Operator-dependent Modifiers in Nilpotent Logical Systems

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Abstract: The purpose of the current study is to consider the main unary operators of a nilpotent logical system in an integral framework and to reveal the underlying general structure of all the previously examined operators in nilpotent logical systems. The unary operators are obtained by repeating the argument in multivariable operators. This enables us to provide a widely applicable system, where all the operators are connected to each other, and where the modalities and hedges are operator-dependent. It becomes possible to describe all the operators by using a generator function and a few parameters. The possibility, necessity and sharpness operators are thoroughly examined and it is also shown how the multivariable operators can be derived from the unary ones.

1 INTRODUCTION

Among other preferable properties, the fulfillment of the law of contradiction and the excluded middle, and the coincidence of the residual and the S-implication (Trillas and L. Valverde, 1981) make the application of nilpotent operators in logical systems feasible. In their pioneer work (Dombi and Csiszár, 2015), Dombi and Csiszár examined connective systems instead of operators themselves. It was shown that a consistent connective system generated by nilpotent operators is not necessarily isomorphic to the Łukasiewicz-system. Using more than one generator function, consistent nilpotent connective systems (so-called bounded systems) can be obtained in a significantly different way with three naturally derived negation operators. Due to the fact that all continuous Archimedean (i.e. representable) nilpotent t-norms are isomorphic to the Łukasiewicz t-norm (Grabisch et al., 2009; Klement et al., 2000), the previously studied nilpotent systems were all isomorphic to the well-known Łukasiewicz-logic.

In the last few years, the most important multivariable operators of general nilpotent systems have been thoroughly examined. In (Dombi and Csiszár, 2014) and in (Dombi and Csiszár, 2016), Dombi and Csiszár examined the implications and equivalence operators in bounded systems. In (Dombi and Csiszár, 2017), a parametric form of the generated operator o_ν was given by using a shifting transformation of the generator function. Here, the parameter can be interpreted semantically as a threshold of expectancy (decision level). This means that nilpotent conjunctive, disjunctive, aggregative (where a high input can compensate for a lower one) and negation operators can be obtained by changing this parameter.

Negation operators were also studied thoroughly in (Dombi and Csiszár, 2015), as they play a significant role in logical systems by building connections between the main operators (De Morgan law) and characterising their basic properties. Despite their significance, about other unary operators (compared to the multivariable ones) there are only limited literature available. In fuzzy theory, modalities (like *possibly, necessarily, ...*) and hedges (like *very, quite, extremely, ...*) are the most studied unary operators, which modify the linguistic variables (Zadeh, 1975c; Zadeh, 1975a; Zadeh, 1975b; Huynh et al., 2002; Türken, 2004; Banks, 1994; Jang et al., 1997; De Cock et al., 2000). In this study, the focus is on the unary operators of a nilpotent logical system. They perform various operations such as incrementing or decrementing a value and they can be widely used for expressing modalities and hedges in human thinking (Liu et al., 2001).

In this paper, our main purpose is to consider the main unary operators of a nilpotent logical system in an integral framework and to reveal the underlying ge-

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neral structure of all the operators considered so far. This enables to provide a widely applicable system, where all operators are connected to each other, and the modalities and hedges are operator-dependent. In such a system, only a few parameters are to be given. By fitting the parameter values, the system can be used to model real life problems.

The article is organized as follows. After recalling some basic preliminaries in Section 2, unary operators in nilpotent logical systems are examined in Section 3. First, in Section 3.1, a possible way of constructing unary operators is considered: repeating the argument in multivariable operators; i.e. by choosing $x_i = x_j$ $(\forall i, j)$ for the arguments of the many-variable operators. This is how it can be ensured that the operators are connected. In Section 3.2, our focus is on the drastic unary operators, in Section 3.3 on the composition rules and then in Section 3.4, it is shown how the multivariable operators can be derived from unary ones. This result underlines the importance of the unary operators in a logical system. In Section 3.5, a general framework is given for all the operators discussed so far.

In Section 4, a future research direction is suggested that could provide the next steps along the path to a practical and widely applicable system (e.g. in neural networks). The main disadvantage of the nilpotent operator family, namely the lack of differentiability can be eliminated by using a continuously differentiable approximation of the cutting function.

Finally, in Section 5, the main results are summarized.

2 PRELIMINARIES

To construct a logical system, we need to define the appropriate logical operators. As in (Dombi and Csiszár, 2015), we consider connective systems where the conjunction and disjunction operators are special types of t-norms and t-conorms, respectively.

A triangular norm (*t-norm* for short) T is a binary operation on the closed unit interval $[0,1]$ such that $([0,1],T)$ is an abelian semigroup with neutral element 1 which is totally ordered, i.e., for all x_1, x_2, y_1, $y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$ we have $T(x_1,y_1) \leq T(x_2,y_2)$, where $\leq$ is the natural order on $[0,1]$.

Standard examples of t-norms are the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$, the Łukasiewicz t-norm $T_{\mathbf{L}}$ given by $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$, and the drastic product $T_{\mathbf{D}}$ with $T_{\mathbf{D}}(1,x) = T_{\mathbf{D}}(x,1) = x$, and $T_{\mathbf{D}}(x,y) = 0$ otherwise.

