Modeling Probability Weighting Functions in Prospect Theory by using a Class of Modifier Operators of Continuous-valued Logic

József Dombi
Institute of Informatics
University of Szeged
Szeged, Hungary
dombi@inf.u-szeged.hu

Tamás Jónás
Institute of Business Economics
Eötvös Loránd University
Budapest, Hungary
jonas@gti.elte.hu

***Abstract*—In this study, it is demonstrated that the application of the Dombi modifier operator class of continuous-valued logic may be viewed as a general approach for modeling probability weighting functions of prospect theory including the well-known ones. Furthermore, a two-phase regression method for fitting a probability weighting function generated by the modifier operator to empirical data is also presented.**

***Index Terms*—probability weighting functions, continuous-valued logic, pliant system, modifier operator, regression**

I. Introduction

In economics, the probability waiting functions describe the phenomenon that people tend to overreact those events that occur with a low probability and underreact the events that have a high probability (see, e.g. [1]–[6]). The large amount of researches related to the probability weighting functions in prospect theory demonstrate that these functions play an essential role in decision making (see, e.g. [7]–[14]). In this paper, we present a novel methodology that can be used to generate parametric probability weighting functions by making the use of Dombi's modifier operators of continuous-valued logic [15]. Here, we will show that the modifier operator $m_{\nu,\nu_0}^{(\lambda)} : [0,1] \to [0,1]$, which is given by

$$m_{\nu,\nu_0}^{(\lambda)}(x) = f^{-1}\left(f(\nu_0)\left(\frac{f(x)}{f(\nu)}\right)^{\lambda}\right), \qquad (1)$$

where $\nu, \nu_0 \in (0,1)$, $\lambda \in \mathbb{R}$ and $f : [0,1] \to [0,\infty]$ is a strictly monotonic (either increasing or decreasing) continuous function with the inverse function $f^{-1} : [0,\infty] \to [0,1]$, is a probability weighting function. In fuzzy theory, function f is called the generator function of the modifier operator (probability weighting function) $m_{\nu,\nu_0}^{(\lambda)}$. The main findings of our research, which we will present in this paper, are as follows.

1) We will demonstrate that the application of the modifier operator in equation (1) may be treated as a general approach for creating probability weighting functions including the well-known ones.
2) We will introduce a two-phase regression method for fitting a probability weighting function generated by the modifier operator in (1) with $\nu = \nu_0$ to empirical data.

The rest of this paper is structured as follows. In Section 2, we will briefly introduce the basic notions of prospect theory including the probability weighting functions. In Section 3, we will revisit the general form of the modifier operators in Dombi's pliant system [15], [16]. Next, in Section 4, we will show how the modifier operator can be utilized to model probability weighting functions. A two-phase regression method for fitting a probability weighting function generated by the modifier operator to empirical data is presented in Section 5. Lastly, in Section 6, we will provide a short summary of our findings.

II. Probability Weighting Functions in Prospect Theory

Here, we will use the concepts of utility functions and probability weighting functions. In prospect theory, these functions are defined as follows [13], [17].

Definition 1 (Utility function): The function $U : [0,\infty] \to \mathbb{R}$ is said to be a utility function, if U satisfies the following requirements:

1) U is continuous and strictly increasing
2) $U(0) = 0$.

The value of the utility function U at x may be viewed as the utility value of the wealth x.

Definition 2 (Probability weighting function): The function $w : [0,1] \to [0,1]$ is said to be a probability weighting function, if w satisfies the following requirements:

1) w is continuous and strictly increasing
2) $w(0) = 0$ and $w(1) = 1$.

Let S be a finite set of possible states (events), and let $E_1, E_2, \ldots, E_n \subseteq S$ be uncertain events, $n \geq 2$. In prospect theory, a prospect is a mapping from S to the real numbers, describing the resulting outcome for every state if that state is the true state [13]. Traditionally, prospects are often denoted as $(E_1 : x_1, E_2 : x_2, \ldots, E_n : x_n)$ meaning that the prospect yields x_i under the event E_i, where $i = 1, 2, \ldots, n$. The prospect $(E_1 : x_1, E_2 : x_2, \ldots, E_n : x_n)$ is evaluated by using the formula:

$$\sum_{i=1}^{n} W(E_i)U(x_i), \tag{2}$$

in which the functions W and U can be interpreted according to the following cases, where each case generalizes the preceding one [17].

