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The generalized sigmoid function and its connection with logical operators

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ABSTRACT

In this study, we present the operator-dependent sigmoid function, which is derived from a universal unary operator called the kappa function. Here, we describe how the generalized sigmoid function is related to representable uninorms (i.e., Dombi's aggregative operator). Namely, we show that the inverse of a generalized sigmoid function is an additive generator of the aggregative operator. We provide the necessary and sufficient conditions for the form of the function that transforms the aggregative operator into a conjunctive or disjunctive logical operator. This transformation is also based on the generalized sigmoid function. Here, we show how conjunctive and disjunctive operators, which form a De Morgan system with a negation, can be derived from the aggregative operator. We point out that, under certain conditions, a set of generalized sigmoid functions is closed under the negation and modifier operators. Lastly, we demonstrate an important connection between the weighted aggregative operator and the generalized sigmoid function. Based on this connection, we provide a new interpretation of the feed-forward neural networks. We show that a perceptron-based neural network can be modeled using the aggregative operator and the generalized sigmoid function.

1. Introduction

There are many branches of science, including mathematics, computer science, economics, biology, engineering and the medical sciences, where the sigmoid function has lots of applications. The following examples cover some recent studies that are connected with the sigmoid function. Extensions of certain families of sigmoid functions and their applications in growth theory were presented by Kyurkchiev et al. [1]. Sigmoid function-based Hausdorff approximations to the Heaviside step function were provided by Iliev et al. [2]. Sharma et al. [3] investigated the performance of a weighted sigmoid function-based noise estimator in speech de-noising. Ezadi et al. [4] introduced a sigmoid function-based method for ranking fuzzy numbers. Using the sigmoid function, Shao et al. [5] presented an integral-derivative observer and its application in autopilot design.

Undoubtedly, in computer science, machine learning, artificial neural networks and optimization are areas where the sigmoid function plays a key role. The following examples are related to some recent results in these areas. Liu et al. [6] presented a novel, sigmoid function-based particle swarm optimizer. Qiao et al. [7] introduced a mutual information-based weight initialization method for sigmoidal feed-forward neural networks. Qin et al. [8] studied optimized deep belief

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networks with improved logistic sigmoid units and presented their applications. The so-called local sigmoid method, which is a non-iterative deterministic learning algorithm for the automatic model construction of neural networks, was provided by Alfarozi et al. [9]. Elfwing et al. [10] presented a paper on sigmoid-weighted linear units for neural network function approximation in reinforcement learning.

The aggregative operator, which was first introduced by Dombi in 1982 (see [11]) is a class of uninorms, which may be regarded as extensions of triangular norms and triangular conorms. The concept of uninorms was introduced by Yager and Rybalov [12]. It is well known that the uninorms are important aggregation operators with an arbitrary neutral element in the unit interval $[0,1]$. Over the past decades, these operators have been intensely studied (see, e.g., the results of Dubois and Prade [13], Fodor [14], Fodor and De Baets [15], and Aşıcı and Mesiar [16]). The flexibility of uninorms also explains why they have many applications in various areas of science including fuzzy logic [17], fuzzy systems [18], artificial neural networks [19] and expert systems [20].

In this paper, we introduce the operator-dependent generalized sigmoid function, which can be derived from the unary operator called the kappa function (see [21]). We present how the generalized sigmoid function is connected with Dombi's aggregative operator. Namely, we show that the aggregative operator is a uninorm, which can be represented by the inverse of a generalized sigmoid function. We demonstrate that in certain domains of the aggregative operator, it is a conjunctive or a disjunctive operator. As an important result of our study, we provide the necessary and sufficient conditions for the form of the function that transforms the aggregative operator into a conjunctive or disjunctive logical operator. Here, we prove that this transformation is based on the generalized sigmoid function. Next, we describe the advantages of using the above results in a Pliant logical system (see [22]). Here, we show how conjunctive and disjunctive operators, which form a De Morgan system with a Pliant negation, can be derived from an aggregative operator. Also, we prove that, under certain conditions, a set of generalized sigmoid functions is closed under the Pliant negation, the aggregative operation and certain modifying operations. Lastly, we demonstrate an important connection between the weighted aggregative operator and the generalized sigmoid function. Based on this connection, we provide a new interpretation of neurons in an artificial neural network. Namely, we show that a feed-forward neural network may be viewed as construction that is based on the aggregative operator and the generalized sigmoid function.

This paper is structured as follows. In Section 2, we briefly summarize the basic notions and notations, which will be used in this study. Next, in Section 3, we describe how the generalized sigmoid function is connected with representable uninorms and with the aggregative operator. In this section, we also show that the generalized sigmoid function plays a crucial role in transforming the aggregative operator into a conjunctive or disjunctive operator. Using the concept of Pliant logical systems, in Section 4, we demonstrate how conjunctive and disjunctive operators, which form a De Morgan system with a negation, can be derived from the aggregative operator. In Section 5, we present some interesting results related to operations over fuzzy sets that have the generalized sigmoid functions as membership functions. Then, we demonstrate an important connection between the weighted aggregative operator and the generalized sigmoid function in Section 6. Here, based on this connection, we provide a new interpretation of neurons in an artificial neural network. Lastly, the key findings of our study are summarized in Section 7.

2. Basic notions and notations

Here, we will present the basic notions and notations, which we will use later on.

2.1. Logical operators, uninorms and the aggregative operator

In continuous-valued logic, the concepts of strict triangular norm (strict t-norm) and strict triangular conorm (strict t-conorm) play an important role mostly in real-life applications. The following definition is based on the application of Aczél's results on the associative functional equation [23] (also see [24]).