A triangular conorm (*t-conorm* for short) S is a binary operation on the closed unit interval $[0,1]$ such that $([0,1],S)$ is an abelian semigroup with neutral element 0 which is totally ordered. Standard examples of t-conorms are the maximum $S_{\mathbf{M}}$, the probabilistic sum $S_{\mathbf{P}}$, the Łukasiewicz t-conorm $S_{\mathbf{L}}$ given by $S_{\mathbf{L}}(x,y) = \min(x+y,1)$, and the drastic sum $S_{\mathbf{D}}$ with $S_{\mathbf{D}}(0,x) = S_{\mathbf{D}}(x,0) = x$, and $S_{\mathbf{D}}(x,y) = 1$ otherwise.

A continuous t-norm T is said to be *Archimedean* if $T(x,x) < x$ holds for all $x \in (0,1)$. A continuous Archimedean T is called *strict* if T is strictly monotone; i.e. $T(x,y) < T(x,z)$ whenever $x \in (0,1]$ and $y < z$, and *nilpotent* if there exist $x, y \in (0,1)$ such that $T(x,y) = 0$.

From the duality between t-norms and t-conorms, we can easily derive the similar properties for t-conorms as well.

As it is well-known, t-norms and t-conorms can be expressed by means of a single real generator function with the following specific properties.

Proposition 1. *(Baczyński, (Baczyński and Jayaram, 2009), Ling, (C. Ling, 1965)) A function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm if and only if it has a continuous additive generator, i.e. there exists a continuous strictly decreasing function $t : [0,1] \to [0,\infty)$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x,y) = t^{-1}(min(t(x)+t(y),t(0)), \quad x,y \in [0,1]. \tag{1}$$

Proposition 2. *(Grabisch et al., 2009)*
A t-norm T is strict if and only if $t(0) = \infty$ holds for each continuous additive generator t of T.
A t-norm T is nilpotent if and only if $t(0) < \infty$ holds for each continuous additive generator t of T.

Due to the duality, additive generators of t-conorms ($s(x)$) can be obtained from the additive generators of their dual t-norms.

Since the generator functions of nilpotent t-norms and t-conorms are bounded and determined up to a multiplicative constant, they can be normalized (Dombi and Csiszár, 2015). Let us use the following notations for the uniquely defined normalized generator functions:

$$f_c(x) := \frac{t(x)}{t(0)}, \qquad f_d(x) := \frac{s(x)}{s(1)}. \tag{2}$$

In order to simplify the notations, we recall the definition of the so-called cutting function.

Definition 1. *(Dombi and Csiszár, 2015; Sabo and Strezo, 2005) Let us define the cutting operation [ ] by*

$$[x] = \begin{cases} 0 & if \quad x < 0 \\ x & if \quad 0 \leq x \leq 1 \\ 1 & if \quad 1 < x \end{cases} \tag{3}$$

and let the notation [ ] *also act as brackets when writing the argument of an operator. Then we can write* $f[x]$ *instead of* $f([x])$.

Definition 2. *(Dombi and Gera, 2005) Let* $a, b \in [0,1]$, $a < b$ *and let us define the generalized cutting operation* $[\ ]_a^b$ *by*

$$[x]_a^b = \begin{cases} 0 & if \quad x < a \\ \frac{x-a}{b-a} & if \quad a \leq x \leq b \\ 1 & if \quad b < x \end{cases} \tag{4}$$

and let the notation [ ] *also act as brackets when writing the argument of an operator. Then we can write* $f[x]$ *instead of* $f([x])$.

Proposition 3. *(Dombi and Csiszár, 2015) With the help of the cutting operator, we can write the conjunction and disjunction operators in the following form, where* $f_c(x)$ *and* $f_d(x)$ *are decreasing and increasing normalized generator functions, respectively.*

$$c(x,y) = f_c^{-1}[f_c(x) + f_c(y)], \tag{5}$$

$$d(x,y) = f_d^{-1}[f_d(x) + f_d(y)]. \tag{6}$$

From now on, the notations $c(x,y)$ and $d(x,y)$ above will be used for the conjunction and disjunction to emphasize the use of the normalized generator functions.

Next, we recall the definition of a nilpotent logical system.

Definition 3. *(Dombi and Csiszár, 2015) The triple* (c,d,n), *where c is a continuous Archimedean t-norm, d is a continuous Archimedean t-conorm and n is a strong negation, is called a connective system.*

Definition 4. *(Dombi and Csiszár, 2015) A connective system is nilpotent if the conjunction c is a nilpotent t-norm, and the disjunction d is a nilpotent t-conorm.*

It was shown in (Dombi and Csiszár, 2015) that to construct a logical system, more than one generator functions can be used without losing consistency. In these systems, $n_c(x)$ and $n_d(x)$, the negations generated by f_c and f_d respectively (also called as the natural negations) do not coincide with the negation operator; i.e. $n_c(x) \neq n_d(x) \neq n(x)$.

Definition 5. *A nilpotent connective system is called a bounded system if* $f_c(x) + f_d(x) > 1$, *or equivalently* $n_d(x) < n(x) < n_c(x)$ *holds for all* $x \in (0,1)$, *where* f_c *and* f_d *are the normalized generator functions of the conjunction and disjunction, and* n_c, n_d *are the natural negations.*

The associativity of t-norms and t-conorms permits us to consider their extensions to the multivariable case. Dombi and Csiszár (Dombi and Csiszár, 2017) examined a general parametric operator $o_\nu(\mathbf{x})$ of nilpotent systems, where the parameter has an important semantic meaning as the threshold of expectation. Nilpotent conjunctive, disjunctive, aggregative and negation operators can be obtained by changing the parameter value.