- Case 1: (Expected value) U is the identity function and W is a probability measure P on the finite set S.
- Case 2: (Expected utility) U is a utility function and W is a probability measure P on the finite set S.
- Case 3: (Probabilistic sophistication (with non-expected utility)) U is a utility function, P is a probability measure on the finite set S, and there exists a probability weighting function w, such that $W = w \circ P$.
- Case 4: (General model) U is a utility function and W is a fuzzy measure (monotone measure) on the finite set S; that is, W satisfies the following requirements: (i) $W(\emptyset) = 0$; (ii) $W(S) = 1$; (iii) for any $A, B \in S$ $A \subset B$ implies $W(A) \leq W(B)$.

In this paper, we will show how certain unariy operators of continuous-valued logic can be applied to modeling probability weighting functions. It should be added here that in prospect theory, the argument of a probability weighting function is traditionally denoted by p, indicating that the probability weighting function is a transformation on a probability measure. Here, we will use the notation x for the argument of function w.

III. General Form of Modifier Operator in Pliant System

In continuous-valued logic, linguistic modifiers over fuzzy sets that have strictly monotonously increasing or decreasing membership functions can be modeled by modifier operators. In Dombi's pliant system [15], [16], the general form of the modifier operator is given as follows.

Definition 3: The modifier operator $m^{(\lambda)}_{\nu,\nu_0} : [0, 1] \to [0, 1]$ is given by

$$m^{(\lambda)}_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0)\left(\frac{f(x)}{f(\nu)}\right)^{\lambda}\right), \tag{3}$$

where $\nu, \nu_0 \in (0, 1)$, $\lambda \in \mathbb{R}$ and $f : [0, 1] \to [0, \infty]$ is a strictly decreasing (or increasing) continuous function with the inverse function $f^{-1} : [0, \infty] \to [0, 1]$. Here, function f is called the generator function of the modifier operator $m^{(\lambda)}_{\nu,\nu_0}$.

Later on, we will show that the value of parameter λ is closely related to the slope of function $m^{(\lambda)}_{\nu,\nu_0}$ at $x = \nu$. Notice that

$$m^{(\lambda)}_{\nu,\nu_0}(\nu) = \nu_0$$

immediately follows from (3). Hence, if $\lambda \neq 1$ and $\nu_0 = \nu$, then ν is the fix point of the transformation $x \longmapsto m^{(\lambda)}_{\nu,\nu_0}(x)$, where $x \in [0, 1]$.

IV. Modeling Probability Weighting Functions

The following proposition lays the foundations for generating probability weighting functions derived from appropriately chosen generator functions.

Proposition 1: Let $f : [0, 1] \to [0, \infty]$ be a strictly decreasing (or increasing) continuous function with the inverse function $f^{-1} : [0, \infty] \to [0, 1]$, where $\nu, \nu_0 \in (0, 1)$, $\lambda > 0$, and let the modifier operator $m^{(\lambda)}_{\nu,\nu_0} : [0, 1] \to [0, 1]$ be induced from the generator function f according to (3). Then, $m^{(\lambda)}_{\nu,\nu_0}$ is a probability weighting function.

Proof: Let $\lambda > 0$, $\nu, \nu_0 \in (0, 1)$ and let $K = f(\nu_0)(f(\nu))^{-\lambda}$. Thus,

$$m^{(\lambda)}_{\nu,\nu_0}(x) = f^{-1}\left(K f^{\lambda}(x)\right)$$

for any $x \in [0, 1]$. If f is strictly decreasing (increasing), then f^{-1} is strictly decreasing (increasing) as well. Noting that $\lambda > 0$, $f^{-1}\left(K f^{\lambda}(x)\right)$ is strictly increasing; that is, $m^{(\lambda)}_{\nu,\nu_0}$ satisfies requirement (1) for a probability weighting function in Definition 2. Next, taking into account the fact that $f^{-1}\left(K f^{\lambda}(x)\right)$ is strictly increasing and $f : [0, 1] \to [0, \infty]$ is strictly monotonic, we have $f^{-1}(f(0)) = 0$ and $f^{-1}(f(1)) = 1$. This means that $m^{(\lambda)}_{\nu,\nu_0}$ satisfies requirement (2) for a probability weighting function in Definition 2. ■

