A Theory for Measuring the Preference of Fuzzy Numbers

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***Abstract*—Here, sigmoid function-based preference measures for intervals and fuzzy numbers are introduced, and their main properties are outlined. Also, formulas for the numerical computation of the proposed preference measures are presented. Next, it is demonstrated that the proposed preference measures for intervals and fuzzy numbers are asymptotically the well-known probability-based preference measures for intervals and fuzzy numbers. Using the new preference measure, two parametric crisp relations, which have common parameters, over a collection of fuzzy numbers are introduced. Next, it is shown that the limits of these relations can be used to rank fuzzy numbers. Namely, it is proved that the limit of one of these relations is a strict order relation, while the limit of the other may be viewed as an indifference relation. This indifference relation can be used to capture situations where the order of two fuzzy numbers cannot be judged; and so, their order may be considered as being indifferent.**

***Index Terms*—preference measure, intervals, fuzzy numbers, ranking**

I. Introduction

In theory and practice of multi-criteria decision making, fuzzy numbers are widely used. This explains why the ranking of fuzzy numbers is an important topic in computer science, especially in fuzzy decision-making. Without doubt, the ranking of fuzzy numbers is a challenging problem, and there are plenty of methods available for tackling this problem (see, e.g. [1]–[9]). To compare and rank fuzzy numbers, Huynh et al. [10] introduced a probability-based methodology. Wang [11] used a relative preference relation for ranking triangular and trapezoidal fuzzy numbers. Chutia and Chutia [12] presented a value and ambiguity-based method for ranking parametric forms of fuzzy numbers. Boulmakoul et al. [13]. proposed an inclusion index and bitset encoding-based approach. Chai et al. [14] used the Dempster–Shafer theory with fuzzy targets to develop a ranking method for fuzzy numbers. A class of signed-distance measures for ranking interval-valued fuzzy numbers was proposed by Akbari and Hesamian [15]. A credibility theory-oriented preference index for ranking fuzzy numbers was introduced by Hesamian and Bahrami [16]. Yatsalo and Martínez [17] proposed an approach for ranking fuzzy numbers and a fuzzy rank acceptability analysis that provides a degree of confidence for all ranks. Roldán López de Hierro et al. [18] presented an interesting application of a fuzzy number ranking method for economic data. Gu and Xuan [19] proposed a possibility theory-based approach for ranking fuzzy numbers. It is worth mentioning that there is a lot of interest in ranking intuitionistic fuzzy numbers (see, e.g. [20]–[25]).

In this study, we introduce sigmoid function-based preference measures for intervals and fuzzy numbers, and describe the main properties of these preference measures. Then, we present formulas for the numerical computation of the proposed preference measures. Here, we show that the proposed preference measures for intervals and fuzzy numbers asymptotically correspond to the well-known probability-based preference measures for intervals and fuzzy numbers. Next, using the new preference measure, we introduce two parametric crisp relations, which have common parameters, over a collection of fuzzy numbers. Then, we prove that the limits of these relations can be used to rank fuzzy numbers. Namely, we show that the limit of one of these relations is a strict order relation, while the limit of the other one may be viewed as an indifference relation. This latter can be used to capture situations where the order of two fuzzy numbers cannot be judged; and then, their order may be viewed as being indifferent.

It should be added that in a recent paper by Zumelzu et al. [26], the authors pointed out that among more than two hundred partial order relations for fuzzy numbers studied, they found just a few that are total orders. They introduced and analyzed the notion of admissible orders for fuzzy numbers with respect to a partial order.

This paper is organized as follows. In Section 2, a sigmoid function-based preference measure for two intervals is introduced. In Section 3, using the sigmoid function-based preference measure for two intervals, a new preference measure for two fuzzy numbers is presented and its main properties are described. Next, in Section 4, we show how the proposed preference measure for two fuzzy numbers can be used to rank fuzzy numbers. Lastly, in Section 5, a short summary of our results is provided.

II. A preference measure for two intervals

Here, first we define a preference measure for two real numbers.

Definition 1: The fuzzy relation y is preferred over x (i.e. $x \prec y$) is given by the membership function $\mu_{\prec}^{(\lambda)}\colon \mathbb{R}^2 \to (0,1)$

$$\mu_{\prec}^{(\lambda)}(x,y) = \frac{1}{1+\mathrm{e}^{-\lambda(y-x)}},$$

where $\lambda > 0$.

The following proposition concerns the main properties of the preference measure $\mu_{\prec}^{(\lambda)}$.

