Modalities based on double negation

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Abstract. Modal operators play an important role in fuzzy theory, and in recent years researchers have devoted more effort on this topic. Here we concentrate on continuous strictly monotonously increasing Archimedian t-norms. In our study, we will construct modal operators related to negation operators and we introduce graded modal operators.

Keywords: negation, modalities, Pliant logic, necessity and possibility operators

1 Introduction

In logic, modal operators have a variety of applications and even from a theoretical perspective it is interesting to study the continuous extension of these operators. Our approach is different from other authors, because we would like to find proper algebric expressions for these operators based on some basic considerations. On the one hand, continuous-valued logic can be studied from a logical point of view (axiomatization, completeness, possible extensions, predicat calculi, etc.). On the other hand it can be studied from algebraic point of view to find the proper operator, as we do it in conjunctive and disjunctive operators and now we have Frank, Hamacher, Einstein mean operators etc. With the latter, he have to solve functional equations. If we have different continuous-valued logical system (i.e. operators), we have to build different modal operators.

Our objective is to find these operators. If a logical operator is given we construct its unary operator. Different unary operators are studied in continuous valued (fuzzy) logic as modal operators (necessity and possibility, hedges (strengthened and weakened operators, truth-value modifiers, truth-stresser, truth-depresser), etc). Here, we will present approaches for obtaining the concrete form of the necessity and possibility operators. These may be expressed in a simple parametrical form. By modifying the parameter value, we get different unary operators, namely modality, hedge and negation operators.

In this paper we deal with continuous valued, Archimedian t-norm based logic. It is Hájek BL with the exception of Gödel logic, since in the latter $x \wedge x = x$ holds. If we use BL formalism (i.e. strong negation to define modalities), then the most general approach to deal with involution in t-norm based logic is the paper of Flaminio, Marchioni [10]. Here the authors set of logical frame to Esteva and Godo monodial logic MTL, which contains BL. So we can say that this is the most general logic from this point of view.

Cintulas paper only deals with involutive expansions of the logic SBL, which includes Gödel logic. We have to mention that Esteva, Godo and Noguera [6] study the probably most general logic for truth-hedges. It is the closest to the system which given in this paper.
Cintula et al. [1] carried out a study on fuzzy logic with an additional involutive negation operator. This was a survey paper and they presented a propositional logic extended with an involutive negation. With this concept, Cintula improved the expressive potential of mathematical logic.
In Hájek's paper [8], a system called basic logic (BL) was defined. Not long ago, a survey paper was published [7] that discussed the state-of-art development of BL. The problem with this logic is that the implication is defined by the residual of the t-norm, the negation operator is defined by $\sim x = x \to 0$ and in the strict operator case, the negation operator is not involutive. In fact it is a drastic negation operator. Neither the implication operator nor the negation operator is continuous. From an application point of view, the continuity property is always indispensable.
In Esteva et al. [5], logics with involutive negation were introduced. This negation is different from implication based negation "$\sim$" and it functions as a basic negation (not$(x) = 1 - x$). But this negation operator is not related to the residual implication in strict monotonous operator case.
Modal logic has been used in rough sets as well, where the sets are approximated by elements of a partition induced by an equivalence relation. A natural choice for rough set logic is S5 (Orlowska [11]). Here, the possibility and necessity modalities express outer and inner approximation operators.
In our previous article [3], we looked for strictly monotonously increasing Archimedian t-norms and t-conorms (called conjunctive and disjunctive operators) for which the De Morgan identity is valid with infinitely many negation operators. In this article, we will denote these operators by $c(x, y)$ and $d(x, y)$, respectively.

2 Basic considerations of negation

Here, we will interpret 1 as the true value and 0 as the false value. Now we will state definitions and properties of negation operator.