Definition 1 *(cf. [24], Sec. 5.1).* We say that the function $o\colon [0,1]^2 \to [0,1]$ is a strict t-norm (strict t-conorm, respectively) if and only if o is continuous, and there exists a continuous and strictly decreasing (increasing, respectively) function $f\colon [0,1] \to [0,\infty]$, called a generator function of o, such that

$$o(x,y) = f^{-1}\left(f(x) + f(y)\right),$$

for any $x, y \in [0,1]$, and

(a) for a strict t-norm c, $f = f_c$ is strictly decreasing with $f_c(1) = 0$ and $\lim_{x\to 0} f_c(x) = \infty$;
(b) for a strict t-conorm d, $f = f_d$ is strictly increasing with $f_d(0) = 0$ and $\lim_{x\to 1} f_d(x) = \infty$.

Note that the strict t-norm and strict t-conorm are special cases of the generalized t-norm and t-conorm classes, respectively.

Remark 1. In this article, we will refer to strict t-norms and t-conorms as conjunctive and disjunctive operators denoted by c and d, respectively. From now on, the mapping $f\colon [0,1] \to [0,\infty]$ will always be a continuous, strictly decreasing (increasing, respectively) generator function of a conjunctive (disjunctive, respectively) operator. Also, if f is strictly decreasing, then we will interpret $f(0)=\infty$ and $f^{-1}(\infty)=0$. Similarly, if f is strictly increasing, then we will interpret the end points $f(1)=\infty$ and $f^{-1}(\infty)=1$. Also, we will use the convention $\frac{1}{0}=\infty$ and $\frac{1}{\infty}=0$.

We will make use of the following aggregative operator, which was introduced by Dombi [11,25].

Definition 2 *(cf. [11,25]).* The function $a_\nu\colon [0,1]^n \to [0,1]$ is an aggregative operator if and only if a_ν is given by

$$a_\nu(x_1,x_2,\dots,x_n) = f^{-1}\left(f(\nu)\prod_{i=1}^{n}\frac{f(x_i)}{f(\nu)}\right), \tag{1}$$

where $\nu \in (0,1)$, $(x_1,x_2,\dots,x_n)\in[0,1]^n$ and f is a generator function of either a conjunctive or a disjunctive operator.

It should be added that the binary aggregative operator is a so-called representable uninorm. The definition of uninorms, originally given by Yager and Rybalov [12] in 1996, is the following:

Definition 3 *(cf. [12]).* We say that the mapping $U\colon [0,1]^2 \to [0,1]$ is a uninorm if and only if U satisfies the following requirements:

(a) For any $x, y \in [0,1]$, $U(x,y)=U(y,x)$ (commutativity)
(b) For any $x_1,x_2,y_1,y_2 \in [0,1]$, if $x_1 \ge x_2$ and $y_1 \ge y_2$, then $U(x_1,y_1) \ge U(x_2,y_2)$ (monotonicity)
(c) For any $x,y,z\in[0,1]$, $U(x,U(y,z)) = U(U(x,y),z)$ (associativity)
(d) There exists a $\nu\in[0,1]$ such that for any $x\in[0,1]$, $U(x,\nu)=x$ (neutral element).

A uninorm is a generalization of t-norms and t-conorms. The uninorms were extensively studied in the seminal paper by Mas et al. (see [26]). By adjusting its neutral element ν, a uninorm is a t-norm if $\nu=1$ and a t-conorm if $\nu=0$. The following representation theorem of strict, continuous on $[0,1]^2\setminus\{(0,1),(1,0)\}$ uninorms (or representable uninorms) was given by Klement et al. [27] (see also Klement et al. [24] and Fodor and De Baets [15]).

Theorem 1 *(cf. [15], Theorem 1). Let $U\colon [0,1]^2\to[0,1]$ be a function and let $\nu\in(0,1)$. The following are equivalent:*

(a) *U is a uninorm with the neutral element ν which is strictly monotone on $(0,1)^2$ and continuous on $[0,1]^2\setminus\{(0,1),(1,0)\}$*
(b) *There exists a strictly increasing bijection $g_u\colon[0,1]\to[-\infty,\infty]$ with $g_u(\nu)=0$ such that for any $(x,y)\in[0,1]^2$, we have*

$$U(x,y) = g_u^{-1}\left(g_u(x)+g_u(y)\right), \tag{2}$$

where, in the case of a conjunctive uninorm U, we use the convention $\infty+(-\infty)=-\infty$, while, in the disjunctive case, we use $\infty+(-\infty)=\infty$.

Remark 2. If Eq. (2) holds, the function g_u is uniquely determined by U up to a positive multiplicative constant, and it is called an additive generator of the uninorm U. If g_u is an additive generator of the uninorm U, then the strictly increasing continuous function $\theta_u\colon[0,1]\to[0,\infty]$, which is given by

$$\theta_u(x) = \mathrm{e}^{g_u(x)},$$

is a multiplicative generator of U, i.e.,

$$U(x,y) = g_u^{-1}\left(g_u(x)+g_u(y)\right) = \theta_u^{-1}\left(\theta_u(x)\theta_u(y)\right)$$

holds for any $(x,y)\in[0,1]^2$.

The parameter ν in Eq. (1) is closely related to the Pliant negation (also known as the Dombi form of the negation), which is defined as follows (see [22,25]).

Definition 4 *(cf. [22], Sec. 5).* Let $f\colon[0,1]\to[0,\infty]$ be a generator function of a conjunctive or disjunctive operator and let $\nu\in(0,1)$. The mapping $\eta_\nu\colon[0,1]\to[0,1]$ given by

$$\eta_\nu(x) = f^{-1}\left(\frac{f^2(\nu)}{f(x)}\right) \tag{3}$$

is a Pliant negation operator with the parameter ν.

Note that parameter ν is the fixed point of the Pliant negation. Using the Pliant negation, a Pliant logical system can be defined as follows (see [22]).

