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Generalizing the sigmoid function using continuous-valued logic

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Abstract

In this study, we present a continuous-valued logical approach to generalize the sigmoid function. Our starting point is the kappa function, which is known as a unary operator in continuous-valued logic. First we extend the kappa function to the (a, b) interval, and then we interpret the generalized sigmoid function as the limit of the extended kappa function when a and b tend to the negative and positive infinity, respectively. Since the extended kappa function is induced by an additive generator of a strict t-norm or strict t-conorm, the generalized sigmoid function is operator dependent. Based on the properties of this new function, we show that it can be viewed as the generalization of the classical sigmoid function. Also, we demonstrate that the classical sigmoid function is a special case of the generalized sigmoid function. Next, we provide a sufficient condition for the equality of two generalized sigmoid functions. It is well known that the classical sigmoid function can be utilized in logistic regression and in preference modeling. Here, we demonstrate how the logistic regression can be generalized using the generator function-based sigmoid function. Also, we show that the generalized sigmoid function can be viewed as a preference measure.

Keywords: Continuous-valued logic; Strict operators; Sigmoid function; Preference measure; Logistic regression

1. Introduction

The sigmoid function, which is also known as the logistic function, has a wide range of applications in many areas of science, including mathematics, computer science, economics, biology, the medical sciences and engineering. The following examples highlight some of the recent studies connected with the sigmoid function. Iliev et al. [1] presented Hausdorff approximations to the Heaviside step function using several sigmoid functions. Han and Tian [2] introduced a new sigmoid-like complementary function to estimate evaporation with physical constraints. A sigmoid function-based integral-derivative observer and its application to autopilot design was presented by Shao et al. [3]. Using the semilog-sigmoid function, Nomura et al. [4] provided a modified expression of the Kozeny-Carman equation that is utilized in the field of fluid dynamics. Kyurkchiev et al. [5] presented extensions of some families of sigmoid functions and their applications in growth theory.

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1.1. Sigmoid functions in machine learning, neural networks and optimization

It is worth mentioning that machine learning, neural networks and optimization are important areas of application of the sigmoid function. The following examples relate to some recent results in these fields. Elfwing et al. [6] presented a paper on sigmoid-weighted linear units for neural network function approximation in reinforcement learning. Alfarozi et al. [7] described a local sigmoid method, which is a non-iterative deterministic learning algorithm for the automatic model construction of neural networks. Optimized deep belief networks with improved logistic sigmoid units and their applications were presented by Qin et al. [8]. In a paper of Liu et al. [9], a novel, sigmoid function-based particle swarm optimizer was described. Qiao et al. [10] presented a mutual information-based weight initialization method for sigmoidal feed-forward neural networks.

1.2. Sigmoid functions in regression analysis

The logistic function is also widely utilized in distribution theory and in logistic regression. Aljarrah et al. [11] presented a generalized asymmetric logistic distribution and its application in regression analysis. Khairunnahar et al. [12] used the logistic regression method to classify malignant and benign tissue. An application of logistic regression and fuzzy logic to traffic management was described in a paper by Tomar et al. [13]. Jagadeesh et al. [14] utilized the logistic regression algorithm to forecast the output of a solar plant. A skewed logistic distribution was applied to statistical temperature post-processing in mountainous areas by Gebetsberger et al. [15].

1.3. Generalization of the sigmoid function

In this study, we present a continuous-valued logical approach to generalize the sigmoid function. This approach includes the following three main stages. (1) Our starting point is a generalized class of unary continuous-valued logical operators, called the kappa function (see [16]). (2) First we extend this function from the $(0,1)$ domain to the (a,b) domain, and then (3) we derive the generalized sigmoid function as a limit of the extended kappa function. In a previous paper of ours [17], we proved that the classical sigmoid function is just the limit of the extended kappa function when the latter is induced by a generator function of the Dombi operators. That is, we interpret the generalized sigmoid function as an asymptotic extended kappa function, which is induced by an arbitrary generator function of a conjunctive or disjunctive operator. This approach leads us to the generalized sigmoid function $\sigma^{(\lambda)}_{x_0,\nu,f}\colon \mathbb{R} \to (0,1)$, which is given by (see later on):

$$\sigma^{(\lambda)}_{x_0,\nu,f}(x) = f^{-1}\left(f(\nu)\,\mathrm{e}^{\lambda \frac{f'(\nu)}{f(\nu)}(x-x_0)} \right),$$

where f is a differentiable generator function of a strict t-norm or a strict t-conorm, $\nu \in (0,1)$, $\lambda, x_0 \in \mathbb{R}$ and $\lambda \neq 0$.

The motivations for developing this new sigmoid function are as follows. The generalized sigmoid function is generator function-dependent, i.e., its properties depend not only on its parameter values, but on the choice of its generator function f as well. This characteristic of the generalized sigmoid function makes it more flexible compared with the classical sigmoid function. On the other hand, if we set $f(x) = \frac{1-x}{x}$, $x \in (0,1)$ and $\nu = \frac{1}{2}$, then we get the classical sigmoid function as a special case of the generalized sigmoid function. For example, as we will present it in Section 7, using the generalized sigmoid function, we can generalize the logistic regression such that depending on the choice of the generator function, we can obtain various regression models with the same number of parameters in each of them. Furthermore, if the generator function is parametric, then its parameters become parameters of the generalized sigmoid function as well increasing the flexibility of this latter further. Next, it is well known that the classical sigmoid function is widely applied in neural networks as activation function. Taking into account the above-mentioned characteristics of the generalized sigmoid function, the latter may be viewed as a new and more flexible alternative activation function.

In our study, we will point out that there are several benefits of the connections between the generalized sigmoid function and continuous valued logic. We will show that this new sigmoid function is closely related to the so-called preference implication (see [18]) and it can be used to model soft inequalities. Here, we will also demonstrate that in a Pliant logical system (see [19]), the generalized sigmoid functions induced by additive generators of a strict triangular norm and a strict triangular conorm coincide if the λ parameters of the two generalized sigmoid functions differ only in sign. Another interesting finding, which we will present here, is that applying the Pliant negation (Dombi form of negation) to a generalized sigmoid function inverts the sign of its λ parameter. It is worth noting that we obtain the n-variate generator function-dependent sigmoid function by applying the n-ary aggregative operator, which is a representable uninorm (see [20] and [21]), to n generalized sigmoid functions.

After describing the main properties of this novel, generator function-dependent sigmoid function, we give a sufficient condition for the equality of two generalized sigmoid functions. Next, we present two applications of the generalized sigmoid function. Namely, we show that this new function may be viewed as a general preference measure. Also, we describe how logistic regression can be generalized using the generator function-dependent sigmoid function.

This paper is structured as follows. In Section 2, we briefly summarize the basic notions and notations, which will be used in this study. Next, we show how the generalized sigmoid function can be derived from the extended kappa function in Section 3. In Section 4, we provide a sufficient condition for equality of generalized sigmoid functions. Some interesting properties of the generalized sigmoid function in Pliant logical systems are introduced in Section 5. In Section 6, we explain why the generalized sigmoid function can be viewed as a preference measure. In Section 7, we present the generalized logistic regression procedure. Lastly, the key findings of our study are summarized in Section 8.

2. Preliminaries

Here, we will present the basic notions and notations, which we will use later on. We will use the following representation theorem of Aczél [22] (also see [23]).

Theorem 1. *A continuous and strictly increasing function $F\colon [a,b]^2 \to [a,b]$ is associative if and only if*

$$F(x,y) = f^{-1}\left(f(x) + f(y)\right),$$

where $f\colon [a,b] \to [0,\infty]$ is a strictly decreasing or strictly increasing continuous function. Here, f is called a generator function of F, and F is uniquely determined up to constant multiplier of f.

In continuous-valued logic, the concepts of strict triangular norm (strict t-norm) and strict triangular conorm (strict t-conorm) play an important role. Since we require that these operators should be associative, based on Theorem 1, we will use the following definition of strict t-norms and strict t-conorms. Note that in this article, we will refer to strict t-norms and t-conorms as conjunctive and disjunctive operators denoted by c and d, respectively.

Definition 1. We say that the function $o\colon [0,1]^2 \to [0,1]$ is a strict t-norm (strict t-conorm, respectively) if and only if o is continuous, and there exists a continuous and strictly decreasing (increasing, respectively) function $f\colon [0,1] \to [0,\infty]$, called a generator function of o, such that

$$o(x,y) = f^{-1}\left(f(x) + f(y)\right),$$

for any $x, y \in [0,1]$, and

(a) for a strict t-norm c, $f = f_c$ is strictly decreasing with $f_c(1) = 0$ and $\lim_{x\to 0} f_c(x) = \infty$;
(b) for a strict t-conorm d, $f = f_d$ is strictly increasing with $f_d(0) = 0$ and $\lim_{x\to 1} f_d(x) = \infty$.

Note that the strict t-norm and strict t-conorm are special cases of the generalized t-norm and t-conorm classes, respectively.