Definition 1. *We say that $\eta(x)$ is a strong negation if $\eta\colon [0,1] \to [0,1]$ satisfies the following conditions:*
C1: $\eta : [0,1] \to [0,1]$ *is continuous (Continuity)*
C2: $\eta(0) = 1$, $\eta(1) = 0$ *(Boundary conditions)*
C3: $\eta(x) < \eta(y)$ *for* $x > y$ *(Monotonicity)*
C4: $\eta(\eta(x)) = x$ *(Involution)*

Remark. *The boundary condition C2 can be inferred by using C1 and C3.*

From *C1*, *C2* and *C3*, it follows that there exists a fixed point (or neutral value) $\nu \in [0, 1]$ of the negation where

$$\eta(\nu) = \nu \tag{1}$$

Later on we will characterise the negation operator in terms of the ν parameter.

Definition 2. *We will say that a negation $\eta_{\nu_1}(x)$ is stricter than $\eta_{\nu_2}(x)$ if $\nu_1 < \nu_2$.*

For the strong negation, two representation theorems are known. Trillas [13] showed that every involutive negation operator has the following form, and here we denoted this negation by $n(x)$.

$$n(x) = g^{-1}(1 - g(x)) \tag{2}$$

where $g : [0, 1] \to [0, 1]$ is a continuous strictly increasing (or decreasing) function. This generator function corresponds to nilpotent operators (nilpotent t-norms [9] [12] [14]). Examples for the negation: $n_\alpha(x) = (1 - x^\alpha)^{\frac{1}{\alpha}}$ (Yager negation), $n_a(x) = \frac{1-x}{1+ax}$ (Hamacher and Sugeno negation). We can express the parameter of the negation operator in terms of its fixed point (or neutral value). The Yager negation operator has the form

$$n_\nu(x) = \left(1 - x^{-\frac{ln\nu}{ln2}}\right)^{-\frac{ln2}{ln\nu}}$$

In a similar way, we get the new form of the Hamacher negation operator:

$$n_\nu(x) = \frac{1}{1 + (\frac{1-\nu}{\nu})^2 \frac{x}{1-x}}.$$

This form of the negation operator can be found in [2].
For the strictly monotonously increasing t-norms, another form of negation operator is given in [3] [4]. It is

$$\eta_\nu(x) = f^{-1}\left(\frac{f^2(\nu)}{f(x)}\right) \tag{3}$$

where $f : [0, 1] \to [0, \infty]$ is a continuous, increasing (or decreasing) function and f is the generator function of a strict monotone t-norm, or t-conorm. This negation operator is an element of the Pliant system [3] [4].

Here we show that (2) and (3) are equivalent, when $f(\nu) = 1$.

Proposition 1. *Let $n(x)$ and $\eta(x)$ be defined by (2) and (3). If $f(\nu) = 1$ and*

$$f(x) = \frac{1 - g(x)}{g(x)}, \qquad g(x) = \frac{1}{1 + f(x)}$$

then

$$n(x) = \eta(x)$$

Proof. The following expression is valid

$$f^{-1}(x) = g^{-1}\left(\frac{1}{x+1}\right).$$

So we get

$$f^{-1}\left(\frac{1}{f(x)}\right) = g^{-1}\left(\frac{1}{1+\frac{1}{f(x)}}\right) =$$
$$= g^{-1}\left(\frac{1}{1+\frac{g(x)}{1-g(x)}}\right) = g^{-1}\left(1-g(x)\right).$$

The properties of the functions (f, g) can be easily verified. □

Next, we will use (3) to represent the negation operator because here we are just considering strict monotone operators.
In Figure 1, we sketch the shape of the negation function and we demonstrate the meaning of the ν value. We can introduce a non continuous negation [3], [4].