Definition 5 *(cf. [22], Sec. 4).* Let the conjunctive operator c and the disjunctive operator d be induced by generator functions f_c and f_d, respectively, and let $\nu \in (0,1)$. We say that the triplet (c, d, η_ν) is a Pliant system if and only if

$$f_c(x) f_d(x) = 1$$

holds for any $x \in (0,1)$, and η_ν is a Pliant negation operator induced either by f_c or by f_d.

Remark 3. Note that in a Pliant system, the Pliant negations induced by f_c and f_d coincide, i.e., in this logical system, the conjunction, disjunction and negation operators are all determined by one generator function (which is uniquely determined up to a multiplicative constant); and c, d and η_ν form a De Morgan system (see Theorem 9 in [22]).

Remark 4. It is worth mentioning that the aggregative operator in Eq. (1) fulfills the self De Morgan identity with the Pliant negation operator given in Eq. (3), i.e., for any $(x_1, x_2, \ldots, x_n) \in (0,1)^n$,

$$\eta_\nu \left(a_\nu(x_1, x_2, \ldots, x_n)\right) = a_\nu \left(\eta_\nu(x_1), \eta_\nu(x_2), \ldots, \eta_\nu(x_n)\right).$$

2.2. The operator-dependent generalized sigmoid function

We will utilize the classical sigmoid function, which is also known as the logistic function.

Definition 6. The sigmoid function $\sigma^{(\lambda)}\colon [-\infty, \infty] \to (0,1)$ with the parameter $\lambda \in \mathbb{R} \setminus \{0\}$ is given by

$$\sigma^{(\lambda)}(x) = \frac{1}{1 + \mathrm{e}^{-4\lambda x}}. \tag{4}$$

Note that the constant 4 in Eq. (4) ensures that the derivative of $\sigma^{(\lambda)}(x)$ at $x = 0$ is equal to λ.

In continuous-valued logic, a generalized class of unary operators, called the kappa function, was introduced by Dombi [21]. This operator class has a number of applications (see e.g., [28–31]). The generator function-dependent kappa function on the interval $[a,b]$ is defined as follows.

Definition 7. The generator function-dependent kappa function $\kappa^{(\lambda)}_{x_\nu,\nu_0}(a,b;\cdot)\colon [a,b] \to [0,1]$ is given by

$$\kappa^{(\lambda)}_{x_\nu,\nu_0}(a,b;x) = f^{-1}\left(f(\nu_0) \left(\frac{f\left(\frac{x-a}{b-a}\right)}{f\left(\frac{x_\nu - a}{b-a}\right)} \right)^{\lambda} \right), \tag{5}$$

where $a < x_\nu < b$, $\nu_0 \in (0,1)$, $\lambda \in \mathbb{R}$ and f is a generator function of a conjunctive or disjunctive operator.

Here, we consider the following assertion.

Theorem 2. *Let f be a differentiable generator function of a conjunctive or disjunctive operator, and let f' be its derivative, $f'\left(\frac{1}{2}\right) \neq 0$ and f' is continuous on $(0,1)$. Let $\kappa^{(\lambda)}_{x_\nu,\nu_0}(a,b;\cdot)$ be a kappa function induced by f, where $a < x_\nu < b$, $\nu_0 \in (0,1)$, $\lambda \in \mathbb{R}$. Let $\lambda_\sigma \in \mathbb{R}$, $\lambda_\sigma \neq 0$,*

$$\frac{a+b}{2} = 0, \quad x_\nu = 0, \quad \lambda = \lambda_\sigma (b-a) \frac{f'(\nu_0)}{f(\nu_0)} \frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}.$$

Then, for any $x \in [a,b]$,

$$\lim_{\substack{a \to -\infty \\ b \to +\infty}} \kappa^{(\lambda)}_{x_\nu,\nu_0}(a,b;x) = f^{-1}\left(f(\nu_0)\, \mathrm{e}^{\lambda_\sigma \frac{f'(\nu_0)}{f(\nu_0)} x} \right), \tag{6}$$

where $\lim_{\substack{a \to -\infty \\ b \to +\infty}}$ is interpreted such that $a \to -\infty$, $b \to +\infty$ and $a + b = 0$.

Proof. Let $x \in [a, b]$ have a fixed value. Since $\frac{a+b}{2} = 0$; we can write $a = -\Delta$, $b = +\Delta$, where $\Delta = \frac{b-a}{2}$. Notice that the condition $a \to -\infty$, $b \to +\infty$ such that $\frac{a+b}{2} = 0$ is equivalent to $\Delta \to \infty$. Using variable Δ and noting the conditions of the theorem, Eq. (6) can be written as

$$\lim_{\Delta\to\infty} f^{-1}\left(f(\nu_0)\left(\left(\frac{f\left(\frac{x+\Delta}{2\Delta}\right)}{f\left(\frac{1}{2}\right)}\right)^{2\Delta\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}}\right)^{\lambda_\sigma \frac{f'(\nu_0)}{f(\nu_0)}}\right) = \\ = f^{-1}\left(f(\nu_0)\, \mathrm{e}^{\lambda_\sigma \frac{f'(\nu_0)}{f(\nu_0)} x}\right). \tag{7}$$

To prove Eq. (7), it is sufficient to show that

$$\lim_{\Delta\to\infty}\left(\frac{f\left(\frac{x+\Delta}{2\Delta}\right)}{f\left(\frac{1}{2}\right)}\right)^{2\Delta\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}} = \mathrm{e}^x. \tag{8}$$

Here, Eq. (8) is equivalent to

$$\lim_{\Delta\to\infty}\left(2\Delta\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}\ln\left(\frac{f\left(\frac{x+\Delta}{2\Delta}\right)}{f\left(\frac{1}{2}\right)}\right)\right) = x. \tag{9}$$