Proposition 1: The preference measure $\mu_{\prec}^{(\lambda)}$ has the following properties:

$$\mu_{\prec}^{(\lambda)}(y,x) < \frac{1}{2} \text{ if and only if } y < x$$
$$\mu_{\prec}^{(\lambda)}(y,x) = \frac{1}{2} \text{ if and only if } y = x$$
$$\mu_{\prec}^{(\lambda)}(y,x) > \frac{1}{2} \text{ if and only if } y > x$$
$$\lim_{(y-x)\to-\infty} \mu_{\prec}^{(\lambda)}(y,x) = 0$$
$$\lim_{(y-x)\to+\infty} \mu_{\prec}^{(\lambda)}(y,x) = 1$$
$$\mu_{\prec}^{(\lambda)}(x,y) + \mu_{\prec}^{(\lambda)}(y,x) = 1. \tag{1}$$

Proof: The first five properties immediately follow from the definition for $\mu_{\prec}^{(\lambda)}$ given in Definition 1. Next, after direct calculation, we get

$$\mu_{\prec}^{(\lambda)}(x,y) + \mu_{\prec}^{(\lambda)}(y,x) = \frac{1}{1+\mathrm{e}^{-\lambda(y-x)}} + \frac{1}{1+\mathrm{e}^{-\lambda(x-y)}} = \\ = \frac{1+\mathrm{e}^{-\lambda(y-x)}+\mathrm{e}^{-\lambda(x-y)}+1}{1+\mathrm{e}^{-\lambda(y-x)}+\mathrm{e}^{-\lambda(x-y)}+1} = 1.$$

∎

Note that (1) is called the reciprocity property of $\mu^{(\lambda)}$.

Using the fuzzy preference relation $\mu_{\prec}^{(\lambda)}$, we will introduce a preference relation for two intervals as follows. Let $\mathbf{I}$ be a collection of intervals on the real line and let $I_1, I_2 \in \mathbf{I}$, $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$. Henceforth, we shall assume that $a_1 < b_1$ and $a_2 < b_2$. We will interpret the preference $I_1 \prec_I I_2$ as the average of the $\mu_{\prec}^{(\lambda)}(x,y)$ values over the rectangle R, which is given by

$$R = \{(x,y) \in \mathbb{R}^2 \colon a_1 \le x \le b_1, a_2 \le y \le b_2\}.$$

Definition 2: The preference measure $M_{I,\prec}^{(\lambda)}\colon \mathbf{I}\times\mathbf{I} \to [0,1]$ is given by

$$M_{I,\prec}^{(\lambda)}(I_1, I_2) = \\ = \frac{1}{(b_1-a_1)(b_2-a_2)} \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} \mu_{\prec}^{(\lambda)}(x,y)\mathrm{d}x \right) \mathrm{d}y, \tag{2}$$

where $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$.

The following proposition concerns the reciprocity property of the preference relation $M_{I,\prec}^{(\lambda)}$.

Proposition 2: For any $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ intervals on the real line and a uniquely determined $\lambda > 0$ parameter value,

$$M_{I,\prec}^{(\lambda)}(I_1, I_2) + M_{I,\prec}^{(\lambda)}(I_2, I_1) = 1.$$

Proof: Using the definition for the preference measure $M_{I,\prec}^{(\lambda)}$ given in Definition 2 and the reciprocity property of the preference measure $\mu_{\prec}^{(\lambda)}$ (see (1)), after direct calculation, we have

$$M_{I,\prec}^{(\lambda)}(I_1, I_2) + M_{I,\prec}^{(\lambda)}(I_2, I_1) = \\ = \frac{1}{(b_1-a_1)(b_2-a_2)} \left(\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} \mu_{\prec}^{(\lambda)}(x,y)\mathrm{d}x \right) \mathrm{d}y + \right. \\ \left. + \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \mu_{\prec}^{(\lambda)}(y,x)\mathrm{d}y \right) \mathrm{d}x \right) = \\ = \frac{1}{(b_1-a_1)(b_2-a_2)} \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} 1\mathrm{d}x \right) \mathrm{d}y = 1.$$

∎

A. Connection with the probability-based approach used to measure the preference of two intervals

The following probability-based approach used to measure the preference of two intervals is well-known.

Definition 3: The probability-based preference intensity index $M_{I,\prec}^{*}\colon \mathbf{I}\times\mathbf{I} \to [0,1]$ is given by

$$M_{I,\prec}^{*}(I_1, I_2) = \frac{\mu(A)}{\mu(\Omega)},$$

where I_1, I_2 are two intervals in the collection $\mathbf{I}$,

$$\Omega = I_1 \times I_2,$$

$$A = \{(x,y)\colon (x,y) \in I_1 \times I_2, x < y\} \subseteq \Omega,$$

and $\mu(R)$ is the area of the two-dimensional region R for any $R \subseteq \Omega$.