From now on, the mapping $f\colon [0,1] \to [0,\infty]$ will always be a continuous, strictly decreasing (increasing, respectively) generator function of a conjunctive (disjunctive, respectively) operator. Also, if f is strictly decreasing, then we will interpret $f(0) = \infty$ and $f^{-1}(\infty) = 0$. Similarly, if f is strictly increasing, then we will interpret the end points $f(1) = \infty$ and $f^{-1}(\infty) = 1$.

In this study, we seek to find connections between the sigmoid function and continuous-valued logic. Therefore, we will utilize the classical sigmoid function, which is also known as the logistic function. This function is defined as follows.

Definition 2. The sigmoid function $\sigma_{x_0}^{(\lambda)}: [-\infty, \infty] \to (0, 1)$ with the parameters $x_0, \lambda \in \mathbb{R}$, $\lambda \neq 0$ is given by

$$\sigma_{x_0}^{(\lambda)}(x) = \frac{1}{1 + \mathrm{e}^{-\lambda(x - x_0)}}. \tag{1}$$

In continuous-valued logic, a generalized class of unary operators, called the kappa function, was introduced by Dombi [16]. This operator class has several applications (see e.g., [24–26]).

We will describe an important connection between sigmoid and kappa functions. In the following, we will show how we can derive a generalized sigmoid function from the kappa function. Also, we will demonstrate that the classical sigmoid function is just a special case of this generalized sigmoid function.

Definition 3. The kappa function $\kappa_{\nu,\nu_0}^{(\lambda)}: (0, 1) \to (0, 1)$ is given by

$$\kappa_{\nu,\nu_0}^{(\lambda)}(x) = f^{-1}\left(f(\nu_0)\left(\frac{f(x)}{f(\nu)}\right)^{\lambda}\right), \tag{2}$$

where $0 < \nu < 1$, $0 < \nu_0 < 1$, $\lambda \in \mathbb{R}$ and f is a generator function of a conjunctive or disjunctive operator.

The generator function of the Dombi conjunction and disjunction operators is the function $f_D: (0, 1) \to (0, \infty)$ that is given by

$$f_D(x) = \left(\frac{1 - x}{x}\right)^{\alpha}, \tag{3}$$

where $\alpha \neq 0$ (see, [27,28]). If $\alpha > 0$, then f_D is a generator function of a conjunctive operator; and if $\alpha < 0$, then f_D is a generator function of a disjunctive operator. Now, let $0 < \nu < 1$, $0 < \nu_0 < 1$ and $\alpha \neq 0$. Then, after direct calculation, we find that the kappa function $\kappa_{\nu,\nu_0}^{(\lambda)}$ induced by f_D is

$$\kappa_{\nu,\nu_0,f_D}^{(\lambda)}(x) = \frac{1}{1 + \frac{1-\nu_0}{\nu_0}\left(\frac{\nu}{1-\nu}\frac{1-x}{x}\right)^{\lambda}}, \tag{4}$$

where $x \in (0, 1)$. Note that $\kappa_{\nu,\nu_0,f_D}^{(\lambda)}$ is independent of the parameter α. That is, the kappa functions induced by generator functions of conjunctive and disjunctive Dombi operators coincide, i.e., the kappa function is independent of the operator type (conjunctive or disjunctive). We will refer to the kappa function in Eq. (4) as the Dombi form of the kappa function.

In a previous article of ours [17], we extended the kappa function $\kappa_{\nu,\nu_0,f_D}^{(\lambda)}$, which is given in Eq. (4), to the interval (a, b) as follows:

$$\kappa_{x_\nu,\nu_0,f_D}^{(\lambda)}(a, b; x) = \frac{1}{1 + \frac{1-\nu_0}{\nu_0}\left(\frac{x_\nu - a}{b - x_\nu}\frac{b-x}{x-a}\right)^{\lambda}}, \tag{5}$$

where $x \in (a, b)$, $a < x_\nu < b$, $\nu_0 \in (0, 1)$, $\lambda \in \mathbb{R}$, and used it to approximate the sigmoid function given in Definition 2. There, we proved the following proposition.

Proposition 1. *Let $\kappa_{x_\nu,\nu_0,f_D}^{(\lambda)}(a, b; \cdot)$ be an extended kappa function given by Eq. (5), and let $\sigma_{x_0}^{(\lambda_\sigma)}$ be a sigmoid function, where $a < x_\nu < b$, $\nu_0 \in (0, 1)$, $x_0, \lambda, \lambda_\sigma \in \mathbb{R}$. If*

$$\frac{a + b}{2} = x_0, \quad x_\nu = x_0, \quad \nu_0 = \frac{1}{2}, \quad \lambda = \lambda_\sigma \frac{b - a}{4},$$

then for any $x \in (a, b)$,

$$\lim_{\substack{a \to -\infty \\ b \to +\infty}} \kappa_{x_\nu,\nu_0,f_D}^{(\lambda)}(a, b; x) = \sigma_{x_0}^{(\lambda_\sigma)}(x),$$

where $\lim_{\substack{a \to -\infty \\ b \to +\infty}}$ is interpreted such that $a \to -\infty$, $b \to +\infty$ and $\frac{a+b}{2} = x_0$.

This theorem tells us that the extended version of the Dombi form of the kappa function can be used to approximate the sigmoid function. Below, we will show how a generalized sigmoid function can be derived from the extended version of the generator function-dependent kappa function defined above in Definition 3.

We will utilize the preference implication operator, which was introduced by Dombi and Baczyński [18]. This operator is defined as follows.

Definition 4. Let $\nu \in (0, 1)$ and let f be a generator function of a conjunctive or disjunctive continuous-valued logical operator. The preference implication $p_\nu : [0, 1]^2 \to [0, 1]$ is given by

$$p_\nu(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \{(0,0), (1,1)\} \\ f^{-1}\left(f(\nu)\frac{f(y)}{f(x)}\right), & \text{otherwise,} \end{cases} \tag{6}$$

where $x, y \in [0, 1]$.

In this implication, the parameter ν can be interpreted as a threshold. That is, if the value of the implication is greater than ν, then it is true. The following proposition (see Proposition 5 in [18]) concerns the ordering principle of the preference implication.

Proposition 2. *For any $x, y \in [0, 1]$ and $\nu \in (0, 1)$, we have $p_\nu(x, y) > \nu$ if and only if $x < y$.*

Moreover, based on this proposition, $p_\nu(x, y)$ can be interpreted as the continuous logical value of the preference $x < y$.

We will make use of the aggregative operator $a_{\nu_*} : [0, 1]^n \to [0, 1]$, which is given by

$$a_{\nu_*}(\mathbf{x}) = f^{-1}\left(f(\nu_*)\prod_{i=1}^{n}\frac{f(x_i)}{f(\nu_*)}\right), \tag{7}$$

where $\nu_* \in (0, 1)$, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in [0, 1]^n$ and f is a generator function of a conjunctive or disjunctive operator. This operator was introduced by Dombi in 1982 (see [20]) and studied further in [21]. The binary aggregative operator is a representable uninorm with the neutral element ν_*, i.e., $a_{\nu_*}(x, \nu_*) = x$, $x \in [0, 1]$. A uninorm is a generalization of t-norms and t-conorms, i.e., adjusting its neutral element ν_*, a uninorm is a t-norm if $\nu_* = 1$ and a t-conorm if $\nu_* = 0$. The definition of uninorms was originally given by Yager and Rybalov [29] in 1996. For more details on (representable) uninorms see Klement et al. [30], Klement et al. [23], Fodor and De Baets [31] and Mas et al. [32].

Later, we will utilize the Pliant negation (also known as the Dombi form of the negation), which is defined as follows (see [19,21]).

Definition 5. Let $f : [0, 1] \to [0, \infty]$ be a generator function of a conjunctive or disjunctive operator and let $\nu \in (0, 1)$. The mapping $\eta_\nu : [0, 1] \to [0, 1]$ given by

$$\eta_\nu(x) = f^{-1}\left(\frac{f^2(\nu)}{f(x)}\right) \tag{8}$$

is a Pliant negation operator with the parameter ν.

Using the Pliant negation, a Pliant logical system can be defined as follows (see [19]).

Definition 6. Let the conjunctive operator c and the disjunctive operator d be induced by generator functions f_c and f_d, respectively, and let $\nu \in (0, 1)$. We say that the triplet (c, d, η_ν) is a Pliant system if and only if

$$f_c(x) f_d(x) = 1$$

holds for any $x \in [0, 1]$, and η_ν is a Pliant negation operator induced by f_c or f_d.

3. The generalized sigmoid function as limit of the kappa function

Here, first we will extend the kappa function to the interval (a, b). Next, we will study the case where $a \to -\infty$ and $b \to \infty$ such that $\frac{a+b}{2}$ is being constant, and we will call the result of this limit the generalized sigmoid function.

Using the definition for a generator function-dependent kappa function in Definition 3, its extension to the interval (a, b) can be defined as follows.

