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Ranking trapezoidal fuzzy numbers using a parametric relation pair

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Abstract

Huynh et al. introduced a probability-based fuzzy relation for comparing fuzzy numbers (see V. Huynh, Y. Nakamori and J. Lawry (2008) [40]), but they did not detail how to compute it. Here, we will consider this fuzzy relation as a probability-based preference intensity index and present closed formulas for the integrals needed to compute this index for fuzzy sets that have trapezoidal membership functions. Also, we will propose an algorithm to compute this index and a numerical method to approximate it. The comparison of two fuzzy numbers should also be able to capture the situation where the order of the fuzzy numbers cannot be judged; and so, their order may be considered as being indifferent. Here, using the probability-based preference intensity index, we will introduce two crisp relations, which have a common parameter, over a collection of fuzzy sets that have trapezoidal membership functions. Next, we will show that - depending on the parameter value - one of them is a strict order relation and the other one may be interpreted as a relation that expresses the order indifference of fuzzy numbers. We will call this latter one the order indifference relation. Lastly, we will demonstrate how these two relations can be utilized to rank a collection of fuzzy sets that have trapezoidal membership functions.

Keywords: Fuzzy relations; Fuzzy numbers; Preference; Ranking intervals; Ranking fuzzy numbers

1. Introduction

Since the ranking of fuzzy numbers plays an important role in fuzzy decision-making, it has been in the focus of a lot of research [1–9]. There is a wide range of approaches available for tackling the problem of raking fuzzy numbers. The diversity of the methods is reflected in the literature of recent years. Wang [10] presented a method for ranking triangular and trapezoidal fuzzy numbers based on a relative preference relation. Chai et al. [11] developed an extended ranking method for fuzzy numbers, which is a synthesis of fuzzy targets and the application of Dempster–Shafer theory. Roldán López de Hierro et al. [12] demonstrated how a ranking method for fuzzy numbers can be applied to economic data. Gu and Xuan [13] proposed an approach for ranking fuzzy numbers by applying the possibility theory. Chutia and Chutia [14] introduced a value and ambiguity-based method for ranking parametric forms of

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fuzzy numbers. Khorshidi and Nikfalazar [15] presented a similarity measure for generalized fuzzy numbers and its application to fuzzy risk analysis. Yu et al. [16] proposed a new epsilon-deviation degree approach which is based on the left and right areas of a fuzzy number and the concept of a centroid point. This method was developed further by Chutia [17]. Rahmani et al. [18] introduced a method for the defuzzification and ranking of fuzzy numbers based on the statistical beta distribution. Boulmakoul et al. [19] approached the ranking of fuzzy numbers by using an inclusion index and bitset encoding. Hesamian and Bahrami [20] introduced a credibility theory oriented preference index for ranking fuzzy numbers. It should be added that there is a lot of interest in ranking intuitionistic fuzzy numbers as well (see, e.g. [21–25]).

Based on the above-mentioned researches, the fuzzy number ranking methods can be classified into two main categories. One of them contains methods that are founded on defuzzification, the other one comprises procedures that compare fuzzy numbers using preference relations. The defuzzification-based approaches are simpler and easier than those which utilize a preference relation for pair-wise comparison of fuzzy numbers. We should note that defuzzification, which is often founded on heuristics, leads to the loss of fuzzy messages, while the preference relation-based pair-wise comparisons are more complex procedures, but they preserve the fuzzy messages. Therefore, Yuan [26] supposed that a fuzzy ranking method had to present preference relation in fuzzy terms. Wang [10] also emphasized that a fuzzy preference relation, through a membership function, represents a preference degree.

It is a well-known fact that characterization of preferences among fuzzy numbers is closely related to ranking of interval-valued quantities. Therefore, there are many publications on the ranking of intervals. Interval numbers can be used to estimate experts' opinions (see, e.g. [27–30]). Also, interval-valued weights are widely applied in various decision making problems (see, e.g. [31–35]). Sengupta and Pal presented a comprehensive overview of the methods for comparing interval numbers [36]. Ordering interval numbers is an important task in multiple attribute decision making processes as well (see, e.g. [37–39]). The idea of utilizing probability measures to quantify the preference intensity between two intervals is not completely new (see, e.g. [28,22,36]).

1.1. The objective of this study

By introducing a probability-based methodology for comparison of fuzzy numbers, Huynh et al. [40] laid the foundations of a novel approach to the problem of raking fuzzy numbers. They presented a probability-based comparison relation for two intervals and two fuzzy sets. Following their results, here, we will utilize the so-called probability-based preference intensity indexes M and M_F for two intervals and two fuzzy sets, respectively. Note that these indexes are the same as the above-mentioned probability-based comparison relations for two intervals and two fuzzy sets.

It should be added that in our study, we will follow a different approach from that presented by Huynh et al. Namely, we will introduce parametric crisp relations that are based on the index M_F, study their algebraic properties and show how these crisp relations can be used to rank fuzzy numbers.

Here, we will present a novel characterization of intervals that results in simpler calculation formulas for the probability-based preference intensity index for two intervals.

Next, we will propose two methods for computing the index M_F for two fuzzy sets that have trapezoidal membership functions. Namely, we will introduce an analytical and a numerical method for computing M_F for fuzzy sets with trapezoidal membership functions. In our analytical method, we will present closed formulas for the integrals needed to compute M_F for fuzzy sets that have trapezoidal membership functions. Also, we will propose an algorithm to compute M_F using the closed formulas for the integrals.

It is worth noting that comparing two fuzzy numbers may not merely mean that one of them is preferred over the other. The comparison should also be able to capture the situation where the order of two fuzzy numbers cannot be judged; and so, the order of these two fuzzy numbers may be considered as being indifferent. Here, using the probability-based preference intensity index M_F for fuzzy sets with trapezoidal membership functions, we will introduce two crisp relations, which have a common parameter, over a collection of such fuzzy sets. Next, we will study the algebraic properties of these relations and show that one of them can be used as a strict order relation, and the other one may be interpreted as a relation that expresses the order indifference of fuzzy numbers. We will call this latter one the order indifference relation. We consider two fuzzy numbers as being comparable, when their order can be unambiguously determined. We will use our strict order relation to rank comparable fuzzy numbers, while the indifference relation is used to express that the order of some fuzzy numbers is indifferent. Lastly, in a comprehensive case study, we will demonstrate how these two relations can be utilized to rank a collection of fuzzy sets that have trapezoidal membership functions.

This paper is organized as follows. In Section 2, we will discuss about the probability-based preference intensity index for intervals and fuzzy numbers. Next, in Section 3, we will present two methods for computing the probability-based preference intensity index for two fuzzy sets with trapezoidal membership functions. In Section 4, we will propose two parametric relations over a collection of such fuzzy sets and demonstrate how these relations can be used to rank trapezoidal fuzzy numbers. Lastly, in Section 5, we will give a short summary of our findings and highlight our future research plans.

2. Probability-based preference intensity index for intervals and fuzzy numbers

Huynh et al. defined the probability-based comparison relation $P(A \preceq B)$ for the fuzzy numbers A and B as

$$P(A \preceq B) = \int_0^1 P(A_\alpha \preceq B_\alpha)\mathrm{d}\alpha, \tag{1}$$

where A_α and B_α are the α-cut intervals of A and B, respectively, and the quantity $P(A_\alpha \preceq B_\alpha)$ is the probability-based comparison relation for the intervals A_α and B_α [40]. In their paper, Huynh et al. also showed how to compute the probability-based comparison relation for two intervals [40].

Following their results, here, we will utilize the so-called probability-based preference intensity indexes M and M_F for two intervals and two fuzzy sets, respectively. Note that this index is the same as the above-mentioned probability-based comparison relation for two intervals. However, we will follow a different approach from that presented by Huynh et al. [40]. Namely, we will introduce crisp relations that are based on index M_F, study their algebraic properties and show how these crisp relations can be used to rank fuzzy numbers. Here, we will briefly summarize the most important properties of the probability-based preference intensity index for two intervals. Also, we will introduce a novel characterization of the intervals, which results in simpler computation formulas.

Let $\mathbf{I}$ be a collection of intervals on the real line and let $I_1, I_2 \in \mathbf{I}$, $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$. Henceforth, we shall assume that $a_1 < b_1$ and $a_2 < b_2$.

Definition 1. The probability-based preference intensity index $M : \mathbf{I} \times \mathbf{I} \to [0, 1]$ is given by

$$M(I_1, I_2) = \frac{\mu(A)}{\mu(\Omega)},$$

where I_1, I_2 are two intervals in the collection $\mathbf{I}$,

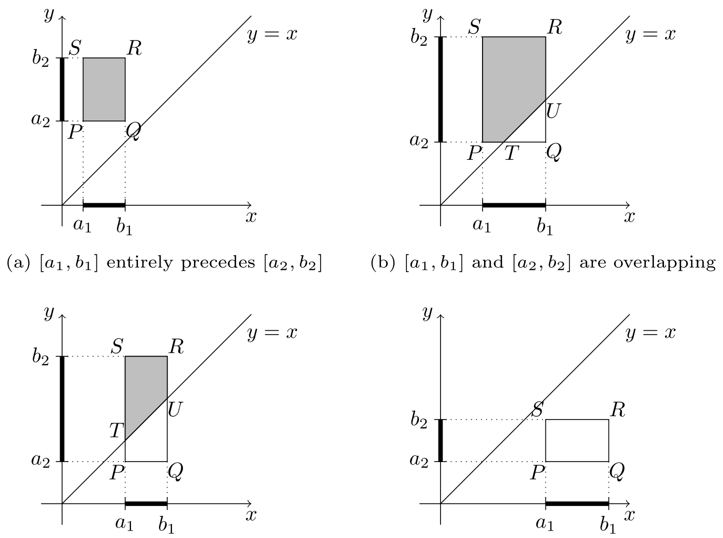
$$\Omega = I_1 \times I_2,$$
$$A = \{(x, y) : (x, y) \in I_1 \times I_2, x < y\} \subseteq \Omega,$$

and $\mu(R)$ is the area of the two-dimensional region R for any $R \subseteq \Omega$.

Here, the function value $M(I_1, I_2)$ represents the probability of $x < y$, where the values of x and y have been randomly chosen from the intervals I_1 and I_2, respectively. The quantity $M(I_1, I_2)$ may be interpreted as a measure of how much the interval I_1 precedes the interval I_2.

Remark 1. The following properties of the probability-based preference intensity index M immediately follow from its definition:

(1) $0 \le M(I_1, I_2) \le 1$ holds for any $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$.
(2) $M(I_1, I_2)$ is a continuous function of the variables a_1, b_1, a_2 and b_2.
(3) If the interval $[a_1, b_1]$ entirely precedes the interval $[a_2, b_2]$; that is, $b_1 \le a_2$, then $A = \Omega$; and so $M(I_1, I_2) = 1$.
(4) If the interval $[a_2, b_2]$ entirely precedes the interval $[a_1, b_1]$; that is, $b_2 \le a_1$, then $A = \emptyset$; and so $M(I_1, I_2) = 0$.
(5) If $I_1 = I_2$, then $M(I_1, I_2) = \frac{1}{2}$.



(a) $[a_1, b_1]$ entirely precedes $[a_2, b_2]$ (b) $[a_1, b_1]$ and $[a_2, b_2]$ are overlapping

(c) $[a_2, b_2]$ includes $[a_1, b_1]$ (d) $[a_2, b_2]$ entirely precedes $[a_1, b_1]$

Fig. 1. A geometric interpretation of the probability-based preference intensity index $M(I_1, I_2)$ for the intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$.

(6) (Reciprocity) If $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ are two intervals on the real line, then

$$M(I_1, I_2) + M(I_2, I_1) = 1.$$

Note that due to the continuity of function M, in (3), we can write $b_1 < a_2$ instead of $b_1 \leq a_2$, and in (4), we can write $b_2 < a_1$ instead of $b_2 \leq a_1$.

It should be mentioned that the definition of the probability-based preference intensity index M is based on the notion that the inequality $x_0 \leq y_0$ can be interpreted geometrically. Namely, $x_0 \leq y_0$ holds if and only if the point (x_0, y_0) is in the upper half plane defined by the line $y = x$, where $x_0, y_0, x, y \in \mathbb{R}$. Fig. 1 shows the geometric interpretation of the probability-based preference intensity index $M(I_1, I_2)$ for the intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$. And it should be added that Chuan Yue used the same approach to define the possibility degree of the preference of two interval-valued intuitionistic fuzzy sets [22].

In Fig. 1, the probability-based preference intensity index $M(I_1, I_2)$ is the ratio between the area of the gray-colored polygon and the area of the rectangle $PQRS$, where the points $P(a_1, a_2)$, $Q(b_1, a_2)$, $R(b_1, b_2)$ and $S(a_1, b_2)$ are determined by the intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$. The gray-colored polygon is that part of the rectangle $PQRS$ located in the upper half plane defined by the line $y = x$. It should be added that if the intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are overlapping, then the line $y = x$ intersects two adjacent sides of the rectangle $PQRS$ (see Fig. 1b). Also, if one of the intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ includes the other one, then the line $y = x$ intersects two opposing sides of the rectangle $PQRS$ (see Fig. 1c).

