Generator-based Modifiers and Membership Functions in Nilpotent Operator Systems

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***Abstract*—We make a suggestion to construct modifiers and membership functions from the generator function of the logical operators in a continuous-valued nilpotent logical system. This approach makes it possible to build a system by using a generator function and only a few parameters. Moreover, it can provide a theoretical foundation for the proper choice of membership functions and modifiers.**

***Index Terms*—membership function, modifier, generator function, nilpotent logical system**

I. Introduction

Among other preferable properties, the fulfillment of the law of contradiction and the excluded middle, and the coincidence of the residual and the S-implication [1], [2] make the application of nilpotent operators in logical systems promising. In their pioneer work [3], Dombi and Csiszár examined connective systems instead of operators themselves. It was shown that a consistent connective system generated by nilpotent operators is not necessarily isomorphic to the Łukasiewicz-system. Using more than one generator function, consistent nilpotent connective systems (so-called bounded systems) can be obtained in a significantly different way with three naturally derived negation operators. Due to the fact that all continuous Archimedean (i.e. representable) nilpotent t-norms are isomorphic to the Łukasiewicz t-norm [4], the previously studied nilpotent systems were all isomorphic to the well-known Łukasiewicz logic.

In the last few years, the most important multivariable operators of general nilpotent systems have been thoroughly examined. In [5] and in [6], Dombi and Csiszár examined the implications and equivalence operators in bounded systems. In [7], a parametric form of the generated operator o_ν was given by using a shifting transformation of the generator function. Here, the parameter has an important semantical meaning as a threshold of expectancy (decision level). This means that nilpotent conjunctive, disjunctive, aggregative (where a high input can compensate for a lower one) and negation operators can be obtained by changing this parameter. Negation operators were also studied thoroughly in [3], as they play a significant role in logical systems by building connections between the main operators (De Morgan law) and characterising their basic properties.

In fuzzy theory, modalities (like *possibly, necessarily, ...*) and hedges (like *very, quite, extremely, ...*) are the most examined unary operators, which modify the linguistic variables. Despite their significance, about other unary operators (compared to the multivariable ones) there are only limited literature available. In [8] Dombi and Csiszár introduced possibility and neccesity operators by choosing $x_i = x_j (\forall i, j)$ for the arguments of the manyvariable operators.

In this study, the focus is on the unary operators of a nilpotent logical system, which perform various operations such as incrementing or decrementing a value and they can be widely used for expressing modalities in human thinking. We introduce a new approach by using compositions of negations. In the early 1970's, Zadeh [9] introduced a class of powering modifiers, which defined the concept of linguistic variables and hedges (like *very, quite, extremely, ...*). He proposed computing with words as an extension of fuzzy sets and logic theory and introduced modifier functions of fuzzy sets called linguistic hedges, which change the meaning of the primary terms. As pointed out by Zadeh, linguistic variables and terms are closer to human thinking and therefore, words and linguistic terms can be used to model human thinking systems [10]. Hedges and also modalities (like *possibly, necessarily, ...*) are the most examined unary operators. From a semantic viewpoint, these unary operators can be viewed as a part of a logical system.

Membership functions, which play a substantial role in the overall performance of fuzzy representation, can also be defined by means of a generator function. In the literature, the membership functions are usually chosen independently of the logical operators of the system. Parameters are normally fine-tuned on the basis of pure experimental results. Now, we make a suggestion, how modifiers and membership functions can be connected to the logical operators of the system. Using operator-dependent membership functions makes it possible to build up a system by using a single generator function and a few parameters. Moreover, it can provide a theoretical explanation for the choice of membership functions and modifiers.

The article is organized as follows. After recalling some basic preliminaries in Section II, we introduce unary operators derived as a composition of negation operators in Section III. In Section IV and V, we suggest a new approach to membership- and nonmembership functions. Finally, in Section VI, the main results are summarized.

II. Preliminaries

First, we recall the basic considerations of negations.

Definition 1. *([11], pp. 13.) A unary operation $n : [0,1] \to [0,1]$ is called a negation if it is non-increasing and compatible with classical logic; i.e. $n(0) = 1$ and $n(1) = 0$. A negation is strict if it is also strictly decreasing and continuous. A negation is strong, if it is also involutive; i.e. $n(n(x)) = x$ for $\forall x \in \mathbb{R}$.*

From the definion follows directly that there exists a fixpoint (or neutral value) $\nu \in [0,1]$ of the negation with $n(\nu) = \nu$.

