familier with fuzzy logic theory will find that the terminology used here is slightly different from that used in standard texts [3,5, 8,13,18,27].
The so-called pan operator concept were introduced by Mesiar and Rybárik [26]. The next theorem is based on this result.
In the following we will show that from any strict continuous t-norm or t-conorm we can derive an aggregative operator by changing addition to multiplication in their additive generator functional forms.
The following theorem can be found in Klement, Mesiar and Pap's paper [19].

Theorem 8. *The following are equivalent:*

1. $o(x,y) = f^{-1}\left(f(x) + f(y)\right)$ *is a strict t-norm or t-conorm.*

2. $a(x,y) = f^{-1}\left(f(x)f(y)\right)$ *is an aggregative operator.*

where f is the generator function of the strict t-norm or t-conorm.

Sketch of the proof. Suppose $o : [0,1]^2 \to [0,1]$ is a strict t-norm or t-conorm with additive generator function $f(x)$. Let $g(x) = \log_s f(x)$, $g^{-1}(x) = f^{-1}(s^x)$. Then $g(x)$ fulfils the conditions of Theorem 1 i.e.

$$a(x,y) = g^{-1}\left(g(x) + g(y)\right) = f^{-1}\left(f(x)f(y)\right) \tag{57}$$

is an aggregative operator.
Similarly, let $a : [0,1]^2 \to [0,1]$ be an aggregative operator with additive generator function g, and let $f(x) = s^{g(x)}$. It is easy to see that $f : [0,1] \to [0,\infty]$ is strictly monotone, continuous and either $f(0) = 0$ or $f(1) = 0$.

■

Definition 8. *Let f be the additive generator of a strict t-norm or t-conorm. Then the corresponding aggregative operator is $a(x,y) = f^{-1}\left(f(x)f(y)\right)$.*

We note that pan-operators, introduced by Wang and Klir [34], with a non-idempotent unit element (see [26] and [32]) have properties not unlike to a corresponding pair of strict t-norm or t-conorm and aggregative operators.

Corollary 1. *A strict t-norm or t-conorm is distributive with its corresponding aggregative operator, i.e.*

$$a(x, c(y, z)) = c(a(x, y), a(x, z)),$$
$$a(x, d(y, z)) = d(a(x, y), a(x, z)). \tag{58}$$

We can find this and some other similar results in the paper by Ruiz and Torrens [30]. The following statement shows the relationship between the set of strict t-norm or t-conorm and the set of aggregative operators.

Corollary 2. *Let $f(x)$ be the additive generator of a strict t-norm or t-conorm. The aggregative operators gene-rated by $f(x)$ and $f_*(x) = cf(x)$ $(c > 0)$ are the same.*

Corollary 3. *Let $f(x)$ be a generator function of a strict t-norm or t-conorm and let $f_\alpha(x) = (f(x))^\alpha$ $(\alpha > 0)$. Then its corresponding aggregative operator is independent of α.*

Proof.

$$a_\alpha(x, y) = f_\alpha^{-1}\left(f_\alpha(x) f_\alpha(y)\right) =$$
$$= f^{-1}\left(((f(x))^\alpha (f(y))^\alpha)^{1/\alpha}\right) = \tag{59}$$
$$= f^{-1}\left(f(x) f(y)\right).$$

∎

By Theorem 8 and Corollaries 2 and 3, every strict t-norm or t-conorm has infinitely many corresponding aggregative operators because its generator function is determined up to a multiplicative constant. Conversely, every aggregative operator has infinitely many corresponding strict t-norm or t-conorm because a generator function on different powers generates different strict t-norm or t-conorm and identical aggregative operators.

A direct consequence of Theorem 3 and Theorem 8 is the following representation theorem of strong negations.

Corollary 4 (Multiplicative form of strong negations). *The function $\eta : [0, 1] \to [0, 1]$ is a strong negation with neutral value ν_* if and only if*

$$\eta(x) = f^{-1}\left(\frac{f^2(\nu_*)}{f(x)}\right), \tag{60}$$

where f is a generator function of a strict t-norm or t-conorm.

Summarizing the above statements, there is a well-defined correspondence between strict t-norm or t-conorm, aggregative operators and strong negations. Every strict t-norm or t-conorm has corresponding aggregative operators, and corresponding strong negations as well. Table 1 lists the corresponding aggregative operators and Table 2 gives the corresponding strong negations of the chief strict t-norm or t-conorm. Note that the Dombi operators have the same corresponding aggregative operator and strong negation.

The next theorem gives a necessary and sufficient condition for a pair of strict t-norm and strict t-conorm to have identical corresponding aggregative operators.