2.1. Computing the probability-based preference intensity index for intervals

In the following, we will use the quantities:

$$l_1 = \frac{b_1 - a_1}{2}, \quad l_2 = \frac{b_2 - a_2}{2} \tag{2}$$

$$d = \frac{a_2 + b_2}{2} - \frac{a_1 + b_1}{2}. \tag{3}$$

Here, $2l_1$ and $2l_2$ are the lengths (measures) of the intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$, respectively, and d is the (signed) distance between the midpoints of I_2 and I_1. The following theorem summarizes how to compute the probability-based preference intensity index for two intervals.

Theorem 1. *Let $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ be two intervals on the real line. Then, the probability-based preference intensity index $M(I_1, I_2)$ for the intervals I_1 and I_2 is*

$$M(I_1, I_2) = \begin{cases} 0, & \text{if } a_2 < b_2 \le a_1 < b_1 \text{ (case 1: precedence)} \\ 1, & \text{if } a_1 < b_1 \le a_2 < b_2 \text{ (case 2: precedence)} \\ 1 - \frac{(l_1+l_2-d)^2}{8l_1l_2}, & \text{if } a_1 < a_2 < b_1 < b_2 \text{ (case 3: overlapping)} \\ \frac{(l_1+l_2+d)^2}{8l_1l_2}, & \text{if } a_2 < a_1 < b_2 < b_1 \text{ (case 4: overlapping)} \\ \frac{1}{2} + \frac{d}{2l_2}, & \text{if } a_2 \le a_1 < b_1 \le b_2 \text{ (case 5: inclusion)} \\ \frac{1}{2} + \frac{d}{2l_1}, & \text{if } a_1 \le a_2 < b_2 \le b_1 \text{ (case 6: inclusion)}, \end{cases} \tag{4}$$

where l_1, l_2 and d are given by Eq. (2) and Eq. (3).

Proof. The theorem immediately follows from the definition of $M(I_1, I_2)$ by utilizing its geometric interpretation in Fig. 1 and the quantities in Eq. (2) and Eq. (3). □

Remark 2. It immediately follows from Theorem 1 that $M(I_1, I_2) = \frac{1}{2}$ holds if and only if $d = 0$.

Remark 3. Let suppose that $d \neq 0$. Then, Eq. (4) can be written as

$$M(I_1, I_2) = \begin{cases} 0, & \text{if } a_2 < b_2 \le a_1 < b_1 \text{ (case 1: precedence)} \\ 1, & \text{if } a_1 < b_1 \le a_2 < b_2 \text{ (case 2: precedence)} \\ 1 - \frac{(x+y-1)^2}{8xy}, & \text{if } a_1 < a_2 < b_1 < b_2 \text{ (case 3: overlapping)} \\ \frac{(x+y+1)^2}{8xy}, & \text{if } a_2 < a_1 < b_2 < b_1 \text{ (case 4: overlapping)} \\ \frac{1}{2} + \frac{1}{2y}, & \text{if } a_2 \le a_1 < b_1 \le b_2 \text{ (case 5: inclusion)} \\ \frac{1}{2} + \frac{1}{2x}, & \text{if } a_1 \le a_2 < b_2 \le b_1 \text{ (case 6: inclusion)}, \end{cases}$$

where

$$x = \frac{l_1}{d}, \quad y = \frac{l_2}{d}.$$

Noting that if $d = 0$, then either $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$ and $M(I_1, I_2) = \frac{1}{2}$, the previous results can be summarized in such a way that the probability-based preference intensity index $M(I_1, I_2)$ depends on the ratios $\frac{l_1}{d}$ and $\frac{l_2}{d}$ if $d \neq 0$, and it is $\frac{1}{2}$ if $d = 0$.

3. Probability-based preference intensity index for trapezoidal fuzzy membership functions

Now, we will show how the probability-based preference intensity index for two intervals given in Definition 1 can be extended to two trapezoidal fuzzy membership functions.

Definition 2. The trapezoidal fuzzy membership function $\mu_A : \mathbb{R} \to [0, 1]$ with the parameters $\underline{x}_A^L < \overline{x}_A^L \le \overline{x}_A^R < \underline{x}_A^R$ is given by

$$\mu_A\left(x; \underline{x}_A^L, \overline{x}_A^L, \overline{x}_A^R, \underline{x}_A^R\right) = \begin{cases} 0, & \text{if } x < \underline{x}_A^L \\ \frac{x-\underline{x}_A^L}{\overline{x}_A^L-\underline{x}_A^L}, & \text{if } \underline{x}_A^L \le x < \overline{x}_A^L \\ 1, & \text{if } \overline{x}_A^L \le x < \overline{x}_A^R \\ \frac{x-\underline{x}_A^R}{\overline{x}_A^R-\underline{x}_A^R}, & \text{if } \overline{x}_A^R \le x < \underline{x}_A^R \\ 0, & \text{if } \underline{x}_A^R \ge x. \end{cases} \tag{5}$$

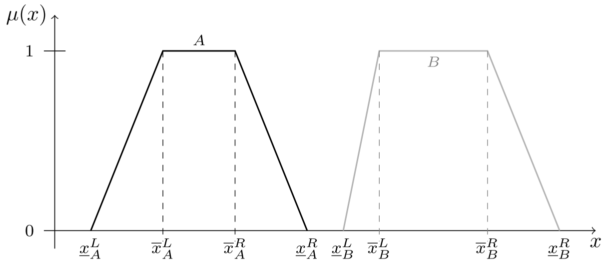


Fig. 2. Plots of trapezoidal fuzzy membership functions of the fuzzy sets A and B.

Fig. 2 shows two examples of trapezoidal fuzzy membership functions. Note that if $\overline{x}_A^L = \overline{x}_A^R$, then the membership function μ_A, which is given in Definition 2, is a triangular membership function. Here, following the proposal of Huynh et al. [40] in Eq. (1), we will define the probability-based preference intensity index for two fuzzy sets that have trapezoidal membership functions.

Definition 3. Let A and B be two fuzzy sets that have the trapezoidal membership functions μ_A and μ_B with the parameters $\underline{x}_A^L < \overline{x}_A^L \le \overline{x}_A^R < \underline{x}_A^R$ and $\underline{x}_B^L < \overline{x}_B^L \le \overline{x}_B^R < \underline{x}_B^R$, respectively. Then, the probability-based preference intensity index $M_F(A, B)$ for the fuzzy sets A and B is given by

$$M_F(A, B) = \int_0^1 M\left(I_A(\alpha), I_B(\alpha)\right) \mathrm{d}\alpha, \tag{6}$$

where $I_A(\alpha)$ and $I_B(\alpha)$ are the α-cuts of the fuzzy sets A and B, respectively.

Since the fuzzy sets A and B have trapezoidal membership functions, the α-cuts $I_A(\alpha)$ and $I_B(\alpha)$ are given by the intervals

$$I_A(\alpha) = [a_A(\alpha), b_A(\alpha)], \quad I_B(\alpha) = [a_B(\alpha), b_B(\alpha)],$$

where

$$\begin{aligned} a_A(\alpha) &= \alpha\overline{x}_A^L + (1-\alpha)\underline{x}_A^L, \quad & b_A(\alpha) &= \alpha\overline{x}_A^R + (1-\alpha)\underline{x}_A^R \\ a_B(\alpha) &= \alpha\overline{x}_B^L + (1-\alpha)\underline{x}_B^L, \quad & b_B(\alpha) &= \alpha\overline{x}_B^R + (1-\alpha)\underline{x}_B^R, \end{aligned} \tag{7}$$

$\alpha \in [0, 1]$. That is, the quantity $M_F(A, B)$ in Definition 3 exits. It also means that we interpret the probability-based preference intensity index $M_F(A, B)$ as the average of the $M\left(I_A(\alpha), I_B(\alpha)\right)$ values, where $\alpha \in [0, 1]$.

Remark 4. Let A and B be two fuzzy sets that have trapezoidal membership functions with the parameters $\underline{x}_A^L < \overline{x}_A^L \le \overline{x}_A^R < \underline{x}_A^R$ and $\underline{x}_B^L < \overline{x}_B^L \le \overline{x}_B^R < \underline{x}_B^R$, respectively. Then, the following properties of the probability-based preference intensity index $M_F(A, B)$ immediately follow from its definition:

(1) $0 \le M_F(A, B) \le 1$.
(2) $M_F(A, B)$ is a continuous function of the variables $\underline{x}_A^L, \overline{x}_A^L, \overline{x}_A^R, \underline{x}_A^R, \underline{x}_B^L, \overline{x}_B^L, \overline{x}_B^R$ and $\underline{x}_B^R$.
(3) If $\underline{x}_A^R \le \underline{x}_B^L$, then $M_F(A, B) = 1$.
(4) If $\underline{x}_B^R \le \underline{x}_A^L$, then $M_F(A, B) = 0$.
(5) If $A = B$, then $M_F(A, B) = \frac{1}{2}$.

From now on, according to Eq. (2) and Eq. (3), $l_A(\alpha)$, $l_B(\alpha)$ and $d(\alpha)$ are given by

$$l_A(\alpha) = \frac{b_A(\alpha) - a_A(\alpha)}{2}, \quad l_B(\alpha) = \frac{b_B(\alpha) - a_B(\alpha)}{2} \tag{8}$$

$$d(\alpha) = \frac{a_B(\alpha) + b_B(\alpha)}{2} - \frac{a_A(\alpha) + b_A(\alpha)}{2} \tag{9}$$

for any $\alpha \in [0, 1]$.

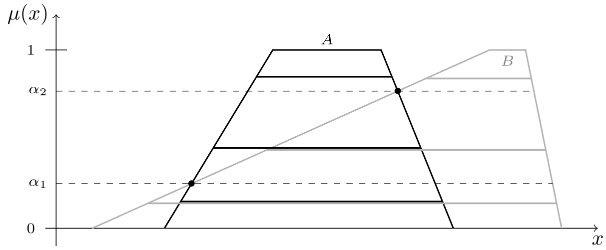


Fig. 3. Examples of α-cuts when $\alpha \in [0, \alpha_1]$, $\alpha \in (\alpha_1, \alpha_2)$ and $\alpha \in [\alpha_2, 1]$.

3.1. Reciprocity of the probability-based preference intensity index for fuzzy sets with trapezoidal membership functions

The following proposition demonstrates the reciprocity property of the probability-based preference intensity index M_F.

Proposition 1. *If A and B are two fuzzy sets that have trapezoidal membership functions, then*

$$M_F(A, B) + M_F(B, A) = 1. \tag{10}$$

Proof. Noting the definition of the function M_F in Definition 3, and the reciprocity property of the probability-based preference intensity index M (see Remark 1), we have

$$M_F(B, A) = \int_0^1 M\left(I_B(\alpha), I_A(\alpha)\right) \mathrm{d}\alpha = \int_0^1 \left(1 - M\left(I_A(\alpha), I_B(\alpha)\right)\right) \mathrm{d}\alpha =$$

$$= 1 - \int_0^1 M\left(I_A(\alpha), I_B(\alpha)\right) \mathrm{d}\alpha = 1 - M_F(A, B),$$

from which Eq. (10) follows. □

3.2. Computing the probability-based preference intensity index for fuzzy sets with trapezoidal membership functions

It should be mentioned here that, based on Theorem 1, the formula for the probability-based preference intensity index $M\left(I_A(\alpha), I_B(\alpha)\right)$ varies depending on whether the intervals $I_A(\alpha)$ and $I_B(\alpha)$ are disjoint, overlapping or including each other. For example, in Fig. 3, we can see that

- if $\alpha \in [0, \alpha_1]$, then $I_A(\alpha) \subseteq I_B(\alpha)$
- if $\alpha \in (\alpha_1, \alpha_2)$, then $I_A(\alpha)$ and $I_B(\alpha)$ are overlapping
- if $\alpha \in [\alpha_2, 1]$, then $I_A(\alpha)$ and $I_B(\alpha)$ are disjoint.