Definition 2. *If $n_1(x)$ and $n_2(x)$ are negations with fixpoints ν_1 and ν_2 respectively, then $n_1(x)$ is called stricter than $n_2(x)$, if $\nu_1(x) < \nu_2(x)$.*

The well-known representation theorem for strong negations was obtained by Trillas [12].

Proposition 1. *n is a strong negation if and only if*

$$n(x) = f_n^{-1}(1 - f_n(x)),$$

where $f_n : [0,1] \to [0,1]$ is an automorphism of $[0,1]$.

Proposition 2. *In Proposition 1 (Trillas) the generator function can also be decreasing.*

Next, we recall the basic concept of the so-called bounded systems [3].

Definition 3. *The triple (c, d, n), where c is a t-norm, d is a t-conorm and n is a strong negation, is called a connective system.*

Definition 4. *A connective system is nilpotent if the conjunction c is a nilpotent t-norm, and the disjunction d is a nilpotent t-conorm.*

Definition 5. *Two connective systems (c_1, d_1, n_1) and (c_2, d_2, n_2) are isomorphic if there exists a bijection $\phi : [0,1] \to [0,1]$ such that*

$$\phi^{-1}\left(c_1\left(\phi(x), \phi(y)\right)\right) = c_2(x, y)$$

$$\phi^{-1}\left(d_1\left(\phi(x), \phi(y)\right)\right) = d_2(x, y)$$

$$\phi^{-1}\left(n_1\left(\phi(x)\right)\right) = n_2(x).$$

In the nilpotent case, the generator functions of the disjunction and the conjunction being determined up to a multiplicative constant can be normalized the following way:

$$f_c(x) := \frac{t(x)}{t(0)}, \qquad f_d(x) := \frac{s(x)}{s(1)}. \tag{1}$$

Remark 1. *Thus, the normalized generator functions are uniquely defined.*

We will use normalized generator functions for conjunctions and disjunctions as well. This means that the normalized generator functions of conjunctions, disjunctions and negations are

$$f_c, f_d, f_n : [0,1] \to [0,1]. \tag{2}$$

We will suppose that f_c, f_d and f_n are continuous and strictly monotonic functions.

Using Proposition 2, two special negations can be generated by the normalized additive generators of the conjunction and the disjunction.

Definition 6. *The negations n_c and n_d generated by f_c and f_d respectively,*

$$n_c(x) = f_c^{-1}(1 - f_c(x))$$

and

$$n_d(x) = f_d^{-1}(1 - f_d(x))$$

are called natural negations of c and d.

This means that for a connective system with normalized generator functions f_c, f_d and f_n we can associate three negations, n_c, n_d and n. In a consistent system, $f_c(x) + f_d(x) \geq 1$ always holds, as the following proposition states.

Proposition 3.
1) *If the connective system (c, d, n) is consistent, then $f_c(x) + f_d(x) \geq 1$ for any $x \in [0,1]$, where f_c and f_d are the normalized generator functions of the conjunction c and the disjunction d respectively.*
2) *If $f_c(x) + f_d(x) \geq 1$ for any $x \in [0,1]$ and the De Morgan law holds, then the connective system (c, d, n) satisfies the classification property as well (which now means that the system is consistent).*

If $f_c(x) + f_d(x) = 1$ (or equivalently, if the two natural negations coincide), we get a system, which is isomorphic to the Łukasiewicz system, otherwise, for $f_c(x) + f_d(x) > 1$, the so-called bounded systems.

Proposition 4. *In a connective system the following equations are equivalent:*

$$f_c(x) + f_d(x) = 1 \tag{3}$$

$$n_c(x) = n_d(x), \tag{4}$$

where f_c, f_d are the normalized generator functions of the conjunction and the disjunction and n_c, n_d are the natural negations.

Definition 7. *A nilpotent connective system is called a bounded system if*

$$f_c(x) + f_d(x) > 1, \text{or equivalently } n_d(x) < n(x) < n_c(x) \tag{5}$$

holds for all $x \in (0,1)$, where f_c and f_d are the normalized generator functions of the conjunction and disjunction, and n_c, n_d are the natural negations.

For examples for consistent bounded systems see [3].

