Information Sciences 283 (2014) 229–240

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Implications in bounded systems

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ARTICLE INFO

Article history:
Received 7 February 2014
Received in revised form 17 June 2014
Accepted 25 June 2014
Available online 7 July 2014

Keywords:
Implication
S-implication
R-implication
Nilpotent operator
Łukasiewicz operator system
Bounded system

ABSTRACT

A consistent connective system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz-system. Using more than one generator function, consistent nilpotent connective systems (so-called bounded systems) can be obtained with the advantage of three naturally derived negations and thresholds. In this paper, implications in bounded systems are examined. Both R- and S-implications with respect to the three naturally derived negations of the bounded system are considered. It is shown that these implications never coincide in a bounded system, as the condition of coincidence is equivalent to the coincidence of the negations, which would lead to Łukasiewicz logic. The formulae and the basic properties of four different types of implications are given, two of which fulfill all the basic properties generally required for implications.

1. Introduction

In our previous article [7], we showed that a consistent connective system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz-system. Using more than one generator function, consistent nilpotent connective systems can be obtained in a significantly different way with three naturally derived negations. As the class of non-strict t-norms has preferable properties that make them useful in constructing logical structures, the advantages of such systems are obvious [14]. Due to the fact that all continuous Archimedean (i.e. representable) nilpotent t-norms are isomorphic to the Łukasiewicz t-norm [11], the previously studied nilpotent systems were all isomorphic to the well-known Łukasiewicz-logic. Those consistent connective systems which are not isomorphic to Łukasiewicz logic are called bounded systems [7].

Fuzzy implications are definitely among the most important operations in fuzzy logic [2,17]. Firstly, other basic logical connectives of the binary logic can be obtained from the classical implication. Secondly, the implication operator plays a crucial role in the inference mechanisms of any logic, like modus ponens, modus tollens, hypothetical syllogism in classical logic. Fuzzy implications all generalize the classical implication with the two possible crisp values from $0, 1$, to the fuzzy concept with truth values from the unit interval $[0, 1]$ [26]. In classical logic the implication can be defined in several ways. The most well-known implications are the usual material implication from the Kleene algebra, the implication obtained as the residuum of the conjunction in Heyting algebra (also called pseudo-Boolean algebra) in the intuitionistic logic framework and the implication in the setting of quantum logic. While all these differently defined implications have identical truth tables in the classical case, the natural generalizations of the above definitions in the fuzzy logic framework are not identical. This fact has led to some throughout research on fuzzy implications [1,3–5,12,18,20,21,24,25].

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http://dx.doi.org/10.1016/j.ins.2014.06.031
0020-0255/

Based on the results of [7], now we focus on residual and S-implication operators [3] in bounded systems. The paper is organized as follows. After some preliminaries in Section 2, we examine the residual implication in Section 3 and S-implications with special attention to the ordering property in Section 4. In Section 6 we show that in a bounded system, the minimum and maximum operators can also be expressed in terms of the conjunction, the implication and the negation. Finally in Section 5 we show that in a bounded system the implications examined in this paper can never coincide. The formulae and the properties of implications are summarized in Section 7.

2. Preliminaries

2.1. t-Norms and conorms

Now we state the basic notations and results for t-norms and t-conorms [13]. A triangular norm (*t-norm* for short) T is a binary operation on the closed unit interval $[0,1]$ such that $([0,1],T)$ is an abelian semigroup with neutral element 1 that is totally ordered; i.e., for all $x_1,x_2,y_1,y_2 \in [0,1]$ with $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$ we have $T(x_1,y_1) \leqslant T(x_2,y_2)$, where $\leqslant$ is the natural order on $[0,1]$.

A triangular conorm (*t-conorm* for short) S is a binary operation on the closed unit interval $[0,1]$ such that $([0,1],S)$ is an abelian semigroup with a neutral element 0 that is totally ordered.

A continuous t-norm T is said to be *Archimedean* if $T(x,x) < x$ holds for all $x \in (0,1)$. A continuous Archimedean T is called *strict* if T is strictly monotone; i.e. $T(x,y) < T(x,z)$ whenever $x \in (0,1]$ and $y < z$, and *nilpotent* if there exist $x,y \in (0,1)$ such that $T(x,y) = 0$.

From the duality between t-norms and t-conorms, we can easily derive the following properties. A continuous t-conorm S is said to be *Archimedean* if $S(x,x) > x$ holds for every $x,y \in (0,1)$. A continuous Archimedean S is called *strict* if S is strictly monotone; i.e. $S(x,y) < S(x,z)$ whenever $x \in [0,1)$ and $y < z$, and *nilpotent* if there exist $x,y \in (0,1)$ such that $S(x,y) = 1$.

The following well-known results provide important single variable representations for t-norms and t-conorms.

Proposition 1 (*[15,18]*). *A function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm iff it has a continuous additive generator; i.e. there exists a continuous strictly decreasing function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x,y) = t^{-1}(\min(t(x) + t(y), t(0))), \quad x,y \in [0,1]. \tag{1}$$

Proposition 2 (*[15,18]*). *A function $S : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-conorm iff it has a continuous additive generator; i.e. there exists a continuous strictly increasing function $s : [0,1] \to [0,\infty]$ with $s(0) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$S(x,y) = s^{-1}(\min(s(x) + s(y), s(1))), \quad x,y \in [0,1]. \tag{2}$$

Proposition 3. [11]

A t-norm T is strict if and only if $t(0) = \infty$ holds for each continuous additive generator t of T.
A t-norm T is nilpotent if and only if $t(0) < \infty$ holds for each continuous additive generator t of T.
A t-conorm S is strict if and only if $s(1) = \infty$ holds for each continuous additive generator s of S.
A t-conorm S is nilpotent if and only if $s(1) < \infty$ holds for each continuous additive generator s of S.