Here, the function value $M_{I,\prec}^{*}(I_1, I_2)$ represents the probability of $x < y$, where the values of x and y have been randomly chosen from the intervals I_1 and I_2, respectively. This notion was utilized by Huynh et al. [10] to measure the preference between two intervals and between two fuzzy numbers. Chuan Yue used the same approach to define the possibility degree of the preference of two interval-valued intuitionistic fuzzy sets [21]. Also, see the papers by Sengupta and Pal [27], Kundu [28], and Dombi and Jónás [29].

In the following proposition, we will demonstrate that the preference measure $M_{I,\prec}^{*}$ is just the limit of the preference measure $M_{I,\prec}^{(\lambda)}$.

Proposition 3: Let $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ be two intervals on the real line. Then,

$$\lim_{\lambda\to\infty} M_{I,\prec}^{(\lambda)}(I_1, I_2) = M_{I,\prec}^{*}(I_1, I_2). \tag{3}$$

Proof: Let $\Omega = I_1 \times I_2$, $A = \{(x,y)\colon (x,y) \in I_1 \times I_2, x < y\} \subseteq \Omega$ and let $\mu(A)$ and $\mu(\Omega)$ be the areas of the two-dimensional regions A and Ω, respectively. Exploiting the properties of the preference measure $\mu_{\prec}^{(\lambda)}$ (see Proposition 1), we have

$$\lim_{\lambda\to\infty} \mu_{\prec}^{(\lambda)}(x,y) = \begin{cases} 0, & \text{if } y < x \\ \frac{1}{2}, & \text{if } y = x \\ 1, & \text{if } y > x. \end{cases}$$

Therefore,

$$\lim_{\lambda\to\infty} \left(\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} \mu_{\prec}^{(\lambda)}(x,y)\mathrm{d}x \right) \mathrm{d}y \right) = \mu(A).$$

Next, by noting this result and the fact that $\mu(\Omega) = (b_1 - a_1)(b_2 - a_2)$, we have

$$\lim_{\lambda\to\infty} M_{I,\prec}^{(\lambda)}(I_1, I_2) =$$

$$= \lim_{\lambda\to\infty} \left(\frac{1}{(b_1-a_1)(b_2-a_2)} \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} \mu_{\prec}^{(\lambda)}(x,y)\mathrm{d}x \right) \mathrm{d}y \right) =$$

$$= \frac{\mu(A)}{\mu(\Omega)} = M_{I,\prec}^{*}(I_1, I_2),$$

which proves (3). ∎

B. Computing the preference measure for two intervals

Now, we will show how the preference measure $M_{I,\prec}^{(\lambda)}$ for two intervals can be computed. For a fixed $y \in [a_2, b_2]$ let $\mathcal{I}(a_1, b_1, \lambda, y)$ be given by

$$\mathcal{I}(a_1, b_1, \lambda, y) = \int_{a_1}^{b_1} \mu_{\prec}^{(\lambda)}(x,y)\mathrm{d}x = \int_{a_1}^{b_1} \frac{1}{1+\mathrm{e}^{-\lambda(y-x)}}\mathrm{d}x.$$

By applying the $u = -\lambda x$ and the $v = \mathrm{e}^{u+ay}+1$ substitutions, we get that

$$\int \frac{1}{1+\mathrm{e}^{-\lambda(y-x)}}\mathrm{d}x = x - \frac{\ln\left(\mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda y}\right)}{\lambda} + C,$$

where C is an arbitrary constant. Hence, $\mathcal{I}(a_1, b_1, \lambda, y)$ is

$$\mathcal{I}(a_1, b_1, \lambda, y) = b_1 - a_1 + \frac{1}{\lambda} \ln\left(\frac{\mathrm{e}^{\lambda a_1} + \mathrm{e}^{\lambda y}}{\mathrm{e}^{\lambda b_1} + \mathrm{e}^{\lambda y}} \right).$$

Next, using (2), after direct calculation, we have

$$M_{I,\prec}^{(\lambda)}(I_1, I_2) = 1 + \frac{\mathcal{I}(a_1, b_1, a_2, b_2, \lambda)}{(b_1-a_1)(b_2-a_2)}, \quad (4)$$

where

$$\mathcal{I}(a_1, b_1, a_2, b_2, \lambda) = \int_{a_2}^{b_2} \left(\frac{1}{\lambda} \ln\left(\frac{\mathrm{e}^{\lambda a_1} + \mathrm{e}^{\lambda y}}{\mathrm{e}^{\lambda b_1} + \mathrm{e}^{\lambda y}} \right) \right) \mathrm{d}y. \quad (5)$$