We can use the L'Hospital rule to prove Eq. (9). Taking into account the continuity and differentiability of f, after direct calculation, we have

$$\lim_{\Delta\to\infty}\left(2\Delta\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}\ln\left(\frac{f\left(\frac{x+\Delta}{2\Delta}\right)}{f\left(\frac{1}{2}\right)}\right)\right) = \\ = 2\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}\lim_{\Delta\to\infty}\left(\frac{\left(\ln\left(\frac{f\left(\frac{x+\Delta}{2\Delta}\right)}{f\left(\frac{1}{2}\right)}\right)\right)'}{\left(\frac{1}{\Delta}\right)'}\right) = \\ = 2\frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}\lim_{\Delta\to\infty}\left(\frac{f'\left(\frac{x+\Delta}{2\Delta}\right)}{f\left(\frac{x+\Delta}{2\Delta}\right)}\frac{x}{2}\right) = x. \quad \square$$

Noting Theorem 2, we define the generalized sigmoid function according to Eq. (6) as follows.

Definition 8. Let f be a differentiable generator function of a conjunctive or disjunctive operator. The generalized sigmoid function induced by f is a mapping $\sigma_\nu^{(\lambda)}: \mathbb{R} \to (0, 1)$, which is given by

$$\sigma_\nu^{(\lambda)}(x) = f^{-1}\left(f(\nu)\, \mathrm{e}^{\lambda \frac{f'(\nu)}{f(\nu)} x}\right), \tag{10}$$

where $\nu \in (0, 1)$, $\lambda \in \mathbb{R} \setminus \{0\}$.

Since the generalized sigmoid function given in Definition 8 is induced by a generator function of a conjunctive or disjunctive operator, we will also refer to this function as the operator-dependent sigmoid function. Here, we will briefly summarize the main properties of the generalized sigmoid function. It should be added that based on these properties, $\sigma_\nu^{(\lambda)}$ can be treated as a generalization of the classical sigmoid function.

2.2.1. Properties of the generalized sigmoid function

Here, we present the key properties of the generator function-dependent sigmoid function.

Proposition 1. *The generator function-dependent sigmoid function $\sigma_\nu^{(\lambda)}$, which is given in Definition 8, has the following properties:*

(a) *$\sigma_\nu^{(\lambda)}$ is a continuous and differentiable function.*

(b) *The derivative function of* $\sigma_\nu^{(\lambda)}$ *is*

$$\left(\sigma_\nu^{(\lambda)}(x)\right)' = \lambda \frac{f'(\nu)}{f(\nu)} \frac{f\left(\sigma_\nu^{(\lambda)}(x)\right)}{f'\left(\sigma_\nu^{(\lambda)}(x)\right)}.$$

(c) *If* $\lambda > 0$ *(*$\lambda < 0$*, respectively), then* $\sigma_\nu^{(\lambda)}$ *is strictly increasing (strictly decreasing, respectively).*

(d) *The values of* $\sigma_\nu^{(\lambda)}(x)$ *and its derivative at* $x = 0$ *are*

$$\sigma_\nu^{(\lambda)}(0) = \nu \quad \textit{and} \quad (\sigma_\nu^{(\lambda)}(x))'\big|_{x=0} = \lambda.$$

(e) *If* $\lambda > 0$ *(i.e.,* $\sigma_\nu^{(\lambda)}$ *is strictly increasing), then*

$$\lim_{x\to-\infty} \sigma_\nu^{(\lambda)}(x) = 0 \quad \textit{and} \quad \lim_{x\to\infty} \sigma_\nu^{(\lambda)}(x) = 1.$$

If $\lambda < 0$ *(i.e.,* $\sigma_\nu^{(\lambda)}$ *is strictly decreasing), then*

$$\lim_{x\to-\infty} \sigma_\nu^{(\lambda)}(x) = 1 \quad \textit{and} \quad \lim_{x\to\infty} \sigma_\nu^{(\lambda)}(x) = 0.$$

Proof. The proofs of the properties (a) - (e) are as follows.
(a) $\sigma_\nu^{(\lambda)}$ is a composition of continuous and differentiable functions and so it is continuous and differentiable as well.
(b) For the sake of simplicity, let $g(x)$ be given by

$$g(x) = f(\nu)\, \mathrm{e}^{\lambda \frac{f'(\nu)}{f(\nu)} x}, \quad x \in \mathbb{R}.$$

With this substitution, $\sigma_\nu^{(\lambda)}(x) = f^{-1}(g(x))$ and $\left(\sigma_\nu^{(\lambda)}(x)\right)'$ can be written as

$$\left(\sigma_\nu^{(\lambda)}(x)\right)' = \left(f^{-1}(g(x))\right)' = \frac{1}{f'\left(f^{-1}(g(x))\right)} g'(x) =$$

$$= \frac{1}{f'\left(\sigma_\nu^{(\lambda)}(x)\right)} f(\nu)\, \mathrm{e}^{\lambda \frac{f'(\nu)}{f(\nu)} x} \lambda \frac{f'(\nu)}{f(\nu)} = \lambda \frac{f'(\nu)}{f(\nu)} \frac{f\left(\sigma_\nu^{(\lambda)}(x)\right)}{f'\left(\sigma_\nu^{(\lambda)}(x)\right)}.$$

(c) Let $\lambda > 0$. If f is strictly increasing (decreasing, respectively), then both f^{-1} and $\mathrm{e}^{\lambda \frac{f'(\nu)}{f(\nu)}(\cdot)}$ are strictly increasing (decreasing, respectively) and so $\sigma_\nu^{(\lambda)}$ is strictly increasing.