Proposition 4 [11]. *Let T be a continuous Archimedean t-norm.*

If T is strict, then it is isomorphic to the product t-norm $T_{\mathbf{P}}$, i.e., there exists an automorphism ϕ of the unit interval such that $T_\phi = \phi^{-1}(T(\phi(x),\phi(y))) = T_{\mathbf{P}}$.

If T is nilpotent, then it is isomorphic to the Łukasiewicz t-norm $T_{\mathbf{L}}$, i.e., there exists an automorphism of the unit interval ϕ such that $T_\phi = \phi^{-1}(T(\phi(x),\phi(y))) = T_{\mathbf{L}}$.

From the definitions of t-norms and t-conorms it follows immediately that t-norms are conjunctive (i.e. $T(x,y) \leqslant \min(x,y)$), while t-conorms are disjunctive (i.e. $S(x,y) \geqslant \max(x,y)$) aggregation functions. This is why they are widely used as conjunctions and disjunctions in multivalued logical structures.

The use of the so-called cutting function makes the formulae simpler.

Definition 1 (*[7,19]*). Let us define the cutting operation [ ] by

$$[x] = \begin{cases} 0 & \text{if} \quad x < 0 \\ x & \text{if} \quad 0 \leqslant x \leqslant 1 \\ 1 & \text{if} \quad 1 < x \end{cases}$$

and let the notation [ ] also act as 'brackets' when writing the argument of an operator, so that we can write $f[x]$ instead of $f([x])$.

2.2. Negations

Definition 2 [2], pp. 13. A unary operation $n : [0,1] \to [0,1]$ is called a negation if it is non-increasing and compatible with classical logic; i.e. $n(0) = 1$ and $n(1) = 0$.

A negation is strict if it is also strictly decreasing and continuous.

A negation is strong, if it is also involutive; i.e. $n(n(x)) = x$ for $\forall x \in \mathbb{R}$.

The well-known representation theorem for strong negations was obtained by Trillas.

Proposition 5 [23]. *n is a strong negation if and only if*

$$n(x) = f_n(x)^{-1}(1 - f_n(x)),$$

where $f_n : [0,1] \to [0,1]$ *is an automorphism of* $[0,1]$.

Remark 1. This result also means that $n(x)$ is a strict negation if and only if

$$n(x) = f_n^{-1}(n'(f_n(x))) \tag{3}$$

where $f_n : [0;1] \to [0;1]$, called the generator function of n, is a strictly monotone, continuous function with $f_n(0) = 0$ and $f_n(1) = 1$ and n' is a strong negation.

Proposition 6. *In Proposition 5 (Trillas) the generator function can also be decreasing.*

Proof. Proof can be found in [7], Proposition 7. □

2.3. Implication operators

A mapping $i : [0,1]^2 \to [0,1]$ is called an implication operator if and only if it satisfies the boundary conditions $i(0,0) = i(0,1) = i(1,1) = 1$ and $i(1,0) = 0$.

The above conditions are the minimum requirements for an implication operator. Other potentially interesting properties of implication operators are listed in [2,6,8,20,22,24]. All fuzzy implications can be obtained by generalizing the implication operator of classical logic. In this sense, Fodor and Roubens [10] established the following definition.

Definition 3. A fuzzy implication is a function $i : [0,1]^2 \to [0,1]$ that satisfies the following properties:

1. The first place antitonicity:

$$\text{for all } x_1, x_2, y \in [0,1] \quad (\text{if} \quad x_1 \leqslant x_2 \text{ then } i(x_1, y) \geqslant i(x_2, y)). \tag{FA}$$

2. The second place isotonicity:

$$\text{for all } x, y_1, y_2 \in [0,1] \quad (\text{if} \quad y_1 \leqslant y_2 \text{ then } i(x, y_1) \leqslant i(x, y_2)). \tag{SI}$$

3. The dominance of falsity of antecedent:

$$i(0, y) = 1 \text{ for all } y \in [0,1]. \tag{DF}$$

4. The dominance of truth of consequent:

$$i(x, 1) = 1 \text{ for all } x \in [0,1]. \tag{DT}$$

5. The boundary condition:

$$i(1,0) = 0 \quad \text{and} \quad i(1,1) = 1. \tag{BC}$$

Other important but usually not required properties of fuzzy implications are defined below [2].

Definition 4. A fuzzy implication i satisfies

1. The left neutrality property (the neutrality of truth), if

$$i(1, y) = y \text{ for all } y \in [0,1]. \tag{NP}$$

2. The exchange principle, if

$$i(x, i(y, z)) = i(y, i(x, z)) \text{ for all } x, y, z \in [0, 1]. \quad \text{(EP)}$$

3. The identity principle, if

$$i(x, x) = 1 \text{ for all } x \in [0, 1]. \quad \text{(IP)}$$

4. The strong negation principle, if the mapping n^* defined as

$$n^*(x) = i(x, 0) \text{ for all } x \in [0, 1] \quad \text{(SN)}$$

is a strong negation.