Note that the integral $\mathcal{I}(a_1, b_1, a_2, b_2, \lambda)$ in (5) has no closed form. We can approximate it quite well by using the trapezoidal rule. That is,

$$\mathcal{I}(a_1, b_1, a_2, b_2, \lambda) \approx$$

$$\approx \frac{\Delta y}{2\lambda} \sum_{i=1}^{n} \left(\ln\left(\frac{\mathrm{e}^{\lambda a_1} + \mathrm{e}^{\lambda(a_2+(i-1)\Delta y)}}{\mathrm{e}^{\lambda b_1} + \mathrm{e}^{\lambda(a_2+(i-1)\Delta y)}} \right) + \right. \quad (6)$$

$$\left. + \ln\left(\frac{\mathrm{e}^{\lambda a_1} + \mathrm{e}^{\lambda(a_2+i\Delta y))}}{\mathrm{e}^{\lambda b_1} + \mathrm{e}^{\lambda(a_2+i\Delta y)}} \right) \right),$$

where $\Delta y = \frac{b_2-a_2}{n}$ and n is sufficiently large (e.g. $n = 1000$).

III. A preference measure for two fuzzy numbers

Here, we will present a concept for measuring the preference between two fuzzy numbers by using the preference measure $M_{I,\prec}^{(\lambda)}$ for two intervals, which we introduced in the previous section. From now on, we will use the following definition for a fuzzy number.

Definition 4: The fuzzy number A is given by the membership function $\mu_A \colon \mathbb{R} \to [0,1]$,

$$\mu_A\left(x; \underline{x}_A^L, \overline{x}_A^L, \overline{x}_A^R, \underline{x}_A^R\right) = \begin{cases} 0, & \textit{if } x < \underline{x}_A^L \\ l_A(x), & \textit{if } \underline{x}_A^L \le x < \overline{x}_A^L \\ 1, & \textit{if } \overline{x}_A^L \le x < \overline{x}_A^R \\ r_A(x), & \textit{if } \overline{x}_A^R \le x < \underline{x}_A^R \\ 0, & \textit{if } \underline{x}_A^R \ge x, \end{cases} \quad (7)$$

where $\underline{x}_A^L < \overline{x}_A^L \le \overline{x}_A^R < \underline{x}_A^R$, and $l_A \colon [\underline{x}_A^L, \overline{x}_A^L) \to [0,1)$ and $r_A \colon [\overline{x}_A^R, \underline{x}_A^R) \to [0,1)$ are continuous, strictly increasing and decreasing functions with the inverse functions $l_A^{-1} \colon [0,1) \to [\underline{x}_A^L, \overline{x}_A^L)$ and $r_A^{-1} \colon [0,1) \to [\overline{x}_A^R, \underline{x}_A^R)$, respectively.
Note that in Definition 4, the functions l_A and r_A determine the left hand side and the right hand side of the membership function of fuzzy number A.

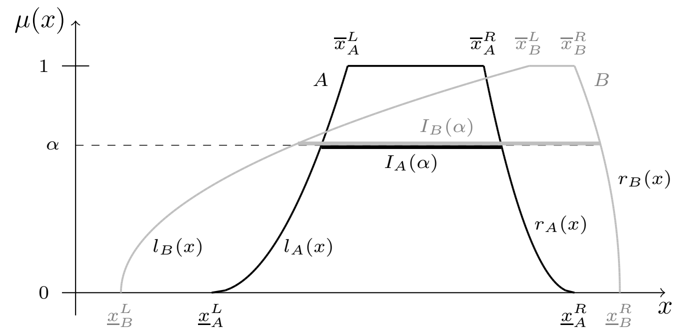

Fig. 1. Two fuzzy numbers with their α-cut intervals

Suppose that A and B are two fuzzy numbers given by Definition 4 (see Figure 1). Then, the α-cut intervals $I_A(\alpha)$ and $I_B(\alpha)$ of the membership functions of A and B, respectively, are

$$I_A(\alpha) = [a_A(\alpha), b_A(\alpha)], \quad I_B(\alpha) = [a_B(\alpha), b_B(\alpha)],$$

where

$$a_A(\alpha) = l_A^{-1}(\alpha), \quad b_A(\alpha) = r_A^{-1}(\alpha)$$
$$a_B(\alpha) = l_B^{-1}(\alpha), \quad b_B(\alpha) = r_B^{-1}(\alpha),$$

and $\alpha \in [0,1]$. Using the preference measure $M_{I,\prec}^{(\lambda)}$ for two intervals, the value of $M_{I,\prec}^{(\lambda)}(I_A(\alpha), I_B(\alpha))$ characterizes how much the fuzzy number B is preferred over the fuzzy number A at the α level. Following this line of thinking, we interpret the preference between the fuzzy numbers A and B as the average of the $M_{I,\prec}^{(\lambda)}(I_A(\alpha), I_B(\alpha))$ values, where $\alpha \in [0,1]$.