Therefore, to compute the integral in Eq. (6), we need to identify those disjoint sub-intervals of the interval [0, 1] in each of which the integrand is the same function. Since the fuzzy sets A and B have trapezoidal membership functions, intersections of the lateral sides of the trapezoids A and B determine sub-intervals in the interval [0, 1] so that in each of these sub-intervals the integrand is the same. In the example given in Fig. 3,

- if $\alpha \in [0, \alpha_1]$, then $I_A(\alpha) \subseteq I_B(\alpha)$ and the integral that we need to calculate is

$$\int_0^{\alpha_1} \left(\frac{1}{2} + \frac{d(\alpha)}{2 l_B(\alpha)}\right) \mathrm{d}\alpha$$

- if $\alpha \in (\alpha_1, \alpha_2)$, then $I_A(\alpha)$ and $I_B(\alpha)$ are overlapping so that $a_A(\alpha) < a_B(\alpha)$; that is, here we need to compute

$$\int_{\alpha_1}^{\alpha_2} \left(1 - \frac{(l_A(\alpha) + l_B(\alpha) - d(\alpha))^2}{8l_A(\alpha)l_B(\alpha)}\right) \mathrm{d}\alpha$$

- if $\alpha \in [\alpha_2, \alpha_1]$, then $I_A(\alpha)$ entirely precedes $I_B(\alpha)$, and so we need to compute the integral

$$\int_{\alpha_2}^{1} 1\mathrm{d}\alpha.$$

And so

$$M_F(A,B) = \int_0^{\alpha_1} \left(\frac{1}{2} + \frac{d(\alpha)}{2l_B(\alpha)}\right) \mathrm{d}\alpha + \\ + \int_{\alpha_1}^{\alpha_2} \left(1 - \frac{(l_A(\alpha) + l_B(\alpha) - d(\alpha))^2}{8l_A(\alpha)l_B(\alpha)}\right) \mathrm{d}\alpha + \int_{\alpha_2}^{1} 1\mathrm{d}\alpha.$$

Thus, more generally, according to Theorem 1, we will need to compute the following types of integrals:

$$\mathcal{I}^p_{a_A \geq b_B}(\alpha', \alpha'') = \int_{\alpha'}^{\alpha''} 0\mathrm{d}\alpha \tag{11}$$

$$\mathcal{I}^p_{a_B \geq b_A}(\alpha', \alpha'') = \int_{\alpha'}^{\alpha''} 1\mathrm{d}\alpha \tag{12}$$

$$\mathcal{I}^o_{a_A < a_B}(\alpha', \alpha'') = \int_{\alpha'}^{\alpha''} \left(1 - \frac{(l_A(\alpha) + l_B(\alpha) - d(\alpha))^2}{8l_A(\alpha)l_B(\alpha)}\right) \mathrm{d}\alpha \tag{13}$$

$$\mathcal{I}^o_{a_A > a_B}(\alpha', \alpha'') = \int_{\alpha'}^{\alpha''} \frac{(l_A(\alpha) + l_B(\alpha) + d(\alpha))^2}{8l_A(\alpha)l_B(\alpha)} \mathrm{d}\alpha \tag{14}$$

$$\mathcal{I}^i_{I_A \subseteq I_B}(\alpha', \alpha'') = \int_{\alpha'}^{\alpha''} \left(\frac{1}{2} + \frac{d(\alpha)}{2l_B(\alpha)}\right) \mathrm{d}\alpha \tag{15}$$

$$\mathcal{I}^i_{I_B \subseteq I_A}(\alpha', \alpha'') = \int_{\alpha'}^{\alpha''} \left(\frac{1}{2} + \frac{d(\alpha)}{2l_A(\alpha)}\right) \mathrm{d}\alpha, \tag{16}$$

where, $\alpha, \alpha', \alpha'' \in [0,1]$ and $\alpha' \leq \alpha''$. Here, the upper index of $\mathcal{I}$ indicates that for any $\alpha \in [\alpha', \alpha''] \subseteq [0,1]$, the intervals $I_A(\alpha)$ and $I_B(\alpha)$ are preceding (p), overlapping (o), or including (i) each other, and the lower index of $\mathcal{I}$ indicates the relative position of the intervals $I_A(\alpha)$ and $I_B(\alpha)$. Using these quantities and notations, the probability-based preference intensity index $M_F(A,B)$ in the example given by Fig. 3 can be written as

$$M_F(A,B) = \mathcal{I}^i_{I_A \subseteq I_B}(0, \alpha_1) + \mathcal{I}^o_{a_A < a_B}(\alpha_1, \alpha_2) + \mathcal{I}^p_{a_B \geq b_A}(\alpha_2, 1).$$

Here, we will show how to compute the integrals in equations from Eq. (11) to Eq. (16).

3.2.1. Computing $\mathcal{I}^p_{a_A \geq b_B}(\alpha', \alpha'')$ and $\mathcal{I}^p_{a_B \geq b_A}(\alpha', \alpha'')$

The integrals in Eq. (11) and Eq. (12) are trivial, namely

$$\mathcal{I}^p_{a_A \geq b_B}(\alpha', \alpha'') = 0 \tag{17}$$

$$\mathcal{I}^p_{a_B \geq b_A}(\alpha', \alpha'') = \alpha'' - \alpha'. \tag{18}$$

In order to compute the integrals in Eq. (13), Eq. (14), Eq. (15) and Eq. (16), we need to use $l_A(\alpha)$, $l_B(\alpha)$ and $d(\alpha)$. Now, by using Eq. (8), Eq. (9) and the α-cuts given in Eq. (7), $l_A(\alpha)$, $l_B(\alpha)$ and $d(\alpha)$ can be written as

$$l_A(\alpha) = p_{l_A}\alpha + q_{l_A} \tag{19}$$

$$l_B(\alpha) = p_{l_B}\alpha + q_{l_B} \tag{20}$$

$$d(\alpha) = p_d\alpha + q_d, \tag{21}$$

where

$$p_{l_A} = \frac{1}{2}\left(\left(\overline{x}_A^R - \underline{x}_A^R\right) - \left(\overline{x}_A^L - \underline{x}_A^L\right)\right) \tag{22}$$

$$q_{l_A} = \frac{1}{2}\left(\underline{x}_A^R - \underline{x}_A^L\right) \tag{23}$$

$$p_{l_B} = \frac{1}{2}\left(\left(\overline{x}_B^R - \underline{x}_B^R\right) - \left(\overline{x}_B^L - \underline{x}_B^L\right)\right) \tag{24}$$

$$q_{l_B} = \frac{1}{2}\left(\underline{x}_B^R - \underline{x}_B^L\right) \tag{25}$$

$$p_d = \frac{1}{2}\left(\left(\overline{x}_B^R - \underline{x}_B^R\right) + \left(\overline{x}_B^L - \underline{x}_B^L\right) - \left(\overline{x}_A^L - \underline{x}_A^L\right) - \left(\overline{x}_A^R - \underline{x}_A^R\right)\right) \tag{26}$$

$$q_d = \frac{1}{2}\left(\underline{x}_B^L + \underline{x}_B^R - \underline{x}_A^L - \underline{x}_A^R\right). \tag{27}$$

3.2.2. Computing $\mathcal{I}^o_{a_A < a_B}(\alpha', \alpha'')$

By using Eq. (19), Eq. (20), Eq. (21), the definitions of p_{l_A}, q_{l_A}, p_{l_B}, q_{l_B}, p_d and q_d, the integral $\mathcal{I}^o_{a_A < a_B}(\alpha', \alpha'')$ in Eq. (13) can be written as

$$\begin{aligned}\mathcal{I}^o_{a_A < a_B}(\alpha', \alpha'') &= \int_{\alpha'}^{\alpha''} \left(1 - \frac{(l_A(\alpha) + l_B(\alpha) - d(\alpha))^2}{8 l_A(\alpha) l_B(\alpha)}\right) \mathrm{d}\alpha = \\ &= \alpha'' - \alpha' - \int_{\alpha'}^{\alpha''} \frac{\left(\left(p_{l_A} + p_{l_B} - p_d\right)\alpha + \left(q_{l_A} + q_{l_B} - q_d\right)\right)^2}{8\left(p_{l_A}\alpha + q_{l_A}\right)\left(p_{l_B}\alpha + q_{l_B}\right)} \mathrm{d}\alpha.\end{aligned} \tag{28}$$

Now, consider the integral

$$\int \frac{(A_0 x + B_0)^2}{(A_1 x + B_1)(A_2 x + B_2)} \mathrm{d}x. \tag{29}$$

By performing polynomial long division and applying partial fraction decomposition, we get that the integral in Eq. (29) is

$$\begin{aligned}&\int \frac{(A_0 x + B_0)^2}{(A_1 x + B_1)(A_2 x + B_2)} \mathrm{d}x = \\ &= \frac{(A_0 B_1 - A_1 B_0)^2 \ln(|A_1 x + B_1|)}{A_1^2 (A_1 B_2 - A_2 B_1)} - \frac{(A_0 B_2 - A_2 B_0)^2 \ln(|A_2 x + B_2|)}{A_2^2 (A_1 B_2 - A_2 B_1)} + \\ &+ \frac{A_0^2 x}{A_1 A_2} + C,\end{aligned} \tag{30}$$

where C is an arbitrary constant. Next, using Eq. (30), after direct calculation, Eq. (28) can be written as

$$\begin{aligned}
&\mathcal{I}^o_{a_A<a_B}(\alpha',\alpha'') = \left(1-\frac{A_0^2}{8A_1A_2}\right)(\alpha''-\alpha')+\\
&+\frac{(A_0B_2-A_2B_0)^2}{8A_2^2(A_1B_2-A_2B_1)}\ln\left|\frac{A_2\alpha''+B_2}{A_2\alpha'+B_2}\right|-\\
&-\frac{(A_0B_1-A_1B_0)^2}{8A_1^2(A_1B_2-A_2B_1)}\ln\left|\frac{A_1\alpha''+B_1}{A_1\alpha'+B_1}\right|,
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
&A_0 = p_{l_A}+p_{l_B}-p_d, \quad B_0 = q_{l_A}+q_{l_B}-q_d\\
&A_1 = p_{l_A}, \quad B_1 = q_{l_A}, \quad A_2 = p_{l_B}, \quad B_2 = q_{l_B}.
\end{aligned}$$

Now, by noting the equations from Eq. (22) to Eq. (27), after direct calculation, we get

$$\begin{aligned}
&A_0 = \left(\overline{x}_A^R-\underline{x}_A^R\right)-\left(\overline{x}_B^L-\underline{x}_B^L\right), \quad B_0 = \underline{x}_A^R-\underline{x}_B^L\\
&A_1 = \frac{1}{2}\left(\left(\overline{x}_A^R-\underline{x}_A^R\right)-\left(\overline{x}_A^L-\underline{x}_A^L\right)\right), \quad B_1 = \frac{1}{2}\left(\underline{x}_A^R-\underline{x}_A^L\right)\\
&A_2 = \frac{1}{2}\left(\left(\overline{x}_B^R-\underline{x}_B^R\right)-\left(\overline{x}_B^L-\underline{x}_B^L\right)\right), \quad B_2 = \frac{1}{2}\left(\underline{x}_B^R-\underline{x}_B^L\right).
\end{aligned} \tag{32}$$

3.2.3. Computing $\mathcal{I}^o_{a_A>a_B}(\alpha',\alpha'')$

By noting Eq. (19), Eq. (20), Eq. (21), the definitions of p_{l_A}, q_{l_A}, p_{l_B}, q_{l_B}, p_d and q_d, the integral $\mathcal{I}^o_{a_A>a_B}(\alpha',\alpha'')$ in Eq. (14) can be written as

$$\begin{aligned}
&\mathcal{I}^o_{a_A>a_B}(\alpha',\alpha'') = \int_{\alpha'}^{\alpha''}\frac{(l_A(\alpha)+l_B(\alpha)+d(\alpha))^2}{8l_A(\alpha)l_B(\alpha)}\mathrm{d}\alpha =\\
&=\int_{\alpha'}^{\alpha''}\frac{\left(\left(p_{l_A}+p_{l_B}+p_d\right)\alpha+\left(q_{l_A}+q_{l_B}+q_d\right)\right)^2}{8\left(p_{l_A}\alpha+q_{l_A}\right)\left(p_{l_B}\alpha+q_{l_B}\right)}\mathrm{d}\alpha.
\end{aligned} \tag{33}$$

Next, by taking into account (30), Eq. (33) can be written as

$$\begin{aligned}
&\mathcal{I}^o_{a_A>a_B}(\alpha',\alpha'') = \frac{A_0^2}{8A_1A_2}(\alpha''-\alpha')+\\
&+\frac{(A_0B_1-A_1B_0)^2}{8A_1^2(A_1B_2-A_2B_1)}\ln\left|\frac{A_1\alpha''+B_1}{A_1\alpha'+B_1}\right|-\\
&-\frac{(A_0B_2-A_2B_0)^2}{8A_2^2(A_1B_2-A_2B_1)}\ln\left|\frac{A_2\alpha''+B_2}{A_2\alpha'+B_2}\right|,
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
&A_0 = p_{l_A}+p_{l_B}+p_d, \quad B_0 = q_{l_A}+q_{l_B}+q_d\\
&A_1 = p_{l_A}, \quad B_1 = q_{l_A}, \quad A_2 = p_{l_B}, \quad B_2 = q_{l_B}.
\end{aligned}$$