5. The law of contraposition (or in other words,the contrapositive symmetry) with respect to a strong negation n, if

$$i(x, y) = i(n^*(y), n^*(x)) \text{ for all } x, y \in [0, 1]. \quad \text{(CP)}$$

6. The ordering property, if

$$i(x, y) = 1 \quad \text{if and only if} \quad x \leqslant y \text{ for all } x, y \in [0, 1]. \quad \text{(OP)}$$

Remark 2. The negation operator n^* is also called the natural negation of the implication i (see [2]).

A detailed study of possible relations between all these properties can be found in [2,6,21]. Notice that other properties can also be found in the literature. In particular, $i(x, n^*(x)) = n^*(x)$ for all $x \in [0, 1]$, where n^* is a strong negation (see [17]).

Three well-established classes of implication operators are (S,N)-, QL- and R-implications.

Definition 5 [2], pp. 57. A function $i : [0, 1]^2 \to [0, 1]$ is called an S-implication, if there exists a t-conorm S and a strong negation n^* such that

$$i_S(x, y) = S(n^*(x), y), \quad x, y \in [0, 1].$$

Definition 6 [2], pp. 90. A function $i : [0, 1]^2 \to [0, 1]$ is called a QL-operation, if there exists a t-conorm S, a t-norm T and a strong negation n^* such that

$$i_Q(x, y) = S(n^*(x), T(x, y)), x, y \in [0, 1].$$

In general, QL-operations violate property (FA). The conditions under which (FA) is satisfied can be found in [9]. When a QL-operation is a fuzzy implication, then it is called a QL-implication.

Definition 7 [2], pp. 68. A function $i : [0, 1]^2 \to [0, 1]$ is called an R-implication, if there exists a t-norm T such that

$$i_R(x, y) = \sup\{z \in [0, 1] \mid T(x, z) \leqslant y\}.$$

In the case where the given t-norm is left-continuous, we will refer to the R-implication defined above as a residual implication [2,12,14]. Note that in this case we have $T(x, y) = inf_z(z \in [0, 1], \mid i(x, z) \geqslant y)$. It is easy to see that both S-implications and R-implications satisfy properties FA, SI, DF, DT, BC, regardless of the t-norm T, the t-conorm S and the strong negation n^* types. Hence, they are implications in the Fodor and Roubens sense. Different characterizations of S-implications, QL-implications and R-implications can be found in the literature (for details, see [3,10,24]). It is worth mentioning here that new characterizations of R and S-implications can also be found in [25].

2.4. Bounded systems

To construct a logical system, we need to define the logical operators. As in [7], we consider connective systems where the conjunction and the disjunction are special types of t-norms and t-conorms, respectively.

Definition 8. The triple (c, d, n), where c is a t-norm, d is a t-conorm and n is a strong negation, is called a connective system.

Definition 9. A connective system is nilpotent, if the conjunction c is a nilpotent t-norm, and the disjunction d is a nilpotent t-conorm.

Definition 10. Two connective systems, (c_1, d_1, n_1) and (c_2, d_2, n_2) are isomorphic, if there exists a monotone bijection $\phi : [0, 1] \to [0, 1]$ such that

$$\phi^{-1}(c_1(\phi(x),\phi(y))) = c_2(x,y)$$
$$\phi^{-1}(d_1(\phi(x),\phi(y))) = d_2(x,y)$$
$$\phi^{-1}(n_1(\phi(x))) = n_2(x).$$

Definition 11. A connective system is called Łukasiewicz system, if it is isomorphic to $([x+y-1],[x+y],1-x)$, i.e. it has the form

$$(\phi^{-1}[\phi(x)+\phi(y)-1], \phi^{-1}[\phi(x)+\phi(y)], \phi^{-1}[1-\phi(x)]).$$

Since the generator functions of the nilpotent t-norms and t-conorms are bounded and determined up to a multiplicative constant (see Proposition 1 and 2), they can be normalized (see [7]). Let us use the following notations for the uniquely defined normalized generator functions:

$$f_c(x) := \frac{t(x)}{t(0)}, \qquad f_d(x) := \frac{s(x)}{s(1)}.$$

Using this concept, we have $f_c, f_d, f_n : [0,1] \to [0,1]$, where f_n is the generator function of the negation used in our system. We will suppose that f_c is continuous and strictly decreasing, f_d is continuous and strictly increasing and f_n is continuous and strictly monotone.

Definition 12. The negations n_c and n_d generated by f_c and f_d respectively,

$$n_c(x) = f_c^{-1}(1 - f_c(x))$$

and

$$n_d(x) = f_d^{-1}(1 - f_d(x))$$

are called natural negations.

Next, we recall certain important properties of connective systems and then give the propositions describing the conditions that a logical system must satisfy in order to have the above properties (see [7]).