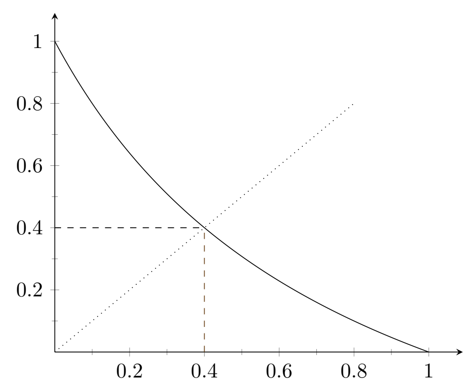

Fig. 1. The shape of the negation function when $f(x) = \frac{1-x}{x}$ and $\nu = 0.4$

Definition 3 (Drastic negation). *We call $\eta_0(x)$ and $\eta_1(x)$ drastic negations when*

$$\eta_0(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases} \quad \eta_1(x) = \begin{cases} 1 \text{ if } x \neq 1 \\ 0 \text{ if } x = 1 \end{cases}$$

Here, $\eta_0(x)$ is the strictest negation, while $\eta_1(x)$ is the least strict negation. They are non-continuous negation operators, so they are not negation operators in the original sense (see Figure 2).
Let $g(x) = \dfrac{1}{1+\frac{1-\nu}{\nu}\frac{x}{1-x}}$ Then

$$g^{-1}(1-g(x)) = \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2 \frac{x}{1-x}} = \frac{1-x}{1+a\,x},$$

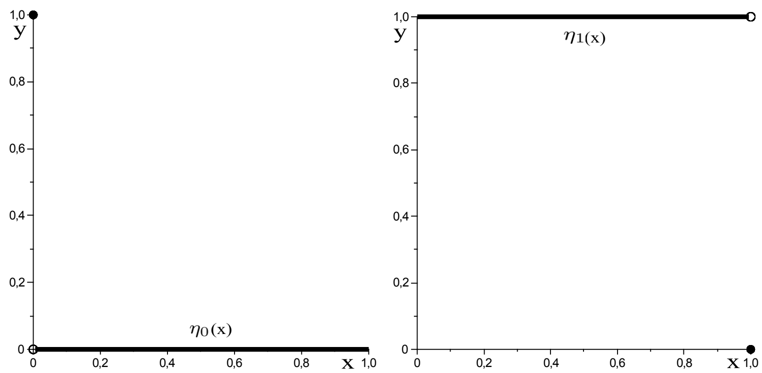

Fig. 2. $\eta_0(x)$ and $\eta_1(x)$ are drastic negation operators

where $a + 1 = \left(\frac{1-\nu}{\nu}\right)^2$ and $\nu \in (0, 1)$.

Let $f(x) = \frac{1-x}{x}$ Then

$$f^{-1}\left(\frac{f^2(\nu)}{f(x)}\right) = \frac{1}{1 + \left(\frac{1-\nu}{\nu}\right)^2 \cdot \frac{x}{1-x}} \tag{4}$$

Remark. *There are strictly monotone operators (t-norm, t-conorm) that build a De-Morgan system with infinitely many negations. This operator is called Pliant operator [3].*

This system is useful for building modal operators. In the Pliant system the negation operator closely related to the t-norm and t-conorm. Based on different negations, in the next we deal with modalities.

3 Modalities induced by two different negation operators

To obtain this structure, we equip it with another type of negation operator. In modal logic, it is called an intuitionistic negation operator. In our system, the modalities induced by a suitable composition of the two negation operators generate a modal system with the full distributivity property of the modal operators. The necessity operator is simultaneously distributive over the conjunctive and disjunctive operators and the possibility operator is also simultaneously distributive over the conjunctive and disjunctive operators.

With this starting point, the necessity and possibility operators used in fuzzy logic are based on an extension of modal logic to the continuous case. We begin with the negation operator and we make use of two types of this operator, one is strict, and one is less strict. We will show that with these two negation operators we can define the modal hedges. Next, we use the classical notation for the sake of convention.