Definition 8. *Let us define the cutting operation $[\ ]$ by*

$$[x] = \begin{cases} 0 & if \quad x < 0 \\ x & if \quad 0 \le x \le 1 \\ 1 & if \quad 1 < x \end{cases}$$

Proposition 5. *With the help of the cutting operator, we can write the conjunction and disjunction in the following form, where f_c and f_d are decreasing and increasing normalized generator functions respectively.*

$$c(x, y) = f_c^{-1}[f_c(x) + f_c(y)], \tag{6}$$

$$d(x, y) = f_d^{-1}[f_d(x) + f_d(y)]. \tag{7}$$

III. Unary operators induced by negation operators

We assume that the possibility and neccessity operators have to fulfill the following conditions:

$$\text{impossible}(x) = \text{necessity}(\text{not}(x)) \tag{8}$$

and

$$\text{possible}(x) = \text{not}(\text{impossible}(x)). \tag{9}$$

In [8] Dombi and Csiszár introduced possibility and neccesity operators by repeating the arguments of manyvariable operators. An alternative option for defining unary operators is to obtain them by means of a suitable composition of negation operators. From a semantic point of view, we can think of the word "impossible" as a stricter (stronger) negation, in a sense that it has a smaller fixpoint (see Figure 1).

If $n_{\nu_2}(x)$ is a negation with a fixpoint ν_2 (with the semantic meaning of *"not"*) and $n_{\nu_1}(x)$ is a stricter negation (see Definition 2) with a fixpoint ν_1 (with the semantic meaning of *"impossible"*), i.e. $\nu_1 < \nu_2$, then the necessity operator can be derived from the following interpretation (see also (8)):

$$\text{"}impossible\text{"} = \text{"}necessarily\ not\text{"};$$

i.e. if we denote the necessity operator by $\tau^N(x)$,

$$n_{\nu_1}(x) = \tau^N(n_{\nu_2}(x)).$$

Based on this interpretation, we can define the necessity operator (by plugging $n_{\nu_2}(x)$ into the equation above and taking into account the fact that $n_{\nu_2}(x)$ is involutive) the following way.

Definition 9. *Let $n_{\nu_1}(x)$ and $n_{\nu_2}(x)$ be negations with fixponts ν_1 and ν_2 respectively, where $\nu_1 < \nu_2$.*

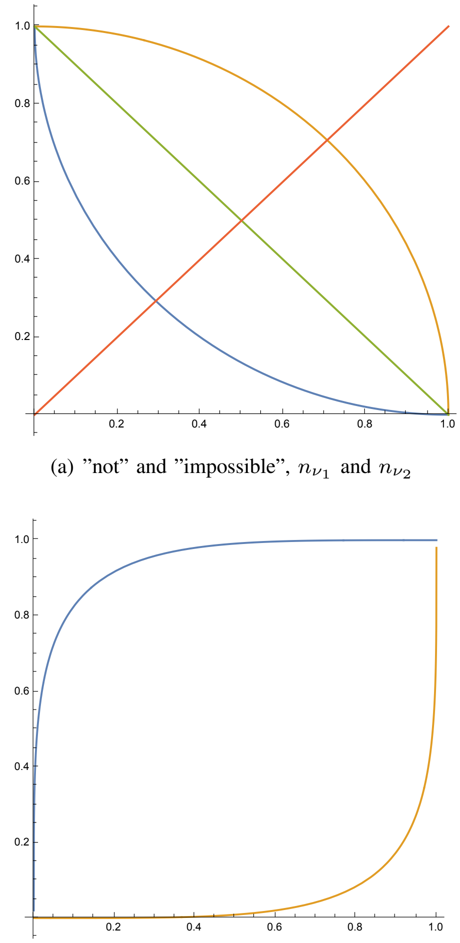
$$\tau^N_{\nu_1,\nu_2}(x) := n_{\nu_1}(n_{\nu_2}(x)) \tag{10}$$

is called the necessity operator.

Similarly, interpreting "possible" as "not impossible" (see also (9)), the possibility operator can be defined in the following way.

Definition 10. *Let $n_{\nu_1}(x)$ and $n_{\nu_2}(x)$ be negations with fixponts ν_1 and ν_2, respectively, where $\nu_1 < \nu_2$. Then*

$$\tau^P_{\nu_1,\nu_2}(x) := n_{\nu_2}(n_{\nu_1}(x)) \tag{11}$$



(a) "not" and "impossible", n_{ν_1} and n_{ν_2}

(b) Possibility and necessity operators $\tau^P_{\nu_1,\nu_2}(x)$ and $\tau^N_{\nu_1,\nu_2}(x)$

Fig. 1. Unary operators "not", "impossible, "possible" and "necessary"

is called the possibility operator.

Remark 2. *Note that the necessity and possibility operators differ only in the order of the negations in the composition. Necessity and possibility can be described by the parameter values ν_1 and ν_2.*

In a bounded system, the natural negations can serve as n_{ν_1} and n_{ν_2}. In this case, necessity and possibility can also be defined in a natural way, and the parameters of the generator functions of the conjunction, disjunction and negation, determine the parameters of the modal operators.