Let $\lambda < 0$. If f is strictly increasing (decreasing, respectively), then f^{-1} is strictly increasing (decreasing, respectively) and $\mathrm{e}^{\lambda \frac{f'(\nu)}{f(\nu)}(\cdot)}$ is strictly decreasing (increasing, respectively) and so $\sigma_\nu^{(\lambda)}$ is strictly decreasing.
(d) Based on the definition of $\sigma_\nu^{(\lambda)}$, we readily get that $\sigma_\nu^{(\lambda)}(0) = \nu$. Taking into account this property and property (b), we get that $(\sigma_\nu^{(\lambda)}(x))'\big|_{x=0} = \lambda$.
(e) Taking into consideration the limit properties of a generator function f and the limit properties of the exponential function, this property immediately follows from the definition of the generator function-dependent sigmoid function $\sigma_\nu^{(\lambda)}$. □

In certain cases, based on practical considerations, it is more convenient to use alternative forms of the generalized sigmoid function. Since $\frac{f'(\nu)}{f(\nu)} \neq 0$, with the substitution

$$\lambda^* = -\lambda \frac{f'(\nu)}{f(\nu)},$$

the generalized sigmoid function in Eq. (10) can be written as

$$\sigma_\nu^{(\lambda^*)}(x) = f^{-1}\left(f(\nu)\, \mathrm{e}^{-\lambda^* x}\right). \tag{11}$$

However, we should add that the generalized sigmoid function in the form given by Eq. (10) has an advantageous property. Namely, based on Proposition 1, property (c), its monotonicity is independent of the type of the generator function (i.e., conjunctive or disjunctive), it depends solely on the sign of the value of parameter λ. If we used the form given by Eq. (11) in itself, i.e., if we left out of consideration that $\lambda^* = -\lambda \frac{f'(\nu)}{f(\nu)}$ and we just required that $\lambda^* \neq 0$, then the monotonicity of $\sigma_\nu^{(\lambda^*)}$ would depend both on the generator f and on the sign of λ^*.

The generalized sigmoid function neither takes the value of zero nor the value of one, these are its limits (see Proposition 1).

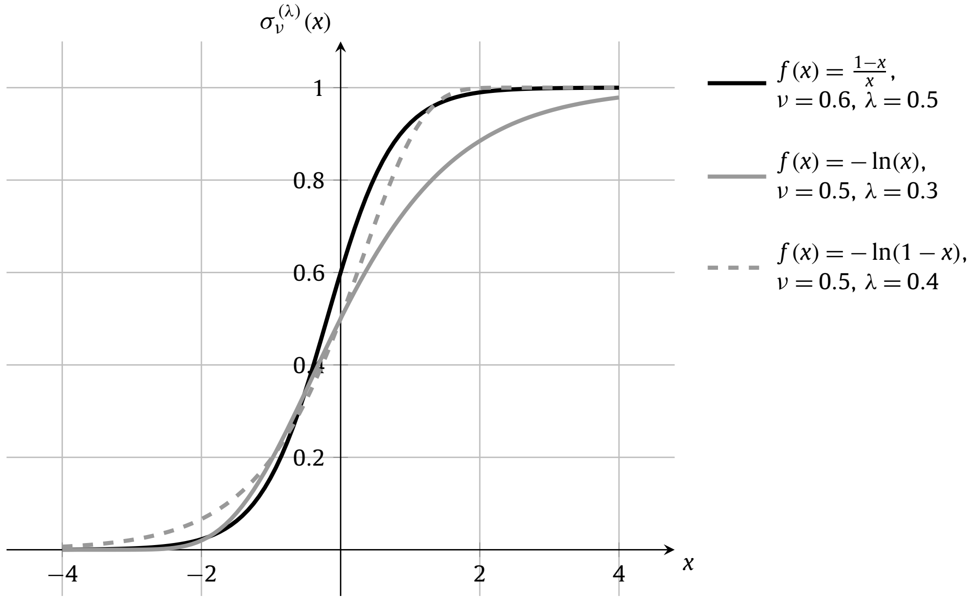

Fig. 1. Example plots of generator function-dependent sigmoid functions.

Remark 5. From now on, we will use the generalized sigmoid function $\sigma_\nu^{(\lambda)}$ with the range $[0,1]$ with the following convention:

(a) If $\lambda > 0$ (i.e., $\sigma_\nu^{(\lambda)}$ is strictly increasing), then

$$\sigma_\nu^{(\lambda)}(-\infty) = \lim_{x\to-\infty} \sigma_\nu^{(\lambda)}(x) = 0 \quad \text{and} \quad \sigma_\nu^{(\lambda)}(\infty) = \lim_{x\to\infty} \sigma_\nu^{(\lambda)}(x) = 1$$

(b) If $\lambda < 0$ (i.e., $\sigma_\nu^{(\lambda)}$ is strictly decreasing), then

$$\sigma_\nu^{(\lambda)}(-\infty) = \lim_{x\to-\infty} \sigma_\nu^{(\lambda)}(x) = 1 \quad \text{and} \quad \sigma_\nu^{(\lambda)}(\infty) = \lim_{x\to\infty} \sigma_\nu^{(\lambda)}(x) = 0,$$

where $\nu \in (0,1)$, $\lambda \in \mathbb{R} \setminus \{0\}$.

Fig. 1 shows sample plots of generator function-dependent sigmoid functions.

The classical sigmoid function is a special case of the generalized sigmoid function. Namely, if we use Eq. (10) with $f(x) = \frac{1-x}{x}$, $x \in (0,1)$ and $\nu = \frac{1}{2}$, then we get the classical sigmoid function, which is given in Definition 6.

3. The generalized sigmoid function, the aggregative operator and the logical operators

In this section, we will show how the generalized sigmoid function is connected with the aggregative operator. Also, we will present a generalized sigmoid function-based transformation that can be used to derive logical operators from the aggregative operator.