Definition 6. *Let* $f : [0,1] \to [0,1]$ *be an increasing bijection,* $\nu \in [0,1]$, *and* $\boldsymbol{x} = (x_1, \ldots, x_n)$, *where* $x_i \in [0,1]$ *and let us define the general operator by*

$$o_\nu(\boldsymbol{x}) = f^{-1}\left[\sum_{i=1}^{n}(f(x_i) - f(\nu)) + f(\nu)\right] = \\ = f^{-1}\left[\sum_{i=1}^{n} f(x_i) - (n-1)f(\nu)\right]. \tag{7}$$

Remark 1. *Note that the general operator for* $\nu = 1$ *is conjunctive, for* $\nu = 0$ *it is disjunctive and for* $\nu = \nu^* = f^{-1}\left(\frac{1}{2}\right)$ *it is self-dual.*

On the basis of Remark 1, the conjunction, the disjunction and the aggregative operator can be defined in the following way.

Definition 7. *Let* $f : [0,1] \to [0,1]$ *be an increasing bijection. Let us define the conjunction, the disjunction and the aggregative operator by*

$$c(\boldsymbol{x}) := o_1(\boldsymbol{x}) = f^{-1}\left[\sum_{i=1}^{n} f(x_i) - (n-1)\right], \tag{8}$$

$$d(\boldsymbol{x}) := o_0(\boldsymbol{x}) = f^{-1}\left[\sum_{i=1}^{n} f(x_i)\right], \tag{9}$$

$$a(\boldsymbol{x}) := o_{\nu^*}(\boldsymbol{x}) = f^{-1}\left[\sum_{i=1}^{n} f(x_i) - \frac{(n-1)}{2}\right], \tag{10}$$

respectively, where $\nu^* = f^{-1}\left(\frac{1}{2}\right)$.

Remark 2. *A conjunction, a disjunction and an aggregative operator differ only in one parameter of the general operator in* (7). *The parameter* ν *has the semantic meaning of the level of expectation: maximal for the conjunction, neutral for the aggregation and minimal for the disjunction.*

Definition 8. *Let* $\mathbf{w} = (w_1, \ldots, w_n)$ *and* $w_i > 0$ *be real parameters,* $f : [0,1] \to [0,1]$ *an increasing bijection with* $\nu \in [0,1]$. *The weighted general operator is defined by*

$$a_{\nu,\boldsymbol{w}}(\boldsymbol{x}) := f^{-1}\left[\sum_{i=1}^{n} w_i(f(x_i) - f(\nu)) + f(\nu)\right]. \tag{11}$$

Definition 9. *The operator*

$$a_w(\boldsymbol{x}) = f^{-1}\left[w\left(\sum_{i=1}^{n} f(x_i) - \frac{n}{2}\right) + \frac{1}{2}\right], \tag{12}$$

where $w > 0$, *is called the weighted aggregative operator.*

Proposition 4. *The weighted general operator $a_{\nu,\boldsymbol{w}}(\boldsymbol{x})$ satisfies*

1. *The boundary condition $a_{\nu,\boldsymbol{w}}(\boldsymbol{0}) = 0$, if and only if $\nu = 0$ or $\sum_{i=1}^{n} w_i \geq 1$ (for a commutative operator: $w \geq \frac{1}{n}$);*
2. *The boundary condition $a_{\nu,\boldsymbol{w}}(1,\ldots,1) = 1$, if and only if $\nu = 1$ or $\sum_{i=1}^{n} w_i \geq 1$ (for a commutative operator: $w \geq \frac{1}{n}$);*
3. *Both of the above-mentioned boundary conditions, if and only if $\sum_{i=1}^{n} w_i \geq 1$ (for a commutative operator: $w \geq \frac{1}{n}$);*
4. *$a_{\nu,\boldsymbol{w}}(\nu,\ldots,\nu) = \nu$.*

3 UNARY OPERATORS IN NILPOTENT LOGICAL SYSTEMS

In the early 1970's, Zadeh (L. A. Zadeh, 1972) introduced a class of powering modifiers, which defined the concept of linguistic variables and hedges (like *very, quite, extremely, ...*). He proposed computing with words as an extension of fuzzy sets and logic theory and introduced modifier functions of fuzzy sets called linguistic hedges, which change the meaning of the primary terms. As pointed out by Zadeh, linguistic variables and terms are closer to human thinking and therefore, words and linguistic terms can be used to model human thinking systems (L. A. Zadeh, 1971). Hedges and also modalities (like *possibly, necessarily, ...*) are the most examined unary operators. From a semantic viewpoint, these unary operators can also be viewed as a part of a logical system. In this section, two possible ways of extending a nilpotent logical system by defining the necessity and possibility operators are examined. The novelty of these two methods lies in the fact that they provide a logical system, where all the operators are connected to each other.

The possibility and neccessity operators have to satisfy the following equations.

$$\text{impossible}(x) = \text{necessity}(\text{not}(x)) \tag{13}$$

and

$$\text{possible}(x) = \text{not}(\text{impossible}(x)). \tag{14}$$

3.1 Possibility and Necessity as Unary Operators Derived from Multivariable Operators

A possible way of obtaining unary operators is by choosing $x_i = x_j$ ($\forall i, j$) for the arguments of the many-variable operators. Based on the De Morgan property of the conjunction and the disjunction, the unary operators derived from them satisfy the equations above.