Remark 3. *If $f_c(x)$ and $f_d(x)$ are the generator functions of a bounded system and $n_c(x)$ and $n_d(x)$ the natural negations of c and d with fixpoints ν_c and ν_d respectively, then the possibilty and necessity operators can be defined by*

$$\tau^N_{\nu_d,\nu_c}(x) = n_d(n_c(x)), \tag{12}$$

$$\tau^P_{\nu_c,\nu_d}(x) = n_c(n_d(x)), \tag{13}$$

since from (7) follows $\nu_d < \nu_c$.

Example 1. *The Dombi functions defined as*

$$f_n(x) = \begin{cases} \dfrac{1}{1 + \frac{\nu}{1-\nu}\frac{1-x}{x}} & x \neq 0, \\ 0 & x = 0; \end{cases}$$

$$f_c(x) = \begin{cases} \dfrac{1}{1 + \frac{\nu_c}{1-\nu_c}\frac{x}{1-x}} & x \neq 1, \\ 0 & x = 1; \end{cases}$$

$$f_d(x) = \begin{cases} \dfrac{1}{1 + \frac{\nu_d}{1-\nu_d}\frac{1-x}{x}} & x \neq 0, \\ 0 & x = 0, \end{cases}$$

where $\nu, \nu_d, \nu_c \in (0,1), \quad \nu_d < \nu < \nu_d,$
generate a bounded system if and only if $\nu_c + \nu_d < 1$ *[3]. Here,*

$$n_c(x) = \frac{1}{1 + \left(\frac{1-\nu_c}{\nu_c}\right)^2 \frac{x}{1-x}}$$

and

$$n_d(x) = \frac{1}{1 + \left(\frac{1-\nu_d}{\nu_d}\right)^2 \frac{x}{1-x}},$$

see Table I.

TABLE I
RATIONAL FUNCTIONS AS NORMALIZED GENERATORS – 2 NATURAL NEGATIONS

$f(x)$	$f^{-1}(x)$	$1 - f(x)$	negation
$\frac{1}{1+\frac{1-\nu_c}{\nu_c}\frac{x}{1-x}}$	$\frac{1}{1+\frac{1-\nu_c}{\nu_c}\frac{x}{1-x}}$	$\frac{1}{1+\frac{\nu_c}{1-\nu_c}\frac{1-x}{x}}$	$\frac{1}{1+\left(\frac{1-\nu_c}{\nu_c}\right)^2\frac{x}{1-x}}$
$\frac{1}{1+\frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu_d}{\nu_d}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu_d}{\nu_d}\frac{x}{1-x}}$	$\frac{1}{1+\left(\frac{1-\nu_d}{\nu_d}\right)^2\frac{x}{1-x}}$

Proposition 6. *For the Dombi functions from Example 1,*

$$\tau^N_{\nu_d,\nu_c}(x) = n_d(n_c(x)) = \frac{1}{1 + A \cdot \frac{1-x}{x}} \tag{14}$$

and

$$\tau^P_{\nu_c,\nu_d}(x) = n_c(n_d(x)) = \frac{1}{1 + \frac{1}{A} \cdot \frac{1-x}{x}}, \tag{15}$$

where $A = \left(\frac{\nu_c}{1-\nu_c}\right)^2 \left(\frac{1-\nu_d}{\nu_d}\right)^2.$

In Figure 2, $\tau^N_{\nu_d,\nu_c}(x)$ and $\tau^P_{\nu_c,\nu_d}(x)$ are illustrated for the Dombi functions with different values of ν_c and ν_d.

Proof. We prove only for the necessity operator. The possibility case can be proven in a similar way.

Let us use the following notations:

$$C := \left(\frac{1-\nu_c}{\nu_c}\right)^2 \text{ and } \quad D := \left(\frac{1-\nu_d}{\nu_d}\right)^2.$$

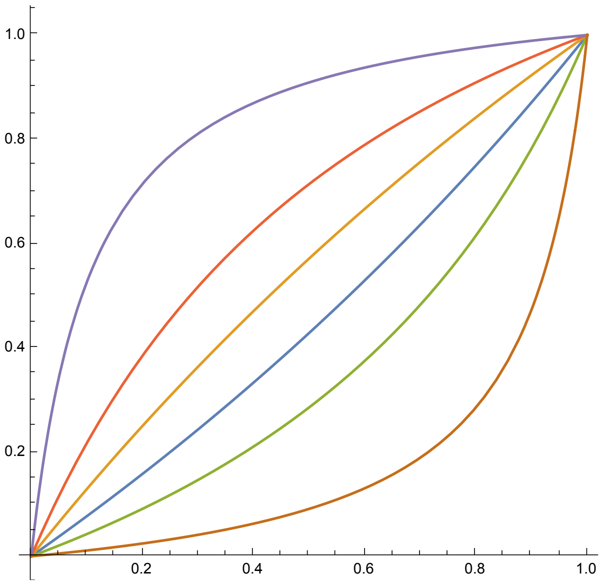


Fig. 2. $\tau^N_{\nu_d,\nu_c}(x)$ and $\tau^P_{\nu_c,\nu_d}(x)$ for $A = 0.1, 0.4$ and 0.75