Definition 5: Let $\mathbf{F}$ be a collection of fuzzy numbers. Let $A, B \in \mathbf{F}$, and let $\lambda > 0$. The preference measure $M_{F,\prec}^{(\lambda)} \colon \mathbf{F}^2 \to (0,1)$ of the preference $A \prec B$ (i.e. $M_{F,\prec}^{(\lambda)}(A,B)$) is given by

$$M_{F,\prec}^{(\lambda)}(A,B) = \int_0^1 M_{I,\prec}^{(\lambda)}(I_A(\alpha), I_B(\alpha))\mathrm{d}\alpha,$$

where $I_A(\alpha)$ and $I_B(\alpha)$ are the α-cut intervals of the membership functions of A and B, respectively, $\alpha \in [0,1]$, and the preference measure $M_{I,\prec}^{(\lambda)}$ for two intervals is given by Definition 2.

The following proposition is about the reciprocity property of the preference measure $M_{F,\prec}^{(\lambda)}$.

Proposition 4: For any fuzzy numbers A, B and a uniquely determined $\lambda > 0$ parameter value

$$M_{F,\prec}^{(\lambda)}(A,B) + M_{F,\prec}^{(\lambda)}(B,A) = 1.$$

Proof: Using the definition for the preference measure $M_{F,\prec}^{(\lambda)}$ given in Definition 5 and the reciprocity property of the measure $M_{I,\prec}^{(\lambda)}$ (see Proposition 2) we have

$$M_{F,\prec}^{(\lambda)}(A,B) + M_{F,\prec}^{(\lambda)}(B,A) = \int_0^1 M_{I,\prec}^{(\lambda)}(I_A(\alpha), I_B(\alpha))\mathrm{d}\alpha +$$
$$+ \int_0^1 \left(1 - M_{I,\prec}^{(\lambda)}(I_A(\alpha), I_B(\alpha))\right)\mathrm{d}\alpha = 1.$$

■

A. Computing the preference measure for two fuzzy numbers

Using (4) and (5), $M_{F,\prec}^{(\lambda)}(A,B)$ can be written as

$$M_{F,\prec}^{(\lambda)}(A,B) =$$
$$= \int_0^1 \left(1 + \frac{\mathcal{I}(a_A(\alpha), b_A(\alpha), a_B(\alpha), b_B(\alpha), \lambda)}{(b_A(\alpha) - a_A(\alpha))(b_B(\alpha) - a_B(\alpha))}\right)\mathrm{d}\alpha =$$
$$= 1 + \int_0^1 \frac{\mathcal{I}(a_A(\alpha), b_A(\alpha), a_B(\alpha), b_B(\alpha), \lambda)}{(b_A(\alpha) - a_A(\alpha))(b_B(\alpha) - a_B(\alpha))}\mathrm{d}\alpha,$$

where

$$\mathcal{I}(a_A(\alpha), b_A(\alpha), a_B(\alpha), b_B(\alpha), \lambda) =$$
$$= \int_{a_B(\alpha)}^{b_B(\alpha)} \left(\frac{1}{\lambda}\ln\left(\frac{\mathrm{e}^{\lambda a_A(\alpha)} + \mathrm{e}^{\lambda y}}{\mathrm{e}^{\lambda b_A(\alpha)} + \mathrm{e}^{\lambda y}}\right)\right)\mathrm{d}y.$$