Definition 7. The generator function-dependent extended kappa function $\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; \cdot)\colon (a, b) \to (0, 1)$ is given by

$$\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; x) = f^{-1}\left(f(\nu_0)\left(\frac{f\left(\frac{x-a}{b-a}\right)}{f\left(\frac{x_\nu-a}{b-a}\right)}\right)^{\lambda}\right), \tag{9}$$

where $a < x_\nu < b$, $\nu_0 \in (0, 1)$, $\lambda \in \mathbb{R}$ and f is a generator function of a conjunctive or disjunctive operator.

Next, we will summarize the main properties of the extended kappa function $\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; \cdot)$.

Proposition 3. *The extended kappa function $\kappa^{(\lambda)}_{x_\nu,\nu_0}(a, b; \cdot)$, which is given in Definition 7, has the following properties:*

(a) $\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; x)$ *is a continuous function in* (a, b).

(b) *If* $\lambda > 0$, *then* $\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; \cdot)$ *is strictly increasing.*
If $\lambda < 0$, *then* $\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; \cdot)$ *is strictly decreasing.*
If $\lambda = 0$, *then* $\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; \cdot)$ *has a constant value of* ν_0.

(c)

$$\lim_{x\to a^+} \kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; x) = \begin{cases} 0, & \text{if } \lambda > 0 \\ 1, & \text{if } \lambda < 0 \end{cases}$$

$$\lim_{x\to b^-} \kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; x) = \begin{cases} 1, & \text{if } \lambda > 0 \\ 0, & \text{if } \lambda < 0. \end{cases}$$

(d) $\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; x_\nu) = \nu_0$.

(e) *The first derivative of* $\kappa^{(\lambda)}_{x_\nu,\nu_0}(a, b; x)$ *at* $x = x_\nu$ *is*

$$\left.\frac{\mathrm{d}\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; x)}{\mathrm{d}x}\right|_{x=x_\nu} =$$

$$= \frac{\lambda}{b-a}\frac{f(\nu_0)}{f'\left(\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; x)\right)}\frac{f^{\lambda}\left(\frac{x-a}{b-a}\right)}{f^{\lambda}\left(\frac{x_\nu-a}{b-a}\right)}\left.\frac{f'\left(\frac{x-a}{b-a}\right)}{f\left(\frac{x-a}{b-a}\right)}\right|_{x=x_\nu} =$$

$$= \frac{\lambda}{b-a}\frac{f(\nu_0)}{f'(\nu_0)}\frac{f'\left(\frac{x_\nu-a}{b-a}\right)}{f\left(\frac{x_\nu-a}{b-a}\right)};$$

that is, the λ *parameter determines the slope of function* $\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a, b; x)$ *at* $x = x_\nu$, *if* ν_0 *is fixed.*

Proof. These properties immediately follow from the definition of the extended kappa function given in Definition 7. □

Fig. 1 shows two plots of extended kappa functions. In this figure, we can see the effect of the parameter ν_0 for a fixed value of the parameter x_ν.

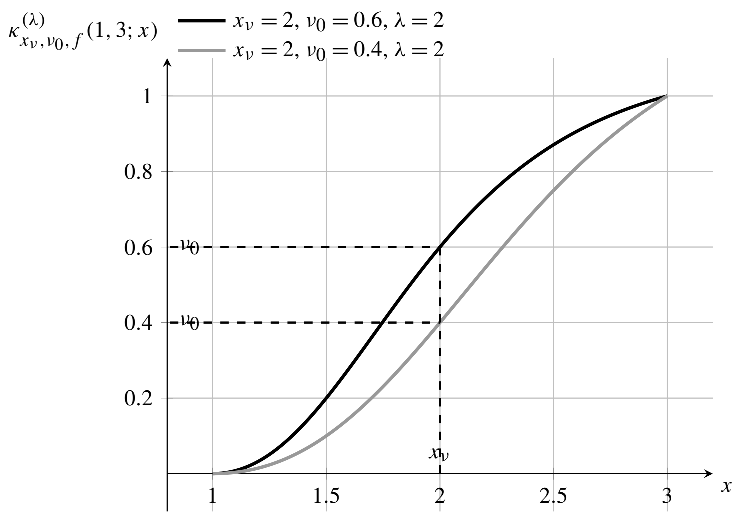

Fig. 1. Example plots of extended kappa functions with the generator $f = f_D$.

We know that if $f = f_D$, where f_D is a generator function of the Dombi operators, then under the conditions of Proposition 1, the classical sigmoid function is the limit of the generator function-dependent extended kappa function $\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a,b,\cdot)$. That is,

$$\lim_{\substack{a\to-\infty\\ b\to+\infty}} \kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a,b;x) = \lim_{\substack{a\to-\infty\\ b\to+\infty}} \kappa^{(\lambda)}_{x_\nu,\nu_0,f_D}(a,b;x) = \sigma^{(\lambda_\sigma)}_{x_0}(x)$$

for any $x \in (a,b)$. Here, $\lim_{\substack{a\to-\infty\\ b\to+\infty}}$ means that $a \to -\infty$, $b \to +\infty$ and $\frac{a+b}{2} = x_0$, i.e., a and b tends to $-\infty$ and $+\infty$, respectively, with the same degree.

Following this line of thinking, we will interpret the generalized sigmoid function as an asymptotic extended kappa function, which is induced by an arbitrary generator function of a conjunctive or disjunctive operator. This idea is based on the following result.

Theorem 2. *Let f be a differentiable generator function of a conjunctive or disjunctive operator, and let f' be its derivative, $f'\left(\frac{1}{2}\right) \neq 0$. Let $\kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a,b;\cdot)$ be an extended kappa function induced by f, where $a < x_\nu < b$, $\nu_0 \in (0,1)$, $x_0, \lambda \in \mathbb{R}$. Let $x_0, \lambda_\sigma \in \mathbb{R}$, $\lambda_\sigma \neq 0$. If*

$$\frac{a+b}{2} = x_0, \quad x_\nu = x_0, \quad \lambda = \lambda_\sigma (b-a)\frac{f'(\nu_0)}{f(\nu_0)}\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)},$$

then for any $x \in (a,b)$,

$$\lim_{\substack{a\to-\infty\\ b\to+\infty}} \kappa^{(\lambda)}_{x_\nu,\nu_0,f}(a,b;x) = f^{-1}\left(f(\nu_0)\,\mathrm{e}^{\lambda_\sigma \frac{f'(\nu_0)}{f(\nu_0)}(x-x_0)}\right), \tag{10}$$

where $\lim_{\substack{a\to-\infty\\ b\to+\infty}}$ is interpreted such that $a \to -\infty$, $b \to +\infty$ and $\frac{a+b}{2} = x_0$.

Proof. Since $\frac{a+b}{2} = x_0$; we can write $a = x_0 - \Delta$, $b = x_0 + \Delta$, where $\Delta = \frac{b-a}{2}$. Notice that the condition $a \to -\infty$, $b \to +\infty$ such that $\frac{a+b}{2} = x_0$ is equivalent to $\Delta \to \infty$. Using variable Δ and noting the conditions of the theorem, Eq. (10) can be written as

$$\lim_{\Delta\to\infty} f^{-1}\left(f(\nu_0)\left(\left(\frac{f\left(\frac{x-x_0+\Delta}{2\Delta}\right)}{f\left(\frac{1}{2}\right)}\right)^{2\Delta\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}}\right)^{\lambda_\sigma\frac{f'(\nu_0)}{f(\nu_0)}}\right) = \tag{11}$$

$$= f^{-1}\left(f(\nu_0)\,\mathrm{e}^{\lambda_\sigma\frac{f'(\nu_0)}{f(\nu_0)}(x-x_0)}\right).$$

To prove Eq. (11), it is sufficient to show that

$$\lim_{\Delta\to\infty}\left(\frac{f\left(\frac{x-x_0+\Delta}{2\Delta}\right)}{f\left(\frac{1}{2}\right)}\right)^{2\Delta\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}} = \mathrm{e}^{x-x_0}. \tag{12}$$

Here, Eq. (12) is equivalent to

$$\lim_{\Delta\to\infty}\left(2\Delta\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}\ln\left(\frac{f\left(\frac{x-x_0+\Delta}{2\Delta}\right)}{f\left(\frac{1}{2}\right)}\right)\right) = x - x_0. \tag{13}$$

We can use the L'Hospital rule to prove Eq. (13). Namely, after direct calculation, we have

$$\lim_{\Delta\to\infty}\left(2\Delta\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}\ln\left(\frac{f\left(\frac{x-x_0+\Delta}{2\Delta}\right)}{f\left(\frac{1}{2}\right)}\right)\right) =$$

$$= 2\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}\lim_{\Delta\to\infty}\left(\frac{\left(\ln\left(\frac{f\left(\frac{x-x_0+\Delta}{2\Delta}\right)}{f\left(\frac{1}{2}\right)}\right)\right)'}{\left(\frac{1}{\Delta}\right)'}\right) =$$

$$= 2\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}\lim_{\Delta\to\infty}\left(\frac{f'\left(\frac{x-x_0+\Delta}{2\Delta}\right)}{f\left(\frac{x-x_0+\Delta}{2\Delta}\right)}\frac{x-x_0}{2}\right) = x - x_0. \quad \square$$

Remark 1. Let us consider the sigmoid function

$$\sigma_{x_0}^{(4\lambda_\sigma)}(x) = \frac{1}{1+\mathrm{e}^{-4\lambda_\sigma(x-x_0)}}.$$

Based on Proposition 1, we know that if

$$\frac{a+b}{2} = x_0, \quad x_\nu = x_0, \quad \nu_0 = \frac{1}{2}, \quad \lambda = 4\lambda_\sigma\frac{b-a}{4},$$

then for any $x \in (a,b)$,

$$\lim_{\substack{a\to-\infty\\ b\to+\infty}} \kappa_{x_\nu,\nu_0,f_D}^{(\lambda)}(a,b;x) = \sigma_{x_0}^{(4\lambda_\sigma)}(x),$$

where $\lim_{\substack{a\to-\infty\\ b\to+\infty}}$ means that $a \to -\infty$, $b \to +\infty$ and $\frac{a+b}{2} = x_0$.