$$\tau^N_{\nu_d,\nu_c}(x) = n_d(n_c(x)) = \frac{1}{1 + D\frac{\frac{1}{1+C\frac{x}{1-x}}}{1-\frac{1}{1+C\frac{x}{1-x}}}} =$$

$$= \frac{1}{1 + \frac{D}{C}\frac{1-x}{x}} = \frac{1}{1 + A \cdot \frac{1-x}{x}}.$$

□

Proposition 7. $\tau^N_{\nu_1,\nu_2}(x)$ *and* $\tau^P_{\nu_1,\nu_2}(x)$ *satisfy the basic properties of modalities:*

$N1 \quad \tau^N_{\nu_1,\nu_2}(1) = 1$
$N2 \quad \tau^N_{\nu_1,\nu_2}(x) \leq x$
$N3 \quad x \leq y$ *if and only if* $\tau^N_{\nu_1,\nu_2}(x) \leq \tau^N_{\nu_1,\nu_2}(y)$

$P1 \quad \tau^P_{\nu_1,\nu_2}(0) = 0$
$P2 \quad x \leq \tau^P_{\nu_1,\nu_2}(x)$
$P3 \quad x \leq y$ *if and only if* $\tau^P_{\nu_1,\nu_2}(x) \leq \tau^P_{\nu_1,\nu_2}(y).$

Proof. We prove only for the necessity operator. The possibility case can be proven in a similar way.

$N1 \quad \tau^N_{\nu_1,\nu_2}(1) = n_{\nu_1}(n_{\nu_2}(1)) = 1$
$N2 \quad n_{\nu_1}(x) \leq n_{\nu_2}(x) \implies x \geq n_{\nu_1}(n_{\nu_2}(x)) = \tau^N_{\nu_1,\nu_2}(x)$
$N3$ Follows from the monotonicity of the negations.

□

The following proposition describes all the possible compositions of the negations n_{ν_1}, n_{ν_2}, possibility $\tau^P_{\nu_1,\nu_2}$ and necessity τ^{ν_1,ν_2}_N.

Proposition 8. 1) *"it is not impossible" = "it is possible"; i.e.* $n_{\nu_2}(n_{\nu_1}(x)) = \tau^P_{\nu_1,\nu_2}(x)$.

2) *"it is not possible" = "it is impossible"; i.e.* $n_{\nu_2}\left(\tau^P_{\nu_1,\nu_2}(x)\right) = n_{\nu_2}(x)$.

3) *"it is not necessary" = "it is possibly not"; i.e.* $n_{\nu_2}\left(\tau^N_{\nu_1,\nu_2}(x)\right) = \tau^P_{\nu_1,\nu_2}(n_{\nu_2}(x))$.

4) *"it is impossible that it is not" = "it is necessary"; i.e.* $n_{\nu_1}(n_{\nu_2}(x)) = \tau^N_{\nu_1,\nu_2}(x)$.

5) *"it is impossible that it is possible" = "it is necessary that it is impossible"; i.e.* $n_{\nu_1}\left(\tau^P_{\nu_1,\nu_2}(x)\right) = \tau^N_{\nu_1,\nu_2}(n_{\nu_1}(x))$.

6) *"it is impossible that it is necessary" = "it is possible that it is impossible"="not"; i.e.* $n_{\nu_1}\left(\tau^N_{\nu_1,\nu_2}(x)\right) = \tau^P_{\nu_1,\nu_2}(n_{\nu_1}(x)) = n_{\nu_2}(x)$.

7) *"it is possible that it is necessary" = "it is necessary that it is possible"; i.e.* $\tau^P_{\nu_1,\nu_2}(\tau^N_{\nu_1,\nu_2}(x)) = \tau^N_{\nu_1,\nu_2}(\tau^P_{\nu_1,\nu_2}(x)) = x$.