Definition 10. *Let $k \in \mathbb{N}, \lambda \in \mathbb{R}^+, \lambda > 1$, $f : [0,1] \to [0,1]$ be an increasing bijection and let us define the so-called necessity operator, $\tau_N^{(k)}(x) : [0,1] \to [0,1]$ in the following way:*

$$\tau_N^{(k)}(x) := c[\underbrace{x, x, \ldots x}_{k-times}] = f^{-1}\left[k(f(x) - 1) + 1\right], \tag{15}$$

and the generalized necessity operator $\tau_N^{(\lambda)}(x) : [0,1] \to [0,1]$ as

$$\begin{aligned}\tau_N^{(\lambda)}(x) &:= f^{-1}\left[\lambda(f(x) - 1) + 1\right] = \\ &= f^{-1}\left[\lambda f(x) - (\lambda - 1)\right],\end{aligned} \tag{16}$$

where c is the conjunction generated by $f_c(x) = 1 - f(x)$.

Similarly, the so-called possibility operator can also be defined by means of the disjunction operator.

Definition 11. *Let $k \in \mathbb{N}, \lambda \in \mathbb{R}^+, \lambda > 1$, $f : [0,1] \to [0,1]$ be an increasing bijection and let us define the so-called possibility operator, $\tau_N^{(k)}(x) : [0,1] \to [0,1]$ in the following way:*

$$\tau_P^{(k)}(x) := d[\underbrace{x, x, \ldots x}_{k-times}] = f^{-1}\left[k f(x)\right], \tag{17}$$

and the generalized possibility operator $\tau_P^{(\lambda)}(x) : [0,1] \to [0,1]$ as

$$\tau_P^{(\lambda)}(x) := f^{-1}\left[\lambda f(x)\right], \tag{18}$$

where d is the disjunction generated by $f(x)$.

Next, the so-called sharpness operator is defined, based on the self-duality of the aggregative operator.

Definition 12. *Let $k \in \mathbb{N}, \lambda \in \mathbb{R}^+, \lambda > 1$, $f : [0,1] \to [0,1]$ be an increasing bijection and let us define the so-called sharpness operator, $\tau_S^{(k)}(x) : [0,1] \to [0,1]$ in the following way:*

$$\tau_S^{(k)}(x) := a[\underbrace{x, x, \ldots x}_{k-times}] = f^{-1}\left[k f(x) - \frac{k-1}{2}\right], \tag{19}$$

and the generalized sharpness operator $\tau_S^{(\lambda)}(x) : [0,1] \to [0,1]$ as

$$\tau_S^{(\lambda)}(x) := f^{-1}\left[\lambda f(x) - \frac{\lambda - 1}{2}\right], \tag{20}$$

where a is the aggregative operator generated by $f(x)$.

The three definitions above can be summarized in a unified formula.

Definition 13. *Let* $\lambda \in \mathbb{R}^+, \lambda > 1, \nu \in [0,1]$, $f : [0,1] \to [0,1]$ *be an increasing bijection. Let us define the unary operator* $\tau_\nu^{(\lambda)}(x)$ *in the following way.*

$$\tau_\nu^{(\lambda)}(x) := f^{-1}\left[\lambda\left(f(x) - f(\nu)\right) + f(\nu)\right]. \tag{21}$$

Remark 3. *For* $\nu = 1$, $\nu = 0$ *and* $\nu = \nu^*$ *(i.e.* $f(\nu) = \frac{1}{2}$*), we get the necessity, the possibility and the sharpness operators, respectively.*

The above-defined unary operators fulfill the following De Morgan identities (see equations 13 and 14).

Proposition 5. *Let* $f : [0,1] \to [0,1]$ *be an increasing bijection and let* $n(x)$ *be the negation generated by* $f(x)$.

$$n\left(\tau_N^{(\lambda)}(x)\right) = \tau_P^{(\lambda)}\left(n(x)\right), \tag{22}$$

$$n\left(\tau_P^{(\lambda)}(x)\right) = \tau_N^{(\lambda)}\left(n(x)\right), \tag{23}$$

$$n\left(\tau_S^{(\lambda)}(x)\right) = \tau_S^{(\lambda)}\left(n(x)\right). \tag{24}$$

Proof. The proof is similar in all three cases. Let us prove the first statement.

Taking into account the fact that $1 - [x] = [1 - x]$, $n\left(\tau_N^{(\lambda)}(x)\right) = f^{-1}\left[1 - [\lambda f(x) - (\lambda - 1)]\right] = f^{-1}\left[\lambda(1 - f(x))\right] = \tau_P^{(\lambda)}\left(n(x)\right)$. □

Proposition 6. $\tau_\nu^{(\lambda)}(x)$ *is for* $\forall \nu \in [0,1]$ *increasing. Let* $x = x_1$ *be the greatest value, for which* $\tau_\nu^{(\lambda)}(x) = 0$, *and let* $x = x_2$ *be the lowest value, for which* $\tau_\nu^{(\lambda)}(x) = 1$. *In this case*

$$x_1 = f^{=1}\left[\frac{\lambda - 1}{\lambda} f(\nu)\right] \tag{25}$$

and

$$x_2 = f^{=1}\left[\frac{\lambda - 1}{\lambda} f(\nu) + \frac{1}{\lambda}\right]. \tag{26}$$

Proof. The monotonicity follows from the monotonicity of $f(x)$. To find x_1 and x_2, the following two equations need to be solved:

$$\lambda\left(f(x) - f(\nu)\right) + f(\nu) = 0,$$

and

$$\lambda\left(f(x) - f(\nu)\right) + f(\nu) = 1.$$

The solution follows from a direct calculation. □

The values x_1 and x_2 in Proposition 6 for $\nu = 1, \nu = 0$ and $\nu = \nu^*$ can be found in Table 1.