Now, by using the equations from Eq. (22) to Eq. (27), after direct calculation, we get

$$\begin{aligned}
&A_0 = \left(\overline{x}_B^R-\underline{x}_B^R\right)-\left(\overline{x}_A^L-\underline{x}_A^L\right), \quad B_0 = \underline{x}_B^R-\underline{x}_A^L\\
&A_1 = \frac{1}{2}\left(\left(\overline{x}_A^R-\underline{x}_A^R\right)-\left(\overline{x}_A^L-\underline{x}_A^L\right)\right), \quad B_1 = \frac{1}{2}\left(\underline{x}_A^R-\underline{x}_A^L\right)\\
&A_2 = \frac{1}{2}\left(\left(\overline{x}_B^R-\underline{x}_B^R\right)-\left(\overline{x}_B^L-\underline{x}_B^L\right)\right), \quad B_2 = \frac{1}{2}\left(\underline{x}_B^R-\underline{x}_B^L\right).
\end{aligned} \tag{35}$$

3.2.4. Computing $\mathcal{I}^{i}_{I_A \subseteq I_B}(\alpha', \alpha'')$

By using Eq. (20), Eq. (21), the definitions of p_{l_B}, q_{l_B}, p_d and q_d, the integral $\mathcal{I}^{i}_{I_A \subseteq I_B}(\alpha', \alpha'')$ in Eq. (15) can be written as

$$\begin{aligned} \mathcal{I}^{i}_{I_A \subseteq I_B}(\alpha', \alpha'') &= \int_{\alpha'}^{\alpha''} \left(\frac{1}{2} + \frac{d(\alpha)}{2 l_B(\alpha)} \right) \mathrm{d}\alpha = \\ &= \frac{1}{2}\left(\alpha'' - \alpha'\right) + \frac{1}{2} \int_{\alpha'}^{\alpha''} \frac{p_d \alpha + q_d}{p_{l_B} \alpha + q_{l_B}} \mathrm{d}\alpha. \end{aligned} \tag{36}$$

Now, consider the integral

$$\int \frac{A_0 x + B_0}{A_1 x + B_1} \mathrm{d}x. \tag{37}$$

It can be shown that the integral in Eq. (37) is

$$\begin{aligned} &\int \frac{A_0 x + B_0}{A_1 x + B_1} \mathrm{d}x = \\ &= \frac{A_0}{A_1} x - \frac{(A_0 B_1 - A_1 B_0) \ln\left(|A_1 x + B_1|\right)}{A_1^2} + C, \end{aligned} \tag{38}$$

where C is an arbitrary constant. Next, using Eq. (38), after direct calculation, Eq. (36) can be written as

$$\begin{aligned} &\mathcal{I}^{i}_{I_A \subseteq I_B}(\alpha', \alpha'') = \\ &= \frac{1}{2}\left(\alpha'' - \alpha'\right)\left(1 + \frac{A_0}{A_1}\right) - \frac{(A_0 B_1 - A_1 B_0)}{2 A_1^2} \ln \left| \frac{A_1 \alpha'' + B_1}{A_1 \alpha' + B_1} \right|, \end{aligned} \tag{39}$$

where

$$A_0 = p_d, \quad B_0 = q_d \quad A_1 = p_{l_B}, \quad B_1 = q_{l_B}.$$

Now, by using Eq. (24), Eq. (25), Eq. (26) and Eq. (27), we have

$$\begin{aligned} A_0 &= \frac{1}{2}\left(\left(\overline{x}_B^R - \underline{x}_B^R\right) + \left(\overline{x}_B^L - \underline{x}_B^L\right) - \left(\overline{x}_A^L - \underline{x}_A^L\right) - \left(\overline{x}_A^R - \underline{x}_A^R\right) \right) \\ B_0 &= \frac{1}{2}\left(\underline{x}_B^L + \underline{x}_B^R - \underline{x}_A^L - \underline{x}_A^R \right) \\ A_1 &= \frac{1}{2}\left(\left(\overline{x}_B^R - \underline{x}_B^R\right) - \left(\overline{x}_B^L - \underline{x}_B^L\right) \right), \quad B_1 = \frac{1}{2}\left(\underline{x}_B^R - \underline{x}_B^L \right). \end{aligned} \tag{40}$$

3.2.5. Computing $\mathcal{I}^{i}_{I_B \subseteq I_A}(\alpha', \alpha'')$

Here, we seek to compute the following integral:

$$\mathcal{I}^{i}_{I_B \subseteq I_A}(\alpha', \alpha'') = \int_{\alpha'}^{\alpha''} \left(\frac{1}{2} + \frac{d(\alpha)}{2 l_A(\alpha)} \right) \mathrm{d}\alpha. \tag{41}$$

Notice that this integral has the same form as the integral $\mathcal{I}^{i}_{I_A \subseteq I_B}(\alpha', \alpha'')$ in Eq. (36). Therefore, by noting Eq. (36) and Eq. (42), we get

$$\begin{aligned} &\mathcal{I}^{i}_{I_B \subseteq I_A}(\alpha', \alpha'') = \\ &= \frac{1}{2}\left(\alpha'' - \alpha'\right)\left(1 + \frac{A_0}{A_1}\right) - \frac{(A_0 B_1 - A_1 B_0)}{2 A_1^2} \ln \left| \frac{A_1 \alpha'' + B_1}{A_1 \alpha' + B_1} \right|, \end{aligned} \tag{42}$$

where

$$A_0 = p_d, \quad B_0 = q_d \quad A_1 = p_{l_A}, \quad B_1 = q_{l_A}.$$

Next, by using Eq. (22), Eq. (23), Eq. (26) and Eq. (27), we have

$$\begin{aligned}
A_0 &= \frac{1}{2}\left(\left(\overline{x}_B^R - \underline{x}_B^R\right) + \left(\overline{x}_B^L - \underline{x}_B^L\right) - \left(\overline{x}_A^L - \underline{x}_A^L\right) - \left(\overline{x}_A^R - \underline{x}_A^R\right)\right) \\
B_0 &= \frac{1}{2}\left(\underline{x}_B^L + \underline{x}_B^R - \underline{x}_A^L - \underline{x}_A^R\right) \\
A_1 &= \frac{1}{2}\left(\left(\overline{x}_A^R - \underline{x}_A^R\right) - \left(\overline{x}_A^L - \underline{x}_A^L\right)\right), \quad B_1 = \frac{1}{2}\left(\underline{x}_A^R - \underline{x}_A^L\right).
\end{aligned} \tag{43}$$

(Note that the discontinuities of $\mathcal{I}^o_{a_A<a_B}(\alpha', \alpha'')$, $\mathcal{I}^o_{a_A>a_B}(\alpha', \alpha'')$, $\mathcal{I}^i_{I_A\subseteq I_B}(\alpha', \alpha'')$ and $\mathcal{I}^i_{I_B\subseteq I_A}(\alpha', \alpha'')$ are all removable.)

3.3. Computation methods

Now we will introduce two methods that can be utilized in practice to compute the probability-based preference intensity index $M_F(A, B)$ for the fuzzy sets A and B when they have trapezoidal membership functions. First, we will introduce an analytical method, and then we will discuss a numerical one.

3.3.1. Analytical approach

Let A and B be two fuzzy sets that have the trapezoidal membership functions μ_A and μ_B with the parameters $\underline{x}_A^L < \overline{x}_A^L \le \overline{x}_A^R < \underline{x}_A^R$ and $\underline{x}_B^L < \overline{x}_B^L \le \overline{x}_B^R < \underline{x}_B^R$, respectively, as defined in Definition 2. Then, the left hand side line $S_A^L(x)$ and the right hand side line $S_A^R(x)$ of the trapezoid A are given by the equations

$$S_A^L(x) = \frac{x - \underline{x}_A^L}{\overline{x}_A^L - \underline{x}_A^L}, \quad S_A^R(x) = \frac{x - \underline{x}_A^R}{\overline{x}_A^R - \underline{x}_A^R}.$$

Similarly, the left hand side line $S_B^L(x)$ and the right hand side line $S_B^R(x)$ of the trapezoid B are given by the equations

$$S_B^L(x) = \frac{x - \underline{x}_B^L}{\overline{x}_B^L - \underline{x}_B^L}, \quad S_B^R(x) = \frac{x - \underline{x}_B^R}{\overline{x}_B^R - \underline{x}_B^R}.$$

The intersection points of the lateral sides of the trapezoids A and B can be derived by solving the equations

$$S_A^L(x) = S_B^L(x), \quad S_A^L(x) = S_B^R(x), \quad S_A^R(x) = S_B^L(x), \quad S_A^R(x) = S_B^R(x). \tag{44}$$

Notice that the equations in Eq. (44) all have the following form:

$$\frac{x-a}{b-a} = \frac{x-c}{d-c}. \tag{45}$$

That is,

- if $a = \underline{x}_A^L$, $b = \overline{x}_A^L$, $c = \underline{x}_B^L$ and $d = \overline{x}_B^L$, then Eq. (45) is $S_A^L(x) = S_B^L(x)$
- if $a = \underline{x}_A^L$, $b = \overline{x}_A^L$, $c = \underline{x}_B^R$ and $d = \overline{x}_B^R$, then Eq. (45) is $S_A^L(x) = S_B^R(x)$
- if $a = \underline{x}_A^R$, $b = \overline{x}_A^R$, $c = \underline{x}_B^L$ and $d = \overline{x}_B^L$, then Eq. (45) is $S_A^R(x) = S_B^L(x)$
- if $a = \underline{x}_A^R$, $b = \overline{x}_A^R$, $c = \underline{x}_B^R$ and $d = \overline{x}_B^R$, then Eq. (45) is $S_A^R(x) = S_B^R(x)$.

(Note that since $\underline{x}_A^L < \overline{x}_A^L \le \overline{x}_A^R < \underline{x}_A^R$ and $\underline{x}_B^L < \overline{x}_B^L \le \overline{x}_B^R < \underline{x}_B^R$ hold, $b \ne a$ and $d \ne c$.) Obviously, if the two sides of Eq. (45) represent two parallel lines; that is, when $b - a = d - c$, then Eq. (45) has no solution. Otherwise, the solution of Eq. (45) is

$$x_\alpha = \frac{ad - bc}{a - b + d - c}$$

and the vertical coordinate of the intersection point corresponding to x_α is

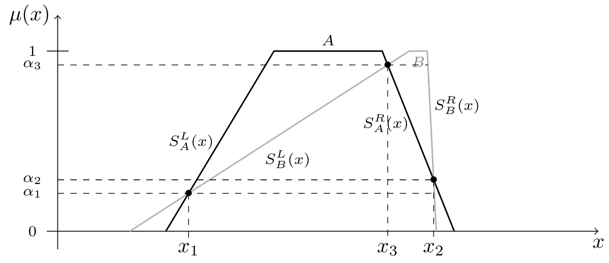


Fig. 4. Two trapezoidal membership functions with three intersections.

$$\alpha = \frac{a-c}{a-b+d-c}.$$

From the intersection points obtained from the equations in Eq. (44), we need only those whose vertical coordinates are in the interval (0, 1). Notice that there can be at most three intersections with vertical coordinates in the interval (0, 1) obtained from the equations in Eq. (44). For example, if the equation $S_A^L(x) = S_B^L(x)$ has such a solution and the equation $S_A^R(x) = S_B^L(x)$ also has such a solution, then the equation $S_A^L(x) = S_B^R(x)$ cannot not have any solution for which $S_A^L(x) \in (0,1)$ (and also $S_B^R(x) \in (0,1)$). An example for this is shown in Fig. 4. Therefore, the lateral sides of the trapezoids A and B can have zero, one, two or three intersections with vertical coordinates in the interval (0, 1). It should also be added that we are interested only in those intersection points that have different vertical coordinates. This is because if the vertical coordinates of two intersection points are the same, then these do not represent any case for computing the integral $M_F(A,B)$. For example, suppose that the intersections (x_1,α_1) and (x_2,α_2) in Fig. 4 had the same vertical coordinates; that is, $\alpha_1 = \alpha_2$. Then, $M_F(A,B)$ would be computed as

$$M_F(A,B) = \mathcal{I}^o_{a_A > a_B}(0,\alpha_1) + \mathcal{I}^o_{a_A < a_B}(\alpha_1,\alpha_3) + \mathcal{I}^p_{a_B \ge b_A}(\alpha_3, 1).$$

Note that if $\alpha_1 < \alpha_2$, as it is originally in Fig. 4, then $M(A,B)$ can be computed as

$$M_F(A,B) = \mathcal{I}^o_{a_A > a_B}(0,\alpha_1) + \mathcal{I}^i_{I_B \subseteq I_A}(\alpha_1,\alpha_2) + \\ + \mathcal{I}^o_{a_A < a_B}(\alpha_2,\alpha_3) + \mathcal{I}^p_{a_B \ge b_A}(\alpha_3, 1).$$

Based on the considerations and results above, a method for computing the probability-based preference intensity index $M_F(A,B)$ for the fuzzy sets A and B when they have trapezoidal membership functions can be summarized as follows.