Proof. All these statements follow from a direct calculation, taking into account the fact that the negation operators n_{ν_1} and n_{ν_2} are involutive. □

IV. Membership Functions

As highly applied membership functions, the triangular membership functions are formed using straight lines. These straight line membership functions have the advantage of simplicity. However, triangle membership functions are non-differentiable in three points, which may lead to problems if using classical optimization methods. Because of their smoothness and concise notation, Gaussian membership functions are popular for specifying fuzzy sets. These curves have the advantage of being smooth and nonzero at all points.

When it comes to application, real life situations have a higher complexity and therefore, special membership functions are usually needed.

Most of the applications use arbitrary functions that suit the given situation regarding simplicity, convenience, speed and efficiency.

The membership functions defined in this section model the truth value of the statement "x is equal to 0". Similarly, by means of an adequate translation, such memebrship functions can be easily obtained modelling the statement "x is equal to a", where a is an arbitrary given value.

Note that in the following definition, the parameter ε has the semantic meaning of tolerance.

Definition 11. *Let* $f_c : [0,1] \to [0,1]$ *be a decreasing bijection,* $\nu \in (0,1)$*,* $\lambda \in \mathbb{R}, \lambda > 1$*,* $\varepsilon \in [0,1]$*, and let us define the operator-dependent membership function as*

$$\delta^{(\lambda)}_{\varepsilon}(x) = f_c^{-1}\left[f_c(\nu)\left|\frac{x}{\varepsilon}\right|^{\lambda}\right]. \tag{16}$$

Proposition 9. 1) $\delta^{(\lambda)}_{\varepsilon}(x)$ *is an even function,*

2) $\delta^{(\lambda)}_{\varepsilon}(\varepsilon) = \nu$,

3) $\delta^{(\lambda)}_{\varepsilon}(0) = 1$.

Proof. Follows from direct calculation. □

In Figures 3 and 4, operator-dependent membership functions are illustrated using the generator function of the Łukasiewicz- and Dombi operators respectively.

For $\lambda = 2$, the absolute value function can be omitted, which proves to be a key step towards differentiability.

Remark 4. *Note that the above-defined construction of membership functions connects the Gauss-curve and probability theory together by providing a Gaussian membership function for* $\lambda = 2$ *and* $f_c(x) = -\ln x$ *(the generator function of the product operator, which belongs to probabilistic reasoning).*

Remark 5. *Note that for* $\lambda = 1$ *and* $f_c(x) = 1 - x$ *(generator function of the Łukasiewicz t-norm), the definition above provides a triangular membership function.*

The following proposition states an important advantage of the above described approach to membership functions and modifiers.

Proposition 10.

Proof. □

V. Non-membership functions

A generalization of fuzzy sets was introduced by Atanassov in 1986 as intuitionistic fuzzy sets (IFSs) [13], including both membership and non-membership of the elements. In a similar way as in Section IV, the non-membership functions can be defined in naturally by using the generator function of the disjunction. These functions can model the truth value of the statement "x is not equal to 0" or, by means of an adequate translation, also the statement "x is not equal to a", where a is an arbitrary given value.

Definition 12. *Let* $f_d : [0,1] \to [0,1]$ *be an increasing bijection,* $\nu \in (0,1)$*,* $\lambda \in \mathbb{R}, \lambda > 1$*,* $\varepsilon \in [0,1]$*, and let us define the operator-dependent membership function as*

$$\hat{\delta}^{(\lambda)}_{\varepsilon}(x) = f_d^{-1}\left[f_d(\nu)\left|\frac{x}{\varepsilon}\right|^{\lambda}\right]. \tag{17}$$

Proposition 11. 1) $\hat{\delta}^{(\lambda)}_{\varepsilon}(x)$ *is an even function,*

2) $\hat{\delta}^{(\lambda)}_{\varepsilon}(\varepsilon) = \nu$,

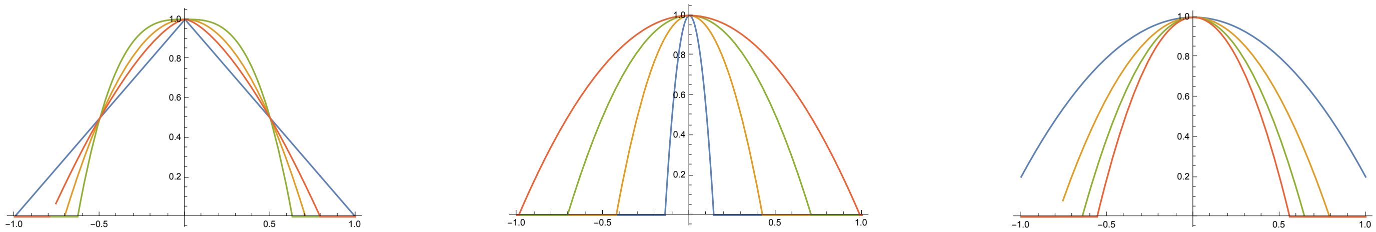
3) $\hat{\delta}^{(\lambda)}_{\varepsilon}(0) = 0$.