Table 1: x_1 and x_2 values for $\nu = 1, \nu = 0$ and $\nu = \nu^*$.

	ν	x_1	x_2
$\tau_N^{(\lambda)}(x)$	1	$f^{-1}\left(1 - \frac{1}{\lambda}\right)$	1
$\tau_P^{(\lambda)}(x)$	0	0	$f^{-1}\left(\frac{1}{\lambda}\right)$
$\tau_S^{(\lambda)}(x)$	ν^*	$f^{-1}\left(\frac{\lambda-1}{2\lambda}\right)$	$f^{-1}\left(\frac{\lambda+1}{2\lambda}\right)$

Table 2: x_1 and x_2 values for $f(x) = x$.

	ν	x_1	x_2
$\tau_N^{(\lambda)}(x)$	1	$1 - \frac{1}{\lambda}$	1
$\tau_P^{(\lambda)}(x)$	0	0	$\frac{1}{\lambda}$
$\tau_S^{(\lambda)}(x)$	ν^*	$\frac{\lambda-1}{2\lambda}$	$\frac{\lambda+1}{2\lambda}$

Proposition 7. *Let* $\nu^* = f^{-1}\left(\frac{1}{2}\right)$.

$$\tau_N^{(\lambda)}(\nu^*) = f^{-1}\left[1 - \frac{\lambda}{2}\right], \tag{27}$$

$$\tau_P^{(\lambda)}(\nu^*) = f^{-1}\left[\frac{\lambda}{2}\right] \tag{28}$$

and

$$\tau_S^{(\lambda)}(\nu^*) = \nu^*. \tag{29}$$

Proof. The statements follow from direct calculations. □

Remark 4. *Note that* ν^* *is a fixpoint of the sharpness operator* $\tau_S^{(\lambda)}(x)$.

Next, let us consider the case $f(x) = x$.

Remark 5. *In particular for* $f(x) = x$,

$$\tau_N^{(\lambda)}(x) = \min\left(1, \max\left(0, \lambda \mathrm{x} - (\lambda - 1)\right)\right), \tag{30}$$

$$\tau_P^{(\lambda)}(x) = \min\left(1, \max\left(0, \lambda \mathrm{x}\right)\right), \tag{31}$$

$$\tau_S^{(\lambda)}(x) = \min\left(1, \max\left(0, \lambda \mathrm{x} - \frac{\lambda - 1}{2}\right)\right). \tag{32}$$

In Figure 1, unary operators generated by $f(x) = x$ are shown. For the values x_1 and x_2, see Table 2.

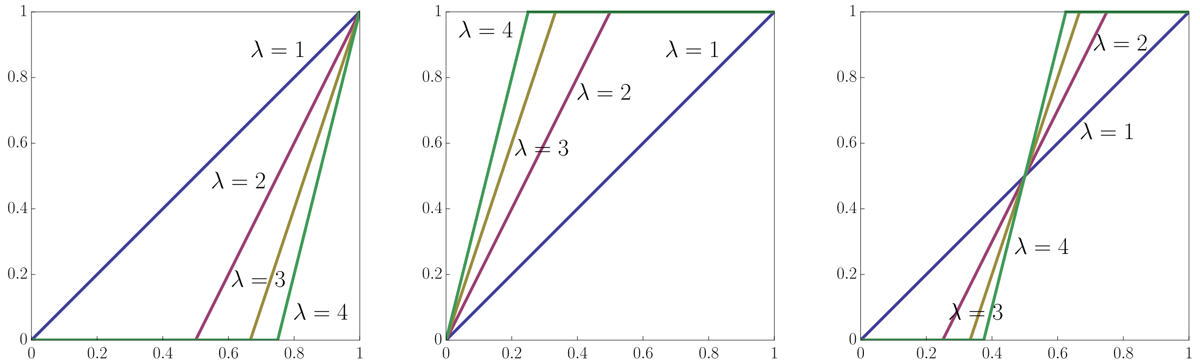
Remark 6. *As can be seen, for* $f(x) = x$, *the unary operators* $\tau_I^{(\lambda)}(x)$ $(I \in \{N, P, S\})$, *have a value in* $(0,1)$ *if and only if* $x \in (x_1, x_2)$. *Note that the length of this interval,* $x_2 - x_1 = \frac{1}{\lambda}$.

3.2 Drastic Unary Operators

Let us now define the so-called drastic unary operators in the following way.