We know that $\mathcal{I}(a_A(\alpha), b_A(\alpha), a_B(\alpha), b_B(\alpha), \lambda)$ has no closed form, but using (6), it can be approximated as follows:

$$\mathcal{I}(a_A(\alpha), b_A(\alpha), a_B(\alpha), b_B(\alpha), \lambda) \approx$$
$$\approx \frac{\Delta y}{2\lambda}\sum_{i=1}^{n}\left(\ln\left(\frac{\mathrm{e}^{\lambda a_A(\alpha)} + \mathrm{e}^{\lambda(a_B(\alpha)+(i-1)\Delta y)}}{\mathrm{e}^{\lambda b_A(\alpha)} + \mathrm{e}^{\lambda(a_B(\alpha)+(i-1)\Delta y)}}\right) + \right.$$
$$\left. + \ln\left(\frac{\mathrm{e}^{\lambda a_A(\alpha)} + \mathrm{e}^{\lambda(a_B(\alpha)+i\Delta y))}}{\mathrm{e}^{\lambda b_A(\alpha)} + \mathrm{e}^{\lambda(a_B(\alpha)+i\Delta y)}}\right)\right),$$

where $\Delta y = \frac{b_B(\alpha) - a_B(\alpha)}{n}$ and n is sufficiently large (e.g. $n = 1000$). Next, using the trapezoidal rule, $M_{F,\prec}^{(\lambda)}(A,B)$ can be approximated by

$$M_{F,\prec}^{(\lambda)}(A,B) \approx 1+$$
$$+\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{\mathcal{I}\left(a_A\left(\frac{i-1}{n}\right), b_A\left(\frac{i-1}{n}\right), a_B\left(\frac{i-1}{n}\right), b_B\left(\frac{i-1}{n}\right), \lambda\right)}{\left(b_A\left(\frac{i-1}{n}\right) - a_A\left(\frac{i-1}{n}\right)\right)\left(b_B\left(\frac{i-1}{n}\right) - a_B\left(\frac{i-1}{n}\right)\right)} + \right.$$
$$\left. + \frac{\mathcal{I}\left(a_A\left(\frac{i}{n}\right), b_A\left(\frac{i}{n}\right), a_B\left(\frac{i}{n}\right), b_B\left(\frac{i}{n}\right), \lambda\right)}{\left(b_A\left(\frac{i}{n}\right) - a_A\left(\frac{i}{n}\right)\right)\left(b_B\left(\frac{i}{n}\right) - a_B\left(\frac{i}{n}\right)\right)}\right).$$

B. A demonstrative example

Here, we will show how the preference measure $M_{F,\prec}^{(\lambda)}$ can be computed for two trapezoidal fuzzy numbers. If the membership function $\mu_A \colon \mathbb{R} \to [0,1]$ of the fuzzy number A is given by (7) and

$$l_A(x) = \frac{x - \underline{x}_A^L}{\overline{x}_A^L - \underline{x}_A^L}, \qquad \textit{if } \underline{x}_A^L \le x < \overline{x}_A^L$$
$$r_A(x) = \frac{x - \underline{x}_A^R}{\overline{x}_A^R - \underline{x}_A^R}, \qquad \textit{if } \overline{x}_A^R \le x < \underline{x}_A^R,$$

then A is a trapezoidal fuzzy number.

Let A and B be two trapezoidal fuzzy numbers with the membership functions μ_A and μ_B, and let the parameters of μ_A and μ_B be $\underline{x}_A^L < \overline{x}_A^L \le \overline{x}_A^R < \underline{x}_A^R$ and $\underline{x}_B^L < \overline{x}_B^L \le \overline{x}_B^R < \underline{x}_B^R$, respectively (see Figure 2).

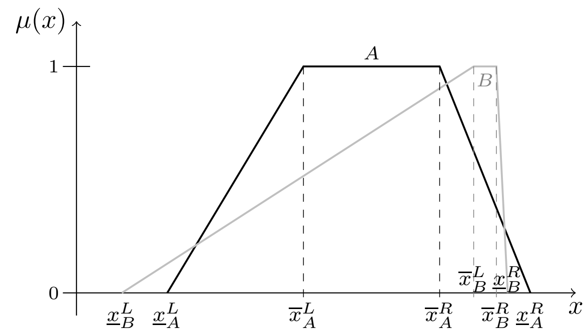

Fig. 2. The membership functions of two trapezoidal fuzzy numbers.

Let the parameter values be

$$\underline{x}_A^L = 3.00 \quad \overline{x}_A^L = 6.00 \quad \overline{x}_A^R = 9.00 \quad \underline{x}_A^R = 11.00$$
$$\underline{x}_B^L = 2.00 \quad \overline{x}_B^L = 9.75 \quad \overline{x}_B^R = 10.25 \quad \underline{x}_B^R = 10.50.$$