Proof. Follows from direct calculation. □

In Figures 5 and 6 operator-dependent non-membership functions are illustrated using the generator function of the Łukasiewicz- and Dombi operators respectively.

VI. Conclusion and future work

The main purpose of this paper was to examine a new approach to modifiers of nilpotent logical systems by using compositions of negations and to suggest choosing membership functions based on the generator function of the logical operators. As a result, a nilpotent logical system can be obtained, in which all operators are connected to each other, and where the modalities, hedges and also the membership functions are operator-dependent. This approach opens the

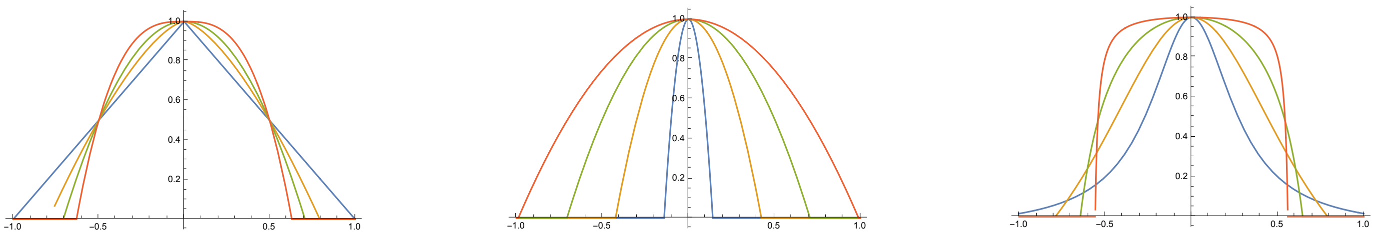


(a) Membership functions $\delta_{\varepsilon}^{(\lambda)}(x)$ for $\lambda = 1, 1.5, 2, 3,\ \varepsilon = 0.5, \nu = 0.5$

(b) Membership functions $\delta_{\varepsilon}^{(\lambda)}(x)$ for $\varepsilon = 0.1, 0.3, 0.5, 0.7,\ \lambda = 2, \nu = 0.5$

(c) Membership functions $\delta_{\varepsilon}^{(\lambda)}(x)$ for $\lambda = 2,\ \varepsilon = 0.5,\ \nu = 0.2, 0.4, 0.6, 0.8$

Fig. 3. Membership funcions $\delta_{\varepsilon}^{(\lambda)}(x)$ generated by $f_c(x) = 1 - x$

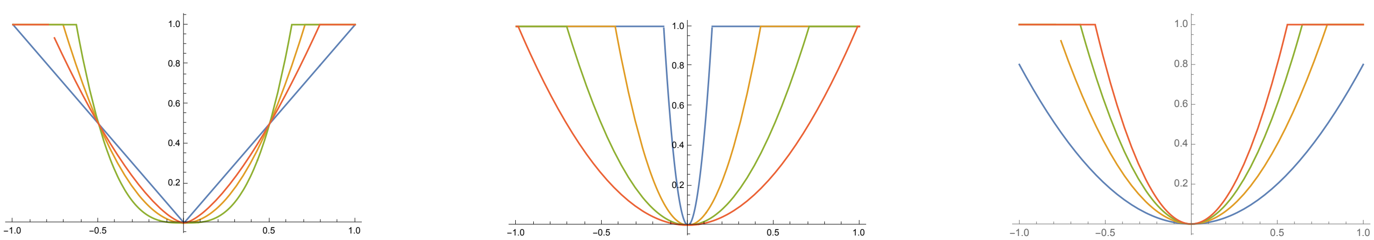


(a) Membership functions $\delta_{\varepsilon}^{(\lambda)}(x)$ for $\lambda = 1, 1.5, 2, 3,\ \varepsilon = 0.5, \nu = 0.5$

(b) Membership functions $\delta_{\varepsilon}^{(\lambda)}(x)$ for $\varepsilon = 0.1, 0.3, 0.5, 0.7,\ \lambda = 2, \nu = 0.5$

(c) Membership functions $\delta_{\varepsilon}^{(\lambda)}(x)$ for $\lambda = 2,\ \varepsilon = 0.5,\ \nu = 0.2, 0.4, 0.6, 0.8$