Here, we will utilize the modifier operator $m^{(\lambda)}_{\nu,\nu_0}$ with the parameter settings $\nu_0 = \nu$ and $\lambda > 0$. This allows us to characterize the generated probability weighting function by its fix point ν and by its sharpness parameter λ. That is, the generated probability weighting function $w^{(\lambda)}_{\nu}$ will always have the form

$$w^{(\lambda)}_{\nu}(x) = m^{(\lambda)}_{\nu}(x),$$

where

$$m^{(\lambda)}_{\nu}(x) = f^{-1}\left(f(\nu)\left(\frac{f(x)}{f(\nu)}\right)^{\lambda}\right), \tag{4}$$

$\nu \in (0, 1)$ and $\lambda > 0$. For the sake of simplicity, from now on, we will use the shortened notation w for the probability weighting function $w^{(\lambda)}_{\nu}$. The following proposition tells us about the shape of the probability weighting function generated by the modifier operator $m^{(\lambda)}_{\nu}$.

Proposition 2: If $\lambda > 0$, $\nu \in (0, 1)$ and the probability weighting function $w : [0, 1] \to [0, 1]$ is generated by the modifier operator $m^{(\lambda)}_{\nu} : [0, 1] \to [0, 1]$, then

1) If $0 < \lambda < 1$, then $w(x)$ is concave in $(0, \nu]$ and $w(x)$ is convex in $[\nu, 1)$;
2) If $\lambda = 1$, then $w(x) = x$ for any $x \in [0, 1]$;
3) If $1 < \lambda$, then $w(x)$ is convex in $(0, \nu]$ and $w(x)$ is concave in $[\nu, 1)$.

Proof: Let $w(x) = m^{(\lambda)}_{\nu}(x)$ for any $x \in [0, 1]$, where $\nu \in (0, 1)$ and $\lambda > 0$. Furthermore, let $g(x)$ be defined by

$$g(x) = (f(\nu))^{1-\lambda}\,(f(x))^{\lambda}.$$

Then, $w(x)$ can be written as

$$w(x) = f^{-1}\left(g(x)\right).$$

Here, we will differentiate two cases: (a) f is strictly increasing, (b) f is strictly decreasing.

(a) In this case, f is a strictly increasing function and so f^{-1} is a strictly increasing function as well.

(a1) If $x \in (0,\nu]$ and $0 < \lambda < 1$, then $(f(\nu))^{1-\lambda} \geq (f(x))^{1-\lambda}$, $g(x) \geq f(x)$ and $w(x) = f^{-1}(g(x)) \geq x$.

(a2) If $x \in [\nu,1)$ and $0 < \lambda < 1$, then $(f(\nu))^{1-\lambda} \leq (f(x))^{1-\lambda}$, $g(x) \leq f(x)$ and $w(x) = f^{-1}(g(x)) \leq x$.

From (a1) and (a2), 1) follows.

(a3) If $x \in (0,\nu]$ and $1 < \lambda$, then $(f(\nu))^{1-\lambda} \leq (f(x))^{1-\lambda}$, $g(x) \leq f(x)$ and $w(x) = f^{-1}(g(x)) \leq x$.

(a4) If $x \in [\nu,1)$ and $1 < \lambda$, then $(f(\nu))^{1-\lambda} \geq (f(x))^{1-\lambda}$, $g(x) \geq f(x)$ and $w(x) = f^{-1}(g(x)) \geq x$.

From (a3) and (a4), 3) follows.

(b) In this case, f is a strictly decreasing function and so f^{-1} is a strictly decreasing function as well.

(b1) If $x \in (0,\nu]$ and $0 < \lambda < 1$, then $(f(\nu))^{1-\lambda} \leq (f(x))^{1-\lambda}$, $g(x) \leq f(x)$ and $w(x) = f^{-1}(g(x)) \geq x$.