3.1. The generalized sigmoid function and the aggregative operator

Here, we will describe how the generalized sigmoid function given in Definition 8 is connected with the aggregative operator given in Definition 2.

Let f be a differentiable generator function of a conjunctive or disjunctive operator, let $\nu \in (0,1)$ and let the aggregative operator $a_\nu\colon [0,1]^2 \to [0,1]$ be induced by f, i.e.,

$$a_\nu(x,y) = f^{-1}\left(f(\nu)\frac{f(x)}{f(\nu)}\frac{f(y)}{f(\nu)}\right). \tag{12}$$

Let $\lambda > 0$ and let the function $\theta\colon [0,1] \to [0,\infty]$ be given by

$$\theta(x) = \left(\frac{f(x)}{f(\nu)}\right)^{\frac{1}{\lambda}\frac{f(\nu)}{f'(\nu)}}. \tag{13}$$

Clearly, θ is continuous, strictly increasing and it can be verified that

$$a_\nu(x, y) = \theta^{-1}\left(\theta(x)\theta(y)\right)$$

holds for any $(x, y) \in [0, 1]^2$. This means that a_ν is a representable uninorm with the neutral element ν and θ is a multiplicative generator of a_ν. Therefore,

$$g(x) = \ln\left(\theta(x)\right) = \frac{1}{\lambda}\frac{f(\nu)}{f'(\nu)}\ln\left(\frac{f(x)}{f(\nu)}\right) \tag{14}$$

is an additive generator of the aggregative operator a_ν. Taking into account that

$$g(x) = \left(\sigma_\nu^{(\lambda)}\right)^{-1}(x),$$

i.e., g is the inverse of the generalized sigmoid function $\sigma_\nu^{(\lambda)}$, we can conclude that the aggregative operator a_ν is a representable uninorm and the inverse of the generalized sigmoid function with a $\lambda > 0$ is an additive generator function of a_ν.

3.2. A proper transformation to derive logical operators from the aggregative operator

Now, we will show that the generalized sigmoid function plays a crucial role in transforming the aggregative operator into a conjunctive or disjunctive operator.

Here again, let f be a differentiable generator function of a conjunctive (disjunctive, respectively) operator, let $\nu \in (0, 1)$ and let the aggregative operator $a_\nu\colon [0, 1]^2 \to [0, 1]$ be induced by f according to Definition 2. For a conjunctive (disjunctive, respectively) generator function f, we seek to find a strictly increasing continuous function $t\colon [0, 1] \to [0, \nu]$ ($t\colon [0, 1] \to [\nu, 1]$, respectively) such that

$$t^{-1}\left(a_\nu(t(x), t(y))\right) = g^{-1}\left(g(x) + g(y)\right) \tag{15}$$

holds for any $x, y \in [0, 1]$, where g is a generator function of a conjunctive (disjunctive, respectively) operator. Here, we will distinguish two cases: (1) f and g are both additive generators of conjunctive operators (i.e., generators of strict t-norms); (2) f and g are both additive generators of disjunctive operators (i.e., generators of strict t-conorms).

(1) First let f be a generator function of a conjunctive operator, and let $p = t(x)$, $q = t(y)$, hence, $(p, q) \in [0, \nu]^2$. Then Eq. (15) is equivalent to

$$a_\nu(p, q) = u^{-1}\left(u(p) + u(q)\right), \tag{16}$$

where $u = -g \circ t^{-1}$, g is a generator function of a conjunctive operator and a_ν is restricted to $[0, \nu]^2$. Notice that since g is strictly decreasing and t is strictly increasing, u is a strictly increasing and continuous function with $u(\nu) = -g(t^{-1}(\nu)) = -g(1) = 0$. That is, u is an additive generator of a_ν in Eq. (16) (with the restriction of a_ν to $[0, \nu]^2$). Let $\theta(p)$ be given by

$$\theta(p) = \left(\frac{f(p)}{f(\nu)}\right)^b, \tag{17}$$

where $b < 0$. Clearly, θ in Eq. (17) is continuous, strictly increasing and it can be verified that

$$a_\nu(p, q) = \theta^{-1}\left(\theta(p)\theta(q)\right)$$

holds for any $(p, q) \in [0, \nu]^2$. That is, θ is a multiplicative generator of a_ν (with the restriction of a_ν to $[0, \nu]^2$). Therefore, $\ln(\theta(p))$ is an additive generator of a_ν. Since u is also an additive generator of a_ν and the additive generator is uniquely determined up to a positive constant multiplier, we have

$$u(p) = cb\ln\left(\frac{f(p)}{f(\nu)}\right),$$

where $c > 0$ is a constant. Since $cb < 0$ and f is differentiable and strictly decreasing, there exists a $\lambda < 0$ such that $\frac{1}{cb} = -\lambda\frac{f'(\nu)}{f(\nu)}$. Using the definition of p, from the previous equation we have

$$u(t(x)) = -\frac{1}{\lambda}\frac{f(\nu)}{f'(\nu)}\ln\left(\frac{f(t(x))}{f(\nu)}\right), \quad x, y \in [0, 1].$$

Taking into account that $u = -g \circ t^{-1}$, we have $u(t(x)) = -g(t^{-1}(t(x))) = -g(x)$ and so