Definition 13. Classification property means that the law of contradiction holds, i.e.

$$c(x, n(x)) = 0 \quad \forall x \in [0,1], \tag{4}$$

and the excluded middle principle holds as well, i.e.

$$d(x, n(x)) = 1 \quad \forall x \in [0,1]. \tag{5}$$

Definition 14. The **De Morgan identity** means that

$$c(n(x), n(y)) = n(d(x,y)) \tag{6}$$

or

$$d(n(x), n(y)) = n(c(x,y)) \quad \forall x, y \in [0,1]. \tag{7}$$

Remark 3. These two forms of the De Morgan law are equivalent, if the negation is involutive. The first De Morgan law holds with a strict negation n if and only if the second holds with n^{-1} (see page 18 in [10]).

Definition 15. A connective system is said to be **consistent**, if the *classification property* (Definition 13) and the *De Morgan identity* (Definition 14) hold.

Proposition 7 (See also [7] Proposition 10, [10] 1.5.4. and 1.5.5., and [2] 2.3.12. and 2.3.15.). *In a connective system the classification property holds if and only if* $n_d(x) \leqslant n(x) \leqslant n_c(x)$, *where* n_c *and* n_d *are the natural negations.*

Proposition 8 (See [7] Proposition 12). *If* f_c *is the normalized generator function of a conjunction in a connective system,* f_d *is a normalized generator function of the disjunction and* n *is a strong negation, then the following statements are equivalent:*

1. *The De Morgan law holds in the connective system. That is,*

$$c(n(x), n(y)) = n(d(x, y)) \quad \forall x, y \in [0, 1]. \tag{8}$$

2. *The normalized generator functions of the conjunction, disjunction and negation operator obey the following equations (which are obviously equivalent to each other):*

$$n(x) = f_c^{-1}(f_d(x)) = f_d^{-1}(f_c(x)), \tag{9}$$
$$f_c(x) = f_d(n(x)) \quad \text{or equivalently} \quad f_d(x) = f_c(n(x)). \tag{10}$$

Proposition 9 (See [7] Proposition 15.).

1. *If the connective system (c, d, n) is consistent, then $f_c(x) + f_d(x) \geqslant 1$ for any $x \in [0, 1]$, where f_c and f_d are the normalized generator functions of the conjunction c and the disjunction d respectively.*
2. *If $f_c(x) + f_d(x) \geqslant 1$ for any $x \in [0, 1]$ and the De Morgan law holds, then the connective system (c, d, n) satisfies the classification property as well (which now means that the system is consistent).*

The following proposition shows that a consistent nilpotent connective system is isomorphic to Łukasiewicz system if and only if the negations coincide.

Proposition 10. *In a connective system $f_c(x) + f_d(x) = 1$ if and only if*

$$n_c(x) = n_d(x).$$

Definition 16 (*See [7]*). A nilpotent connective system is called a bounded system, if

$$f_c(x) + f_d(x) > 1 \quad (\text{or equivalently } n_d(x) < n(x) < n_c(x))$$

holds for all $x \in (0, 1)$, where f_c and f_d are the normalized generator functions of the conjunction and disjunction, and n_c, n_d are the natural negations.

Remark 4. Note that Łukasiewicz system is characterized by $n_d(x) = n_c(x)$ or equivalently,

$$f_c(x) + f_d(x) = 1.$$

Next we examine the implications in bounded systems.

3. R-implications in bounded systems

For implications i in nilpotent connective systems we use the notation i. For the residual implication, we easily get the following formula (see [2], Theorem 2.5.21.).

Proposition 11. *In a nilpotent connective system (c, d, n) the residual implication has the following form.*

$$i_R(x, y) = f_c^{-1}[f_c(y) - f_c(x)],$$

where f_c is the generator function of c, and $[\,]$ is the cutting operator defined in Definition 1.

Proof. From the definition of residual implication,

$$i_R(x, y) = max\{z : \ c(x, z) \leqslant y\},$$

where

$$c(x, z) = f_c^{-1}[f_c(x) + f_c(z)] \leqslant y.$$

From this, we have $z = f_c^{-1}[f_c(y) - f_c(x)]$. $\square$

Proposition 12. *We can also express i_R by using the negation operator and the normalized generator function of d.*

Proof. From $n(x) = f_c^{-1}(f_d(x))$, we have

$$f_c(x) = f_d(n(x)) \quad \text{and} \quad f_c^{-1}(x) = n^{-1}\left(f_d^{-1}(x)\right),$$

$$i_R(x, y) = n^{-1}\left(f_d^{-1}[f_d(n(x)) - f_d(n(y))]\right). \quad \square$$

The notation H is introduced below for further applications. A new formula for i_R is given in (12) by using H.

$$H(x) = 1 - f_d(n(x)), \tag{11}$$

so $n^{-1}(f_d^{-1}(x)) = H^{-1}(1 - x)$. From this we have

$$i_R(x,y) = H^{-1}(1 - [1 - H(x) - (1 - H(y))]) = H^{-1}[H(y) - H(x) + 1]. \tag{12}$$

Next, we examine the properties given in Definition 4 to see whether they are compatible with the R-implication in a nilpotent connective system.

Remark 5. Note that the following results regarding the properties of i_R correspond with Section 2.5. in [2].