(b2) If $x \in [\nu,1)$ and $0 < \lambda < 1$, then $(f(\nu))^{1-\lambda} \geq (f(x))^{1-\lambda}$, $g(x) \geq f(x)$ and $w(x) = f^{-1}(g(x)) \leq x$.

From (b1) and (b2), 1) follows.

(b3) If $x \in (0,\nu]$ and $1 < \lambda$, then $(f(\nu))^{1-\lambda} \geq (f(x))^{1-\lambda}$, $g(x) \geq f(x)$ and $w(x) = f^{-1}(g(x)) \leq x$.

(b4) If $x \in [\nu,1)$ and $1 < \lambda$, then $(f(\nu))^{1-\lambda} \leq (f(x))^{1-\lambda}$, $g(x) \leq f(x)$ and $w(x) = f^{-1}(g(x)) \geq x$.

From (b3) and (b4), 3) follows.

If $\lambda = 1$, then $w(x) = x$ trivially holds for any $x \in [0,1]$. ■

Suppose that the probability weighting function $w : [0,1] \to [0,1]$ is generated by the modifier operator $m_\nu^{(\lambda)} : [0,1] \to [0,1]$, $\lambda > 0$ and $\nu \in (0,1)$. Then, the effect of the parameters λ and ν on the shape of the probability weighting function w can be summarized as follows.

- Parameter λ determines the sharpness and the shape of w. The more the value of λ differs from 1, the more the shape of w differs from that of the identity function. If $0 < \lambda < 1$, then w is inverse S-shaped, and if $1 < \lambda$, then w is S-shaped.
- Parameter ν determines the point where w intersects the diagonal line; that is, parameter ν may be viewed as the elevator parameter of w.

Figure 1 shows the effect of the values of parameters λ and ν on the shape of the probability weighting function w that was generated from the same generator function via the modifier operator $m_\nu^{(\lambda)}$.

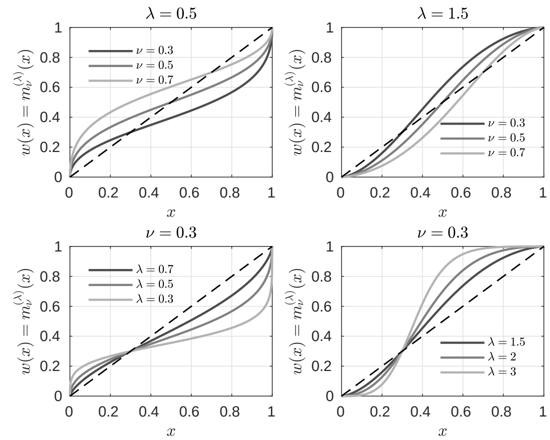

Fig. 1. The role of parameters ν and λ

One can easily see that the modifier operator $m_\nu^{(\lambda)}$ with the generator function $f(x) = -\ln(x)$, where $x \in (0,1]$, generates the probability weighting function

$$w(x) = \left(\mathrm{e}^{-(-\ln(x))^{\lambda}}\right)^{(-\ln(\nu))^{1-\lambda}}. \tag{5}$$

Now, if the parameters λ and ν are set as

$$\lambda = a,$$
$$\nu = \mathrm{e}^{-b^{\frac{1}{1-a}}},$$

where $0 < a < 1$ and $b > 0$, then the equation in (5) can be written as

$$w(x) = \left(\mathrm{e}^{-(-\ln(x))^{a}}\right)^{b},$$

which is the well-known Prelec's probability weighting function [10].

It can also be shown that the modifier operator $m_\nu^{(\lambda)}$ with the generator function $f(x) = \frac{1-x}{x}$, where $x \in (0,1]$, generates the probability weighting function

$$w(x) = \frac{1}{1 + \frac{1-\nu}{\nu}\left(\frac{1-x}{x}\frac{\nu}{1-\nu}\right)^{\lambda}}. \tag{6}$$

Now, if the parameters λ and ν are set as

$$\lambda = a,$$
$$\nu = \frac{1}{1 + \left(\frac{1}{b}\right)^{\frac{1}{1-a}}},$$

where $0 < a < 1$ and $b > 0$, then the equation in (6) can be written as

$$w(x) = \frac{bx^a}{bx^a + (1-x)^a}.$$

In prospect theory, this function is known as the Ostaszewski, Green and Myerson probability weighting function [12]. Note that this probability weighting function family was introduced independently by Lattimore, Baker and Witte as well in 1992 (see [11]).