In intuitionistic logic, another kind of negation operator also has to be taken into account. Here $\sim x$ means the negated value of x. $\sim_1 x$ and $\sim_2 x$ are two negation operators. We will construct linguistic modal hedges called necessity and possibility hedges. The construction is based on the fact that modal operators can be realized by combining two kinds of negation operators. In our modal logic, $\sim_1 x$ means x is impossible. In other words, $\sim_1$ is a stronger negation than not(x), i.e. $\sim_2 x$.
We can write

$$\sim_1 x := \text{impossible}(x)$$
$$\sim_2 x := \text{not}(x)$$

As we mentioned above, in modal logic we have two more operators than the classical logic case, namely necessity and possibility; and in modal logic there are two basic identities. These are:

$$\sim_1 x = \text{impossible}(x) = \text{necessity}(\text{not}(x)) = \Box \sim_2 x \tag{5}$$

$$\Diamond x = \text{possible}(x) = \text{not}(\text{impossible}(x)) = \sim_2(\sim_1 x) \tag{6}$$

In our context, we model impossible(x) with a stricter negation operator than not(x). Eq.(6) also serves as a definition for the possibility operator.

If in Eq.(5) we replace x by $\sim_2 x$ and using the fact that $\sim_2 x$ is involutive, we get

$$\Box x = \sim_1(\sim_2 x), \tag{7}$$

and with Eq.(6), we have

$$\Diamond x = \sim_2(\sim_1 x). \tag{8}$$

The necessity and possibility operators have a common form, i.e. they can be expressed by double negation equipped by different neutral values. Here, "not" and "impossible" are two different negations.
If ν is small, we can say that the negation operator is strict; otherwise it is not strict. "Impossible" is a stricter negation compared with "not".
Based on the above considerations, we can formally define the necessity and possibility modifiers.

Definition 4. *The general form of the modal operator is*

$$\tau_{\nu_1,\nu_2}(x) = \eta_{\nu_1}\left(\eta_{\nu_2}(x)\right) \qquad \textit{or} \qquad \tau_{\nu_1,\nu_2}(x) = n_{\nu_1}(n_{\nu_2}(x)) \tag{9}$$

and ν_1, ν_2 *are neutral values of the negation operator. If* $\nu_1 < \nu_2$, *then* $\tau_\nu(x)$ *is a necessity operator, and if* $\nu_2 < \nu_1$, *then* $\tau_\nu(x)$ *is a possibility operator. If* $\nu_1 = \nu_2$ *then* $\tau_\nu(x)$ *is the identity operator and* $\tau_\nu(x) = x$.

With this notion, we can make use of Eq.(7) and Eq.(8)

The necessity and possibility operators using the representation of the negation operator (τ) have a common form

$$\tau_{\nu_1,\nu_2}(x) = \eta_{\nu_1}(\eta_{\nu_2}(x)) = f^{-1}\left(\frac{f^2(\nu_1)}{f^2(\nu_2)} f(x)\right)$$
$$\tau_{\nu_1,\nu_2}(x) = n_{\nu_1}(n_{\nu_2}(x)) = f_1^{-1}\left(2\nu_1 - f_1\left(f_2^{-1}\left(2\nu_2 - f_2(x)\right)\right)\right)$$

We can define the dual possibility and necessiy operators like so:

Definition 5. *A necessity operator and a possibility operator are dual if*

$$\nu_1 = \eta(\nu_2) \qquad\qquad \nu_1 = f^{-1}\left(\frac{1}{f(\nu_2)}\right)$$

We will show that both modal operators belong to the same class of unary operators, and also show that because they have a common form, we can denote both of them by $\tau_\nu(x)$. Depending on the ν value, we get the necessity operator or the possibility operator.

