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Short communication

Remarks on two representations of strong negations and a connection between nilpotent and strict triangular norms

József Dombi [a], Tamás Jónás [b,*]

[a] *Institute of Informatics, University of Szeged, Szeged, Hungary*
[b] *Faculty of Economics, Eötvös Loránd University, Budapest, Hungary*

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Abstract

In this short communication, we will present a connection between two well-known representations of the strong negations (involutive negations). Namely, we will provide a necessary and sufficient condition for the equality of Trillas and Dombi forms of negations. We will also show a connection between the additive generators of nilpotent and strict triangular norms.

Keywords: Strong negations; Nilpotent t-norms; Strict t-norms

1. Preliminaries

Here, we will give a brief overview of the concepts that we will utilize later. The Archimedean triangular norms (t-norms in short) and triangular conorms (t-conorms in short) as well as their strict and nilpotent classes play important roles in continuous-valued logic (for more details see [1] or [2]). These norms are defined as follows.

Definition 1. We say that a continuous t-norm $T\colon [0,1] \to [0,1]$ (t-conorm $S\colon [0,1] \to [0,1]$, respectively) is Archimedean, if $T(x,x) < x$ ($S(x,x) > x$, respectively) holds for any $x \in (0,1)$.

Definition 2. We say that a continuous Archimedean t-norm T (t-conorm S, respectively) is a strict t-norm (strict t-conorm, respectively), if $T(x,y) < T(x,z)$ whenever $x \in (0,1]$ and $y < z$ (if $S(x,y) < S(x,z)$ whenever $x \in [0,1)$ and $y < z$, respectively).

Another important class of Archimedean t-norms and t-conorms is the class of nilpotent t-norms and nilpotent t-conorms.

* Corresponding author.
E-mail addresses: dombi@inf.u-szeged.hu (J. Dombi), jonas@gtk.elte.hu (T. Jónás).

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Definition 3. We say that a continuous Archimedean t-norm T (t-conorm S, respectively) is a nilpotent t-norm (nilpotent t-conorm, respectively), if there exists $x, y \in (0, 1)$ such that $T(x, y) = 0$ ($S(x, y) = 1$, respectively).

The Archimedean t-norms and t-conorms can be represented as follows (see [3]).

Theorem 1. *A function $T\colon [0, 1] \to [0, 1]$ ($S\colon [0, 1] \to [0, 1]$, respectively) is a continuous Archimedean t-norm (t-conorm, respectively) if and only if there exists a continuous, strictly decreasing (increasing, respectively) function $t\colon [0, 1] \to [0, \infty]$ ($s\colon [0, 1] \to [0, \infty]$, respectively) with $t(1) = 0$ ($s(0) = 0$, respectively), which is uniquely determined up to a positive constant multiplier, such that for any $x, y \in [0, 1]$,*

$$T(x, y) = t^{-1}\left(\min(t(x) + t(y), t(0))\right)$$
$$\left(S(x, y) = s^{-1}\left(\min(s(x) + s(y), s(1))\right), \textit{respectively}\right).$$

In Theorem 1, the function t (s, respectively) is called an additive generator function of the Archimedean t-norm (t-conorm, respectively). Using Theorem 1, the strict t-norms, strict t-conorms and the nilpotent t-norms and nilpotent t-conorms can be characterized as follows (see [4]).

Theorem 2. *The following are valid.*

(a) *A t-norm T is strict if and only if $t(0) = \infty$ holds for each continuous additive generator t of T.*
(b) *A t-norm T is nilpotent if and only if $t(0) < \infty$ holds for each continuous additive generator t of T.*
(c) *A t-conorm S is strict if and only if $s(1) = \infty$ holds for each continuous additive generator s of S.*
(d) *A t-conorm S is nilpotent if and only if $s(1) < \infty$ holds for each continuous additive generator s of S.*

Remark 1. We will operate on the extended real line and use the conventions $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$, $\mathrm{e}^{-\infty} = 0$ and $\ln(0) = -\infty$.

Here, we will use the following definition of a strong negation (see, e.g., Definition 1.2 in Fodor and Rubens [5], or Definition 11.3 in Klement et al. [1]).

Definition 4. We say that $\eta\colon [0, 1] \to [0, 1]$ is a strong negation if and only if η satisfies the following requirements:

(a) η is continuous	(Continuity)
(b) $\eta(0) = 1$, $\eta(1) = 0$	(Boundary conditions)
(c) $\eta(x) < \eta(y)$ for $x > y$	(Monotonicity)
(d) $\eta(\eta(x)) = x$ for any $x \in [0, 1]$	(Involution).

Remark 2. It should be added that the requirements (a) and (b) in Definition 4 can be omitted (see Theorem 3.1 in the book of Klir and Yuan [6]).

1.1. Two representations of strong negations

Here, we will make use of the concept of automorphism of the interval $[a, b]$.

Definition 5. We say that $\varphi\colon [a, b] \to [a, b]$ is an automorphism of $[a, b]$ if and only if φ is a continuous, strictly increasing function with the boundary conditions $\varphi(a) = a$ and $\varphi(b) = b$.