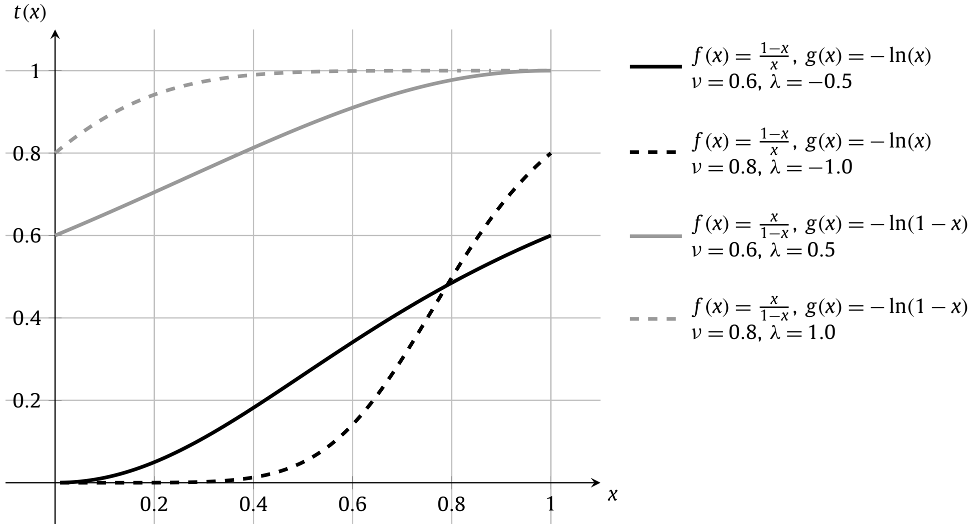

Fig. 2. Example plots of the transformation function t.

$$g(x) = \frac{1}{\lambda}\frac{f(\nu)}{f'(\nu)}\ln\left(\frac{f(t(x))}{f(\nu)}\right),$$

from which

$$t(x) = f^{-1}\left(f(\nu)\,\mathrm{e}^{\lambda\frac{f'(\nu)}{f(\nu)}g(x)}\right) = \sigma_\nu^{(\lambda)}(g(x)), \tag{18}$$

where $\lambda < 0$.

(2) In the case where f and g are both additive generators of disjunctive operators, function $t\colon [0,1] \to [\nu, 1]$ can be derived in a similar way to that of case (1) with the difference that $u = g \circ t^{-1}$, $b > 0$ and $\lambda > 0$. That is, in this case, $t\colon [0,1] \to [\nu, 1]$ has the form given by Eq. (18) with a $\lambda > 0$.

Remark 6. We should add that it can be verified by direct calculation that if the function t is given by Eq. (18), then it satisfies Eq. (15).

Based on the results above, we may conclude that we have proven the following theorem.

Theorem 3. *Let f be a differentiable generator function of a conjunctive (disjunctive, respectively) operator, let $\nu \in (0,1)$ and let the aggregative operator $a_\nu\colon [0,1]^2 \to [0,1]$ be induced by f according to Definition 2.*

For a conjunctive (disjunctive, respectively) generator function f, let $t\colon [0,1] \to [0,\nu]$ ($t\colon [0,1] \to [\nu,1]$, respectively) be a strictly increasing continuous function and let g be a generator function of a conjunctive (disjunctive, respectively) operator.

Then, for a conjunctive (disjunctive, respectively) generator function f, the equation

$$t^{-1}\left(a_\nu(t(x), t(y))\right) = g^{-1}\left(g(x) + g(y)\right)$$

holds for any $x, y \in [0,1]$ if and only if the function t has the form

$$t(x) = \sigma_\nu^{(\lambda)}\left(g(x)\right) = f^{-1}\left(f(\nu)\,\mathrm{e}^{\lambda\frac{f'(\nu)}{f(\nu)}g(x)}\right),$$

$x \in [0,1]$, with a $\lambda < 0$ (conjunctive case) ($\lambda > 0$ (disjunctive case), respectively).

Fig. 2 shows example plots of the transformation function t.

Remark 7. Note that if the neutral element ν of a representable uninorm is in the interval $(0,1)$, then a typical approach to transforming the uninorm into a t-norm or t-conorm is the application of a linear transformation to the uninorm. Here, in Theorem 3, we demonstrated that the adequate transformation is the generalized sigmoid function.

4. The generalized sigmoid function in Pliant systems

We have seen that the aggregative operator depends on its generator function. This means that generator functions of conjunctive and disjunctive logical operators may induce different aggregative operators. The generalized sigmoid function has the same characteristic, i.e., generalized sigmoid functions induced by conjunctive and disjunctive generator functions may differ. Here, we will give the sufficient condition for the equality of two aggregative operators (see Theorem 4). Also, we will give the sufficient condition for the equality of two generalized sigmoid functions (see Theorem 5). We will see that, interestingly, these two conditions are the same. Afterwards, using Theorem 4, Theorem 5 and the concept of Pliant logical systems, we will demonstrate how conjunctive and disjunctive operators, which form a De Morgan system with a Pliant negation, can be derived from an aggregative operator (see Theorem 6).

The following theorem, which can be verified via direct calculation, provides the sufficient condition for the equality of two aggregative operators.

Theorem 4. *Let $\nu \in (0,1)$ and let f and g be generator functions of conjunctive or disjunctive operators. Let $a_{\nu,f}\colon [0,1]^2 \to [0,1]$ and $a_{\nu,g}\colon [0,1]^2 \to [0,1]$ be two aggregative operators induced by f and g, respectively, according to Definition 2. Then,*

$$a_{\nu,f}(x,y) = a_{\nu,g}(x,y) \tag{19}$$

holds for any $x, y \in [0,1]$, if

$$f(x) = \beta g^{\alpha}(x) \tag{20}$$

holds for any $x \in [0,1]$, where $\alpha \neq 0$ and $\beta > 0$.

The following sufficient condition for the equality of two generalized sigmoid functions can be proven by direct calculation.