Based on the results presented in this section, the application of the modifier operator in (4) can be treated as a general approach for creating probability weighting functions including the most important ones.

V. A Two-phase Regression Method

In this section, we will show how a probability weighting function generated by the modifier operator in (4) can be fitted to empirical data. Suppose that the probability weighting function $w : (0,1] \to (0,1]$ is induced from the generator function $f : (0,1] \to (0,\infty]$ by the modifier operator $m_\nu^{(\lambda)}$ given in (4), where $\nu \in (0,1)$ and $\lambda > 0$. That is, we have

$$w(x) = f^{-1}\left(f(\nu)\left(\frac{f(x)}{f(\nu)}\right)^{\lambda}\right) \tag{7}$$

for any $x \in (0,1]$.

Let A be an uncertain event in the finite event space S, and let $P : \mathcal{P}(S) \to [0,1]$ be a probability measure on S. Suppose that $x = P(A)$ is the known probability of event A. Let Y be a dichotomous random variable such that

$$Y = \begin{cases} 1, & \text{if } A \text{ happens} \\ 0, & \text{if } \overline{A} \text{ happens,} \end{cases}$$

where $\overline{A}$ denotes the complement of event A. In practice, one may have a perceived probability $P_p(A)$ of the event A, which may differ from $P(A)$. Here, we seek to model the perceived probability $P_p(A)$, when the probability $P(A)$ is assumed to be known. In other words, we wish to model the conditional probability $P_p(Y=1|x)$ which represents the perceived probability of event A given that the probability of event A is equal to x. Let $(x_1,Y_1),(x_2,Y_2),\ldots,(x_n,Y_n)$ be a sample of independent observation pairs on the variable x and the dichotomous random variable Y, $x_i \in (0,1)$, $Y_i \in \{0,1\}$, $n \geq 2$. Assume that

$$P_p(Y=1|x) = w(x),$$

that is, the conditional perceived probability $P_p(Y=1|x)$ is modeled by a probability weighting function w. Recall that the function w is parametric, it has the parameters λ and ν. Now, we will introduce a two-phase method for fitting the probability weighting function w to empirical data:

1) In the first phase of our method, we will transform the function w to a linear function, and then we will apply linear regression to get the estimates $\hat{\nu}_0$ and $\hat{\lambda}_0$ of the parameters ν and λ, respectively.
2) In the second phase of our method, we will give the maximum likelihood estimation of the parameters ν and λ by using a numeric optimization method in which the unknown parameter values are initialized with $\hat{\nu}_0$ and $\hat{\lambda}_0$ that were obtained in the first phase.

A. Phase 1: Linearization and Linear Regression

Let $y = w(x)$, where $x \in (0,1]$. Then, by noting (7), we have

$$f(y) = f(\nu)\left(\frac{f(x)}{f(\nu)}\right)^{\lambda}.$$

Since both members of the previous equation are positive, after taking the logarithm of its both sides, we get

$$\ln(f(y)) = \ln(f(\nu)) + \lambda\ln(f(x)) - \lambda\ln(f(\nu)).$$

The last equation can be written in the form

$$v = \alpha u + \beta,$$

where $u = \ln(f(x))$, $v = \ln(f(y))$, $\alpha = \lambda$ and $\beta = \ln(f(\nu))(1-\alpha)$. Hence, the values of parameters α and β can be obtained by applying a linear regression. Once we have the estimated values of $\hat{\alpha}$ and $\hat{\beta}$ for the parameters α and β, respectively, the estimates $\hat{\lambda}_0$ and $\hat{\nu}_0$ of the parameters λ and ν are

$$\begin{aligned} \hat{\lambda}_0 &= \hat{\alpha} \\ \hat{\nu}_0 &= f^{-1}\left(\mathrm{e}^{\frac{\hat{\beta}}{1-\hat{\alpha}}}\right), \end{aligned} \tag{8}$$

respectively.