Method 1. Computing the probability-based preference intensity index $M_F(A,B)$ for the fuzzy sets A and B when they have trapezoidal membership functions.

INPUTS: parameters $\underline{x}_A^L < \overline{x}_A^L \le \overline{x}_A^R < \underline{x}_A^R$ and $\underline{x}_B^L < \overline{x}_B^L \le \overline{x}_B^R < \underline{x}_B^R$ of the trapezoidal membership functions of the fuzzy sets A and B, respectively.

Step 1: $\mathcal{S} = \emptyset$

Step 2:

- If $\overline{x}_A^L - \underline{x}_A^L \ne \overline{x}_B^L - \underline{x}_B^L$, then

$$\alpha_1 = \frac{\underline{x}_A^L - \underline{x}_B^L}{\underline{x}_A^L - \overline{x}_A^L + \overline{x}_B^L - \underline{x}_B^L};$$

- If $\overline{x}_A^L - \underline{x}_A^L \ne \overline{x}_B^R - \underline{x}_B^R$, then

$$\alpha_2 = \frac{\underline{x}_A^L - \underline{x}_B^R}{\underline{x}_A^L - \overline{x}_A^L + \overline{x}_B^R - \underline{x}_B^R};$$

- If $\overline{x}_A^R - \underline{x}_A^R \neq \overline{x}_B^L - \underline{x}_B^L$, then

$$\alpha_3 = \frac{\underline{x}_A^R - \underline{x}_B^L}{\underline{x}_A^R - \overline{x}_A^R + \overline{x}_B^L - \underline{x}_B^L};$$

- If $\overline{x}_A^R - \underline{x}_A^R \neq \overline{x}_B^R - \underline{x}_B^R$, then

$$\alpha_4 = \frac{\underline{x}_A^R - \underline{x}_B^R}{\underline{x}_A^R - \overline{x}_A^R + \overline{x}_B^R - \underline{x}_B^R};$$

Step 3:
For all $i \in \{1, 2, 3, 4\}$
if $\alpha_i \in (0, 1)$ and $\alpha_i \notin \mathcal{S}$, then $\mathcal{S} := \mathcal{S} \cup \{\alpha_i\}$;
End for

Step 4:
$n := |\mathcal{S}|$;
Let the ordered set $\{\alpha'_1, \ldots, \alpha'_n\}$ contain all the elements of $\mathcal{S}$ set in increasing order;
$\alpha'_0 := 0, \alpha'_{n+1} = 1$;

Step 5:
$M_F(A, B) := 0$.
For all $i \in \{0, \ldots, n\}$

$$\alpha_i^* := \frac{\alpha'_i + \alpha'_{i+1}}{2}$$

$$a_A(\alpha_i^*) := \alpha_i^* \overline{x}_A^L + (1 - \alpha_i^*)\underline{x}_A^L, \quad b_A(\alpha_i^*) = \alpha_i^* \overline{x}_A^R + (1 - \alpha_i^*)\underline{x}_A^R$$

$$a_B(\alpha_i^*) := \alpha_i^* \overline{x}_B^L + (1 - \alpha_i^*)\underline{x}_B^L, \quad b_B(\alpha_i^*) = \alpha_i^* \overline{x}_B^R + (1 - \alpha_i^*)\underline{x}_B^R$$

$$M_i := \begin{cases} \mathcal{I}^p_{a_A \geq b_B}(\alpha'_i, \alpha'_{i+1}), & \text{if } a_B(\alpha_i^*) < b_B(\alpha_i^*) \leq a_A(\alpha_i^*) < b_A(\alpha_i^*) \\ \mathcal{I}^p_{a_B \geq b_A}(\alpha'_i, \alpha'_{i+1}), & \text{if } a_A(\alpha_i^*) < b_A(\alpha_i^*) \leq a_B(\alpha_i^*) < b_B(\alpha_i^*) \\ \mathcal{I}^o_{a_A < a_B}(\alpha'_i, \alpha'_{i+1}), & \text{if } a_A(\alpha_i^*) < a_B(\alpha_i^*) < b_A(\alpha_i^*) < b_B(\alpha_i^*) \\ \mathcal{I}^o_{a_A > a_B}(\alpha'_i, \alpha'_{i+1}), & \text{if } a_B(\alpha_i^*) < a_A(\alpha_i^*) < b_B(\alpha_i^*) < b_A(\alpha_i^*) \\ \mathcal{I}^i_{I_A \subseteq I_B}(\alpha'_i, \alpha'_{i+1}), & \text{if } a_B(\alpha_i^*) \leq a_A(\alpha_i^*) < b_A(\alpha_i^*) \leq b_B(\alpha_i^*) \\ \mathcal{I}^i_{I_B \subseteq I_A}(\alpha'_i, \alpha'_{i+1}), & \text{if } a_A(\alpha_i^*) \leq a_B(\alpha_i^*) < b_B(\alpha_i^*) \leq b_A(\alpha_i^*) \end{cases}$$

$$M_F(A, B) := M_F(A, B) + M_i$$

End for

OUTPUT: $M_F(A, B)$.

The formulas for the integrals needed to calculate M_i; that is, $\mathcal{I}^p_{a_A \geq b_B}(\alpha'_i, \alpha'_{i+1})$, $\mathcal{I}^p_{a_B \geq b_A}(\alpha'_i, \alpha'_{i+1})$, $\mathcal{I}^o_{a_A < a_B}(\alpha'_i, \alpha'_{i+1})$, $\mathcal{I}^o_{a_A > a_B}(\alpha'_i, \alpha'_{i+1})$, $\mathcal{I}^i_{I_A \subseteq I_B}(\alpha'_i, \alpha'_{i+1})$ and $\mathcal{I}^i_{I_B \subseteq I_A}(\alpha'_i, \alpha'_{i+1})$ were computed in Section 3.2.

3.3.2. Numerical approximation

Here again, let A and B be two fuzzy sets that have the trapezoidal membership functions μ_A and μ_B with the parameters $\underline{x}_A^L < \overline{x}_A^L \leq \overline{x}_A^R < \underline{x}_A^R$ and $\underline{x}_B^L < \overline{x}_B^L \leq \overline{x}_B^R < \underline{x}_B^R$, respectively, as defined in Definition 2. Now, we will introduce a numerical approach to approximate the probability-based preference intensity index $M_F(A, B)$. Let $n > 0$ be a fixed integer and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n equidistant points in the interval $[0, 1]$ such that $\alpha_i = \frac{i}{n}$ for

$i = 1, 2, \ldots, n$. Recall that since the fuzzy sets A and B have trapezoidal membership functions, the α_i-cuts of A and B are the intervals $I_A(\alpha_i)$ and $I_B(\alpha_i)$, respectively, where

$$I_A(\alpha_i) = [a_A(\alpha_i), b_A(\alpha_i)], \quad I_B(\alpha_i) = [a_B(\alpha_i), b_B(\alpha_i)],$$

and

$$a_A(\alpha_i) = \alpha_i \overline{x}_A^L + (1-\alpha_i)\underline{x}_A^L, \quad b_A(\alpha_i) = \alpha \overline{x}_A^R + (1-\alpha_i)\underline{x}_A^R$$
$$a_B(\alpha_i) = \alpha_i \overline{x}_B^L + (1-\alpha_i)\underline{x}_B^L, \quad b_B(\alpha_i) = \alpha \overline{x}_B^R + (1-\alpha_i)\underline{x}_B^R.$$

Next, if n is sufficiently large, for example $n = 1000$, then the integral

$$M_F(A, B) = \int_0^1 M\left(I_A(\alpha), I_B(\alpha)\right) \mathrm{d}\alpha$$

can be approximated quite well by the following average:

$$\frac{1}{n} \sum_{i=1}^{n} M\left(I_A(\alpha_i), I_A(\alpha_i)\right),$$

where

$$M\left(I_A(\alpha_i), I_A(\alpha_i)\right) =$$
$$= \begin{cases} 0, & \text{if } a_B(\alpha_i) < b_B(\alpha_i) \le a_A(\alpha_i) < b_A(\alpha_i) \\ 1, & \text{if } a_A(\alpha_i) < b_A(\alpha_i) \le a_B(\alpha_i) < b_B(\alpha_i) \\ 1 - \frac{(l_A(\alpha_i)+l_B(\alpha_i)-d(\alpha_i))^2}{8 l_A(\alpha_i) l_B(\alpha_i)}, & \text{if } a_A(\alpha_i) < a_B(\alpha_i) < b_A(\alpha_i) < b_B(\alpha_i) \\ \frac{(l_A(\alpha_i)+l_B(\alpha_i)+d(\alpha_i))^2}{8 l_A(\alpha_i) l_B(\alpha_i)}, & \text{if } a_B(\alpha_i) < a_A(\alpha_i) < b_B(\alpha_i) < b_A(\alpha_i) \\ \frac{1}{2} + \frac{d(\alpha_i)}{2 l_B(\alpha_i)}, & \text{if } a_B(\alpha_i) \le a_A(\alpha_i) < b_A(\alpha_i) \le b_B(\alpha_i) \\ \frac{1}{2} + \frac{d(\alpha_i)}{2 l_A(\alpha_i)}, & \text{if } a_A(\alpha_i) \le a_B(\alpha_i) < b_B(\alpha_i) \le b_A(\alpha_i) \end{cases}$$

and

$$l_A(\alpha_i) = \frac{b_A(\alpha_i) - a_A(\alpha_i)}{2}, \quad l_B(\alpha_i) = \frac{b_B(\alpha_i) - a_B(\alpha_i)}{2}$$
$$d(\alpha_i) = \frac{a_B(\alpha_i) + b_B(\alpha_i)}{2} - \frac{a_A(\alpha_i) + b_A(\alpha_i)}{2}$$

for any $\alpha_i \in [0, 1]$. Therefore, a method for approximating the probability-based preference intensity index $M_F(A, B)$ for the fuzzy sets A and B when they have trapezoidal membership functions can be summarized as follows.

Method 2. Computing the probability-based preference intensity index $M(A, B)$ for the fuzzy sets A and B when they have trapezoidal membership functions.

<u>INPUTS:</u> parameters $\underline{x}_A^L < \overline{x}_A^L \le \overline{x}_A^R < \underline{x}_A^R$ and $\underline{x}_B^L < \overline{x}_B^L \le \overline{x}_B^R < \underline{x}_B^R$ of the trapezoidal membership functions of the fuzzy sets A and B, respectively.

$S = 0$;
$n = 1000$;
For all $i \in \{1, \ldots, n\}$

$$\alpha_i := \frac{i}{n}$$
$$a_A(\alpha_i) := \alpha_i \overline{x}_A^L + (1-\alpha_i)\underline{x}_A^L;$$
$$b_A(\alpha_i) := \alpha \overline{x}_A^R + (1-\alpha_i)\underline{x}_A^R;$$

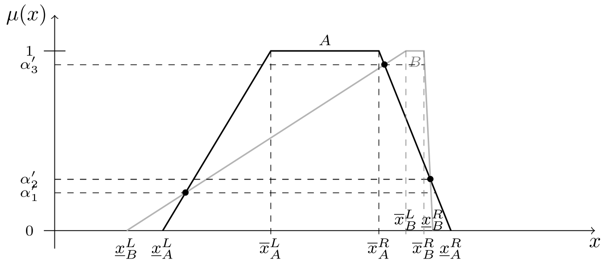


Fig. 5. The trapezoidal membership functions of the fuzzy sets A and B.

$$a_B(\alpha_i) := \alpha_i \overline{x}_B^L + (1-\alpha_i)\underline{x}_B^L;$$
$$b_B(\alpha_i) := \alpha \overline{x}_B^R + (1-\alpha_i)\underline{x}_B^R;$$
$$l_A(\alpha_i) := \frac{b_A(\alpha_i) - a_A(\alpha_i)}{2};$$
$$l_B(\alpha_i) := \frac{b_B(\alpha_i) - a_B(\alpha_i)}{2};$$
$$d(\alpha_i) := \frac{a_B(\alpha_i) + b_B(\alpha_i)}{2} - \frac{a_A(\alpha_i) + b_A(\alpha_i)}{2};$$
$$M_i := \begin{cases} 0, & \text{if } a_B(\alpha_i) < b_B(\alpha_i) \le a_A(\alpha_i) < b_A(\alpha_i) \\ 1, & \text{if } a_A(\alpha_i) < b_A(\alpha_i) \le a_B(\alpha_i) < b_B(\alpha_i) \\ 1 - \frac{(l_A(\alpha_i)+l_B(\alpha_i)-d(\alpha_i))^2}{8 l_A(\alpha_i) l_B(\alpha_i)}, & \text{if } a_A(\alpha_i) < a_B(\alpha_i) < b_A(\alpha_i) < b_B(\alpha_i) \\ \frac{(l_A(\alpha_i)+l_B(\alpha_i)+d(\alpha_i))^2}{8 l_A(\alpha_i) l_B(\alpha_i)}, & \text{if } a_B(\alpha_i) < a_A(\alpha_i) < b_B(\alpha_i) < b_A(\alpha_i) \\ \frac{1}{2} + \frac{d(\alpha_i)}{2 l_B(\alpha_i)}, & \text{if } a_B(\alpha_i) \le a_A(\alpha_i) < b_A(\alpha_i) \le b_B(\alpha_i) \\ \frac{1}{2} + \frac{d(\alpha_i)}{2 l_A(\alpha_i)}, & \text{if } a_A(\alpha_i) \le a_B(\alpha_i) < b_B(\alpha_i) \le b_A(\alpha_i) \end{cases}$$
$$S := S + M_i;$$

End for
$M_F(A,B) := \frac{S}{n};$

OUTPUT: $M_F(A,B)$.