Definition 14. *Let* $f : [0,1] \to [0,1]$ *be an increasing bijection. Let*

$$\tau_N^{(\infty)}(x) := \lim_{\lambda \to \infty} \tau_N^{(\lambda)}(x), \tag{33}$$



(a) Necessity operators $\tau_N^{(\lambda)}$ for $\lambda = 1,2,3,4$

(b) Possibility operators $\tau_P^{(\lambda)}$ for $\lambda = 1,2,3,4$

(c) Sharpness operators $\tau_S^{(\lambda)}$ for $\lambda = 1,2,3,4$

Figure 1: Unary operators generated by $f(x) = x$.

$$\tau_P^{(\infty)}(x) := \lim_{\lambda\to\infty} \tau_P^{(\lambda)}(x), \quad (34)$$

and

$$\tau_S^{(\infty)}(x) := \lim_{\lambda\to\infty} \tau_S^{(\lambda)}(x). \quad (35)$$

$\tau_N^{(\infty)}(x)$, $\tau_P^{(\infty)}(x)$ and $\tau_S^{(\infty)}(x)$ are called drastic unary operators.

Proposition 8.

$$\tau_N^{(\infty)}(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad x = 1, \end{cases} \quad (36)$$

and

$$\tau_P^{(\infty)}(x) = \begin{cases} 0 & if \quad x = 0 \\ 1 & if \quad x > 0, \end{cases} \quad (37)$$

and

$$\tau_S^{(\infty)}(x) = \begin{cases} 0 & if \quad x < \nu \\ \nu & if \quad x = \nu \\ 1 & if \quad x > \nu. \end{cases} \quad (38)$$

Proof. The statement follows from a direct calculation. □

3.3 Composition Rules

In human thinking and languages, emphasis is often expressed by repeating modalities and hedges, such as "very-very". The following proposition shows that the necessity, possibility and sharpness operators are all closed under composition. The parameter of the compositon is the product of the input parameters.

Proposition 9. *Let $f : [0,1] \to [0,1]$ be an increasing bijection and let $n(x)$ be the negation generated by $f(x)$.*

$$\tau_N^{(\lambda_1)}\left(\tau_N^{(\lambda_2)}(x)\right) = \tau_N^{(\lambda_1\lambda_2)}(x), \quad (39)$$

$$\tau_P^{(\lambda_1)}\left(\tau_P^{(\lambda_2)}(x)\right) = \tau_P^{(\lambda_1\lambda_2)}(x), \quad (40)$$

$$\tau_S^{(\lambda_1)}\left(\tau_S^{(\lambda_2)}(x)\right) = \tau_S^{(\lambda_1\lambda_2)}(x). \quad (41)$$

Proof. 1. It has to be shown that $\tau_N^{(\lambda_1)}\left(\tau_N^{(\lambda_2)}(x)\right) = f^{-1}\left[\lambda_1 f\left(f^{-1}[\lambda_2 f(x) - (\lambda_2 - 1)]\right) - (\lambda_2 - 1)\right] = f^{-1}\left[\lambda_1\left[\lambda_2 f(x) - (\lambda_2 - 1)\right] - (\lambda_2 - 1)\right]$.

(a) For $\lambda_2 f(x) - (\lambda_2 - 1) \le 0$; i.e. for $f(x) \le 1 - \frac{1}{\lambda_2}$, we obtain $\tau_N^{(\lambda_1)}\left(\tau_N^{(\lambda_2)}(x)\right) = 0$. In this case, $\tau_N^{(\lambda_1\lambda_2)}(x) = 0$ as well, since from $f(x) \le 1 - \frac{1}{\lambda_2}$ follows $f(x) \le 1 - \frac{1}{\lambda_2} - \frac{1}{\lambda_1}$; i.e. $\lambda_1\lambda_2 f(x) - ((\lambda_1 - 1)(\lambda_2 - 1) - 1) \le 0$.

(b) For $0 < \lambda_2 f(x) - (\lambda_2 - 1) \le 0 < 1$; i.e. for $f(x) > 1 - \frac{1}{\lambda_2}$, the cutting function can be omitted and the statement follows from a direct calculation.

(c) Taking into account the fact that $\lambda_2 > 1$ and $0 \le f(x) \le 1$, $\lambda_2 f(x) - (\lambda_2 - 1) > 1$ is impossible.

2. It has to be shown that $\tau_P^{(\lambda_1)}\left(\tau_P^{(\lambda_2)}(x)\right) = f^{-1}\left[\lambda_1\left[\lambda_2 f(x)\right]\right] = f^{-1}\left[\lambda_1\lambda_2 f(x)\right]$.

(a) If $f(x) \ge \frac{1}{\lambda_2}$, then $f^{-1}\left[\lambda_1\left[\lambda_2 f(x)\right]\right] = f^{-1}\left[\lambda_1\lambda_2 f(x)\right] = 1$.

(b) If $0 < f(x) < \frac{1}{\lambda_2}$, then the cutting function can be omitted and the statement is trivial.

3. It has to be shown that $\tau_S^{(\lambda_1)}\left(\tau_S^{(\lambda_2)}(x)\right) = f^{-1}\left[\lambda_1\left[\lambda_2 f(x) - \frac{\lambda_2 - 1}{2}\right] - \frac{\lambda_1 - 1}{2}\right] = f^{-1}\left[\lambda_1\lambda_2 f(x) - \frac{\lambda_1\lambda_2 - 1}{2}\right]$.