Let $x_1^*, x_2^*, \ldots, x_m^*$ be the unique values among the values $x_1, x_2, \ldots, x_n$ in the sample $(x_1,Y_1),(x_2,Y_2),\ldots,(x_n,Y_n)$, $m \leq n$. Furthermore, let y_r^* be the estimated value of the conditional perceived probability $P_p(Y=1|x=x_r^*)$ computed from the sample $(x_1,Y_1),(x_2,Y_2),\ldots,(x_n,Y_n)$, where $r = 1,2,\ldots,m$. That is,

$$y_r^* = \frac{k_r}{n_r}$$

where

$$k_r = \left|\{(x_i,Y_i) : x_i = x_r^*, Y_i = 1, i = 1,2,\ldots,n\}\right|, \tag{9}$$

$$n_r = \left|\{(x_i,Y_i) : x_i = x_r^*, i = 1,2,\ldots,n\}\right| \tag{10}$$

and $r = 1,2,\ldots,m$. In other words, k_r^* and y_r^* are, respectively, the conditional frequency and the conditional relative frequency of event A given the condition $x = x_r^*$. Next, following the line of thinking presented above, the unknown parameters λ and ν of the function w can be estimated by fitting the linear regression model $v = \alpha u + \beta$ to the transformed data pairs (u_r, v_r), where

$$\begin{aligned} u_r &= \ln(f(x_r^*)) \\ v_r &= \ln(f(y_r^*)), \end{aligned}$$

and then applying the equations in (8) with the estimates $\hat{\alpha}$ and $\hat{\beta}$ of the parameters α and β, respectively.

B. Phase 2: Maximum Likelihood Estimation

By utilizing the sample $(x_1,Y_1),(x_2,Y_2),\ldots,(x_n,Y_n)$ the unknown parameters ν and λ of the probability weighting function w can be estimated by maximizing the perceived likelihood function $L : (0,1) \times (0,\infty] \to (0,1)$

$$\begin{aligned} L(\nu,\lambda) &= \prod_{i=1}^{n} P_p(Y=Y_i|x_i) = \\ &= \prod_{i=1}^{n} w^{Y_i}(x_i;\nu,\lambda)\left(1 - w(x_i;\nu,\lambda)\right)^{1-Y_i}, \end{aligned} \tag{11}$$

where $w(x_i;\nu,\lambda)=w(x_i)=w_\nu^{(\lambda)}(x_i)$, $i=1,2,\ldots,n$. Obviously, maximizing the likelihood function in (11) is equivalent to maximizing the log-likelihood function $l:(0,1)\times(0,\infty)\to(0,-\infty]$, which is given by

$$l(\nu,\lambda)=\sum_{i=1}^{n} Y_i \ln\left(w(x_i;\nu,\lambda)\right)+ \\ +\sum_{i=1}^{n}(1-Y_i)\ln\left(1-w(x_i;\nu,\lambda)\right). \quad (12)$$

By making the use of the frequencies k_r and n_r given in (9) and (10) the log-likelihood function in (12) can be written as

$$l(\nu,\lambda)=\sum_{i=1}^{m} k_r \ln\left(w(x_i;\nu,\lambda)\right)+ \\ +\sum_{i=1}^{m}(n_r-k_r)\ln\left(1-w(x_i;\nu,\lambda)\right). \quad (13)$$

The maxima of the log-likelihood function in (13) can be determined by using the so-called GLOBAL method which is a stochastic global optimization procedure introduced by Csendes (see [18], [19]). The GLOBAL method was implemented in the MATLAB 2019a numerical computing environment. In the optimization procedure, the initial values of the parameters ν and λ can be set to those determined in the first phase of our regression method (see (8)). This approach increases the speed of convergence of the GLOBAL method.

C. A Demonstrative Example

100 people were surveyed in 9 runs about if, in their opinion, an uncertain event will happen or not. The empirical results from the survey are summarized in Table I, the first column of which (r) contains the run identifier ($r=1,2,\ldots,9$). The known likelihood x_r of the event was different in each run. In Table I, for each run, column k_r contains the number of people who taught that the event will happen, while column n_r-k_r indicates the number of those participants who believed that the event will not happen ($n_r=100$ is the number of survey participants). The estimate of the perceived conditional likelihood of the event computed from the survey results for each run is in column y_r of Table I.