Here, the α-cut intervals $I_A(\alpha)$ and $I_B(\alpha)$ of the membership functions of A and B, respectively, are

$$I_A(\alpha) = [a_A(\alpha), b_A(\alpha)], \quad I_B(\alpha) = [a_B(\alpha), b_B(\alpha)],$$

where

$$a_A(\alpha) = l_A^{-1}(\alpha) = \alpha\overline{x}_A^L + (1-\alpha)\underline{x}_A^L$$
$$b_A(\alpha) = r_A^{-1}(\alpha) = \alpha\overline{x}_A^R + (1-\alpha)\underline{x}_A^R$$
$$a_B(\alpha) = l_B^{-1}(\alpha) = \alpha\overline{x}_B^L + (1-\alpha)\underline{x}_B^L$$
$$b_B(\alpha) = r_B^{-1}(\alpha) = \alpha\overline{x}_B^R + (1-\alpha)\underline{x}_B^R$$

and $\alpha \in [0,1]$. Using the expressions for the α-cut intervals and the approximation formulas presented in this section with $n = 1000$, we computed the values of the preference measure $M^{(\lambda)}_{F,\prec}(A,B)$ for certain λ values. The computation results are listed in Table I.

TABLE I
VALUES OF $M^{(\lambda)}_{F,\prec}(A,B)$ FOR VARIOUS VALUES OF λ

λ	$M^{(\lambda)}_{F,\prec}(A,B)$
1	0.6412
2	0.6715
10	0.6873
30	0.6879
50	0.6880
75	0.6880

Notice that the value of $M^{(\lambda)}_{F,\prec}(A,B)$ stabilizes as the value of λ increases. In the next section, Proposition 5 will provide an explanation for this phenomenon.

IV. RANKING FUZZY NUMBERS

In [29], we used the so-called probability-based preference intensity index for two fuzzy numbers to derive a crisp strict order relation over fuzzy numbers. The probability-based preference intensity index for two fuzzy numbers is defined as follows.

Definition 6: Let $\mathbf{F}$ be a collection of fuzzy numbers. Let $A, B \in \mathbf{F}$, and let $\lambda > 0$. The probability-based preference intensity index $M^*_{F,\prec}\colon \mathbf{F}^2 \to [0,1]$ of the preference $A \prec B$ (i.e. $M^*_{F,\prec}(A,B)$) is given by

$$M^*_{F,\prec}(A,B) = \int_0^1 M^*_{I,\prec}(I_A(\alpha), I_B(\alpha))\mathrm{d}\alpha,$$

where $I_A(\alpha)$ and $I_B(\alpha)$ are the α-cut intervals of the membership functions of A and B, respectively, $\alpha \in [0,1]$, and the probability-based preference intensity index $M^*_{I,\prec}$ for two intervals is given by Definition 3.

The following proposition states an important connection between the preference measures $M^{(\lambda)}_{F,\prec}$ and $M^*_{F,\prec}$.

Proposition 5: Let $\mathbf{F}$ be a collection of fuzzy numbers and let $A, B \in \mathbf{F}$. Then,

$$\lim_{\lambda\to\infty} M^{(\lambda)}_{F,\prec}(A,B) = M^*_{F,\prec}(A,B). \tag{8}$$

Proof: This proposition immediately follows from Proposition 3. ∎

Now, we will introduce a parametric crisp relation over a collection of fuzzy numbers and show that the limit of this relation is a strict order relation.

Definition 7: Let $\mathbf{F}$ be a collection of fuzzy numbers. The binary relation $\prec^{(\lambda,\delta)}_F$ over the collection $\mathbf{F}$ is given by

$$\prec^{(\lambda,\delta)}_F = \left\{(A,B) \in \mathbf{F}\times\mathbf{F}\colon M^{(\lambda)}_{F,\prec}(A,B) \geq \frac{1}{2} + \delta\right\},$$

where $\delta \in (0,1/2]$ and $\lambda > 0$.

Theorem 1: Let $\mathbf{F}$ be a collection of fuzzy numbers. If $\lambda \to \infty$, then there exists a $\delta \in (0,1/2]$ such that $\prec^{(\lambda,\delta)}_F$ is a strict order relation over $\mathbf{F}$.

Proof: Based on (8), we have that if $\lambda \to \infty$, then $\prec^{(\lambda,\delta)}_F = \prec^{(\delta)}_F$, where

$$\prec^{(\delta)}_F = \left\{(A,B) \in \mathbf{F}\times\mathbf{F}\colon M^*_{F,\prec}(A,B) \geq \frac{1}{2} + \delta\right\},$$

$\delta \in (0,1/2]$, and $M^*_{F,\prec}$ is the probability-based preference intensity index for two fuzzy numbers given in Definition 6. In [29, Theorem 2], we proved that there exists a $\delta \in (0,1/2]$ such that $\prec^{(\delta)}_F$ is a strict order relation over $\mathbf{F}$. Therefore, the statement of this theorem is valid. ∎