Fig. 4. Membership funcions $\delta_{\varepsilon}^{(\lambda)}(x)$ generated by $f_c(x) = \frac{1}{1+\frac{1-\nu_c}{\nu_c}\frac{x}{1-x}}$

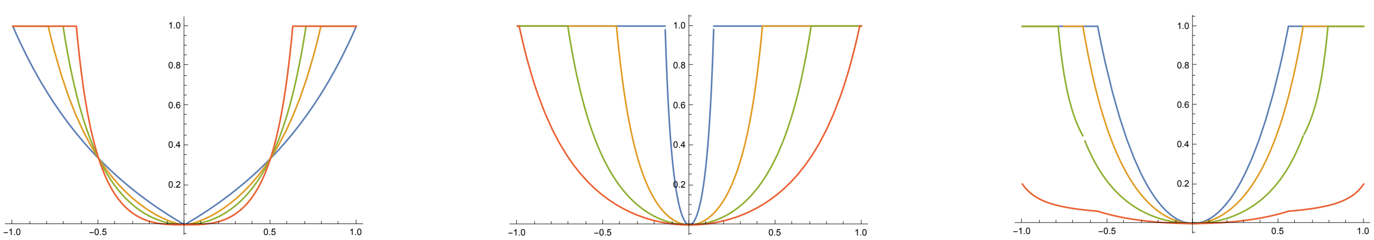


(a) Non-membership functions $\hat{\delta}_{\varepsilon}^{(\lambda)}(x)$ for $\lambda = 1, 1.5, 2, 3,\ \varepsilon = 0.5, \nu = 0.5$

(b) Non-membership functions $\hat{\delta}_{\varepsilon}^{(\lambda)}(x)$ for $\varepsilon = 0.1, 0.3, 0.5, 0.7,\ \lambda = 2, \varepsilon = 0.5$

(c) Membership functions $\delta_{\varepsilon}^{(\lambda)}(x)$ for $\lambda = 2,\ \varepsilon = 0.5,\ \nu = 0.2, 0.4, 0.6, 0.8$

Fig. 5. Non-membership funcions $\hat{\delta}_{\varepsilon}^{(\lambda)}(x)$ generated by $f_d(x) = x$



(a) Non-membership functions $\hat{\delta}_{\varepsilon}^{(\lambda)}(x)$ for $\lambda = 1, 1.5, 2, 3,\ \varepsilon = 0.5, \nu = 0.5$

(b) Non-membership functions $\hat{\delta}_{\varepsilon}^{(\lambda)}(x)$ for $\varepsilon = 0.1, 0.3, 0.5, 0.7,\ \lambda = 2, \varepsilon = 0.5$

(c) Membership functions $\delta_{\varepsilon}^{(\lambda)}(x)$ for $\lambda = 2,\ \varepsilon = 0.5,\ \nu = 0.2, 0.4, 0.6, 0.8$

Fig. 6. Non-membership funcions $\hat{\delta}_{\varepsilon}^{(\lambda)}(x)$ generated by $f_d(x) = \frac{1}{1+\frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}$

door to an easy to handle system. It becomes possible to define all the operators by a single generator function and a few parameters. By fitting the parameter values, the system can be used to model real-life problems.

The main disadvantage of the Łukasiewicz operator family is the lack of differentiability, which would be necessary for numerous practical applications. Although most fuzzy applications (e.g. embedded fuzzy control) use piecewise linear membership functions due to their easy handling, there are significant areas, where the parameters are learned by a gradient based optimization method. In this case, the lack of continuous derivatives makes the application impossible. For example, the membership functions have to be differentiable for every input in order to fine tune a fuzzy control system by a simple gradient based technique.

This problem could be easily solved by using the so-called squashing function (see Dombi and Gera, [14]), which provides a solution to the above mentioned problem by a continuously differentiable approximation of the cut function. This approximation could be the next step along the path to a practical and widely applicable system.

In deep neural networks, which are rapidly becoming a fundamental component of high performance speech systems and image recognition systems, a faster and more effective training can be provided by using rectifiers [15] (compared to the widely used activation functions prior to 2011, such as the logistic sigmoid inspired by probability theory and its more practical counterpart, the hyperbolic tangent). Based on a strong biological motivation, the rectifier was first introduced to a dynamical network by Hahnloser et al. in 2000 in Nature [16], [17]. Today, the rectifier function is the most popular activation function used for deep neural networks [18], [19].

A possible reason for the better performance of the rectifier might be the Łukasiewicz logic in the background. In our next paper we are planning to show how a nilpotent logical system can be modelled by rectifiers in neural network calculations.

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