Now, let $f(x) = \frac{1-x}{x}$, where $x \in (0,1)$, i.e., f is a generator function of the Dombi operators in Eq. (3) with $\alpha = 1$. In this case,

$$\kappa_{x_\nu,\nu_0,f}^{(\lambda)}(a,b;x) = \kappa_{x_\nu,\nu_0,f_D}^{(\lambda)}(a,b;x)$$

for any $x \in (a, b)$. Here, for $\nu_0 = \frac{1}{2}$, we have $f(\nu_0) = 1$, $f'(\nu_0) = -4$ and so

$$\lambda = \lambda_\sigma (b-a) \frac{f'(\nu_0)}{f(\nu_0)} \frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)} = \lambda_\sigma (b-a)$$

and

$$f^{-1}\left(f(\nu_0)\, \mathrm{e}^{\lambda_\sigma \frac{f'(\nu_0)}{f(\nu_0)}(x-x_0)} \right) = \frac{1}{1+\mathrm{e}^{-4\lambda_\sigma (x-x_0)}}.$$

Therefore, applying Theorem 2 with $f(x) = \frac{1-x}{x}$ and $\nu_0 = \frac{1}{2}$, we can say that if

$$\frac{a+b}{2} = x_0, \quad x_\nu = x_0, \quad \lambda = \lambda_\sigma (b-a),$$

then for any $x \in (a, b)$,

$$\lim_{\substack{a \to -\infty \\ b \to +\infty}} \kappa^{(\lambda)}_{x_\nu, \nu_0, f}(a, b; x) = \frac{1}{1+\mathrm{e}^{-4\lambda_\sigma (x-x_0)}}.$$

This means that Proposition 1 is just a special case of Theorem 2.

Now, based on Theorem 2, we will define the generalized sigmoid function according to Eq. (10) as follows.

Definition 8. Let f be a differentiable generator function of a conjunctive or disjunctive operator. The generalized sigmoid function induced by f is a mapping $\sigma^{(\lambda)}_{x_0,\nu,f} : \mathbb{R} \to (0, 1)$, which is given by

$$\sigma^{(\lambda)}_{x_0,\nu,f}(x) = f^{-1}\left(f(\nu)\, \mathrm{e}^{\lambda \frac{f'(\nu)}{f(\nu)}(x-x_0)} \right), \tag{14}$$

where $\nu \in (0, 1)$, $\lambda, x_0 \in \mathbb{R}$ and $\lambda \neq 0$.

We will also refer to this function as the generator function-dependent sigmoid function because the generalized sigmoid function given in Definition 8 is induced by a generator function f. Here, we will briefly summarize the main properties of the generalized sigmoid function. It should be added that based on these properties, $\sigma^{(\lambda)}_{x_0,\nu,f}$ can be treated as a generalization of the classical sigmoid function.

Proposition 4. *The generator function-dependent sigmoid function $\sigma^{(\lambda)}_{x_0,\nu,f}$, which is given in Definition 8, has the following properties:*

(a) *$\sigma^{(\lambda)}_{x_0,\nu,f}$ is a continuous and differentiable function.*

(b) *The derivative function of $\sigma^{(\lambda)}_{x_0,\nu,f}$ is*

$$\left(\sigma^{(\lambda)}_{x_0,\nu,f}(x)\right)' = \lambda \frac{f'(\nu)}{f(\nu)} \frac{f\left(\sigma^{(\lambda)}_{x_0,\nu,f}(x)\right)}{f'\left(\sigma^{(\lambda)}_{x_0,\nu,f}(x)\right)}.$$

(c) *If $\lambda > 0$ ($\lambda < 0$, respectively), then $\sigma^{(\lambda)}_{x_0,\nu,f}$ is strictly increasing (strictly decreasing, respectively).*

(d) *The values of $\sigma^{(\lambda)}_{x_0,\nu,f}(x)$ and its derivative at $x = x_0$ are*

$$\sigma^{(\lambda)}_{x_0,\nu,f}(x_0) = \nu \quad \text{and} \quad (\sigma^{(\lambda)}_{x_0,\nu,f}(x))'_{|x=x_0} = \lambda.$$

(e) *If $\lambda > 0$, then*

$$\lim_{x \to -\infty} \sigma^{(\lambda)}_{x_0,\nu,f}(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} \sigma^{(\lambda)}_{x_0,\nu,f}(x) = 1.$$

Table 1
Examples of generator function-dependent sigmoid functions.

$f(x)$	$\sigma^{(\lambda)}_{x_0,\nu,f}(x)$	$\sigma^{(\lambda^*)}_{x_0^*,f}(x)$
$\frac{1-x}{x}$	$\left(1+\frac{1-\nu}{\nu}\,\mathrm{e}^{\frac{-\lambda}{\nu(1-\nu)}(x-x_0)}\right)^{-1}$	$\left(1+\mathrm{e}^{-\lambda^*\left(x-x_0^*\right)}\right)^{-1}$
$\frac{x}{1-x}$	$\left(1+\frac{1-\nu}{\nu}\,\mathrm{e}^{\frac{-\lambda}{\nu(1-\nu)}(x-x_0)}\right)^{-1}$	$\left(1+\mathrm{e}^{-\lambda^*\left(x-x_0^*\right)}\right)^{-1}$
$-\ln(x)$	$\nu^{\,\mathrm{e}^{\frac{\lambda}{\nu\ln(\nu)}(x-x_0)}}$	$\mathrm{e}^{-\mathrm{e}^{-\lambda^*\left(x-x_0^*\right)}}$
$-\ln(1-x)$	$1-(1-\nu)^{\mathrm{e}^{-\frac{\lambda(x-x_0)}{(1-\nu)\ln(1-\nu)}}}$	$1-\mathrm{e}^{-\mathrm{e}^{-\lambda^*\left(x-x_0^*\right)}}$

If $\lambda < 0$, then

$$\lim_{x\to-\infty}\sigma^{(\lambda)}_{x_0,\nu,f}(x)=1 \quad \textit{and} \quad \lim_{x\to\infty}\sigma^{(\lambda)}_{x_0,\nu,f}(x)=0.$$

Proof. These properties immediately follow from Definition 8 and from the properties of a generator function f. □

In certain cases, based on practical considerations, it is more convenient to use alternative forms of the generalized sigmoid function.

Remark 2. Since $\frac{f'(\nu)}{f(\nu)} \neq 0$, with the substitution

$$\lambda^* = -\lambda\frac{f'(\nu)}{f(\nu)},$$

the generalized sigmoid function in Eq. (14) can be written as

$$\sigma^{(\lambda^*)}_{x_0,\nu,f}(x) = f^{-1}\left(f(\nu)\,\mathrm{e}^{-\lambda^*(x-x_0)}\right). \tag{15}$$

Noting that $\lambda^* \neq 0$, using the substitution

$$x_0^* = x_0 + \frac{\ln(f(\nu))}{\lambda^*},$$

Eq. (15) we can be written as

$$\sigma^{(\lambda^*)}_{x_0^*,f}(x) = f^{-1}\left(\mathrm{e}^{-\lambda^*(x-x_0^*)}\right). \tag{16}$$

We can use the alternative forms of the generalized sigmoid function given by Eq. (15) and Eq. (16).

3.1. Examples of generator function-dependent sigmoid functions

Table 1 shows some examples of generalized sigmoid functions induced by various generator functions. The expressions for the corresponding λ^* and x_0^* parameters are shown in Table 2.

Fig. 2 shows sample plots of generator function-dependent sigmoid functions.

Remark 3. Notice that the generalized sigmoid functions induced by the generator functions $f(x)=\frac{1-x}{x}$ and $f(x)=\frac{x}{1-x}$ are centrally symmetric to the x_0 locus, while other generator functions can produce asymmetric generalized sigmoid functions. This feature of the generalized sigmoid function is useful in practice; for example, we can apply a generalized sigmoid function as an asymmetric logistic regression function (see Section 7).