Theorem 9. *Let f_c be an additive generator function of a strict t-norm, and f_d be an additive generator function of a strict t-conorm. Their corresponding aggregative operators a_c and a_d are equivalent on $[0, 1]^2 \setminus \{(0, 1), (1, 0)\}$ if and only if $f_d(x) = f_c^k(x)$, where $k \in \mathbb{R}^-$.*

Proof. The equivalence means that

$$f_c^{-1}\left(f_c(x)f_c(y)\right) = f_d^{-1}\left(f_d(x)f_d(y)\right) \tag{61}$$

This equation can be transformed to

$$f_d\left(f_c^{-1}(x'y')\right) = f_d(f_c^{-1}(x'))f_d(f_c^{-1}(y')), \tag{62}$$

by substituting $\xi' = f_c(\xi)$, $\xi \in \{x, y\}$, and multiplying f_d to both sides. Substituting $h(x) = f_d(f_c^{-1}(x))$

$$h(x'y') = h(x')h(y'). \tag{63}$$

This is the well-known power-law Cauchy equation, which has a solution of $h(x) = x^k$, where k is a constant, thus

$$h(x') = f_d(f_c^{-1}(x')) = (x')^k \tag{64}$$

$$f_d(x) = f_c^k(x). \tag{65}$$

Note that if $k \geq 0$ then $f_c^k(x)$ is also a conjunctive generator function, and if $k < 0$ then f_d is indeed a disjunctive generator function.

■

Corollary 5. *Let c and d be a strict t-norm and a strict t-conorm with additive generator functions f_c and f_d. Let a_c and a_d be their corresponding aggregative operators, and let η_c and η_d be their corresponding strong negations. The strong negations η_c and η_d are equivalent if and only if $f_d(x) = f_c^k(x)$, $k \in \mathbb{R}^-$.*

6 Pliant operators

If the condition $f_d(x) = f_c^k(x)$ with $k < 0$ is fulfilled then the strict t-norm or t-conorm have a common aggregative operator and strong negation. This set of strict t-norm or t-conorm is still general. DeMorgan's law is a condition which must be fulfilled by a "good" triplet of connectives. Persanding that they satisfy of DeMorgan's law further restricts the given set of strict t-norm or t-conorm.

Theorem 10. *Let c and d be a strict t-norm and a strict t-conorm with additive generator functions f_c and f_d. Suppose their corresponding strong negations are equivalent (i.e. $f_d(x) = f_c^k(x)$, $k < 0$), denoted by η ($\eta(\nu_*) = \nu_*$). The three connectives c, d and n form a DeMorgan triplet if and only if $k = -1$.*

Proof. DeMorgan's law is the following:

$$c(x, y) = \eta(d(\eta(x), \eta(y))). \tag{66}$$

Using the generator functions of the operators the equation becomes:

$$f_c^{-1}\left(f_c(x) + f_c(y)\right) =$$
$$= \eta\left(f_c^{-1}\left(\left(f_c^k(\eta(x)) + f_c^k(\eta(y))\right)^{1/k}\right)\right). \tag{67}$$

According to Corollary 4,

$$\eta(x) = f_c^{-1}\left(\frac{f_c^2(\nu_*)}{f_c(x)}\right) = f_d^{-1}\left(\frac{f_d^2(\nu_*)}{f_d(x)}\right), \tag{68}$$

hence

$$\eta\left(f_c^{-1}\left(\left(f_c^k(\eta(x))+f_c^k(\eta(y))\right)^{1/k}\right)\right)=$$

$$=f_c^{-1}\left(f_c^2(\nu_*)\left(\left(\frac{f_c^2(\nu_*)}{f_c(x)}\right)^k+\left(\frac{f_c^2(\nu_*)}{f_c(y)}\right)^k\right)^{-1/k}\right)= \tag{69}$$

$$=f_c^{-1}\left(\left(f_c^{-k}(x)+f_c^{-k}(y)\right)^{-1/k}\right).$$

So Eq.(67) can be written as

$$f_c(x)+f_c(y)=\left((f_c(x))^{-k}+(f_c(y))^{-k}\right)^{-1/k}. \tag{70}$$

Let $x'=f_c(x)$ and $y'=f_c(y)$. Then we have

$$(x'+y')^{-k}=x'^{-k}+y'^{-k}. \tag{71}$$

Let $x'=y'=1/2$, then the only solution is $k=-1$. It is straightforward to see that $k=-1$ fullfils Eq. (70), so c, d and η form a DeMorgan triplet if and only if $k=-1$.

∎

Definition 9. *A system of strict t-norm or t-conorm which have the property $f_c(x)f_d(x)=1$ is called a multiplicative pliant system.*

In multiplicative pliant systems the corresponding aggregative operators of the strict t-norm and strict t-conorm are equivalent, and DeMorgan's law is obeyed with the (common) corresponding strong negation of the strict t-norm or t-conorm.

