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Short communication

Lower and upper bounds for the probabilistic Poincaré formula using the general Poincaré formula for λ-additive measures

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Abstract

Following our previous paper Dombi and Jónás (2019) [16], we will now present new inequalities using the general Poincaré formula for λ-additive measures. These inequalities represent bounds for the well-known Poincaré formula of probability theory.

Keywords: λ-additive measure; Poincaré formula

1. Introduction

It is a familiar fact that the monotone (fuzzy) measures have been applied in many areas of science (see, e.g. [1–5]). The λ-additive measures (Sugeno λ-measures) [6], which are a specific class of monotone measures, play an important role in describing and modeling uncertainty (see, e.g. [7–11]). It is well known that the λ-additive measure is strongly connected with the belief- and plausibility measures and these may be viewed as lower- and upper probabilities, respectively (see, e.g. [12–14]). In [15], we presented the general formula for the λ-additive measure of the union of n sets, which we called the general Poincaré formula for λ-additive measures. Using this formula, we introduced novel inequalities for λ-additive measures in [16]. Following these results, by applying the general Poincaré formula for λ-additive measures, we will now present bounds for the well-known Poincaré formula of probability theory.

2. Preliminaries

In this study, we will use the common notations $\cap$ and $\cup$ for the intersection and union operations over sets, respectively. Also, will use the notation $\overline{A}$ for the complement of set A, and $\mathcal{P}(X)$ will denote the power set of the set X.

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The familiar Poincaré formula of probability theory is

$$Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} \sum_{1 \le i_1 < \cdots < i_k \le n} (-1)^{k-1} a_k, \tag{1}$$

where Pr is a probability measure on the set X, $A_1, \ldots, A_n \in \mathcal{P}(X)$ and $a_k = Pr\left(A_{i_1} \cap \cdots \cap A_{i_k}\right)$.

The λ-additive measures were first proposed by Sugeno in 1974 [6].

Definition 1. The function $Q_\lambda : \mathcal{P}(X) \to [0, 1]$ is a λ-additive measure (Sugeno λ-measure) on the finite set X, iff Q_λ satisfies the following requirements:

(1) $Q_\lambda(X) = 1$
(2) For any $A, B \in \mathcal{P}(X)$ and $A \cap B = \emptyset$,

$$Q_\lambda(A \cup B) = Q_\lambda(A) + Q_\lambda(B) + \lambda Q_\lambda(A) Q_\lambda(B),$$

where $\lambda \in (-1, \infty)$ and $\mathcal{P}(X)$ is the power set of X.

Note that if X is an infinite set, then the continuity of function Q_λ is also required. From now on, here Q_λ will always denote a λ-additive measure on X.

In an earlier paper (see [15]), we introduced the general Poincaré formula for λ-additive measures, which is given by Eq. (2). This formula allows us to compute the λ-additive measure of the union of n sets.

Proposition 1. *If X is a finite set, Q_λ is a λ-additive measure on X, $\lambda \in (-1, \infty)$, $\lambda \neq 0$, $A_1, \ldots, A_n \in \mathcal{P}(X)$ and $n \ge 2$, then*

$$Q_\lambda\left(\bigcup_{i=1}^{n} A_i\right) = \frac{1}{\lambda}\left(\prod_{k=1}^{n}\left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 + \lambda c_{\lambda,k}\right)\right)^{(-1)^{k-1}} - 1\right), \tag{2}$$

where $c_{\lambda,k} = Q_\lambda\left(A_{i_1} \cap \cdots \cap A_{i_k}\right)$.

Proof. See the proof of Theorem 1 in [15]. □

Note that we also gave an elementary proof of Proposition 1 in [17], and we proved the following Proposition in [16]:

Proposition 2. *If X is a finite set, Q_λ is a λ-additive measure on X, $\lambda \in (-1, \infty)$, $\lambda \neq 0$, $A_1, \ldots, A_n \in \mathcal{P}(X)$ and $n \ge 2$, then*

$$Q_\lambda\left(\bigcap_{i=1}^{n} A_i\right) = \frac{1}{\lambda}\left(\prod_{k=1}^{n}\left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 + \lambda d_{\lambda,k}\right)\right)^{(-1)^{k-1}} - 1\right), \tag{3}$$

where $d_{\lambda,k} = Q_\lambda\left(A_{i_1} \cup \cdots \cup A_{i_k}\right)$.