Definition 6. *The dual possibility and necessity operators are*

$$\Box_\nu(x) = \tau_\nu^N(x) = f^{-1}\left(\frac{f(x)}{f^2(\nu)}\right) \qquad \textit{and} \tag{10}$$
$$\Diamond_\nu(x) = \tau_\nu^P(x) = f^{-1}\left(f^2(\nu) f(x)\right) \tag{11}$$

when $f(\nu) < 1$

Previously we defined the drastic negation operator. Here we will define the drastic necessity and possibility operators by using the drastic negation operators:

Definition 7. *Drastic model operators are the following:*

$$\textit{Drastic necessity} \qquad \Box_1(x) = \tau_1^N(x) = \begin{cases} 1 & \textit{if } x = 1 \\ 0 & \textit{if } x \neq 1 \end{cases} \tag{12}$$

$$\textit{Drastic possibility} \qquad \Diamond_0(x) = \tau_0^P(x) = \begin{cases} 0 & \textit{if } x = 0 \\ 1 & \textit{if } x \neq 0 \end{cases} \tag{13}$$

See Figure 3 below

Remark. *Drastic necessity and possibility operators can be obtained by using drastic negations. (See Definition 3.)*

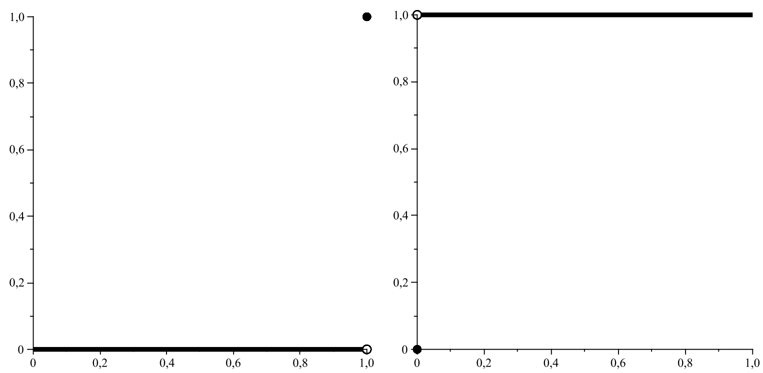

Fig. 3. The drastic necessity $\tau_1(x)$ and the drastic possibility $\tau_0(x)$

(12) and (13) are known as Baas-Monteiro Δ operator and its dual ∇, respectively. Δ and ∇ are definable by an involution and strict negation [7].

The necessity operator ($\Box$) will be denoted by $\tau^N(x)$, and the possibility operator ($\Diamond$) will be denoted by $\tau^P(x)$. Because both operators can be deduced from each other, we handled them together.

Now let $\tau^N[0,1] \to [0,1]$ and $\tau^P[0,1] \to [0,1]$ be two unary operators that satisfy the following conditions for the necessity operator and the possibility operator:

$N1. \quad \tau^N(1) = 1$	$P1. \quad \tau^P(0) = 0$
$N2. \quad \tau^N(x) \leq x$	$P2. \quad x \leq \tau^P(x)$
$N3. \quad x \leq y \quad$ *implies* $\quad \tau^N(x) \leq \tau^N(y)$	$P3. \quad x \leq y \quad$ *implies* $\quad \tau^P(x) \leq \tau^P(y)$
$N4. \quad \tau^P(x) = \eta\left(\tau^N\left(\eta(x)\right)\right)$	$P4. \quad \tau^N(x) = \eta\left(\tau^P(\eta(x))\right)$
$[N5. \quad \tau^P(x) = \tau^N\left(\tau^P(x)\right)$	$P5. \quad \tau^N(x) = \tau^P\left(\tau^N(x)\right)]$

Remark. *In our system,* $(N5)$ *and* $(P5)$ *are not required. Only a special parametrical form of* τ^P *and* τ^N *satisfies* $(N5)$ *and* $(P5)$.
Instead of $(N5)$ *and* $(P5)$*, our demand is the so-called neutrality principle, i.e.*

$$N'(5) \quad \tau^N\left(\tau^P(x)\right) = x \qquad\qquad P'(5) \quad \tau^P\left(\tau^N(x)\right) = x$$

Next, we will show that the basic properties are fulfilled.