In practice, for a given finite collection $\mathbf{F}$, the smallest value of $\delta \in (0, \frac{1}{2}]$, for which relation $\prec^{(\delta)}_F$ is transitive, can be numerically determined by using searching methods such as a binary search. Now, suppose that A and B are two different elements of $\mathbf{F}$ such that $\frac{1}{2} - \delta < M^*_{F,\prec}(A,B) < \frac{1}{2} + \delta$ holds. This means that neither $A \prec^{(\delta)}_F B$ nor $B \prec^{(\delta)}_F A$ nor $A = B$ holds; that is, $\prec^{(\delta)}_F$ is not a total order. In this case, the order of A and B may be viewed as being indifferent. With the purpose of expressing the fact that the order of two fuzzy numbers is really indifferent, we introduce the following indifference relation.

Definition 8: Let $\mathbf{F}$ be a collection of fuzzy numbers. The indifference relation relation $\lesseqqgtr^{(\lambda,\delta)}_F$ over the collection $\mathbf{F}$ is given by

$$\lesseqqgtr^{(\lambda,\delta)}_F = \left\{(A,B) \in \mathbf{F}\times\mathbf{F} : \left|M^{(\lambda)}_{F,\prec}(A,B) - \frac{1}{2}\right| < \delta\right\},$$

where $\delta \in (0,1/2]$ and $\lambda > 0$.

The following proposition is about the limit of the indifference relation $\lesseqqgtr^{(\lambda,\delta)}_F$.

Proposition 6: Let $\mathbf{F}$ be a collection of fuzzy numbers. Then,

$$\lim_{\lambda\to\infty} \lesseqqgtr^{(\lambda,\delta)}_F = \lesseqqgtr^{(\delta)}_F,$$

where

$$\lesseqqgtr^{(\delta)}_F = \left\{(A,B) \in \mathbf{F}\times\mathbf{F} : \left|M^*_{F,\prec}(A,B) - \frac{1}{2}\right| < \delta\right\},$$

$\delta \in (0,1/2]$, and $M^*_{F,\prec}$ is the probability-based preference intensity index for two fuzzy numbers given in Definition 6.

Proof: This proposition immediately follows from Proposition 5. ∎

Theorem 2: Let $\mathbf{F}$ be a collection of fuzzy numbers and let the relations $\prec^{(\lambda,\delta)}_F$ and $\lesseqqgtr^{(\lambda,\delta)}_F$ over the set $\mathbf{F}$ be given by Definition 7 and Definition 8, respectively, where $\delta \in (0,1/2]$ has a fixed value and $\lambda > 0$ has a fixed value as well. If $\lambda \to \infty$, then there exists a $\delta \in (0,1/2]$ such that $\prec^{(\lambda,\delta)}_F$ is a strict order relation over $\mathbf{F}$, and for any $A, B \in \mathbf{F}$, either $A \prec^{(\lambda,\delta)}_F B$, or $B \prec^{(\lambda,\delta)}_F A$ or $A \lesseqqgtr^{(\lambda,\delta)}_F B$ holds.

Proof: The theorem immediately follows from Theorem 1, Proposition 6 and the reciprocity property of $M^{(\lambda)}_{F,\prec}$. ∎

Remark 1: An important practical consequence of Theorem 1 and Theorem 2 is that the limits ($\lambda \to \infty$) of the relations

$\prec_F^{(\lambda,\delta)}$ and $\lesseqgtr_F^{(\lambda,\delta)}$ can be used to rank fuzzy numbers. Namely, the limit of $\prec_F^{(\lambda,\delta)}$ is a strict order relation and it can be used to rank comparable fuzzy numbers, while the limit of the indifference relation $\lesseqgtr_F^{(\lambda,\delta)}$ can be used to express the fact that the order of some fuzzy numbers is indifferent.

V. Conclusions

The main findings of this study can be summarized as follows.

(a) We introduced sigmoid function-based preference measures for intervals and fuzzy numbers, and described the main properties of these preference measures.
(b) We presented formulas for the numerical computation of the proposed preference measures.
(c) We showed that the proposed preference measures for intervals and fuzzy numbers are, asymptotically, the well-known probability-based preference measures for intervals and fuzzy numbers.
(d) Using the new preference measure, we introduced two parametric crisp relations, which have common parameters, over a collection of fuzzy numbers.
(e) Then we proved that the limits of these relations can be used to rank fuzzy numbers. Here, we showed that the limit of one of these relations is a strict order relation, while the limit of the other may be viewed as an indifference relation. This latter can be used to capture the situations where the order of two fuzzy numbers cannot be judged; and so, their order may be viewed as being indifferent.

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