It is worth noting that according to Maes and De Baets (see [7]), the strict negations and automorphisms are prevalently used to fuzzify the Boolean negation.

Trillas in 1979 presented the following representation theorem of strong negations (see [8]).

Theorem 3 *(Trillas representation). The function $n\colon [0, 1] \to [0, 1]$ is a strong negation if and only if for any $x \in [0, 1]$,*

$$n(x) = g^{-1}(1 - g(x)), \tag{1}$$

where $g\colon [0,1] \to [0,1]$ *is an automorphism of* $[0,1]$.

If Eq. (1) holds, then we say that the strong negation n is induced by g. Noting Theorem 1 and Theorem 2, the function g is an additive generator of a nilpotent t-conorm. Moreover, if the function g_* is given by $g_*(x) = 1 - g(x)$, $x \in [0,1]$, then g_* is strictly decreasing, $g_*(0) = 1$ and $g_*(1) = 0$. That is, g_* is an additive generator of a nilpotent t-norm. On the other hand, after direct calculation, we get that

$$g_*^{-1}(1 - g_*(x)) = g^{-1}(1 - g(x)) = n(x), \quad x \in [0,1],$$

which means that the strong negation n can be induced by g_* as well. That is, Eq. (1) may be viewed as a representation of strong negations for the nilpotent class of t-norms and t-conorms. It is worth adding that if a function t (s, respectively) is an additive generator of a nilpotent t-norm (t-conorm, respectively), then the function $g_t(x) = \frac{t(x)}{t(0)}$ ($g_s(x) = \frac{s(x)}{s(1)}$, respectively) is a generator of a strong negation according to Eq. (1).

Another representation theorem of strong negations was presented by Dombi in 2011 (see [9]).

Theorem 4 *(Dombi representation). The function* $\eta_A\colon [0,1] \to [0,1]$ *is a strong negation if and only if for any* $x \in [0,1]$,

$$\eta_A(x) = f^{-1}\left(\frac{A}{f(x)}\right), \tag{2}$$

where $f\colon [0,1] \to [0,\infty]$ *is an additive generator function of some strict t-norm or strict t-conorm,* $A \in \mathbb{R}$ *and* $A > 0$.

If Eq. (2) holds, then we say that the strong negation η_A is induced by f. Since f is an additive generator of a strict t-norm or strict t-conorm, Eq. (2) may be treated as a representation of strong negations for the strict class of t-norms and t-conorms. It is worth noting that if f is an additive generator of a strict t-norm (t-conorm, respectively), and f_* is given by $f_*(x) = \frac{1}{f(x)}$, $x \in [0,1]$, then f_* is an additive generator of a strict t-conorm (t-norm, respectively) and after direct calculation, we get

$$f_*^{-1}\left(\frac{A}{f_*(x)}\right) = f^{-1}\left(\frac{A}{f(x)}\right) = \eta_A(x), \quad x \in [0,1].$$

2. Connection between the Trillas and Dombi representations of strong negations

Here, using the generator functions of Trillas and Dombi representations of strong negations, we will present a necessary and sufficient condition for the equality of these representations.

Theorem 5. *Let* $g\colon [0,1] \to [0,1]$ *be an automorphism of* $[0,1]$ *and let* $f\colon [0,1] \to [0,\infty]$ *be an additive generator function of a strict t-norm or strict t-conorm. Let the strong negations* $n\colon [0,1] \to [0,1]$ *and* $\eta_A\colon [0,1] \to [0,1]$ *be given by Eq.* (1) *and Eq.* (2)*, respectively, where* $A > 0$*. Then, for any* $x \in [0,1]$,

$$\eta_A(x) = n(x) \tag{3}$$

if and only if

$$f(x) = \sqrt{A}\mathrm{e}^{h(x)} \tag{4}$$

with some strictly monotonic function $h\colon [0,1] \to [-\infty,\infty]$*, where*

$$h(x) = \Phi(g(x), 1 - g(x)) \tag{5}$$

and $\Phi\colon \mathbb{R}^2 \to [-\infty,\infty]$ *is an antisymmetric two-variable function, i.e.,* $\Phi(p,q) = -\Phi(q,p)$ *for any* $p, q \in \mathbb{R}$.

Proof. We will utilize the following result (see [10]). A function $y\colon \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$y(x)y(a-x) = b^2 \tag{6}$$

with some constants $a, b \in \mathbb{R}$ if and only if

$$y(x) = \pm b\mathrm{e}^{\Phi(x,a-x)}, \tag{7}$$

where Φ is an antisymmetric two-variable function on $\mathbb{R}$, i.e., $\Phi(x,z) = -\Phi(z,x)$ for any $x, z \in \mathbb{R}$.