Remark 4. It can be proven that if f is a generator function of a conjunctive or disjunctive operator, then

$$f_{\alpha,\gamma}(x) = \ln(1+\gamma f^{\alpha}(x)),$$

Table 2
The λ^* and x_0^* values for the sigmoid functions in Table 1.

$f(x)$	λ^*	x_0^*
$\frac{1-x}{x}$	$\frac{\lambda}{\nu(1-\nu)}$	$x_0 + \frac{\ln\left(\frac{1-\nu}{\nu}\right)}{\lambda^*}$
$\frac{x}{1-x}$	$\frac{\lambda}{\nu(1-\nu)}$	$x_0 + \frac{\ln\left(\frac{1-\nu}{\nu}\right)}{\lambda^*}$
$-\ln(x)$	$-\frac{\lambda}{\nu\ln(\nu)}$	$x_0 + \frac{\ln(-\ln(\nu))}{\lambda^*}$
$-\ln(1-x)$	$\frac{\lambda}{(1-\nu)\ln(1-\nu)}$	$x_0 + \frac{\ln(-\ln(1-\nu))}{\lambda^*}$

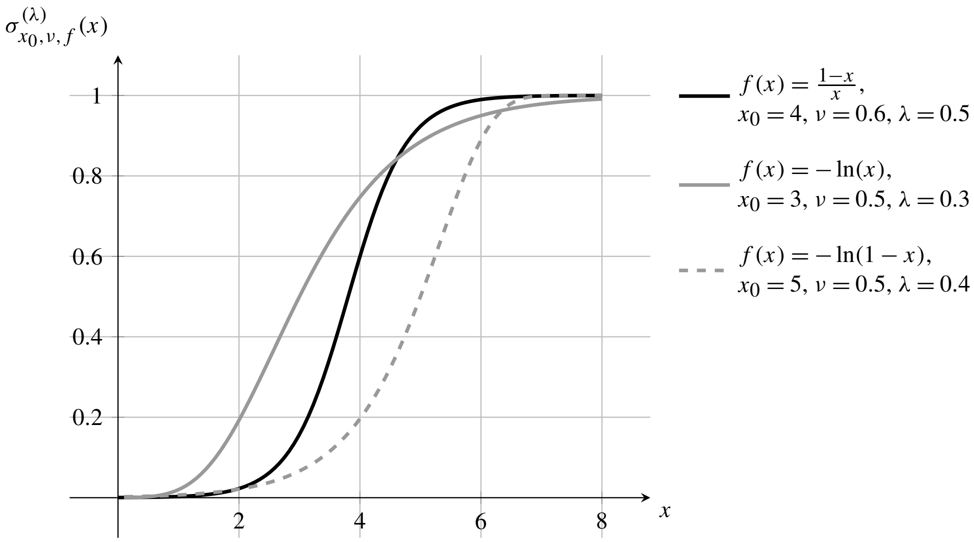

Fig. 2. Example plots of generator function-dependent sigmoid functions.

is a generator function as well, where $\alpha \in \mathbb{R} \setminus \{0\}$, $\gamma \in \mathbb{R}$ and $\gamma > 0$ (see [28]). This result allows us to generate additional generator function-dependent sigmoid functions with more parameters, which increases the flexibility of the generalized sigmoid function. Namely, the generator function $f_{\alpha,\gamma}$ induces the following parametric generalized sigmoid function:

$$\sigma^{(\lambda^*)}_{x_0,\nu,f}(\alpha,\gamma;x) = f^{-1}\left(\left(\frac{1}{\gamma}\left(\mathrm{e}^{f(\nu)\mathrm{e}^{-\lambda^*(x-x_0)}} - 1\right)\right)^{\frac{1}{\alpha}}\right),$$

where

$$\lambda^* = -\lambda\frac{\alpha\gamma f^{\alpha-1}(\nu) f'(\nu)}{1+\gamma f^{\alpha}(\nu)}\frac{1}{\ln(1+\gamma f^{\alpha}(\nu))},$$

and $\nu \in (0,1)$, $\lambda, x_0 \in \mathbb{R}$ and $\lambda \neq 0$.

4. Condition for equality of generalized sigmoid functions

Here, we will provide a sufficient condition for the equality of two generator function-dependent sigmoid functions.

Theorem 3. *Let f and g be differentiable generator functions of conjunctive or disjunctive operators. Let the generator function-dependent sigmoid functions $\sigma^{(\lambda)}_{x_0,\nu,f}$ and $\sigma^{(\lambda)}_{x_0,\nu,g}$ be induced by f and g, respectively, where $\nu \in (0,1)$, $\lambda, x_0 \in \mathbb{R}$ and $\lambda \neq 0$. Then,*

$$\sigma^{(\lambda)}_{x_0,\nu,f}(x) = \sigma^{(\lambda)}_{x_0,\nu,g}(x) \tag{17}$$

holds for any $x \in \mathbb{R}$, *if*

$$f(x) = \beta g^{\alpha}(x) \tag{18}$$

holds for any $x \in (0, 1)$, *where* $\alpha \neq 0$ *and* $\beta > 0$.

Proof. Let us assume that Eq. (18) holds for any $x \in (0, 1)$ and $\alpha \neq 0$, $\beta > 0$. From this equation, we have

$$f^{-1}(x) = g^{-1}\left(\left(\frac{x}{\beta}\right)^{\frac{1}{\alpha}}\right)$$

for any $x \in [0, \infty]$. Therefore, noting the definition for a generator function-dependent sigmoid function in Definition 8, after direct calculation, we get

$$\sigma^{(\lambda)}_{x_0,\nu,f}(x) = g^{-1}\left(\left(\frac{1}{\beta}\beta g^{\alpha}(\nu)\,\mathrm{e}^{\lambda\frac{\beta\alpha g^{\alpha-1}(\nu)g'(\nu)}{\beta g^{\alpha}(\nu)}(x-x_0)}\right)^{\frac{1}{\alpha}}\right) =$$
$$= g^{-1}\left(g(\nu)\,\mathrm{e}^{\lambda\frac{g'(\nu)}{g(\nu)}(x-x_0)}\right) = \sigma^{(\lambda)}_{x_0,\nu,g}(x)$$

for any $x \in \mathbb{R}$. □

Remark 5. It can be proven that the sufficient condition for the equality of two kappa functions is the same as that for two generalized sigmoid functions. That is, if two kappa functions with pairwise identical parameter values are induced by the generator functions f and g, respectively, then these two kappa functions are equal for any $x \in (0, 1)$ if Eq. (18) holds for any $x \in (0, 1)$.

Remark 6. Theorem 3 explains why the generator function-dependent sigmoid functions induced by $f(x) = \frac{1-x}{x}$ and $f(x) = \frac{x}{1-x}$ coincide ($x \in (0, 1)$). Also, owing to this theorem, these generator function-dependent sigmoid functions coincide with that induced by $f_D(x) = \left(\frac{1-x}{x}\right)^{\alpha}$, where $x \in (0, 1)$ and $\alpha \neq 0$. Therefore, the generator function-dependent sigmoid function induced by f_D is independent of α. This finding can also be verified by a direct calculation.

5. The generalized sigmoid function in Pliant systems

Here, we will show that in a Pliant system, the generalized sigmoid functions induced by a conjunctive and a disjunctive operator coincide if their λ parameters differ only in sign.

Proposition 5. *Let the conjunctive operator c and the disjunctive operator d be induced by the differentiable generator functions* f_c *and* f_d*, respectively. Let* $\nu \in (0, 1)$ *and let* $\lambda_c, \lambda_d \in \mathbb{R}$, $\lambda_c, \lambda_d \neq 0$. *If c and d form a Pliant system with a Pliant negation and*

$$\lambda_c = -\lambda_d,$$

then

$$f_c^{-1}\left(f_c(\nu)\,\mathrm{e}^{\lambda_c x}\right) = f_d^{-1}\left(f_d(\nu)\,\mathrm{e}^{\lambda_d x}\right)$$

for any $x \in \mathbb{R}$.

Proof. Noting that the operators c and d form a Pliant system (see Definition 6), we have

$$f_c(x) f_d(x) = 1 \tag{19}$$

for any $x \in [0, 1]$. From Eq. (19), we have

$$f_d^{-1}(x) = f_c^{-1}\left(\frac{1}{x}\right)$$

for any $x \in [0, \infty]$. Therefore, after direct calculation, we get

$$f_d^{-1}\left(f_d(\nu)\, e^{\lambda_d x}\right) = f_c^{-1}\left(\frac{1}{f_d(\nu)\, e^{\lambda_d x}}\right) = f_c^{-1}\left(f_c(\nu)\, e^{-\lambda_d x}\right) = \\ = f_c^{-1}\left(f_c(\nu)\, e^{\lambda_c x}\right)$$

for any $x \in \mathbb{R}$. □

It is an interesting finding that applying the Pliant negation to a generalized sigmoid function inverts the sign of its λ parameter.