(a) If $\lambda_2 f(x) - \frac{\lambda_2 - 1}{2} \le 0$, then taking into account the fact that $\lambda_i > 1$, the left hand side of the equation is 0. Since in this case $f(x) \le \frac{\lambda_2 - 1}{2\lambda_2}$, $2\lambda_1\lambda_2 f(x) \le \lambda_1\lambda_2 - 1 \le \lambda_1\lambda_2 - 1$. Therefore, the value in the cutting function on the right hand side is less than or equal to 0, which means that the equation holds.

(b) If $0 \le \lambda_2 f(x) - \frac{\lambda_2 - 1}{2} \le 1$, then the cutting function can be omitted and the statement is trivial.

(c) If $\lambda_2 f(x) - \frac{\lambda_2 - 1}{2} > 1$, then $f(x) > \frac{\lambda_2+1}{2\lambda_2} > 1$, which means that the left hand side of the equation is 1. Since in this case $\frac{1+\lambda_1\lambda_2}{2} < \frac{\lambda_1\lambda_2+\lambda_1}{2} < \lambda_1\lambda_2 f(x)$, the value in the cutting function on the right hand side is greater than 1, which means that the equation is valid.

□

Proposition 10. *1. For the drastic operators*

$$\tau_I^{(\infty)}\left(\tau_J^{(\infty)}(x)\right) = \tau_J^{(\infty)}(x), \tag{42}$$

where $I, J \in \{N, P, S\}$.

Proof. This statement follows from a direct calculation. □

3.4 Multivariable Operators Derived from Unary Operators

Proposition 11 tells us how the conjunction and the disjunction can be expressed in terms of the unary operators and the arithmetic mean operator.

First, let us recall the definition of the arithmetic mean operator.

Definition 15.

$$m(\boldsymbol{x}) := f^{-1}\left(\frac{1}{k}\sum_{i=1}^{k}(f(x_i))\right), \tag{43}$$

where $f : [0,1] \to [0,1]$ *is an increasing bijection.*

Proposition 11. *The unary operators satisfy the following equation:*

$$\tau_\nu^{(k)}(m(\boldsymbol{x})) = o_\nu(\boldsymbol{x}). \tag{44}$$

In particular,

1. $\tau_P^{(k)}(m(\boldsymbol{x})) = d(\boldsymbol{x})$,
2. $\tau_N^{(k)}(m(\boldsymbol{x})) = c(\boldsymbol{x})$,
3. $\tau_S^{(k)}(m(\boldsymbol{x})) = a(\boldsymbol{x})$.

Proof. The statements follow from a direct calculation. □

Proposition 12. *The necessity and the possibility operator have the following property:*

1. $d\left(\tau_P^{(k)}(x_1), \tau_P^{(k)}(x_2), \ldots \tau_P^{(k)}(x_k)\right) = \tau_P^{(k)}\left(d(\boldsymbol{x})\right)$.
2. $c\left(\tau_N^{(k)}(x_1), \tau_N^{(k)}(x_2), \ldots \tau_N^{(k)}(x_k)\right) = \tau_N^{(k)}\left(c(\boldsymbol{x})\right)$,

Proof. 1. The following statement has to be proven: $f^{-1}\left[\sum_{i=1}^{k}[\lambda f(x_i)]\right] = f^{-1}\left[\lambda \sum_{i=1}^{k}[f(x_i)]\right]$. If $\lambda f(x_i) \le 1$ for $\forall i$, then the statement is trivial. If $\exists i$, for which $\lambda f(x_i) > 1$, then both sides of the equations have the same value (i.e. a value of 1).

2. This follows from the first statement by applying the De Morgen law.

□

3.5 A General Framework: The α, β, γ-Model

All basic operators discussed so far can be handled in a common framework, since they all can be described by the following parametric form.

Definition 16. *Let* $x, y \in [0,1]$, $\alpha, \beta, \gamma \in \mathbb{R}$ *and let* $f : [0,1] \to [0,1]$ *be a strictly increasing bijection. Let the general parametric operator be*

$$o_{\alpha,\beta,\gamma}(x,y) := f^{-1}[\alpha f(x) + \beta f(y) + \gamma]. \tag{45}$$

The most commonly used operators for special values of α, β and γ are listed in Table 3.

Now let us focus on the unary (1-place) case.

Definition 17. *Let* $x \in [0,1]$, $\alpha, \gamma \in \mathbb{R}$ *and let* $f : [0,1] \to [0,1]$, *a strictly increasing bijection. Then*

$$o_{\alpha,\gamma}(x) := f^{-1}[\alpha f(x) + \gamma]. \tag{46}$$

For special γ values, see Table 4.

In this framework it becomes possible to define all the operators by a single generator function and a few parameters.

4 FUTURE WORK AND APPLICATION

The main disadvantage of the Łukasiewicz operator family is the lack of differentiability, which would be necessary for numerous practical applications. Although most fuzzy applications (e.g. embedded fuzzy control) use piecewise linear membership functions owing to their easy handling, there are areas where the parameters are learned by a gradient-based optimization method. In this case, the lack of continuous derivatives makes the application impossible. For example, the membership functions have to be differentiable for each input in order to fine-tune a fuzzy control system by a simple gradient-based technique. This problem could be easily solved by using the so-called squashing function (see Dombi and Gera (Dombi and Gera, 2005)), which provides a solution to the above-mentioned problem by a continuously differentiable approximation of the cut function. This approximation could be the next step to realizing a practical and widely applicable system.