Proposition 2. $\tau_\nu^N(x)$ *and* $\tau_\nu^P(x)$ *statisfies the basic properties of modalities: {N principle, T principle, K principle, DF$\Diamond$ principle,* N^* *principle, P principle, T principle, K principle, DF$\Box$ principle,* P^* *principle}*

Proof. We prove only the Necessity case. The possibility case can be proven by a similar way. We will assume that f is strictly decreasing and that $f(\nu) < 1$

N1. $$\tau_\nu^N(1) = f^{-1}\left(\frac{f(1)}{f^2(\nu)}\right) = f^{-1}(0) = 1$$

N2. $\tau_\nu^N(x) < x \qquad x \in (0,1) \qquad$ So:
$$f^{-1}\left(\frac{f(x)}{f^2(\nu)}\right) < x$$
$$1 < f^2(\nu)$$

N3. if $x < y$ then
$$f^{-1}\left(\frac{f(x)}{f^2(\nu)}\right) < f^{-1}\left(\frac{f(y)}{f^2(\nu)}\right),$$
so $f(x) > f(y)$

N4. $$\tau_\nu^N(x) = f^{-1}\left(\frac{f(x)}{f^2(\nu)}\right),\ \tau_\nu^N(\eta(x)) = f^{-1}\left(\frac{1}{f(x)f^2(\nu)}\right)$$
$$\eta(\tau_\nu^N(\eta(x))) = f^{-1}\left(f^2(\nu)f(x)\right) = \tau_\nu^P(x)$$

N'5 $$\tau_\nu^N(\tau_\nu^P(x)) = x,\ f^{-1}\left(\frac{f^2(\nu)f(x)}{f^2(\nu)}\right) = x$$

□

Proposition 3. *For the composition of the drastic modal operator the following are valid:*

A.) *i* $\Box_1(\Diamond_\nu(x)) = \tau_1^N(\tau_\nu^P(x)) = \tau_1^N(x) = \Box_1(x)$
ii $\Box_\nu(\Diamond_0(x)) = \tau_\nu^N(\tau_0^P(x)) = \tau_0^P(x) = \Diamond_0(x)$
B.) *i* $\Diamond_0(\Box_\nu(x)) = \tau_0^P(\tau_\nu^N(x)) = \tau_0^P(x) = \Diamond_0(x)$
ii $\Diamond_\nu(\Box_1(x)) = \tau_\nu^P(\tau_1^N(x)) = \tau_1^N(x) = \Box_1(x)$
C.) *i* $\Box_1(\Diamond_0(x)) = \tau_1^N(\tau_0^P(x)) = \tau_0^P(x) = \Diamond_0(x)$
ii $\Diamond_0(\Box_1(x)) = \tau_0^P(\tau_1^N(x)) = \tau_1^N(x) = \Box_1(x)$

The proofs are based on the definition of drastic modal operators stated above.

Remark. *N5 and P5 are also valid when the possibility operator is drastic.*

For $N5$ *see:* A/ii*, and* C/i
For $P5$ *see:* B/ii*, and* C/ii

By making use of (2) and (2), we can define the concrete forms of the necessity and possibility operators.

Example 1.
Here, we use the Yager operator and the representation theorem of Trillas [13]

$$\tau(x) = n_{\nu_1}(n_{\nu_2}(x)) = \left(1 - \left(1 - x^{-\frac{\ln 2}{ln\nu_2}}\right)^{\frac{ln\nu_2}{ln\nu_1}}\right)^{-\frac{\ln \nu_1}{ln2}}$$

Example 2.

$$\tau_\nu^P(x) = \frac{1}{1 + \left(\frac{1-\nu}{\nu}\right)^2 \frac{1-x}{x}} \quad \text{or} \quad \tau_\nu^N(x) = \frac{1}{1 + \left(\frac{\nu}{1-\nu}\right)^2 \frac{1-x}{x}} \quad \text{when } \nu > \frac{1}{2} \tag{14}$$