3.3.3. A demonstrative example

Here, we will demonstrate how our analytical and numerical methods can be applied in practice to compute the probability-based preference intensity index $M_F(A,B)$ for the fuzzy sets A and B when they have trapezoidal membership functions.

Let A and B be two fuzzy sets that have the trapezoidal membership functions μ_A and μ_B with the parameters $\underline{x}_A^L < \overline{x}_A^L \le \overline{x}_A^R < \underline{x}_A^R$ and $\underline{x}_B^L < \overline{x}_B^L \le \overline{x}_B^R < \underline{x}_B^R$, respectively, as defined in Definition 2. Let the parameter values be

$$\underline{x}_A^L = 3, \quad \overline{x}_A^L = 6, \quad \overline{x}_A^R = 9, \quad \underline{x}_A^R = 11$$
$$\underline{x}_B^L = 2, \quad \overline{x}_B^L = 9.75, \quad \overline{x}_B^R = 10.25, \quad \underline{x}_B^R = 10.5$$

The trapezoidal membership functions of the fuzzy sets A and B are shown in Fig. 5. Firstly, we will compute $M_F(A,B)$ according to our analytical method. With this method, the vertical coordinates of the intersections of the lateral sides of the trapezoids A and B are

$$\alpha_1 = \frac{\underline{x}_A^L - \underline{x}_B^L}{\underline{x}_A^L - \overline{x}_A^L + \overline{x}_B^L - \underline{x}_B^L} = 0.2105$$

$$\alpha_2 = \frac{\underline{x}_A^L - \underline{x}_B^R}{\underline{x}_A^L - \overline{x}_A^L + \overline{x}_B^R - \underline{x}_B^R} = 2.3077$$
$$\alpha_3 = \frac{\underline{x}_A^R - \underline{x}_B^L}{\underline{x}_A^R - \overline{x}_A^R + \overline{x}_B^L - \underline{x}_B^L} = 0.9231$$
$$\alpha_4 = \frac{\underline{x}_A^R - \underline{x}_B^R}{\underline{x}_A^R - \overline{x}_A^R + \overline{x}_B^R - \underline{x}_B^R} = 0.2857.$$

Here, the set $\mathcal{S}$ is

$$\mathcal{S} = \{\alpha_i : \alpha_i \in (0, 1), i = 1, 2, 3, 4\} = \{0.2105, 0.9231, 0.2857\},$$

and its cardinality is $n = 3$. That is, the trapezoids A and B have three intersection points. Therefore the ordered set $\{\alpha'_1, \alpha'_2, \alpha'_3\}$ which contains all the elements of set $\mathcal{S}$ is

$$\{\alpha'_1, \alpha'_2, \alpha'_3\} = \{0.2105, 0.2857, 0.9231\}.$$

That is, $\alpha'_1 = 0.2105$, $\alpha'_2 = 0.2857$ and $\alpha'_3 = 0.9231$; and we also have $\alpha'_0 = 0$ and $\alpha'_4 = 1$. With our analytical method, we have

$$\alpha_0^* = \frac{\alpha'_0 + \alpha'_1}{2} = 0.1053$$
$$a_A(\alpha_0^*) = 3.3158, \quad b_A(\alpha_0^*) = 10.7895$$
$$a_B(\alpha_0^*) = 2.8158, \quad b_B(\alpha_0^*) = 10.4737$$

and since $a_B(\alpha_i^*) < a_A(\alpha_i^*) < b_B(\alpha_i^*) < b_A(\alpha_i^*)$ holds, we need to compute $M_0 = \mathcal{I}^o_{a_A > a_B}(0, \alpha'_1)$. Using the formula for the integral $\mathcal{I}^o_{a_A > a_B}(\alpha', \alpha'')$, we get

$$M_0 = \mathcal{I}^o_{a_A > a_B}(0, \alpha'_1) = 0.0945.$$

By continuing the method for $i = 1, 2, 3$, we get

$$M_1 = \mathcal{I}^i_{I_B \subseteq I_A}(\alpha'_1, \alpha'_2) = 0.0383$$
$$M_2 = \mathcal{I}^o_{a_A < a_B}(\alpha'_2, \alpha'_3) = 0.4783$$
$$M_3 = \mathcal{I}^p_{a_B \geq b_A}(\alpha'_3, 1) = 0.0769,$$

and so $M_F(A, B) = M_0 + M_1 + M_2 + M_3 = 0.6880$.

If we execute our numerical method with the parameter value $n = 1000$, then we get $M_F(A, B) \approx 0.6883$, which is very close to the analytically computed value of $M_F(A, B)$. It is worth mentioning here that if we run the approximation with the parameter value $n = 100$, then we get $M_F(A, B) \approx 0.6909$, which is still quite a good approximation of the exact value.

4. Ranking fuzzy sets with trapezoidal membership functions using the probability-based preference intensity index

Now, using the probability-based preference intensity index for fuzzy sets with trapezoidal membership functions, we will introduce a relation over a collection of such fuzzy sets and demonstrate that this relation is irreflexive and antisymmetric.

Definition 4. Let $\mathbf{F}$ be a collection of fuzzy sets with trapezoidal membership functions. The binary relation $\prec_F$ over the collection $\mathbf{F}$ is given by

$$\prec_F := \left\{ (A, B) \in \mathbf{F} \times \mathbf{F} : M_F(A, B) > \frac{1}{2} \right\}.$$

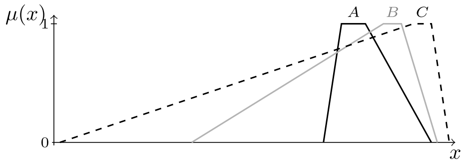


Fig. 6. Intransitivity of $\prec_F$: $A \prec_F B$, $B \prec_F C$, but $A \not\prec_F C$.

In other words, based on Definition 4, the relation $A \prec_F B$ holds if and only if $M_F(A, B) > \frac{1}{2}$, where A and B are two fuzzy sets with trapezoidal membership functions. The following proposition demonstrates the irreflexivity and antisymmetry properties of relation $\prec_F$ over a collection of fuzzy sets that have trapezoidal membership functions.

Proposition 2. *The relation $\prec_F$ given in Definition 4 is irreflexive and antisymmetric.*

Proof. Let $\mathbf{F}$ be a collection of fuzzy sets with trapezoidal membership functions. Here, we intend to show that relation $\prec_F$ is

(1) irreflexive, i.e., $A \not\prec_F A$ holds for any $A \in \mathbf{F}$
(2) antisymmetric, i.e., if $A \prec_F B$, then $B \not\prec_F A$ for any $A, B \in \mathbf{F}$.

(1) (Irreflexivity.) Since $M_F(A, A) = \frac{1}{2}$ for any $A \in \mathbf{F}$, based on Definition 4, $A \prec_F A$ does not hold.

(2) (Antisymmetry.) Let $A, B \in \mathbf{F}$ such that $A \prec_F B$. Then, based on Definition 4, $M_F(A, B) > \frac{1}{2}$. Now, by noting the reciprocity property of function M_F (see Proposition 1), we have $M_F(A, B) + M_F(B, A) = 1$ and so $M_F(B, A) < \frac{1}{2}$, which means that $B \not\prec_F A$. □

Note that if the relation $\prec_F$ given in Definition 4 were also transitive; i.e., if $A \prec_F B$ and $B \prec_F C$, then $A \prec_F C$ for any $A, B, C \in \mathbf{F}$, then $\prec_F$ would meet all the criteria for a strict order. In Fig. 6, we can see the trapezoidal membership functions of three fuzzy sets, A, B and C, with the following parameters:

$$\underline{x}_A^L = 45, \quad \overline{x}_A^L = 48, \quad \overline{x}_A^R = 52, \quad \underline{x}_A^R = 63$$
$$\underline{x}_B^L = 23, \quad \overline{x}_B^L = 55, \quad \overline{x}_B^R = 58, \quad \underline{x}_B^R = 64$$
$$\underline{x}_C^L = 1, \quad \overline{x}_C^L = 60, \quad \overline{x}_C^R = 63, \quad \underline{x}_C^R = 66$$

By using Method 1 in Section 3.3, we get

$$M_F(A, B) = 0.5094, \quad M_F(B, C) = 0.5070, \quad M_F(A, C) = 0.4845,$$

meaning that $A \prec_F B$ and $B \prec_F C$ both hold, but $A \prec_F C$ does not hold. This example tells us that the relation $\prec_F$ over a collection of fuzzy sets with trapezoidal membership functions is not transitive. It should be added that in this example, there are very small differences among the M_F values, which is in line with the human perception that judging the order of these fuzzy sets is difficult; and so, we tend to consider their order as being indifferent. In the following, using the probability-based preference intensity index for fuzzy sets with trapezoidal membership functions, we will introduce a parametric relation over a collection of such fuzzy sets and show that this relation can be turned into a strict order relation.

Definition 5. Let $\mathbf{F}$ be a collection of fuzzy sets with trapezoidal membership functions. The binary relation $\prec_F^{(\delta)}$ over the collection $\mathbf{F}$ is given by

$$\prec_F^{(\delta)} := \left\{ (A, B) \in \mathbf{F} \times \mathbf{F} : M_F(A, B) \geq \frac{1}{2} + \delta \right\},$$

where $\delta \in (0, 1/2]$.

Theorem 2. *Let* $\mathbf{F}$ *be a collection of fuzzy sets with trapezoidal membership functions and let the binary relation* $\prec_F^{(\delta)}$ *over the collection* $\mathbf{F}$ *be given by Definition 5. Then, there exists a* $\delta \in (0, 1/2]$ *such that* $\prec_F^{(\delta)}$ *is a strict order relation over* $\mathbf{F}$.

Proof. We will show that if $\delta = \frac{1}{2}$, then $\prec_F^{(\delta)}$ is a strict order relation. In order to prove this, we need to show that $\prec_F^{(\delta)}$ is

(1) irreflexive, i.e., $A \not\prec_F^{(\delta)} A$ holds for any $A \in \mathbf{F}$
(2) antisymmetric, i.e., if $A \prec_F^{(\delta)} B$, then $B \not\prec_F^{(\delta)} A$ holds for any $A, B \in \mathbf{F}$
(3) transitive, i.e., if $A \prec_F^{(\delta)} B$ and $B \prec_F^{(\delta)} C$, then $A \prec_F^{(\delta)} C$ holds for any $A, B, C \in \mathbf{F}$.

(1) (Irreflexivity.) Since $M_F(A, A) = \frac{1}{2}$ for any $A \in \mathbf{F}$, based on Definition 5, $A \prec_F^{(\delta)} A$ does not hold.

(2) (Antisymmetry.) Let $A, B \in \mathbf{F}$ such that $A \prec_F^{(\delta)} B$. Then, based on Definition 5, $M_F(A, B) \geq \delta + \frac{1}{2} > \frac{1}{2}$. Now, by noting the reciprocity property of function M_F (see Proposition 1), we have $M_F(A, B) + M_F(B, A) = 1$; and so $M_F(B, A) < \frac{1}{2}$, which means that $B \not\prec_F^{(\delta)} A$.