Proposition 6. *Let f be a differentiable generator function of a conjunctive or disjunctive operator. Let $\nu \in (0, 1)$ and let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Let the generalized sigmoid function $\sigma_{\nu,f}^{(\lambda)} : \mathbb{R} \to (0, 1)$ be given by*

$$\sigma_{\nu,f}^{(\lambda)}(x) = f^{-1}\left(f(\nu)\, e^{\lambda x}\right), \tag{20}$$

and let the Pliant negation η_ν be induced by f as well. Then

$$\eta_\nu\left(\sigma_{\nu,f}^{(\lambda)}(x)\right) = \sigma_{\nu,f}^{(-\lambda)}(x)$$

holds for any $x \in \mathbb{R}$.

Proof. Noting the definition of the Pliant negation in Definition 5, we can write

$$\eta_\nu\left(\sigma_{\nu,f}^{(\lambda)}(x)\right) = f^{-1}\left(\frac{f^2(\nu)}{f\left(\sigma_{\nu,f}^{(\lambda)}(x)\right)}\right) = f^{-1}\left(\frac{f^2(\nu)}{f(\nu)\, e^{\lambda x}}\right) = f^{-1}\left(f(\nu)\, e^{-\lambda x}\right). \quad \square$$

6. A preference measure based on the generalized sigmoid function

Here, we will describe how the generalized sigmoid function can be used to model soft inequalities, and by doing so characterize preferences. A soft inequality can be represented by a strictly increasing membership function of a fuzzy set. Hence, we will interpret a soft inequality as follows.

Interpretation 1. A continuous, strictly increasing function $\mu : \mathbb{R} \to [0, 1]$ represents the soft inequality $0 < x$ such that

$$\mu(x) = truth(0 < x).$$

That is, we interpret the value of $\mu(x)$ as the measure of the truth value of the inequality $0 < x$.

The following theorem explains why a generator function-dependent sigmoid function can be used to model soft inequalities, and because of this, it can be viewed as a preference measure.

Theorem 4. *Let $\nu \in (0, 1)$ and let f be a generator function of the preference implication*

$$p_\nu(x, y) = f^{-1}\left(f(\nu)\frac{f(y)}{f(x)}\right),$$

where $x, y \in (0, 1)$ (see Definition 4). Let the fuzzy relation $r_\nu : \mathbb{R}^2 \to (0, 1)$ be given by

$$r_\nu(x, y) = p_\nu\left(\mu(x), \mu(y)\right), \tag{21}$$

where $\mu : \mathbb{R} \to (0, 1)$ is a continuous mapping.

The mapping μ is strictly increasing and it satisfies the equation

$$r_\nu(x, y) = \mu(y - x) \tag{22}$$

for any $x, y \in \mathbb{R}$, if and only if μ has the form

$$\mu(x) = f^{-1}\left(f(\nu)\,\mathrm{e}^{-\lambda x}\right), \tag{23}$$

with $\lambda > 0$ ($\lambda < 0$, respectively) for a strictly decreasing (strictly increasing, respectively) generator function f.

Proof. Let $\nu \in (0, 1)$ and let $\mu : \mathbb{R} \to (0, 1)$ be a strictly increasing, continuous function that satisfies Eq. (22) for any $x, y \in \mathbb{R}$. Then, based on the definition for the preference implication (see Definition 4), we have

$$f^{-1}\left(f(\nu)\frac{f(\mu(y))}{f(\mu(x))}\right) = \mu(y - x),$$

for any $x, y \in \mathbb{R}$. Applying the function f to both sides, we get

$$f(\nu)\frac{f(\mu(y))}{f(\mu(x))} = f(\mu(y - x)),$$

which can also be written as

$$\frac{\frac{f(\mu(y))}{f(\nu)}}{\frac{f(\mu(x))}{f(\nu)}} = \frac{f(\mu(y - x))}{f(\nu)}. \tag{24}$$

Since both sides of Eq. (24) are positive, by taking the logarithm of both sides, we get

$$\ln\left(\frac{f(\mu(y))}{f(\nu)}\right) - \ln\left(\frac{f(\mu(x))}{f(\nu)}\right) = \ln\left(\frac{f(\mu(y - x))}{f(\nu)}\right). \tag{25}$$

Now, let the function $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \ln\left(\frac{f(\mu(x))}{f(\nu)}\right). \tag{26}$$

Then, by utilizing function g, Eq. (25) can be written as

$$g(y) - g(x) = g(y - x)$$

or alternatively,

$$g((y - x) + x) = g(y - x) + g(x). \tag{27}$$

Notice that Eq. (27) is Cauchy's functional equation; hence, g has the form $g(x) = cx$, where c is an arbitrary fixed constant. Next, by substituting $g(x) = cx$ into Eq. (26), after direct calculation, we get

$$f(\mu(x)) = f(\nu)\,\mathrm{e}^{cx}$$

and by applying f^{-1} to both sides of this equation, we have

$$\mu(x) = f^{-1}\left(f(\nu)\,\mathrm{e}^{cx}\right). \tag{28}$$

Here, Eq. (28) can be written as

$$\mu(x) = f^{-1}\left(f(\nu)\,\mathrm{e}^{-\lambda x}\right).$$

Since μ is a strictly increasing function, for a strictly decreasing generator function f, λ has to be positive; and for a strictly increasing generator function f, λ has to be negative.

Conversely, let μ have the form given by Eq. (23) with $\lambda > 0$ ($\lambda < 0$, respectively) for a strictly decreasing (strictly increasing, respectively) generator function f. Then, using the definition for the preference implication and Eq. (21), we can write

$$r_\nu(x, y) = f^{-1}\left(f(\nu)\frac{f(\nu)\,\mathrm{e}^{-\lambda y}}{f(\nu)\,\mathrm{e}^{-\lambda x}}\right) = f^{-1}\left(f(\nu)\,\mathrm{e}^{-\lambda(y-x)}\right) = \mu(y - x)$$

for any $x, y \in \mathbb{R}$. Under these conditions, we immediately get that μ is strictly increasing. □

Remark 7. The preference implication p_ν is an operator over ordered pairs of continuous-logical values, while the function r_ν measures the preference of values that are on the real line. We should add that the results of Theorem 4 are closely related to continuous-valued logic and fuzzy set theory. Namely, the preference implication p_ν can be utilized as a preference measure (see [18]) and this preference measure can also be used to rank fuzzy numbers (see [33]).

Now, we will consider an application of the generalized sigmoid function in probability theory.

7. Generalized logistic regression

It is well known that the logistic regression has a wide range of applications in various areas of science including economics, business, biology, the medical sciences and engineering. Here, we will show how the traditional logistic regression can be generalized by utilizing the generator function-dependent sigmoid function.

Let Y be a dichotomous response random variable, and let 0 and 1 code its possible values. In the generalized linear model (GLM), the conditional expected value $E(Y|\mathbf{x})$ of Y is given by the following function of the independent variable vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$:

$$E(Y|\mathbf{x}) = g^{-1}\left(\sum_{i=1}^{n} \beta_i x_i + \beta_0\right), \tag{29}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\beta_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, and g^{-1} is the inverse of a strictly monotonic function called the link function (see [34]). In logistic regression, the link function $g\colon (0, 1) \to \mathbb{R}$ called the logit transformation is given by

$$g(x) = \ln\left(\frac{x}{1-x}\right). \tag{30}$$

Since

$$E(Y|\mathbf{x}) = 1 \cdot P(Y=1|\mathbf{x}) + 0 \cdot P(Y=0|\mathbf{x}) = P(Y=1|\mathbf{x}), \tag{31}$$

(29) and (30) give the

$$\ln\left(\frac{P(Y=1|\mathbf{x})}{1-P(Y=1|\mathbf{x})}\right) = \sum_{i=1}^{n} \beta_i x_i + \beta_0 \tag{32}$$

equation. From this equation, we get

$$P(Y=1|\mathbf{x}) = \frac{1}{1+\mathrm{e}^{-\sum_{i=1}^{n}\beta_i x_i - \beta_0}}, \tag{33}$$

which is the well-known multiple logistic regression function.

Here, we define the multivariate generator function-dependent sigmoid function as follows.