TABLE I
EMPIRICAL DATA

r	x_r	k_r	n_r-k_r	n_r	y_r	$\ln(f(x_r))$	$\ln(f(y_r))$
1	0.1	15	85	100	0.15	2.1972	1.7346
2	0.2	23	77	100	0.23	1.3863	1.2083
3	0.3	29	71	100	0.29	0.8473	0.8954
4	0.4	35	65	100	0.35	0.4055	0.6190
5	0.5	41	59	100	0.41	0.0000	0.3640
6	0.6	48	52	100	0.48	-0.4055	0.0800
7	0.7	55	45	100	0.55	-0.8473	-0.2007
8	0.8	65	35	100	0.65	-1.3863	-0.6190
9	0.9	76	24	100	0.76	-2.1972	-1.1527

A probability weighting function w induced by the generator function $f:(0,1]\to[0,\infty]$, $f(x)=\frac{1-x}{x}$ by using the modifier operator in (3) was fitted to the empirical data in Table I. Recall that this generator function induces the probability weighting function given in (6). Here, we applied our two-phase regression method to determine the estimates of the parameters ν and λ of function w. In the first phase, by applying a linear regression to the $(\ln(f(x_r)),\ln(f(y_r)))$ pairs and using the formulas in (8), we got the following estimates of the parameters ν and λ:

$$\hat{\nu}_0=0.2793, \qquad \hat{\lambda}_0=0.6567.$$

In the second phase, by initializing the parameters ν and λ with $\hat{\nu}_0$ and $\hat{\lambda}_0$, respectively, and then by applying the GLOBAL optimization method to maximize the log-likelihood function, we received the following $\hat{\nu}$ and $\hat{\lambda}$ estimates of the parameters ν and λ:

$$\hat{\nu}=0.2753, \qquad \hat{\lambda}=0.6591.$$

The maximum value of the log-likelihood function is -546.8268. Figure 2 shows the plots of the probability weighting functions fitted to the empirical data. Since the phase 1 and phase 2 estimates of the parameters are very close, the two regression function plots in Figure 2 almost completely coincide. It should be added that when the generator function is $f(x)=\frac{1-x}{x}$, then the modifier operator generates the so-called kappa function. In this case, the regression is called the kappa regression (see in press [20]), and the log-likelihood function is concave. Therefore, the negative log-likelihood function can be minimized by using gradient descend methods.

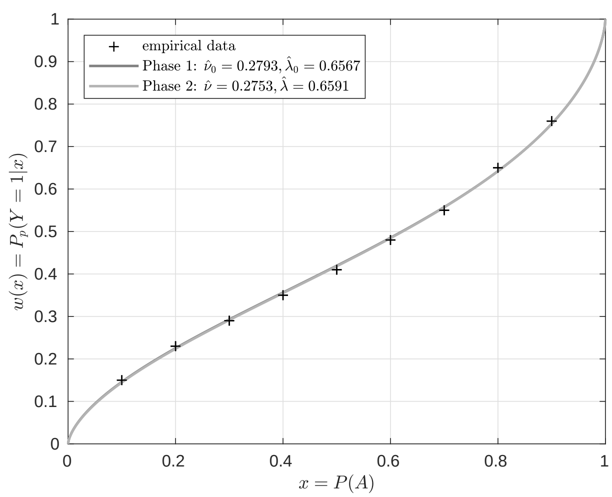

Fig. 2. Plots of regression functions

VI. CONCLUSIONS

The key findings of our study can be summarized as follows.

- The modifier operator $m_{\nu,\nu_0}^{(\lambda)}$ given in (3) satisfies the requirements for a probability weighting function.
- Application of the modifier operator $m_\nu^{(\lambda)}$ (see (4)) can be interpreted as a general approach for generating probability weighting functions, and this includes the well-known ones.
- Since the probability weighting functions generated by the modifier operator $m_\nu^{(\lambda)}$ can be transformed to linear

functions, these probability weighting functions are easy-to-use in regression problems.