See both plots below,

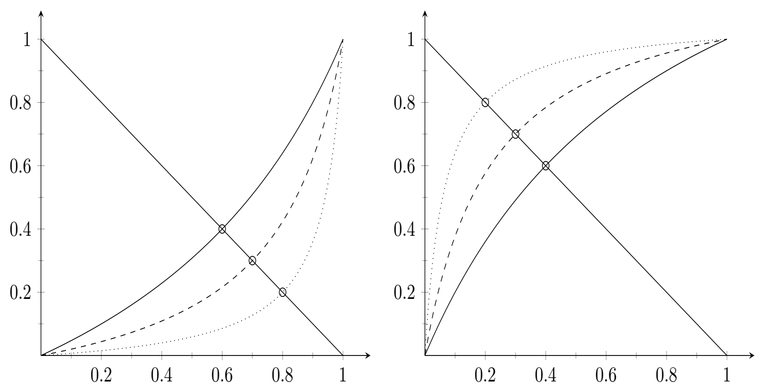

Fig. 4. The *necessity* and *possibility* operators with different ν values ($\nu = 0.2, 0.3, 0.4$)

If we use the definition of $\tau(x) = \Box x$ or $\Diamond x$ (i.e. necessity, or possibility x), then we can introduce different necessity and possibility operators.

$$\Box_\nu^{(2)} x = \Box_\nu\left(\Box_\nu(x)\right) = \tau_\nu^N(\tau_\nu^N(x))$$
$$\Diamond_\nu^{(2)} x = \Diamond_\nu\left(\Diamond_\nu(x)\right) = \tau_\nu^P(\tau_\nu^P(x))$$

Definition 8. *We call graded modalities a k composition of the modalities.*

$$\Box_\nu^{(k)}(x) = \Box_\nu(\underbrace{\Box_\nu(\ldots \quad \Box_\nu}_{k}(x))\ldots) = \tau_\nu^N\left(\tau_\nu^N(\ldots \quad \tau_\nu^N(x))\right) \tag{15}$$

$$\Diamond_\nu^{(k)}(x) = \Diamond_\nu(\underbrace{\Diamond_\nu(\ldots \quad \Diamond_\nu)}_{k}(x))\ldots) = \tau_\nu^P\left(\tau_\nu^P(\ldots \quad \tau_\nu^P(x))\right) \tag{16}$$

Proposition 4. *The composition of a modal operator is a closed operation.*

$$\Box_\nu^{(k)}(x) = \Box_{\nu^*}(x) \qquad \Diamond_\nu^{(k)}(x) = \Diamond_{\nu^*}(x)$$

where

$$\nu^* = f^{-1}\left(f^{2k}(\nu)\right) \tag{17}$$

Proof. In the possible modal operator case:

$$f^{-1}\left(f^2(\nu)\left(f^2(\nu)\dots\left(f^2(\nu)f(x)\right)\dots\right)\right)=f^{-1}\left(f^{2k}(\nu)f(x)\right)$$

□

Proposition 5. *The following properties hold for the composition of modal operators*