Definition 9. Let f be a differentiable generator function of a conjunctive or disjunctive operator and let the generator function-dependent sigmoid function $\sigma^{(\lambda_i)}_{x_{0_i},f}\colon \mathbb{R} \to (0, 1)$ with the parameters $\lambda_i, x_{0_i} \in \mathbb{R}$, $\lambda_i \neq 0$ be induced by f, i.e.,

$$\sigma^{(\lambda_i)}_{x_{0_i},f}(x_i) = f^{-1}\left(\mathrm{e}^{-\lambda_i(x_i - x_{0_i})}\right). \tag{34}$$

Let a_{ν_*} be an aggregative operator induced by f according to Eq. (7) with the parameter $\nu_* = f^{-1}(1)$. The n-variate generator function-dependent sigmoid function $\sigma^{(\boldsymbol{\lambda})}_{\mathbf{x}_0,f}\colon \mathbb{R}^n \to (0, 1)$ induced by f is given by

$$\sigma^{(\boldsymbol{\lambda})}_{\mathbf{x}_0,f}(\mathbf{x}) = a_{\nu_*}\left(\sigma^{(\lambda_1)}_{x_{0_1},f}(x_1), \sigma^{(\lambda_2)}_{x_{0_2},f}(x_2), \ldots, \sigma^{(\lambda_n)}_{x_{0_n},f}(x_n)\right), \tag{35}$$

where $\mathbf{x}_0 = (x_{0_1}, x_{0_2}, \ldots, x_{0_n})$ and $\boldsymbol{\lambda}_0 = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Note that $\sigma^{(\lambda_i)}_{x_{0_i},f}$ in Eq. (34) has an alternative form of the generator function-dependent sigmoid function, which is stated in Eq. (16).

Using the definition for an aggregative operator induced by f in Eq. (7), from Definition 9, we readily find that

$$\begin{aligned}\sigma^{(\boldsymbol{\lambda})}_{\mathbf{x}_0,f}(\mathbf{x}) &= f^{-1}\left(\prod_{i=1}^{n} f\left(f^{-1}\left(e^{-\lambda_i(x_i-x_{0_i})}\right)\right)\right) = \\ &= f^{-1}\left(e^{-\sum\limits_{i=1}^{n}\lambda_i(x_i-x_{0_i})}\right).\end{aligned} \tag{36}$$

Applying the substitutions

$$\beta_i = \lambda_i, \quad \text{and} \quad \beta_0 = -\sum_{i=1}^{n}\lambda_i x_{0_i},$$

the n-variate generator function-dependent sigmoid function induced by f can be written in the following alternative form:

$$\sigma_{\boldsymbol{\beta},f}(\mathbf{x}) = f^{-1}\left(e^{-\sum\limits_{i=1}^{n}\beta_i x_i - \beta_0}\right).$$

In our generalized multiple logistic regression model, we will use this alternative form of the n-variate generator function-dependent sigmoid function.

Now, let us consider the link function $g_f\colon (0,1) \to \mathbb{R}$ that is given by

$$g_f(x) = \ln\left(\frac{1}{f(x)}\right), \tag{37}$$

where f is a differentiable generator function of a conjunctive or disjunctive operator. We will call the link function g_f in Eq. (37), the generalized logit transformation. Notice that the traditional logit transformation given by Eq. (30) is a special case of the generalized logit transformation defined in Eq. (37). Namely, if the generator function f is $f(x) = \frac{1-x}{x}$, $x \in (0,1)$, then the generalized logit transformation becomes the traditional logit transformation.

Noting Eq. (29) and Eq. (31), with the link function g_f, the generalized linear model gives us

$$\ln\left(\frac{1}{f(P(Y=1|\mathbf{x}))}\right) = \sum_{i=1}^{n}\beta_i x_i + \beta_0. \tag{38}$$

From Eq. (38), we have

$$P(Y=1|\mathbf{x}) = f^{-1}\left(e^{-\sum\limits_{i=1}^{n}\beta_i x_i - \beta_0}\right) = \sigma_{\boldsymbol{\beta},f}(\mathbf{x}). \tag{39}$$

This result means that if we apply the generalized linear model with the generalized logit transformation as the link function, then we get the multivariate generator function-dependent sigmoid function as the regression function. We shall call this model the generalized multiple logistic regression model. Notice that if the generator function f is $f(x) = \frac{1-x}{x}$, $x \in (0,1)$, then Eq. (39) becomes Eq. (33). That is, the traditional multiple logistic regression may be viewed as a special case of the generalized multiple logistic regression. Table 3 summarizes the logistic and generalized logistic regression models presented in this article.

7.1. Estimation of model parameters

Here, we present a two-step method for estimating the model parameters in a generalized logistic regression model. For the sake of simplicity we will just do it for the univariate case. The method for a multi-variate case is similar to that for the univariate case.

Table 3
The logistic and generalized logistic regression models.

Model	Link function $x \in (0,1)$	Regression function $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$
Logistic regression	$g(x) = \ln\left(\frac{x}{1-x}\right)$	$P(Y=1 \mid \mathbf{x}) = \left(1 + e^{-\sum_{i=1}^{n} \beta_i x_i - \beta_0}\right)^{-1}$
Generalized logistic regression	$g_f(x) = \ln\left(\frac{1}{f(x)}\right)$	$P(Y=1 \mid \mathbf{x}) = f^{-1}\left(e^{-\sum_{i=1}^{n} \beta_i x_i - \beta_0}\right)$

Let $(x_1, Y_1), (x_2, Y_2), \ldots, (x_n, Y_n)$ be a sample of independent observation pairs on the variable x and the dichotomous random variable Y, where $x_i \in \mathbb{R}$, $Y_i \in \{0, 1\}$, $n \geq 2$. Assume that according to Eq. (39), we have the generalized regression model

$$P(Y=1 \mid x) = f^{-1}\left(e^{-\beta_1 x - \beta_0}\right) = \sigma_{\beta_0, \beta_1, f}(x), \tag{40}$$

where $x, \beta_0, \beta_1 \in \mathbb{R}$ and f is a generator function of a conjunctive or disjunctive operator.

Let $x_1^*, x_2^*, \ldots, x_m^*$ be the unique values among the values $x_1, x_2, \ldots, x_n$ in the sample $(x_1, Y_1), (x_2, Y_2), \ldots,$ (x_n, Y_n), $m \leq n$. Note that in practice, m is much less than n. Next, let n_r be the number of sample elements (x_i, Y_i) for which $x_i = x_r^*$, and let k_r^* be the conditional frequency of the event $Y = 1$ given the condition $x = x_r^*$, where $i = 1, 2, \ldots, n$ and $r = 1, 2, \ldots, m$. That is,

$$n_r = \left|\{(x_i, Y_i) : x_i = x_r^*, i = 1, 2, \ldots, n\}\right| \tag{41}$$

and

$$k_r = \left|\{(x_i, Y_i) : x_i = x_r^*, Y_i = 1, i = 1, 2, \ldots, n\}\right|, \tag{42}$$

where $r = 1, 2, \ldots, m$. Then,

$$y_r^* = \frac{k_r}{n_r},$$

is the estimated value of the conditional probability $P(Y = 1 \mid x = x_r^*)$ computed from the sample $(x_1, Y_1), (x_2, Y_2), \ldots,$ (x_n, Y_n).

Our method consists of the following two steps.

(1) First, we linearize the regression function in Eq. (40) and then we apply linear regression to get the estimates $\hat{\beta_0}_0$ and $\hat{\beta_1}_0$ of the parameters β_0 and β_1, respectively. We call $\hat{\beta_0}_0$ and $\hat{\beta_1}_0$ the initial estimates of β_0 and β_1, respectively.
(2) In the second step of our method, we provide the maximum likelihood estimation of the parameters β_0 and β_1 by using a numeric optimization method in which the unknown parameter values are initialized with $\hat{\beta_0}_0$ and $\hat{\beta_1}_0$ that were obtained in the first phase.

Step 1 From Eq. (40), we have

$$\ln\left(\frac{1}{f(P(Y=1 \mid x))}\right) = \beta_1 x + \beta_0, \tag{43}$$

which is the univariate instance of Eq. (38). Utilizing the link function in Eq. (37), Eq. (43) can be written as

$$u = \beta_1 x + \beta_0, \tag{44}$$

where $u = g_f(y)$ and $y = P(Y = 1 \mid x)$. Recall that based on our sample, we have the ordered pair (x_r^*, y_r^*), and so we also have the ordered pair (x_r^*, u_r), where $u_r = g_f(y_r^*)$ and $r = 1, 2, \ldots, m$. Therefore, using the ordered data pairs (x_r^*, u_r) and Eq. (44), by applying linear regression, we get the initial estimates $\hat{\beta_0}_0$ and $\hat{\beta_1}_0$ of the parameters β_0 and β_1, respectively.