Proposition 13. *In a nilpotent connective system, i_R satisfies*

1. *the left neutrality property (the neutrality of truth),* (NP) i.e. $i_R(1,y) = y$ for all $y \in [0,1]$,
2. *the exchange principle,* (EP) *i.e.* $i_R(x, i_R(y,z)) = i_R(y, i_R(x,z))$ *for all* $x,y,z \in [0,1]$,
3. *the identity principle,* (IP) *i.e.* $i_R(x,x) = 1$ *for all* $x \in [0,1]$,
4. *the strong negation principle,* (SN)*, since* $n_R^*(x) = i_R(x,0) = n_c(x)$ *for all* $x,y \in [0,1]$ *is a strong negation,*
5. *the law of contraposition (contrapositive symmetry),* (CP) *with respect to the strong negation in* (SN)*; i.e.* $i_R(x,y) = i_R(n_c(y), n_c(x))$ *forall* $x,y \in [0,1]$,
6. *the ordering principle,* (OP) *is valid for* $i_R(x,y)$*, i. e.* $i_R(x,y) = 1$ if and only if $x \leqslant y$.

Proof. NP, EP, IP and OP always hold for an R-implication derived from a continuous t-norm (see [2], Theorem 2.5.7.).

CP follows directly from the definition of n_c.

EP and OP together always imply SN for continuous implications (see [2], Corollary 1.4.19.). □

Remark 6. Note that the **law of contraposition** (contrapositive symmetry), (CP) with respect to the strong negation n; i.e. $i_R(x,y) = i_R(n(y), n(x))$ for all $x,y \in [0,1]$, never holds in a bounded system (see also Corollary 1.5.12. in [2]).

Proof. We prove that $i_R(x,y) = i_R(n(y), n(x))$ holds for all $x,y \in [0,1]$ if and only if $f_c(x) + f_d(x) = 1$; i.e. the system is a Łukasiewicz logical system.

If $x \leqslant y$, then $n(y) \leqslant n(x)$, and therefore from the ordering property we get that both sides are equal to 1.

If $x > y$, then the two sides of the equality are equal if and only if $f_c(y) - f_c(x) = f_d(x) - f_d(y)$, i.e. $f_c(x) + f_d(x) = f_c(y) + f_d(y)$ for all $x,y \in [0,1]$, which means that $f_c(x) + f_d(x)$ is constant.

Since $f_c(0) + f_d(0) = 1, \quad f_c(x) + f_d(x) = 1.$ □

A different form of the residual implication is also given in the following section.

4. S-implications in bounded systems

In a nilpotent connective system (c,d,n) we can define different types of S-implications.

Definition 17.

1. $i_{S_n}(x,y) = d(n(x), y), \quad x,y \in [0,1]$,
2. $i_{S_d}(x,y) = d(n_d(x), y), \quad x,y \in [0,1]$,
3. $i_{S_c}(x,y) = d(n_c(x), y), \quad x,y \in [0,1]$,

where n_c and n_d are the natural negations of c and d.

Replacing the disjunction in the definitions above by an appropriate composition of negations and the conjunction leads us to further possible definitions of implications. Since in a bounded system the negations n, n_c and n_d never coincide, negations different from n can also be used similarly to the De Morgan identity.

Definition 18. In a nilpotent connective system (c,d,n)

1. $i^c_{S_n}(x,y) = n(c(x, n(y))), \quad x,y \in [0,1]$,
2. $i^c_{S_d}(x,y) = n_d(c(x, n_d(y))), \quad x,y \in [0,1]$,
3. $i^c_{S_c}(x,y) = n_c(c(x, n_c(y))), \quad x,y \in [0,1]$,

where n_c and n_d are the natural negations of c and d.

Note that from the De Morgan identity it follows immediately that.

$i^c_{S_n}(x,y) = i_{S_n}(x,y)$ and as the following proposition shows, $i^c_{S_c}$ is the residual implication.

Proposition 14. *In a nilpotent connective system* (c,d,n) $i^c_{S_c}(x,y) = f_c^{-1}[f_c(y) - f_c(x)] = i_R(x,y)$, *where* f_c *is the normalized generator function of c.*

Proof.

$$\begin{aligned} i^c_{S_c}(x,y) &= n_c(c(x, n_c(y))) \\ &= n_c\left(f_c^{-1}[f_c(x) + 1 - f_c(y)]\right) \\ &= f_c^{-1}[1 - (1 - f_c(y) + f_c(x))] \\ &= f_c^{-1}[f_c(y) - f_c(x)]. \quad \square \end{aligned}$$

4.1. Properties of i_{S_n}, i_{S_d} *and* i_{S_c}

First the formulae for the S-implications defined above are given.

Proposition 15. *In a nilpotent connective system* (c,d,n)

1. $i_{S_n}(x,y) = f_d^{-1}[f_c(x) + f_d(y)]$,
2. $i_{S_d}(x,y) = f_d^{-1}[1 - f_d(x) + f_d(y)]$,
3. $i_{S_c}(x,y) = f_d^{-1}[f_d(y) + f_d(n_c(x))]$,

where f_c *and* f_d *are the normalized generator functions of c and d, respectively.*

Proof. All the three formulae are easy to verify. □

Next, the basic properties of the S-implications in a nilpotent connective system are stated. Note that the following results are consistent with those described in Section 2.5. of [2].