Properties using classical notations

1. $\Box_\nu^{(n)}(\Box_\nu^{(m)}(x))=(\tau_\nu^N(\tau_\nu^N(x))^{(m)})^{(n)}=\left(\tau_\nu^N(x)\right)^{(n+m)}=\Box_\nu^{(n+m)}(x)$
2. $\Diamond_\nu^{(n)}(\Diamond_\nu^{(m)}(x))=(\tau_\nu^P(\tau_\nu^P(x))^{(m)})^{(n)}=\left(\tau_\nu^P(x)\right)^{(n+m)}=\Diamond_\nu^{(n+m)}(x)$
3. $\Diamond_\nu^{(n)}(\Box_\nu^{(m)}(x))=(\tau_\nu^P(\tau_\nu^N(x))^{(m)})^{(n)}=\begin{cases}\Diamond_\nu^{(n-m)}(x) & \text{if} \quad n-m>0\\ x & \text{if} \quad n=m=0\\ \Box_\nu^{(m-n)}(x) & \text{if} \quad n-m<0\end{cases}$
4. $\Box_\nu^{(n)}(\Diamond_\nu^{(m)}(x))=(\tau_\nu^N(\tau_\nu^P(x))^{(m)})^{(n)}=\begin{cases}\Box_\nu^{(n-m)}(x) & \text{if} \quad n-m>0\\ x & \text{if} \quad n=m=0\\ \Diamond_\nu^{(m-n)}(x) & \text{if} \quad n-m<0\end{cases}$
5. $\lim_{K\to\infty}(\tau_\nu^N(x))^{(K)}=\tau_1(x)$
6. $\lim_{K\to\infty}(\tau_\nu^P(x))^{(K)}=\tau_0(x)$

Proof: They follow from direct calculation.

Using the Dombi operator when $\nu>\frac{1}{2}$, we have

$$\tau_\nu^{(N)}(x)=\frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^{2k}\frac{1-x}{x}} \qquad \tau_\nu^{(P)}(x)=\frac{1}{1+\left(\frac{\nu}{1-\nu}\right)^{2k}\frac{1-x}{x}}$$

4 Conclusions

We defined necessity and possibility operators using double negation where the fixed points are different. The composition of the modal operators is closed. So we have series of modal operators with different degrees. We defined these modal operators using a generator function of the operator system and these functions are the generator functions of the operators of the Pliant logical system. There are several open questions as how can we apply this approach to the nilpotent operator class, or how the logic behind this structure can be characterized, etc. It seems that the Pliant structure has an outstanding position. Particularly its special case namely the Dombi operator plays an important role in the practical applications.

References

1. P. Cintula, E.P. Klement, R. Mesiar, and M. Navara. Fuzzy logics with an additional involutive negation. *Fuzzy Sets and Systems*, 161:390–411, 2010.
2. J. Dombi. Towards a general class of operators for fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 16:477–484, 2008.
3. J. Dombi. De Morgan systems with an infinitely many negations in the strict monotone operator case. *Information Sciences*, 181:1440–1453, 2011.
4. J. Dombi. On a certain class of aggregative operators. *Information Sciences*, 245(1):313–328, 2013.
5. F. Esteva, L Godo, P. Hájek, and M. Navara. Residuated fuzzy logics with an involutive negation. *Arch. Math. Log.*, 39(2):103–124, 2000.
6. C. Noguera F. Esteva, L. Godo. A logical approach to fuzzy truth hedges. *Inf. Sci.*, 232:366–385, 2013.
7. S. Gottwald and P. Hájek. Triangular norm-based mathematical fuzzy logics. *Logical Algebraic, Analytic and Probabilistic Aspects of Triangular Norms*, pages 275–299, 2005.
8. P. Hájek. *Metamathematics of fuzzy logic.* Kluwer Academic Publishers, Dordrecht, 1998.
9. J. Łukasiewicz. Two-valued logic. *Przeglad Filozoficzny*, 13:189–205, 1921.
10. T. Flaminio. E. Marchioni. T-norm-based logics with an independent involutive negation. *Fuzzy Sets and Systems*, 157(24):3125–3144, 2006.
11. E. Orlowska. A logic of indiscernibility relations. *LNCS, Springer-Verlag, Berlin*, 208:177–186, 1985.
12. J. Pavelka. On fuzzy logic I. Many-valued rules of inference. *Z. Math. Logik Grundlag. Math.*, 25:45–52, 1979.
13. E. Trillas. Sobre functiones de negacion en la teoria de conjuntas difusos. *Stochastica*, 3:47–60, 1979.
14. L. A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning, part 1. *Information Sciences*, 8:199–249, 1975.