Table 4
Data from a medical study.

r	1	2	3	4	5	6	7	8	9	10	11	12	13
x_r	20	25	30	35	40	45	50	55	60	65	70	75	80
k_r	1	2	5	11	15	22	27	30	45	65	72	84	96
$n_r - k_r$	99	98	95	89	85	78	73	70	55	35	28	16	4

Step 2 By utilizing the sample $(x_1, Y_1), (x_2, Y_2), \ldots, (x_n, Y_n)$, the unknown parameters β_0 and β_1 can be estimated by maximizing the likelihood function $L\colon \mathbb{R}^2 \to (0, 1)$

$$L(\beta_0, \beta_1) = \prod_{i=1}^{n} P(Y = Y_i | x_i) = \prod_{i=1}^{n} \sigma_{\beta_0,\beta_1,f}^{Y_i}(x_i)\left(1 - \sigma_{\beta_0,\beta_1,f}(x_i)\right)^{1-Y_i}, \tag{45}$$

where $i = 1, 2, \ldots, n$. Obviously, maximizing the likelihood function in (45) is equivalent to maximizing the log-likelihood function $l\colon \mathbb{R}^2 \to (0, -\infty]$, which is given by

$$l(\beta_0, \beta_1) = \sum_{i=1}^{n} Y_i \ln\left(\sigma_{\beta_0,\beta_1,f}(x_i)\right) + \sum_{i=1}^{n}(1 - Y_i) \ln\left(1 - \sigma_{\beta_0,\beta_1,f}(x_i)\right). \tag{46}$$

By making use of the frequencies n_r and k_r given in (41) and (42), the log-likelihood function in (46) can be written as

$$l(\beta_0, \beta_1) = \sum_{r=1}^{m} k_r \ln\left(\sigma_{\beta_0,\beta_1,f}(x_r^*)\right) + \sum_{r=1}^{m}(n_r - k_r) \ln\left(1 - \sigma_{\beta_0,\beta_1,f}(x_r^*)\right). \tag{47}$$

The maxima of the log-likelihood function in Eq. (47) can be determined by using the so-called GLOBAL method which is a stochastic global optimization procedure introduced by Csendes (see [35,36]). In the optimization procedure, we initialize the parameters β_0 and β_1 with the values determined in the first phase of our regression method. This approach increases the speed of convergence of the GLOBAL method.

7.2. A numerical example

In a medical study, the relationship between age and the presence of chronic diseases was analyzed. Altogether 1300 people participated in a survey, in which, based on their age, the participants were grouped into 13 groups, with 100 participants in each group. In Table 4, the first row contains the group index ($r = 1, 2, \ldots, 13$), while the second row indicates the age (in terms of years) (x_r) of the survey participants in each group. The third row in Table 4 contains the number of participants who suffer from some chronic disease (k_r), while $n_r - k_r$ denotes the number of those participants who do not suffer from any chronic disease ($n_r = 100$).

Let the dichotomous random variable Y be 1, if a person suffers from some chronic disease, and let it be 0, if he or she does not suffer from any chronic disease. With this notation, our goal was to model the conditional probability $P(Y = 1|x)$, where x denotes the age of a person. In order to model this conditional probability, we applied the generalized logistic regression with three various generator functions. Namely, we applied the model in Eq. (40) with the generator functions: (1) $f(x) = \frac{1-x}{x}$, (2) $f(x) = -\ln(x)$ and (3) $f(x) = -\ln(1 - x)$, where $x \in (0, 1)$. The maximum likelihood estimations $\hat{\beta}_0$ and $\hat{\beta}_1$ of the model parameters β_0 and β_1, respectively, and the value of the log-likelihood function $l(\hat{\beta}_0, \hat{\beta}_1)$ computed for each of these three regressions are summarized in Table 5.

Recall that Model 1 in Table 5 is the classical logistic regression. For this data set, the generalized sigmoid function induced by the generator function $f(x) = -\ln(1 - x)$ (Model 3) resulted in the best fitting model (highest log-likelihood value) among the three models applied.

We should mention that in the generalized logistic regression method, various generator functions can be utilized to create regression functions. This means that by using this approach, we can achieve a higher level of flexibility and modeling capability than that for the classical logistic regression. In this example, each of the three models has the same number of parameters, the models differ in the generator function which the applied regression function (i.e., a generalized sigmoid function) is based on.

Table 5
Regression results.

	$f(x)$	$\sigma_{\beta_0,\beta_1,f}(x)$	$\hat{\beta}_0$	$\hat{\beta}_1$	$l(\hat{\beta}_0,\hat{\beta}_1)$
Model 1	$\frac{1-x}{x}$	$\left(1+e^{-\beta_1 x-\beta_0}\right)^{-1}$	−6.1430	0.1033	−545.6349
Model 2	$-\ln(x)$	$e^{-e^{-\beta_1 x-\beta_0}}$	−2.9338	0.0572	−557.4650
Model 3	$-\ln(1-x)$	$1-e^{-e^{-\beta_1 x-\beta_0}}$	5.0961	−0.0772	−542.2779

Table 6
Empirical probabilities, regression function values and fitting goodness.

r	Empirical prob.	Model 1	Model 2	Model 3
1	0.01	0.0167	0.0025	0.0282
2	0.02	0.0276	0.0111	0.0413
3	0.05	0.0455	0.0341	0.0601
4	0.11	0.0739	0.0789	0.0871
5	0.15	0.1180	0.1484	0.1255
6	0.22	0.1832	0.2386	0.1790
7	0.27	0.2732	0.3407	0.2518
8	0.30	0.3864	0.4454	0.3474
9	0.45	0.5135	0.5446	0.4662
10	0.65	0.6389	0.6335	0.6028
11	0.72	0.7478	0.7097	0.7429
12	0.84	0.8325	0.7729	0.8644
13	0.96	0.8928	0.8240	0.9471
SSE		0.0208	0.0602	0.0100

Table 6 shows the empirical probabilities, the regression function values for the three applied models and the sum of squared errors (SSE) for each model. Based on the three SSE values, Model 3 has the best performance among the three models. This is in line with our previous finding that Model 3 yields the highest log-likelihood value (see Table 5). Fig. 3 shows the plots of the regression functions and the empirical probabilities.

8. Conclusions

The main findings of our study can be summarized as follows.

1. In this paper, we presented a continuous-valued logical approach to generalize the sigmoid function.
2. As a starting point, we utilized a generalized class of unary operators, which is known as the kappa function in continuous-valued logic.
3. First, we extended the kappa function from the (0, 1) domain to the (a, b) interval, and then we showed how a generalized sigmoid function can be derived from the extended kappa function.
4. In a previous study of ours, we proved that the classical sigmoid function is just the limit of the extended kappa function when the latter is induced by a generator function of the Dombi operators.
5. Following this result, we interpreted the generalized sigmoid function as an asymptotic extended kappa function, which is induced by an arbitrary generator function of a conjunctive or disjunctive operator.
6. The generalized sigmoid function is generator function-dependent. Here, we gave a sufficient condition for the equality of two generalized sigmoid functions.
7. We presented two useful applications of the generalized sigmoid function. Namely, we showed that this new function may be viewed as a general preference measure and we described how logistic regression can be generalized using the generator function-dependent sigmoid function.
8. It is well known that the classical sigmoid function is widely applied in neural networks as activation function. Since the traditional sigmoid function is just a special case of the generalized sigmoid function, the latter may be viewed as a new and more flexible alternative activation function.

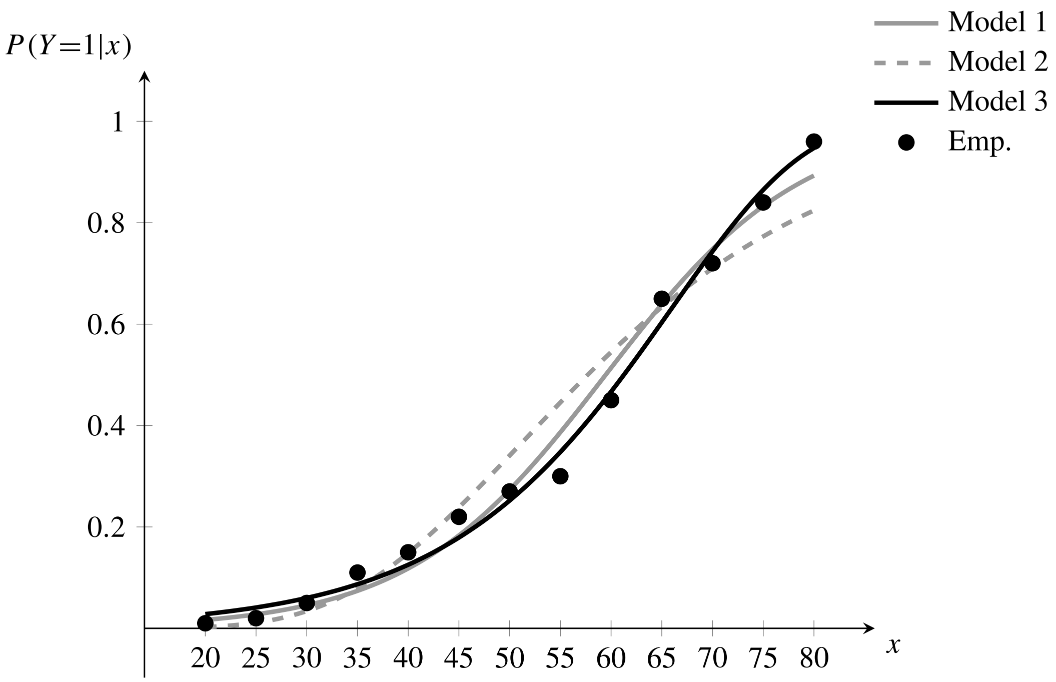

Fig. 3. Plots of the regression functions and the empirical probabilities.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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