Proof. See the proof of Proposition 2 in [16]. □

We will also utilize the following proposition.

Proposition 3. *Let X be a finite set, $Pr : \mathcal{P}(X) \to [0, 1]$ a probability measure on the set X and for any $\lambda \in (-1, \infty)$, let the function $h_\lambda : [0, 1] \to [0, 1]$ be given by*

$$h_\lambda(x) = \begin{cases} \dfrac{(1+\lambda)^x - 1}{\lambda}, & \text{if } \lambda \neq 0 \\ x, & \text{if } \lambda = 0. \end{cases}$$

Then, the set function $Q_\lambda : \mathcal{P}(X) \to [0, 1]$, which is given by

$$Q_\lambda = h_\lambda \circ Pr, \tag{4}$$

is a λ-additive measure on the set X, and for any $A \in \mathcal{P}(X)$

(a) *if $\lambda < 0$, then $Q_\lambda(A) \geq Pr(A)$*
(b) *if $\lambda = 0$, then $Q_\lambda(A) = Pr(A)$*
(c) *if $\lambda > 0$, then $Q_\lambda(A) \leq Pr(A)$.*

Proof. For the proof that the set function Q_λ in Eq. (4) is a λ-additive measure, see [9] or the proof of Theorem 4.11 in [3].

Let $A \in \mathcal{P}(X)$. Since $Pr(A) \in [0, 1]$, by noting Bernoulli's inequality, we have

$$(1+\lambda)^{Pr(A)} \leq 1 + \lambda Pr(A) \tag{5}$$

for any uniquely determined $\lambda \in (-1, \infty)$. Next from Eq. (5), we have

$$\frac{(1+\lambda)^{Pr(A)} - 1}{\lambda} \geq Pr(A), \quad \text{if } \lambda < 0 \tag{6}$$

$$\frac{(1+\lambda)^{Pr(A)} - 1}{\lambda} \leq Pr(A), \quad \text{if } \lambda > 0 \tag{7}$$

Next, by noting Eq. (4), from Eq. (6) and Eq. (7) we get statement (a) and (c), respectively. Also, statement (b) trivially follows from Eq. (4). □

Remark 1. Note that Eq. (4) means that the λ-additive measure Q_λ is represented by the (Pr, h_λ) pair (see [9]).

3. Bounds for the probabilistic Poincaré formula

Utilizing the results above, we can state the following theorem.

Theorem 1. *Let X be a finite set and let $Pr : \mathcal{P}(X) \to [0, 1]$ be a probability measure on the set X. Then, there exists a λ-additive measure $Q_\lambda : \mathcal{P}(X) \to [0, 1]$ such that*

1) *if $\lambda > 0$, then for any $A_1, A_2, \ldots, A_n \in \mathcal{P}(X)$ and $n \geq 2$,*

$$\begin{aligned} &\frac{1}{\lambda}\left(\prod_{k=1}^{n}\left(\prod_{1\leq i_1<\cdots<i_k\leq n}(1+\lambda c_{\lambda,k})\right)^{(-1)^{k-1}} - 1\right) \leq \\ &\leq \sum_{k=1}^{n}\sum_{1\leq i_1<\cdots<i_k\leq n}(-1)^{k-1}a_k \leq \\ &\leq 1 - \frac{1}{\lambda}\left(\prod_{k=1}^{n}\left(\prod_{1\leq i_1<\cdots<i_k\leq n}(1+\lambda \overline{d}_{\lambda,k})\right)^{(-1)^{k-1}} - 1\right); \end{aligned} \tag{8}$$

and

2) *if* $\lambda < 0$, *then for any* $A_1, A_2, \ldots, A_n \in \mathcal{P}(X)$ *and* $n \geq 2$,

$$1 - \frac{1}{\lambda}\left(\prod_{k=1}^{n}\left(\prod_{1\leq i_1<\cdots<i_k\leq n}\left(1+\lambda \overline{d}_{\lambda,k}\right)\right)^{(-1)^{k-1}} - 1\right) \leq \\ \leq \sum_{k=1}^{n}\sum_{1\leq i_1<\cdots<i_k\leq n}(-1)^{k-1}a_k \leq \\ \leq \frac{1}{\lambda}\left(\prod_{k=1}^{n}\left(\prod_{1\leq i_1<\cdots<i_k\leq n}\left(1+\lambda c_{\lambda,k}\right)\right)^{(-1)^{k-1}} - 1\right), \tag{9}$$