We can summarize the properties of multiplicative pliant system like so

$$c(\mathbf{x})=f^{-1}\left(\sum_{i=1}^n f(x_i)\right) \qquad\qquad c(\mathbf{w},\mathbf{x})=f^{-1}\left(\sum_{i=1}^n w_i f(x_i)\right) \tag{72}$$

$$d(\mathbf{x})=f^{-1}\left(\frac{1}{\sum_{i=1}^n \frac{1}{f(x_i)}}\right) \qquad\qquad d(\mathbf{w},\mathbf{x})=f^{-1}\left(\frac{1}{\sum_{i=1}^n \frac{w_i}{f(x_i)}}\right) \tag{73}$$

$$a_{\nu_*}(\mathbf{x})=f^{-1}\left(f(\nu_*)\prod_{i=1}^n\frac{f(x_i)}{f(\nu_*)}\right) \qquad a_{\nu_*}(\mathbf{w},\mathbf{x})=f^{-1}\left(f(\nu_*)\prod_{i=1}^n\left(\frac{f(x_i)}{f(\nu_*)}\right)^{w_i}\right) \tag{74}$$

$$a(\mathbf{x})=f^{-1}\left(\prod_{i=1}^n f(x_i)\right) \qquad\qquad a(\mathbf{w},\mathbf{x})=f^{-1}\left(\prod_{i=1}^n f^{w_i}(x_i)\right) \tag{75}$$

$$\eta(\mathbf{x})=f^{-1}\left(\frac{f^2(\nu_*)}{f(x)}\right) \tag{76}$$

where $f(x)$ is the generator function of the strict t-norm.

It was shown in [11] that the multiplicative pliant system fulfils the DeMorgan identity and the correct strong negation is defined by Eq.(76).

For example, let $f_c(x)=-\ln x$, the additive generator of the product operator. Assuming we have a pliant system, $f_d(x)=(-\ln x)^{-1}$ is a valid generator of a strict t-conorm. Their

corresponding strong negations are the same as $\eta_c(x) = \eta_d(x) = \eta(x) = \exp[\frac{(\ln(\nu_*))^2}{\ln x}]$, so that $\eta(1) = \lim_{x \to 1} \eta(x)$, for which $c(x,y) = xy$ and

$$d(x,y) = \exp\left[\frac{\ln x \ln y}{\ln xy}\right] \tag{77}$$

form a DeMorgan triplet.

6.1 The Dombi operator system

In another example, the Dombi operators form a pliant system. The operators are

$$c(\mathbf{x}) = \frac{1}{1 + \left(\sum\limits_{i=1}^{n} \left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{1/\alpha}} \qquad c(\mathbf{x}) = \frac{1}{1 + \left(\sum\limits_{i=1}^{n} w_i \left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{1/\alpha}} \tag{78}$$

$$d(\mathbf{x}) = \frac{1}{1 + \left(\sum\limits_{i=1}^{n} \left(\frac{1-x_i}{x_i}\right)^{-\alpha}\right)^{-1/\alpha}} \qquad d(\mathbf{x}) = \frac{1}{1 + \left(\sum\limits_{i=1}^{n} w_i \left(\frac{1-x_i}{x_i}\right)^{-\alpha}\right)^{-1/\alpha}} \tag{79}$$

$$a_{\nu_*}(\mathbf{x}) = \frac{1}{1 + \frac{1-\nu_*}{\nu_*} \prod\limits_{i=1}^{n} \left(\frac{1-x_i}{x_i} - \frac{\nu_*}{1-\nu_*}\right)} \qquad a_{\nu_*}(\mathbf{x}) = \frac{1}{1 + \left(\frac{1-\nu_*}{\nu_*}\right) \prod\limits_{i=1}^{n} \left(\frac{1-x_i}{x_i} - \frac{1-\nu_*}{\nu_*}\right)^{w_i}} \tag{80}$$

$$\eta(x) = \frac{1}{1 + \left(\frac{1-\nu_*}{\nu_*}\right)^2 \frac{x}{1-x}} \tag{81}$$

where $\nu_* \in]0,1[$, with generator functions

$$f_c(x) = \left(\frac{1-x}{x}\right)^{\alpha} \qquad\qquad f_d(x) = \left(\frac{1-x}{x}\right)^{-\alpha} \tag{82}$$

where $\alpha > 0$. The operators c, d and n fulfil the DeMorgan identity for all ν, a and n fulfil the self DeMorgan identity for all ν and the aggregative operator is distributive with the strict t-norm or t-conorm.

Eqs.(78), (79), (80), (81) can be found in various articles of Dombi. Eq.(78) and (79) can be found in [10], Eq.(80) in [9] and Eq.(81) can be found in [11].

Eq.(80) called 3Π operator because it can be written in the following form:

$$a(\mathbf{x}) = \frac{\prod\limits_{i=1}^{n} x_i}{\prod\limits_{i=1}^{n} x_i + \prod\limits_{i=1}^{n} (1 - x_i)} \tag{83}$$

7 Conclusions

In this paper we demonstrated the equivalence of the class of representable uninorms and the class of uninorms that are also aggregative operators. In addition, three new representation theorems of strong negations were given with two explicitly containing the neutral value. After, the relationships for strict, continuous operators, aggregative operators and strong negations were clarified, showing the correspondence between the elements of the three classes. We study a certain class of weighted aggregative operators (representable uninorms) which build a self-DeMorgan class wih infinitely many negations. Lastly, the concept of multiplicative pliant systems was presented, and characterized by necessary and sufficient conditions.

Acknowledgements

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