Proposition 16. *In a nilpotent connective system,* i_{S_n}, i_{S_d} *and* i_{S_c} *satisfy*

1. *the left neutrality property (the neutrality of truth), (NP), i.e.* $i(1,y) = y$ *for all* $y \in [0,1]$,
2. *the exchange principle, (EP), i.e.* $i(x, i(y,z)) = i(y, i(x,z))$ *for all* $x,y,z \in [0,1]$,
3. *the identity principle, (IP), i.e.* $i(x,x) = 1$ *for all* $x \in [0,1]$,
4. *the strong negation principle, (SN) since* $i_S(x,0)$ *for all* $x,y \in [0,1]$ *is a strong negation,*
5. *the law of contraposition (contrapositive symmetry), (CP) with respect to the strong negation in SN.*

Proof.

1. NP holds for every S-implication (see [2], Proposition 2.4.3.).
2. EP holds for every S-implication (see [2], Proposition 2.4.3.).
3. IP holds as well, because of the consistency property and the use of nilpotent operators (see [2], Theorem 2.4.17.).
4. For SN,
   (a) $n_n^*(x) = i_{S_n}(x,0) = d(n(x),0) = f_d^{-1}[f_d(n(x)) + 0] = n(x)$,
   (b) $n_d^*(x) = i_{S_d}(x,0) = d(n_d(x),0) = f_d^{-1}[f_d(n_d(x)) + 0] = n_d(x)$,
   (c) $n_c^*(x) = i_{S_c}(x,0) = d(n_c(x),0) = f_d^{-1}[f_d(n_c(x)) + 0] = n_c(x)$.
5. CP is trivial. □

4.1.1. S-implications and the ordering property

First, we define the so-called weak ordering principle for implications. Although the ordering principle plays an important role, as we will see, only the weak ordering property can be required in general.

Definition 19. The implication i satisfies the **weak ordering principle (WOP)**, if the following statement holds:

$$i(x,y) = 1 \text{ if and only if } x \leqslant \tau(y),$$

where τ is a strictly increasing function from $[0,1] \to [0,1]$ with $\tau(0) = 0$ and $\tau(1) = 1$.

Remark 7. In the terminology of Maes and De Baets, τ from Definition 19 is an affirmation (see [16]).

Remark 8. Note that for $\tau(x) = x$ we get the original ordering property (OP).

Henceforth we use the following notations for the composition of two negation operators.

Definition 20. In a connective system (c, d, n)

$$\tau_{n,d}(x) := n(n_d(x)),$$

and

$$\tau_{c,d}(x) := n_c(n_d(x)),$$

where n_c and n_d are the natural negations of c and d respectively.

Remark 9. Note that in a consistent connective system $\tau_{d,n} = \tau_{n,c}$ and similarly, $\tau_{c,n} = \tau_{n,d}$.

Proposition 17. *In a nilpotent connective system i_{S_d} satisfies the ordering principle (OP), while i_{S_n} and i_{S_c} satisfy the weak ordering principle (WOP).*

Proof. For i_{S_d} we have the following:

$$i_{S_d}(x, y) = 1 \text{ if and only if } f_d^{-1}[f_d(n_d(x)) + f_d(y)] = 1,$$

which means that $f_d(n_d(x)) + f_d(y) \geqslant 1$, from which we get $n_d(x) \geqslant n_d(y)$, which holds if and only if $x \leqslant y$.

For i_{S_c}, let $\tau(x) = \tau_{c,d}(x) = n_c(n_d(x))$.

$$i_{S_c}(x, y) = 1 \text{ if and only if } f_d^{-1}[f_d(n_c(x)) + f_d(y)] = 1,$$

which means that $f_d(n_c(x)) + f_d(y) \geqslant 1$, from which we get $n_c(x) \geqslant n_d(y)$, so $x \leqslant n_c(n_d(y)) = \tau_{c,d}(y)$.

Similarly, for i_{S_n}, let $\tau(x) = \tau_{n,d}(x) = n(n_d(x))$.

$$i_{S_n}(x, y) = 1 \text{ if and only if } f_d^{-1}[f_d(n(x)) + f_d(y)] = 1,$$

which means that $f_d(n(x)) + f_d(y) \geqslant 1$, from which we get $n(x) \geqslant n_d(y)$, so $x \leqslant n(n_d(y)) = \tau_{n,d}(y)$. □

Next we give an example for a bounded system illustrating that i_{S_n} does not satisfy the ordering property. For $f_c(x) = 1 - x^2$; $f_d(x) = 1 - (1 - x)^2$; $n(x) = 1 - x$, there exist an x and a y for which $i_{S_n}(x, y) = 1$ and $y < x$, i.e. the ordering principle does not hold, because $i_{S_n}(x, y) = 1$ if and only if $d(n(x), y) = 1$.

For $x = 0.5$ and $y = 0.4$ we get $f_c(0.5) + f_d(0.4) = (1 - 0.5^2) + (1 - (1 - 0.4)^2) = 0.75 + (1 - 0.36) = 1.39$, so $i(0.5, 0.4) = 1$ and $(y < x)$.

Remark 10. Note that the following statements are equivalent:

$$i_{S_c}(x, y) = 1 \text{ if and only if } x \leqslant y \tag{13}$$

$$f_c(x) + f_d(x) = 1 \text{ for all } x \in [0, 1]. \tag{14}$$

In other words, the **ordering property**, (OP) never holds in a bounded system.

We show that the ordering property holds if and only if $f_c(x) + f_d(x) = 1$. We have $n_c(x) \geqslant n_d(y)$. This means that the ordering property for i_{S_c} (and also similarly for i_{S_n}) is equivalent to the followings: $n_c(x) \geqslant n_d(y)$ if and only if $x \leqslant y$.