- A two-phase regression method for fitting generated probability weighting functions to empirical data was introduced. In its first phase, the probability weighting function is transformed to a linear function and linear regression is applied to estimate the function parameters. In the second phase of the presented method, the maximum likelihood estimations of the regression function parameters are provided by using a numeric optimization method in which the unknown parameter values are initialized with those obtained in the first phase.

REFERENCES

[1] C. F. Camerer, "Neuroeconomics: Using neuroscience to make economic predictions," *The Economic Journal*, vol. 117, no. 519, pp. C26–C42, 2007.

[2] V. Köbberling and P. P. Wakker, "Preference foundations for nonexpected utility: A generalized and simplified technique," *Mathematics of Operations Research*, vol. 28, no. 3, pp. 395–423, 2003.

[3] B. Koszegi and M. Rabin, "A model of reference-dependent preferences," *The Quarterly Journal of Economics*, vol. 121, no. 4, pp. 1133–1165, 2006.

[4] H. Bleichrodt, "Probability weighting in choice under risk: An empirical test," *Journal of Risk and Uncertainty*, vol. 23, no. 2, pp. 185–198, Sep 2001.

[5] A. Chateauneuf, J. Eichberger, and S. Grant, "Choice under uncertainty with the best and worst in mind: Neo-additive capacities," *Journal of Economic Theory*, vol. 137, no. 1, pp. 538–567, 2007.

[6] G. Loomes, P. G. Moffatt, and R. Sugden, "A microeconometric test of alternative stochastic theories of risky choice," *Journal of Risk and Uncertainty*, vol. 24, no. 2, pp. 103–130, Mar 2002.

[7] M. Abdellaoui, H. Bleichrodt, and O. l'Haridon, "A tractable method to measure utility and loss aversion under prospect theory," *Journal of Risk and uncertainty*, vol. 36, no. 3, p. 245, 2008.

[8] D. Kahneman and A. Tversky, "Prospect theory: An analysis of decision under risk," in *Handbook of the fundamentals of financial decision making: Part I.* World Scientific, 2013, pp. 99–127.

[9] A. Tversky and D. Kahneman, "Advances in prospect theory: Cumulative representation of uncertainty," *Journal of Risk and uncertainty*, vol. 5, no. 4, pp. 297–323, 1992.

[10] D. Prelec, "The probability weighting function," *Econometrica*, vol. 66, pp. 497–528, 1998.

[11] P. K. Lattimore, J. R. Baker, and A. D. Witte, "The influence of probability on risky choice: A parametric examination," *Journal of Economic Behavior & Organization*, vol. 17, no. 3, pp. 377–400, 1992.

[12] P. Ostaszewski, L. Green, and J. Myerson, "Effects of inflation on the subjective value of delayed and probabilistic rewards," *Psychonomic Bulletin & Review*, vol. 5, no. 2, pp. 324–333, 1998.

[13] P. P. Wakker, *Prospect theory: For risk and ambiguity*. Cambridge University Press, 2010.

[14] P. P. Wakker and J. Yang, "A powerful tool for analyzing concave/convex utility and weighting functions," *Journal of Economic Theory*, vol. 181, pp. 143–159, 2019.

[15] J. Dombi, "On a certain type of unary operators," in *2012 IEEE International Conference on Fuzzy Systems*, June 2012, pp. 1–7.

[16] J. Dombi, "Towards a general class of operators for fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 2, pp. 477–484, 2008.

[17] T. Offerman, J. Sonnemans, G. Van de Kuilen, and P. P. Wakker, "A truth serum for non-Bayesians: Correcting proper scoring rules for risk attitudes," *The Review of Economic Studies*, vol. 76, no. 4, pp. 1461–1489, 2009.

[18] T. Csendes, "Nonlinear parameter estimation by global opitmization - efficiency and reliability," *Acta Cybernetica*, vol. 8, no. 4, pp. 361–372, 1988.

[19] T. Csendes, L. Pál, J. Sendin, and J. Banga, "The global optimization method revisited," *Optimization Letters*, vol. 2, no. 4, pp. 445–454, 2008.

[20] J. Dombi and T. Jónás, "Kappa regression: an alternative to logistic regression," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems,*, vol. accepted paper, in press, 2019.