The squashing function defined below is a continuously differentiable approximation of the generalized cutting function (see Definition 2) by means of sigmoid functions (see Figure 2).

Table 3: Special values for α, β and γ.

	α	β	γ	$o_{\alpha,\beta,\gamma}(x,y)$	Notation
disjunction	1	1	0	$f^{-1}[f(x)+f(y)]$	$d(x,y)$
conjunction	1	1	-1	$f^{-1}[f(x)+f(y)-1]$	$c(x,y)$
implication	-1	1	1	$f^{-1}[f(y)-f(x)+1]$	$i(x,y)$
arithmetic mean	0.5	0.5	0	$f^{-1}\left[\frac{f(x)+f(y)}{2}\right]$	$m(x,y)$
preference	-0.5	0.5	0.5	$f^{-1}\left[\frac{f(y)-f(x)+1}{2}\right]$	$p(x,y)$
aggregative operator	1	1	-0.5	$f^{-1}\left[f(x)+f(y)-\frac{1}{2}\right]$	$a(x,y)$

Table 4: Special values for γ.

	α	γ	$o_{\alpha,\gamma}(x,y)$	Notation
possibility	α	0	$f^{-1}[\alpha f(x)]$	$\tau_P(x)$
necessity	α	$1-\alpha$	$f^{-1}[\alpha f(x)-(\alpha-1)]$	$\tau_N(x)$
sharpness	α	$\frac{\alpha-1}{2}$	$f^{-1}[\alpha f(x)-\frac{(\alpha-1)}{2}]$	$\tau_S(x)$

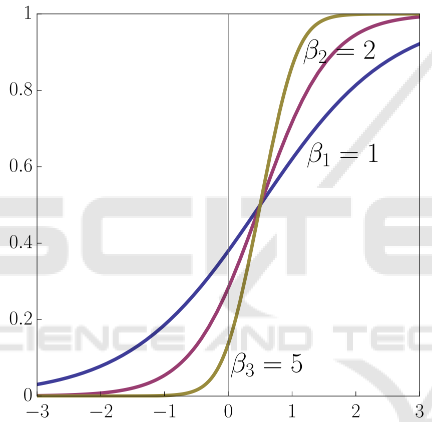


Figure 2: Squashing functions for $b = 1$, $a = 0$, for different β values ($\beta_1 = 1$, $\beta_2 = 2$ and $\beta_3 = 5$).

Definition 18. *(Dombi and Gera, 2005) Let the squashing function over the interval $[a,b]$ be*

$$S_{a,b}^{\beta}(x) = \frac{1}{b-a} \ln \left(\frac{1+e^{\beta(x-a)}}{1+e^{\beta(x-b)}} \right)^{\frac{1}{\beta}} \quad (47)$$

where $a,b \in \mathbb{R}$, $a < b$, $\beta \in \mathbb{R}^+$.

The parameters a and b affect the placement of the squashing function, while the parameter β drives the precision of the approximation.

The reason for choosing the sigmoid function is its significant role in applications such as artificial neural networks, optimization methods, economic and biological models.

In Figure 3, "squashed" unary operators are illustrated; i.e. unary operators, in which the cutting function is approximated with a squashing function.

5 CONCLUSIONS

The main purpose of this paper was to examine the main unary operators of a nilpotent logical system. An integral framework was introduced to reveal the underlying structure of all the operators considered so far. As a result, a nilpotent logical system can be obtained, in which all operators are connected to each other, and the modalities and hedges are operator-dependent. This is how it becomes possible to define all the operators via a single generator function and a few parameters. By fitting the parameter values, the system can be used to model real-life problems.

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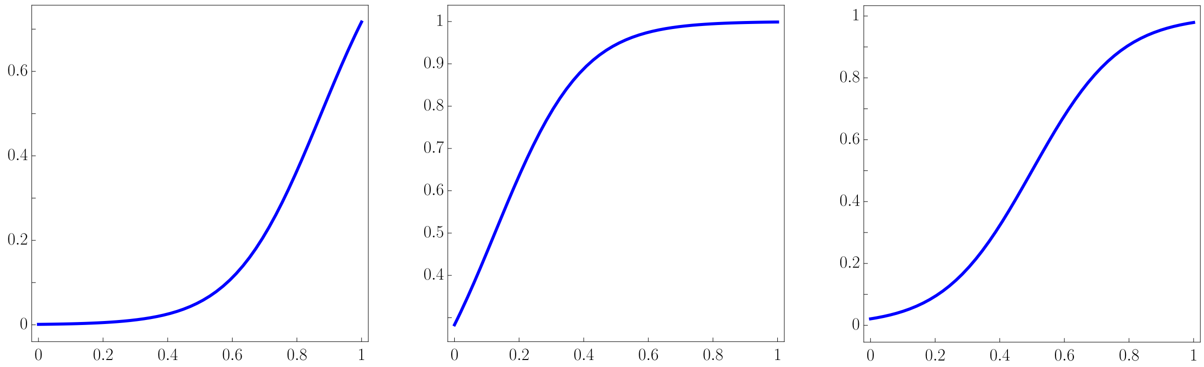

(a) necessity (b) possibility (c) sharpness

Figure 3: Unary operators approximated with the squashing function, $a = 4, \beta = 2$.

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