(3) (Transitivity.) Let $A, B \in \mathbf{F}$ such that $A \prec_F^{(\delta)} B$ and $B \prec_F^{(\delta)} C$. Then, based on Definition 5, noting that $\delta = \frac{1}{2}$, we have $M_F(A, B) \geq 1$ and $M_F(B, C) \geq 1$. Since $M_F(A, B) \leq 1$ and $M_F(B, C) \leq 1$ (see Remark 4), we get that $M_F(A, B) = 1$ and $M_F(B, C) = 1$. Noting the fact that

$$M_F(A, B) = \int_0^1 M\left(I_A(\alpha), I_B(\alpha)\right) \mathrm{d}\alpha,$$

where $I_A(\alpha)$ and $I_B(\alpha)$ are the α-cut intervals of A and B, respectively, and the fact that $M\left(I_A(\alpha), I_B(\alpha)\right) \leq 1$ for any $\alpha \in [0, 1]$, we get that $M_F(A, B) = 1$ can hold only if $M\left(I_A(\alpha), I_B(\alpha)\right) = 1$ for any $\alpha \in [0, 1]$. Now, by noting Theorem 1, we know that $M\left(I_A(\alpha), I_B(\alpha)\right) = 1$ holds only if the interval $I_A(\alpha) = [a_A(\alpha), b_A(\alpha)]$ entirely precedes the interval $I_B(\alpha) = [a_B(\alpha), b_B(\alpha)]$. That is, $b_A(\alpha) < a_B(\alpha)$ holds for any $\alpha \in [0, 1]$. Similarly, $M_F(B, C) = 1$ implies that $b_B(\alpha) < a_C(\alpha)$ holds for any $\alpha \in [0, 1]$, where $I_C(\alpha) = [a_C(\alpha), b_C(\alpha)]$ is the α-cut interval of C. Therefore, by noting that $b_A(\alpha) < a_B(\alpha) < b_B(\alpha) < a_C(\alpha)$ for any $\alpha \in [0, 1]$, we have that the interval $I_A(\alpha)$ entirely precedes the interval $I_C(\alpha)$ for any $\alpha \in [0, 1]$, which, based on Theorem 1, means that $M\left(I_A(\alpha), I_C(\alpha)\right) = 1$ for any $\alpha \in [0, 1]$; and so,

$$M_F(A, C) = \int_0^1 M\left(I_A(\alpha), I_C(\alpha)\right) \mathrm{d}\alpha = 1. \tag{46}$$

Considering Definition 5 and the fact that $\delta = \frac{1}{2}$, Eq. (46) tells us that $A \prec_F^{(\delta)} C$ holds. That is, if $\delta = \frac{1}{2}$, then $A \prec_F^{(\delta)} B$ and $B \prec_F^{(\delta)} C$ imply $A \prec_F^{(\delta)} C$. □

In Theorem 2, we demonstrated that the relation $\prec_F^{(\delta)}$ is transitive if $\delta = \frac{1}{2}$. In practice, for a given finite collection $\mathbf{F}$, the smallest value of $\delta \in (0, \frac{1}{2}]$, for which relation $\prec_F^{(\delta)}$ is transitive, can be numerically determined by using searching methods such as the binary search or the simple brute-force search.

Remark 5. Note that the fact that the relation $\prec_F^{(\delta)}$ is a strict order over a finite collection $\mathbf{F}$ of fuzzy numbers with trapezoidal membership functions does not necessarily imply that $\prec_F^{(\delta)}$ is a total order. That is, it does not necessarily hold for any $A, B \in \mathbf{F}$ that either $A \prec_F^{(\delta)} B$ or $B \prec_F^{(\delta)} A$ or $A = B$.

Suppose that A and B are two incomparable elements of $\mathbf{F}$; that is, neither $A \prec_F^{(\delta)} B$ nor $B \prec_F^{(\delta)} A$ nor $A = B$ holds. This means that $\frac{1}{2} - \delta < M_F(A, B) < \frac{1}{2} + \delta$. Here, we may assume that in practice, the value of parameter δ is close to zero; and so, the previous inequality suggests that the order of A and B may be viewed as being indifferent.

Table 1
Parameters of the trapezoidal membership functions.

	$\underline{x}_A^L$	$\overline{x}_A^L$	$\overline{x}_A^R$	$\underline{x}_A^R$
A_1	33	65	68	74
A_2	55	58	62	73
A_3	67	76	125	130
A_4	38	67	83	107
A_5	72	85	88	91
A_6	65	68	72	83
A_7	22	80	83	86
A_8	28	31	34	61
A_9	2	5	20	66
A_{10}	11	70	73	76

Based on this line of thinking, now, we will introduce the following indifference relation over a collection of fuzzy sets with trapezoidal membership functions.

Definition 6. Let $\mathbf{F}$ be a collection of fuzzy sets with trapezoidal membership functions. The binary relation $\lesseqqgtr_F^{(\delta)}$ (indifference relation) over the collection $\mathbf{F}$ is given by

$$\lesseqqgtr_F^{(\delta)} := \left\{ (A, B) \in \mathbf{F} \times \mathbf{F} : \left| M_F(A, B) - \frac{1}{2} \right| < \delta \right\},$$

where $\delta \in (0, 1/2]$.

Corollary 1. *Let $\mathbf{F}$ be a collection of fuzzy sets with trapezoidal membership functions and let the relations $\prec_F^{(\delta)}$ and $\lesseqqgtr_F^{(\delta)}$ over the set $\mathbf{F}$ be given by Definition 5 and Definition 6, respectively, where $\delta \in (0, 1/2]$ has a fixed value. Then, for any $A, B \in \mathbf{F}$, if $A \nprec_F^{(\delta)} B$, then either $B \prec_F^{(\delta)} A$ or $A \lesseqqgtr_F^{(\delta)} B$ holds.*

Proof. The corollary immediately follows from Definition 5 and Definition 6. $\square$

The following example demonstrates how the relations $\prec_F^{(\delta)}$ and $\lesseqqgtr_F^{(\delta)}$ can be used to rank fuzzy sets that have trapezoidal membership functions.

4.1. A demonstrative example

Suppose that we wish to rank the elements of the collection

$$\mathbf{F} = \{A_1, A_2, \ldots, A_n\}$$

of fuzzy sets that have trapezoidal membership functions. The parameter values of $A_1, A_2, \ldots, A_n$ are shown in Table 1.

Fig. 7 shows the plots of the trapezoidal membership functions of the fuzzy sets in $\mathbf{F}$.

Method 1 and Method 2, which were presented in Section 3.3, had been implemented in MATLAB 2019a computing environment (see https://github.com/dombijozsef). Using Method 2, the probability-based preference intensity index $M_F(A_i, A_j)$ was computed for every (A_i, A_j) pair and stored in the probability-based preference intensity index matrix in Table 2, where $A_i, A_j \in \mathbf{F}$, $i, j = 1, 2, \ldots, 10$. In this matrix, we can see the reciprocity property of the index M_F; that is, $M_F(A_i, A_j) = 1 - M_F(A_j, A_i)$ holds for any $i, j = 1, 2, \ldots, 10$.

Note that the computed values in Table 2 have been rounded to four digits. Here, we have

$$M_F(A_{10}, A_2) = 0.5155, \quad M_F(A_2, A_1) = 0.5094, \quad M_F(A_{10}, A_1) = 0.4929,$$

which means that the relations $A_{10} \prec_F A_2$ and $A_2 \prec_F A_1$ hold, but the relation $A_{10} \prec_F A_1$ does not hold. That is, the relation $\prec_F$ over the set $\mathbf{F}$ (see Definition 4) is not transitive. Also, we have $A_2 \prec_F A_1$ and $A_1 \prec_F A_{10}$, but $A_2 \nprec_F A_{10}$; and $A_1 \prec_F A_{10}$ and $A_{10} \prec_F A_2$, but $A_1 \nprec_F A_2$.

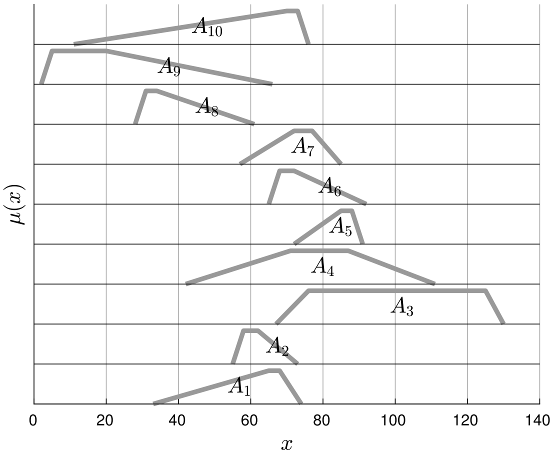


Fig. 7. Plots of the membership functions in collection **F**.

Table 2
Probability-based preference intensity index matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	0.5000	0.4906	0.9984	0.9008	1.0000	0.9740	0.9438	0.0596	0.0360	0.5071
A_2	0.5094	0.5000	0.9984	0.8660	1.0000	0.9834	0.9294	0.0022	0.0043	0.4845
A_3	0.0016	0.0016	0.5000	0.1559	0.2229	0.0714	0.0478	0.0000	0.0000	0.0033
A_4	0.0992	0.1340	0.8441	0.5000	0.6804	0.3700	0.3654	0.0105	0.0080	0.1047
A_5	0.0000	0.0000	0.7771	0.3196	0.5000	0.1086	0.0453	0.0000	0.0000	0.0006
A_6	0.0260	0.0166	0.9286	0.6300	0.8914	0.5000	0.4943	0.0000	0.0000	0.1175
A_7	0.0562	0.0706	0.9522	0.6346	0.9547	0.5057	0.5000	0.0003	0.0012	0.0834
A_8	0.9404	0.9978	1.0000	0.9895	1.0000	1.0000	0.9997	0.5000	0.1372	0.8486
A_9	0.9640	0.9957	1.0000	0.9920	1.0000	1.0000	0.9988	0.8628	0.5000	0.9167
A_{10}	0.4929	0.5155	0.9967	0.8953	0.9994	0.8825	0.9166	0.1514	0.0833	0.5000

Table 3
The n_i, p_i, u_k and o_i values.

i	1	2	3	4	5	6	7	8	9	10
n_i	5	5	0	2	1	3	3	8	9	5
p_i	5	5	10	8	9	7	7	2	1	5
k	1	2	3	4	5	6	7			
u_k	1	2	5	7	8	9	10			
o_i	3	3	7	5	6	4	4	2	1	3

Next, by using a simple incremental search, starting from 0.01 with the increment of 0.01, we identified $\delta = 0.02$ as a value for which the relation $\prec_F^{(\delta)}$ over the set **F** is transitive.

Let n_i denote the cardinality of the set $\{j \in \{1, 2, \dots, 10\} : A_i \prec_F^{(\delta)} A_j\}$; that is, n_i is the number of those A_j fuzzy sets for which $M_F(A_i, A_j) \geq \frac{1}{2} + \delta$ holds, $i \in \{1, 2, \dots, 10\}$ (see Table 3). Let p_i be given by $p_i = 10 - n_i$ for $i = 1, 2, \dots 10$, and let the sequence $u_1 < u_2 < \cdots < u_m$ contain the unique values of $p_1, p_2, \dots, p_n$ in increasing order, $(m \leq n)$. Now, for every $i \in \{1, 2, \dots, 10\}$ let o_i be given by $o_i = k$ such that $u_k = p_i$, where $k \in \{1, 2, \dots, m\}$. Then, o_i may be interpreted as the order index of the fuzzy set A_i, where $i = 1, 2, \dots, 10$. Table 3 summarizes the n_i, p_i, u_k and o_i values for our case.

Notice that the fuzzy sets A_1, A_2 and A_{10} have the same order index (3); that is, we consider the order of these three fuzzy sets as being indifferent. From Table 2, we can see that

$$M_F(A_1, A_2) = 0.4906, \quad M_F(A_2, A_{10}) = 0.4845, \quad M_F(A_1, A_{10}) = 0.5071,$$

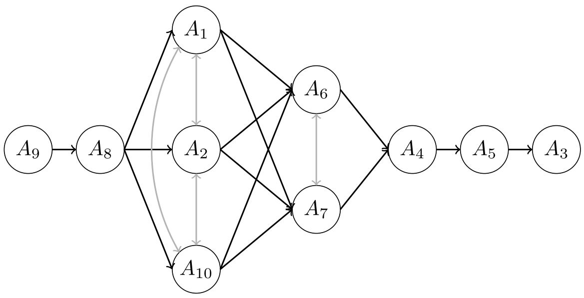


Fig. 8. Graph representation of $\prec_F^{(\delta)}$ and $\lesseqqgtr_F^{(\delta)}$ over the set $\mathbf{F}$.