It is easy to see that the condition above holds if and only if $n_d(x) = n_c(x)$, i.e. $f_c(x) + f_d(x) = 1$.

5. Comparison of implications in bounded systems

Now we prove that in a bounded system, the different types of implications considered so far never coincide.

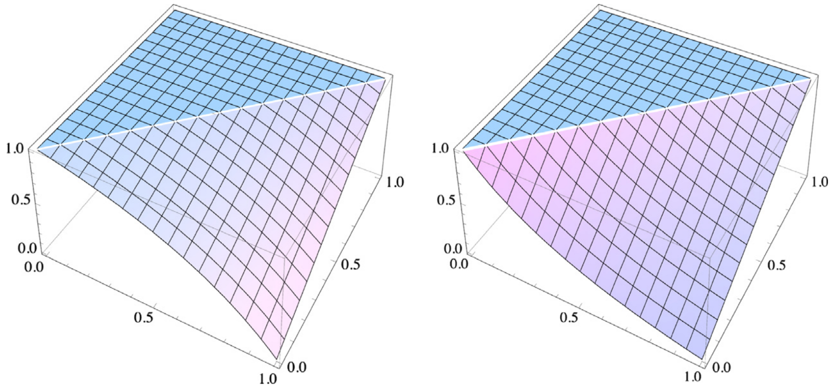
Proposition 18. *In a connective system (c, d, n), any two of the implications defined so far coincide if and only if $f_c(x) + f_d(x) = 1$, where f_c and f_d are the normalized generator functions of c and d respectively.*

Proof. It was shown in Section 4 that in a bounded system (where n_c, n_d and n are different) the natural negations of the implications in question are the same only in the case of i_R and i_{S_d}, which simply means that it is sufficient to examine their equality.

Since i_R satisfies *OP* while i_{S_c} for $f_c(x) + f_d(x) \neq 1$ does not (see Table 1), we see that in a bounded system they cannot be equal. □

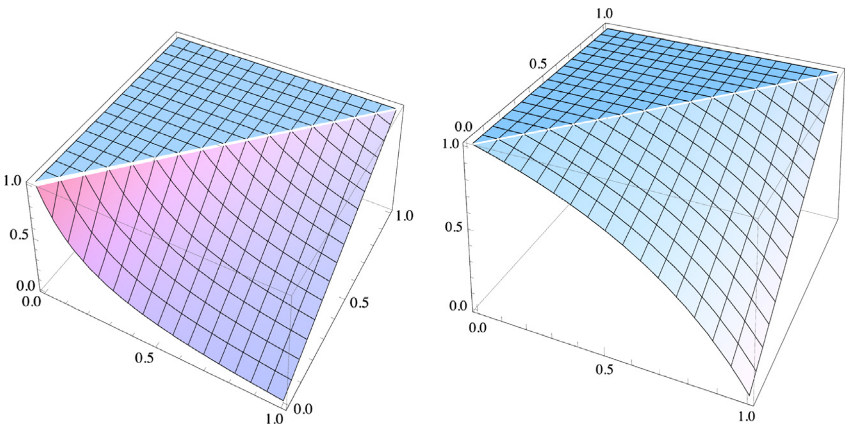
Table 1
Properties of implications in bounded systems.

	Formula	NP	EP	IP	SN	CP	WOP	OP
$\mathbf{i_c} = i_R$	$f_c^{-1}[f_c(y) - f_c(x)]$	✓	✓	✓	✓ $n_c(x)$	✓	✓	✓
$\mathbf{i_d} = i_{S_d}$	$f_d^{-1}[1 - f_d(x) + f_d(y)]$	✓	✓	✓	✓ $n_d(x)$	✓	✓	✓
i_{S_n}	$f_d^{-1}[f_c(x) + f_d(y)]$	✓	✓	✓	✓ $n(x)$	✓	✓ $\tau_{n,d}(x)$	–
i_{S_c}	$f_d^{-1}[f_d(y) + f_d(n_c(x))]$	✓	✓	✓	✓ $n_c(x)$	✓	✓ $\tau_{c,d}(x)$	–



(a) i_c for $\nu_c = 0.4$ (b) i_c for $\nu_c = 0.6$

Fig. 1. i_c (residual) implications for rational generators.



(a) i_d for $\nu_d = 0.3$ (b) i_d for $\nu_d = 0.6$

Fig. 2. i_d implications for rational generators.

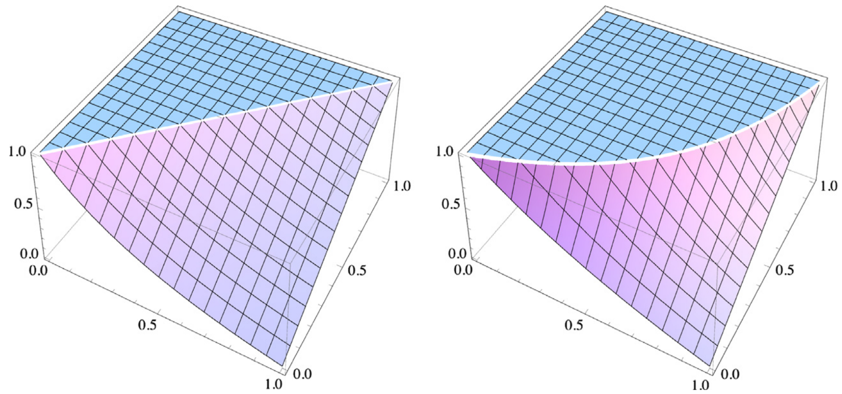
Remark 11. It is clear that in a Łukasiewicz logical system (where $f_c(x) + f_d(x) = 1$), all the implications considered in this paper coincide.