Proof of necessity. Let us assume that the conditions of the theorem hold and Eq. (3) holds for any $x \in [0,1]$. This means that we have

$$f^{-1}\left(\frac{A}{f(x)}\right) = g^{-1}(1-g(x)) \tag{8}$$

for any $x \in [0,1]$. Applying f to both sides of Eq. (8), we get

$$\frac{A}{f(x)} = f\left(g^{-1}(1-g(x))\right). \tag{9}$$

Now, let the function $F\colon [0,1] \to [0,\infty]$ and the variable X be given by

$$F(x) = f\left(g^{-1}(x)\right) \quad \text{and} \quad X = g(x), \tag{10}$$

for any $x \in [0,1]$, respectively. Then, $X \in [0,1]$, $F(X) = F(g(x)) = f(x)$ and so Eq. (9) can be written as

$$F(X)F(1-X) = A. \tag{11}$$

Noting Eq. (6), the solution of the functional equation in Eq. (11) is

$$F(X) = \sqrt{A}\mathrm{e}^{\Phi(X,1-X)}, \tag{12}$$

which, based on Eq. (10) is equivalent to

$$f(x) = \sqrt{A}\mathrm{e}^{\Phi(g(x),1-g(x))}, \tag{13}$$

where $x \in [0,1]$ and $\Phi\colon \mathbb{R}^2 \to [-\infty,\infty]$ is an antisymmetric two-variable function. Now, let $h(x) = \Phi(g(x), 1-g(x))$, where $x \in [0,1]$. Then, from Eq. (13) we get Eq. (4) and since $g\colon [0,1] \to [0,1]$ is a strictly increasing function and $f\colon [0,1] \to [0,\infty]$ is a strictly monotonic function, based on Eq. (13), h is necessarily a strictly monotonic function with the range $[-\infty,\infty]$.

Proof of sufficiency. Suppose that the conditions of the theorem hold. Let us assume that Eq. (4) holds for any $x \in [0,1]$, with a strictly monotonic function $h\colon [0,1] \to [-\infty,\infty]$, where h is given by $h(x) = \Phi(g(x), 1-g(x))$ and $\Phi\colon \mathbb{R}^2 \to [-\infty,\infty]$ is an antisymmetric two-variable function. That is, we have Eq. (13) for any $x \in [0,1]$. Here again, let the function $F\colon [0,1] \to [0,\infty]$ and the variable X be given by Eq. (10), where $x \in [0,1]$. Using the function F and the variable X, from Eq. (13), we get Eq. (12). Next, based on the introduction, note that regarding Eq. (6), we know that the function F is the solution of the functional equation in Eq. (11). Using F and X, Eq. (11) can be written as

$$f(x)f\left(g^{-1}(1-g(x))\right) = A,$$

for any $x \in [0,1]$, from which Eq. (8) (and Eq. (3)) follows. □

The following corollary may be viewed as a practical application of Theorem 5.

Corollary 1. *Let $g\colon [0,1] \to [0,1]$ be an automorphism of $[0,1]$, and let the function f be given by*

$$f(x) = \sqrt{A}\left(\frac{g(x)}{1-g(x)}\right)^c, \tag{14}$$

for any $x \in [0,1]$, where $A, c \in \mathbb{R}$ are arbitrarily fixed constants, $A > 0$ and $c \neq 0$. Then,

(a) *f is an additive generator function of a strict t-norm or a strict t-conorm*
(b) *the strong negation $n: [0,1] \to [0,1]$ induced by the function g according to Eq. (1) and the strong negation $\eta_A: [0,1] \to [0,1]$ induced by the function f according to Eq. (2) are equal for any $x \in [0,1]$.*

Proof. Noting Eq. (14) with $A > 0$, $c \neq 0$, and the fact that g is an automorphism of $[0,1]$, we immediately get that f is a strictly monotonic function on $[0,1]$ with the range $[0,\infty]$. If $c < 0$, then f is strictly decreasing, $f(0) = \infty$ and $f(1) = 0$, i.e., f is an additive generator of a strict t-norm. If $c > 0$, then f is strictly increasing, $f(0) = 0$ and $f(1) = \infty$, i.e., f is an additive generator of a strict t-conorm. This means that (a) holds.

Now, let $\Phi(x, y) = c(\ln(x) - \ln(y))$ for any $x, y \in \mathbb{R}$ and let $h(x) = \Phi(g(x), 1 - g(x))$ for any $x \in [0,1]$. Then,

$$h(x) = \ln\left(\frac{g(x)}{1-g(x)}\right)^c$$

for any $x \in [0,1]$, and h is strictly monotonic with the range $[-\infty,\infty]$. Therefore, noting Theorem 5, we have that for any $x \in [0,1]$, if

$$f(x) = \sqrt{A}\mathrm{e}^{h(x)} = \sqrt{A}\left(\frac{g(x)}{1-g(x)}\right)^c, \tag{15}$$

then $\eta_A(x) = n(x)$. This means that (b) holds. □

Certainly, it can also be verified via direct calculation that if Eq. (14) holds, then $\eta_A(x) = n(x)$ holds for any $x \in [0,1]$.

3. Conclusions

In this short communication, using the generator functions of Trillas and Dombi representations of strong negations, we presented a necessary and sufficient condition for the equality of these representations. Furthermore, since an additive generator of an Archimedean triangular norm is uniquely determined up to a positive constant multiplier, Eq. (15) may be viewed as a connection between the additive generators of the nilpotent and strict triangular norms.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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