where a_k, $c_{\lambda,k}$ *and* $\overline{d}_{\lambda,k}$ *are respectively given by* $a_k = Pr\left(A_{i_1} \cap \cdots \cap A_{i_k}\right)$, $c_{\lambda,k} = Q_\lambda\left(A_{i_1} \cap \cdots \cap A_{i_k}\right)$ *and* $\overline{d}_{\lambda,k} = Q_\lambda\left(\overline{A}_{i_1} \cup \cdots \cup \overline{A}_{i_k}\right)$.

Proof. Here, we will prove Eq. (8). Note that the proof of Eq. (9) is similar to that of Eq. (8). Let the function $h_\lambda : [0, 1] \to [0, 1]$ be given by

$$h_\lambda(x) = \frac{(1+\lambda)^x - 1}{\lambda}, \tag{10}$$

where $\lambda \in (-1, \infty)$ and $\lambda \neq 0$. Now, let $\lambda > 0$. Then, based on Proposition 3, the set function $Q_\lambda : \mathcal{P}(X) \to [0, 1]$ given by $Q_\lambda = h_\lambda \circ Pr$ is a λ-additive measure and for any $B \in \mathcal{P}(X)$

$$Q_\lambda(B) \leq Pr(B). \tag{11}$$

Now, let λ' be given by

$$\lambda' = -\frac{\lambda}{1+\lambda}. \tag{12}$$

Then, based on Corollary 4.5 in [3], the dual measure $Q_{\lambda'}$ of Q_λ, which is given by

$$Q_{\lambda'}(A) = 1 - Q_\lambda(\overline{A}) \tag{13}$$

for any $A \in \mathcal{P}(X)$, is also a λ additive measure with the parameter λ'. Notice that Eq. (12) is a bijection; and as $\lambda > 0$, we have $-1 < \lambda' < 0$. Then, based on Proposition 3, the set function $Q_{\lambda'} : \mathcal{P}(X) \to [0, 1]$ given by $Q_{\lambda'} = h_{\lambda'} \circ Pr$ is a λ-additive measure and for any $B \in \mathcal{P}(X)$

$$Pr(B) \leq Q_{\lambda'}(B). \tag{14}$$

Next, by noting Eq. (11), Eq. (13) and Eq. (14), we have

$$Q_\lambda(B) \leq Pr(B) \leq 1 - Q_\lambda(\overline{B}) \tag{15}$$

for any $B \in \mathcal{P}(X)$, $\lambda > 0$. Now, let $B = \bigcup_{i=1}^{n} A_i$. Then, by utilizing the De Morgan law, from Eq. (15), we have

$$Q_\lambda\left(\bigcup_{i=1}^{n} A_i\right) \leq Pr\left(\bigcup_{i=1}^{n} A_i\right) \leq 1 - Q_\lambda\left(\bigcap_{i=1}^{n} \overline{A}_i\right). \tag{16}$$

Here, by noting the probabilistic Poincaré formula in Eq. (1), the general Poincaré formula for λ-additive measures in Eq. (2) (see Proposition 1), and Eq. (3) (see Proposition 2), from Eq. (16), we get Eq. (8). □

Remark 2. Because $\lim\limits_{\lambda\to 0} \frac{(1+\lambda)^x - 1}{\lambda} = x$, we see that

$$\lim_{\lambda\to 0} Q_\lambda(B) = \lim_{\lambda\to 0} h_\lambda(Pr(B)) = Pr(B).$$

So, the inequalities in Eq. (8) and Eq. (9) furnish sharp (i.e. small) approximation intervals for the midterm in case λ is very close to zero.

Remark 3. Eq. (15) and Eq. (16) represent a well-known property of the λ-additive measure. Namely, if $\lambda > 0$, then the λ-additive measure Q_λ is a belief measure, and its dual measure, which is given by $1 - Q_\lambda(\overline{A})$ for any $A \in \mathcal{P}(X)$, is a plausibility measure (see Theorem 4.21 in [3]).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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