From the results of Sections 3 and 4, we can say that in a bounded system we have two different implications (namely i_R and i_{S_d}) that satisfy all of the properties NP, EP, IP, SN, CP, OP (see Table 1). Hence, the notations i_c and i_d are used, to coincide with the generator functions f_c and f_d used in the formulae of the implications, respectively (see Table 1). Henceforth let us use the following notation for sake of simplicity.

$$i_d(x, y) := i_{S_d}(x, y)$$

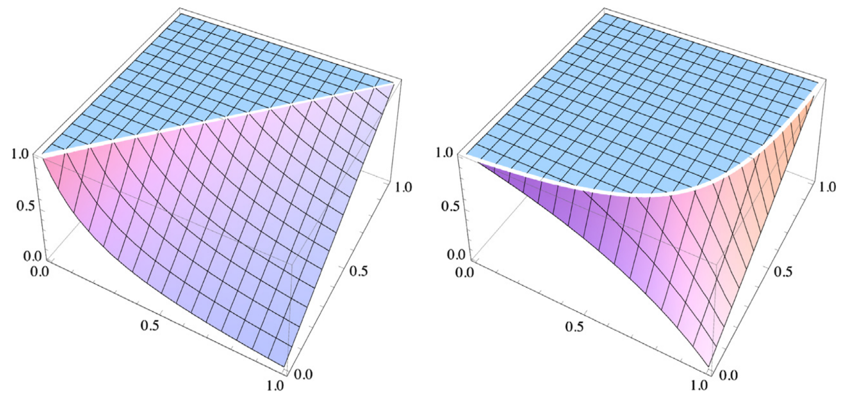
and

$$i_c(x, y) := i_R(x, y).$$



(a) $\nu_c = 0.6$ and $\nu_d = 0.4$ (b) $\nu_c = 0.4$ and $\nu_d = 0.3$

Fig. 3. S_n-implications for rational generators.



(a) $\nu_c = 0.7$ and $\nu_d = 0.3$ (b) $\nu_c = 0.4$ and $\nu_d = 0.3$

Fig. 4. S_c-implications for rational generators.

Table 2
Rational generator functions.

	$f(x)$ (generator)	$f^{-1}(x)$	$1-f(x)$	Negation
Negation	$\frac{1}{1+\frac{\nu}{1-\nu}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu}{\nu}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu}{\nu}\frac{x}{1-x}}$	$n(x) = \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2\frac{x}{1-x}}$
Conjunction	$\frac{1}{1+\frac{\nu_c}{1-\nu_c}\frac{x}{1-x}}$	$\frac{1}{1+\frac{1-\nu_c}{\nu_c}\frac{x}{1-x}}$	$\frac{1}{1+\frac{1-\nu_c}{\nu_c}\frac{1-x}{x}}$	$n_c(x) = \frac{1}{1+\left(\frac{\nu_c}{1-\nu_c}\right)^2\frac{x}{1-x}}$
Disjunction	$\frac{1}{1+\frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu_d}{\nu_d}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu_d}{\nu_d}\frac{x}{1-x}}$	$n_d(x) = \frac{1}{1+\left(\frac{1-\nu_d}{\nu_d}\right)^2\frac{x}{1-x}}$

6. Min and max operators in nilpotent connective systems

In this section we show that in a nilpotent connective system, the minimum and maximum operators can be expressed in terms of the conjunction, the disjunction and the negation.

Proposition 19. $c(x, i_c(x, y)) = \mathrm{Min}(x, y), \quad x, y \in [0, 1]$.

Proof. $c(x, i_c(x, y)) = f_c^{-1}[f_c(x) + [f_c(y) - f_c(x)]]$.
For $x \leqslant y$ $f_c(x) \geqslant f_c(y)$, which means that $c[x, i_c(x, y)] = x$.
Similarly, for $x \geqslant y$ $f_c(x) \leqslant f_c(y)$, which means that $c(x, i_c(x, y)) = y$. $\square$

Proposition 20. $n(c(n(x), i_c(n(x), n(y)))) = \mathrm{Max}(x, y), \quad x, y \in [0, 1].$

Proof. The statement follows immediately from the previous proposition (or also can been proved similarly). □

7. Summary

In this paper implications in nilpotent connective systems were examined. The concept of a weak ordering property was defined. In bounded systems two different implications, i_c and i_d were introduced, both of which fulfill all the basic features generally required for implications. In Table 1, the results concerning the properties of each implication are listed below. For rational generator functions, the implications have been plotted in Figs. 1a–4b. The formulae of the generators and the implications are summarized in Tables 1 and 2 (see [7]).

Acknowledgements

The authors are grateful to all anonymous referees whose comments and suggestions have significantly improved our original version of this paper. This study was partially supported by SROP-4.2.2.C-11/1/KONV-2012-0005, Well-being in the Information Society.

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