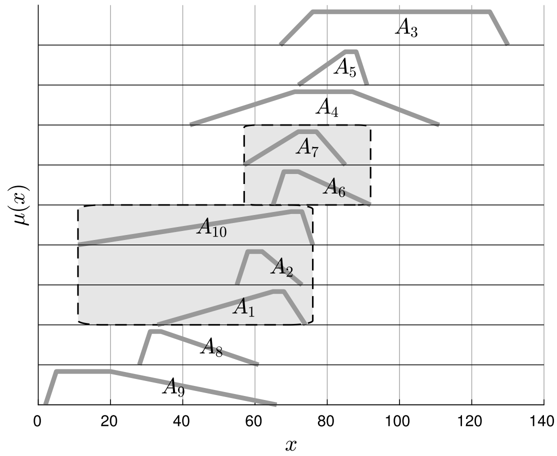


Fig. 9. Ranked fuzzy sets.

which, after considering the definition for the relation $\lesseqqgtr_F^{(\delta)}$ over the set $\mathbf{F}$ and the fact that $\delta = 0.02$, means that the relations

$$A_1 \lesseqqgtr_F^{(\delta)} A_2, \quad A_2 \lesseqqgtr_F^{(\delta)} A_{10}, \quad A_1 \lesseqqgtr_F^{(\delta)} A_{10}$$

hold. Also, the fuzzy sets A_6 and A_7 have the same order index (4), which means that the relation

$$A_6 \lesseqqgtr_F^{(\delta)} A_7$$

hold. Hence, the fuzzy sets with the same order index o_i form groups such that in each of these groups, we consider the order of the sets as being indifferent. Using the relations $\prec_F^{(\delta)}$ and $\lesseqqgtr_F^{(\delta)}$, the order of the fuzzy sets in $\mathbf{F}$ can be represented by the following relation chain:

$$A_9 \prec_F^{(\delta)} A_8 \prec_F^{(\delta)} A_1 \lesseqqgtr_F^{(\delta)} A_2 \lesseqqgtr_F^{(\delta)} A_{10} \prec_F^{(\delta)} A_6 \lesseqqgtr_F^{(\delta)} A_7 \prec_F^{(\delta)} A_4 \prec_F^{(\delta)} A_5 \prec_F^{(\delta)} A_3.$$

Fig. 8 shows a graph representation of this relation chain. In this figure, the black and gray-colored arrows represent the relations $\prec_F^{(\delta)}$ and $\lesseqqgtr_F^{(\delta)}$, respectively.

Fig. 9 shows the fuzzy sets when they are ranked using the relations $\prec_F^{(\delta)}$ and $\lesseqqgtr_F^{(\delta)}$. In this figure, each of the gray-colored areas contain a group of fuzzy numbers whose order is considered as being indifferent: group $\{A_1, A_2, A_{10}\}$ and group $\{A_6, A_7\}$.

5. Conclusions and future work

The chief results of this study can be summarized as follows.

- Following the approach proposed by Huynh et al. [40], utilizing a probability-based preference intensity index M for two intervals, we presented a probability-based preference intensity index M_F for two fuzzy sets that have trapezoidal membership functions.
- We proposed two methods for computing the index M_F for two fuzzy sets that have trapezoidal membership functions. Namely, we introduced an analytical and a numerical method for computing M_F for fuzzy sets with trapezoidal membership functions. In our analytical method, we presented closed formulas for the integrals needed to compute M_F. Also, we proposed an algorithm to compute M_F using the closed formulas for the integrals.
- We introduced two crisp relations, which have a common parameter, over a collection of fuzzy sets with trapezoidal membership functions. Then, we studied the algebraic properties of these relations and show that - depending on the parameter value - one of them is a strict order relation, and the other one may be interpreted as a relation that expresses the order indifference of fuzzy numbers. We called this latter one the order indifference relation. Here, we considered two fuzzy numbers as being comparable, when their order can be unambiguously determined. Next, we showed that our strict order relation can be used to rank comparable fuzzy numbers, while the indifference relation can be utilized to express that the order of some fuzzy numbers is indifferent.
- In a comprehensive case study, we demonstrated how these two relations can be used to rank a collection of fuzzy sets that have trapezoidal membership functions.

As part of our future work, we intend to study how the probability-based preference intensity index M_F can be computed for fuzzy numbers that have membership functions from well-known membership function classes. We would also like to know how the value of the δ parameter can be efficiently determined such that the relation $\prec_F^{(\delta)}$ is a strict order relation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] R. Jain, Decision making in the presence of fuzzy variables, IEEE Trans. Syst. Man Cybern. 6 (1976) 698–703.
[2] R.R. Yager, A procedure for ordering fuzzy subsets of the unit interval, Inf. Sci. 24 (1981) 143–161, https://doi.org/10.1016/0020-0255(81)90017-7.
[3] S.-H. Chen, Ranking fuzzy numbers with maximizing set and minimizing set, Fuzzy Sets Syst. 17 (1985) 113–129, https://doi.org/10.1016/0165-0114(85)90050-8.
[4] C.-H. Cheng, A new approach for ranking fuzzy numbers by distance method, Fuzzy Sets Syst. 95 (1998) 307–317, https://doi.org/10.1016/S0165-0114(96)00272-2.
[5] T.-C. Chu, C.-T. Tsao, Ranking fuzzy numbers with an area between the centroid point and original point, Comput. Math. Appl. 43 (2002) 111–117, https://doi.org/10.1016/S0898-1221(01)00277-2.
[6] B. Asady, A. Zendehnam, Ranking fuzzy numbers by distance minimization, Appl. Math. Model. 31 (2007) 2589–2598, https://doi.org/10.1016/j.apm.2006.10.018.
[7] Y.-M. Wang, J.-B. Yang, D.-L. Xu, K.-S. Chin, On the centroids of fuzzy numbers, Fuzzy Sets Syst. 157 (2006) 919–926, https://doi.org/10.1016/j.fss.2005.11.006.
[8] S. Abbasbandy, B. Asady, Ranking of fuzzy numbers by sign distance, Inf. Sci. 176 (2006) 2405–2416, https://doi.org/10.1016/j.ins.2005.03.013.
[9] Y.-M. Wang, Y. Luo, Area ranking of fuzzy numbers based on positive and negative ideal points, Comput. Math. Appl. 58 (2009) 1769–1779, https://doi.org/10.1016/j.camwa.2009.07.064.
[10] Y.-J. Wang, Ranking triangle and trapezoidal fuzzy numbers based on the relative preference relation, Appl. Math. Model. 39 (2015) 586–599, https://doi.org/10.1016/j.apm.2014.06.011.
[11] K.C. Chai, K.M. Tay, C.P. Lim, A new method to rank fuzzy numbers using Dempster–Shafer theory with fuzzy targets, Inf. Sci. 346 (2016) 302–317, https://doi.org/10.1016/j.ins.2016.01.066.
[12] A.F.R.L. de Hierro, C. Roldán, F. Herrera, On a new methodology for ranking fuzzy numbers and its application to real economic data, in: Theme: Fuzzy Intervals, Fuzzy Sets Syst. 353 (2018) 86–110, https://doi.org/10.1016/j.fss.2018.04.003.
[13] Q. Gu, Z. Xuan, A new approach for ranking fuzzy numbers based on possibility theory, J. Comput. Appl. Math. 309 (2017) 674–682, https://doi.org/10.1016/j.cam.2016.05.017.
[14] R. Chutia, B. Chutia, A new method of ranking parametric form of fuzzy numbers using value and ambiguity, Appl. Soft Comput. 52 (2017) 1154–1168, https://doi.org/10.1016/j.asoc.2016.09.013.

[15] H.A. Khorshidi, S. Nikfalazar, An improved similarity measure for generalized fuzzy numbers and its application to fuzzy risk analysis, Appl. Soft Comput. 52 (2017) 478–486, https://doi.org/10.1016/j.asoc.2016.10.020.

[16] V.F. Yu, H.T.X. Chi, C. wen Shen, Ranking fuzzy numbers based on epsilon-deviation degree, Appl. Soft Comput. 13 (2013) 3621–3627, https://doi.org/10.1016/j.asoc.2013.03.016.

[17] R. Chutia, Ranking of fuzzy numbers by using value and angle in the epsilon-deviation degree method, Appl. Soft Comput. 60 (2017) 706–721, https://doi.org/10.1016/j.asoc.2017.07.025.

[18] A. Rahmani, F. Hosseinzadeh Lotfi, M. Rostamy-Malkhalifeh, T. Allahviranloo, A new method for defuzzification and ranking of fuzzy numbers based on the statistical beta distribution, Adv. Fuzzy Syst. 2016 (2016), https://doi.org/10.1155/2016/6945184.

[19] A. Boulmakoul, M.H. Laarabi, R. Sacile, E. Garbolino, An original approach to ranking fuzzy numbers by inclusion index and bitset encoding, Fuzzy Optim. Decis. Mak. 16 (2017) 23–49, https://doi.org/10.1007/s10700-016-9237-9.

[20] G. Hesamian, F. Bahrami, Credibility theory oriented preference index for ranking fuzzy numbers, Iran. J. Fuzzy Syst. 14 (2017) 103–117, https://doi.org/10.22111/ijfs.2017.3500.

[21] H. Garg, K. Kumar, Improved possibility degree method for ranking intuitionistic fuzzy numbers and their application in multiattribute decision-making, Granul. Comput. 4 (2019) 237–247, https://doi.org/10.1007/s41066-018-0092-7.

[22] C. Yue, A geometric approach for ranking interval-valued intuitionistic fuzzy numbers with an application to group decision-making, Comput. Ind. Eng. 102 (2016) 233–245, https://doi.org/10.1016/j.cie.2016.10.027.

[23] G. Uthra, K. Thangavelu, S. Shunmugapriya, Ranking generalized intuitionistic fuzzy numbers, Int. J. Math. Trends Technol. 56 (2018).

[24] V.L.G. Nayagam, S. Jeevaraj, P. Dhanasekaran, An improved ranking method for comparing trapezoidal intuitionistic fuzzy numbers and its applications to multicriteria decision making, Neural Comput. Appl. 30 (2018) 671–682, https://doi.org/10.1007/s00521-016-2673-1.

[25] J. Lin, F. Meng, R. Chen, Q. Zhang, Preference attitude-based method for ranking intuitionistic fuzzy numbers and its application in renewable energy selection, Complexity 2018 (2018), https://doi.org/10.1155/2018/6251384.

[26] Y. Yuan, Criteria for evaluating fuzzy ranking methods, Fuzzy Sets Syst. 43 (1991) 139–157, https://doi.org/10.1016/0165-0114(91)90073-Y.

[27] T.L. Saaty, L.G. Vargas, Uncertainty and rank order in the analytic hierarchy process, Eur. J. Oper. Res. 32 (1987) 107–117, https://doi.org/10.1016/0377-2217(87)90275-X.

[28] S. Kundu, Min-transitivity of fuzzy leftness relationship and its application to decision making, Fuzzy Sets Syst. 86 (1997) 357–367, https://doi.org/10.1016/S0165-0114(96)00122-4.

[29] F. Herrera, L. Martınez, P.J. Sánchez, Managing non-homogeneous information in group decision making, Eur. J. Oper. Res. 166 (2005) 115–132, https://doi.org/10.1016/j.ejor.2003.11.031.

[30] F. Liu, L.-H. Pan, Z.-L. Liu, Y.-N. Peng, On possibility-degree formulae for ranking interval numbers, Soft Comput. 22 (2018) 2557–2565, https://doi.org/10.1007/s00500-017-2509-7.

[31] S. Lipovetsky, A. Tishler, Interval estimation of priorities in the AHP, Eur. J. Oper. Res. 114 (1999) 153–164, https://doi.org/10.1016/S0377-2217(98)00012-5.

[32] K. Sugihara, H. Ishii, H. Tanaka, Interval priorities in AHP by interval regression analysis, Eur. J. Oper. Res. 158 (2004) 745–754, https://doi.org/10.1016/S0377-2217(03)00418-1.

[33] Y.-M. Wang, J.-B. Yang, D.-L. Xu, A two-stage logarithmic goal programming method for generating weights from interval comparison matrices, Fuzzy Sets Syst. 152 (2005) 475–498, https://doi.org/10.1016/j.fss.2004.10.020.

[34] Z.-J. Wang, A two-stage linear goal programming approach to eliciting interval weights from additive interval fuzzy preference relations, Soft Comput. 20 (2016) 2721–2732, https://doi.org/10.1007/s00500-015-1673-x.

[35] Z. Zhang, Logarithmic least squares approaches to deriving interval weights, rectifying inconsistency and estimating missing values for interval multiplicative preference relations, Soft Comput. 21 (2017) 3993–4004, https://doi.org/10.1007/s00500-016-2049-6.

[36] A. Sengupta, T.K. Pal, On comparing interval numbers: a study on existing ideas, in: Fuzzy Preference Ordering of Interval Numbers in Decision Problems, Springer Berlin Heidelberg, 2009, pp. 25–37.

[37] Q. Zhang, Z. Fan, D. Pan, et al., A ranking approach for interval numbers in uncertain multiple attribute decision making problems, Syst. Eng. Theory Pract. 5 (1999).

[38] S.A. Badri, M. Ghazanfari, K. Shahanaghi, A multi-criteria decision-making approach to solve the product mix problem with interval parameters based on the theory of constraints, Int. J. Adv. Manuf. Technol. 70 (2014) 1073–1080, https://doi.org/10.1007/s00170-013-5360-8.

[39] Y. Ye, N. Yao, Q. Wang, Q. Wang, A method of ranking interval numbers based on degrees for multiple attribute decision making, J. Intell. Fuzzy Syst. 30 (2016) 211–221, https://doi.org/10.3233/IFS-151747.

[40] V. Huynh, Y. Nakamori, J. Lawry, A probability-based approach to comparison of fuzzy numbers and applications to target-oriented decision making, IEEE Trans. Fuzzy Syst. 16 (2008) 371–387, https://doi.org/10.1109/TFUZZ.2007.896315.