Theorem 5. *Let f and g be differentiable generator functions of conjunctive or disjunctive operators. Let the generalized sigmoid functions $\sigma_{\nu,f}^{(\lambda)}$ and $\sigma_{\nu,g}^{(\lambda)}$ be induced by f and g, respectively, where $\nu \in (0,1)$, $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. Then,*

$$\sigma_{\nu,f}^{(\lambda)}(x) = \sigma_{\nu,g}^{(\lambda)}(x) \tag{21}$$

holds for any $x \in \mathbb{R}$, if

$$f(x) = \beta g^{\alpha}(x) \tag{22}$$

holds for any $x \in [0,1]$, where $\alpha \neq 0$ and $\beta > 0$.

Theorem 6 tells us, how conjunctive and disjunctive operators, which form a De Morgan system with a Pliant negation, can be derived from an aggregative operator.

Theorem 6. *Let $\nu \in (0,1)$. Let f be a differentiable generator function of a conjunctive (disjunctive, respectively) operator, let g be a generator function of a conjunctive (disjunctive, respectively) operator and let η_ν be a Pliant negation operator induced by f. Then the following are valid:*

(a) *f and $\frac{1}{f}$ induce a common aggregative operator a_ν;*
(b) *f and $\frac{1}{f}$ induce a common generalized sigmoid function $\sigma_\nu^{(\lambda)}$;*
(c) *If f is a conjunctive (disjunctive, respectively) generator function, $\lambda < 0$ ($\lambda > 0$, respectively) and the functions*

$$t_1\colon [0,1] \to [0,\nu]\ (t_1\colon [0,1] \to [\nu,1], \textit{respectively})$$

and

$$t_2\colon [0,1] \to [\nu,1]\ (t_2\colon [0,1] \to [0,\nu], \textit{respectively})$$

are given by

$$t_1(x) = \sigma_\nu^{(\lambda)}\left(g(x)\right) \tag{23}$$

and

$$t_2(x) = \sigma_\nu^{(-\lambda)}\left(\frac{1}{g(x)}\right), \tag{24}$$

then
$o_1\colon [0,1]^2 \to [0,1]$, *which is given by*

$$o_1(x,y) = t_1^{-1}\left(a_\nu(t_1(x), t_1(y))\right),$$

is a conjunctive (disjunctive, respectively) operator,
$o_2\colon [0,1]^2 \to [0,1]$, *which is given by*

$$o_2(x,y) = t_2^{-1}\left(a_\nu(t_2(x), t_2(y))\right),$$

is a disjunctive (conjunctive, respectively) operator; and o_1, o_2 and η_ν form a De Morgan system.

Proof. Let the function f_* be given by

$$f_*(x) = \frac{1}{f(x)} \tag{25}$$

for any $x \in [0,1]$. It is obvious that if f is a generator function of a conjunctive (disjunctive, respectively) operator, then f_* is a generator function of a disjunctive (conjunctive, respectively) operator.

It is well known that if Eq. (25) holds for any $x \in [0,1]$, then the Pliant negation operators induced by f and f_* coincide (see Theorem 8 in [22]). That is, η_ν is the common negation operator induced by f and f_*.

Here, Eq. (25) can be written as

$$f_*(x) = \beta f^\alpha(x) \tag{26}$$

for any $x \in [0,1]$, where $\beta = 1$ and $\alpha = -1$. Therefore, noting Theorem 4 and Theorem 5, we immediately find that f and $\frac{1}{f}$ induce a common aggregative operator a_ν, and f and $\frac{1}{f}$ induce a common generalized sigmoid function $\sigma_\nu^{(\lambda)}$. This means that both (a) and (b) hold.

Based on the conditions of this theorem, f and g are either both conjunctive or both disjunctive generator functions. Here, we will prove (c) in the case where f is a conjunctive generator function (and so g is a conjunctive generator function as well). The proof of the disjunctive case is similar to that of the conjunctive case. Now, let $\lambda < 0$, let $t_1(x)$ be given by Eq. (23) for any $x \in [0,1]$ and let $t_2(x)$ be given by Eq. (24) for any $x \in [0,1]$. Then, taking into account Theorem 3, we readily get that

$$t_1^{-1}\left(a_\nu(t_1(x), t_1(y))\right) = g^{-1}\left(g(x) + g(y)\right)$$

and

$$t_2^{-1}\left(a_\nu(t_2(x), t_2(y))\right) = g^{-1}\left(\frac{1}{\frac{1}{g(x)} + \frac{1}{g(y)}}\right)$$

hold for any $(x,y) \in [0,1]^2$. This means that o_1 is a conjunctive operator induced by g and o_2 is a disjunctive operator induced by $\frac{1}{g}$. Moreover, noting the definition for a Pliant system in Definition 5, we conclude that (o_1, o_2, η_ν) is a Pliant system. Next, taking into account Remark 3 (see also Theorem 9 in [22]), we get that o_1, o_2 and η_ν form a De Morgan system. □

Example 1. The generator function of the Dombi conjunction and disjunction operators is the function $f_D\colon [0,1] \to [0,\infty]$ that is given by

$$f_D(x) = \left(\frac{1-x}{x}\right)^\alpha, \tag{27}$$

where $\alpha \neq 0$ (see, [32,33]). If $\alpha > 0$, then f_D is a generator function of a conjunctive operator; and if $\alpha < 0$, then f_D is a generator function of a disjunctive operator.

Let the function $f\colon [0,1] \to [0,\infty]$ be given by $f(x) = f_D(x)$, where $\alpha = 1$, i.e.,

$$f(x) = \frac{1-x}{x},$$

with the convention that $f(0) = \infty$. Then, f is a generator function of a conjunctive Dombi operator. Similarly, let the function $f_*\colon [0,1] \to [0,\infty]$ be given by $f_*(x) = f_D(x)$, where $\alpha = -1$, i.e.,

$$f_*(x) = \frac{x}{1-x},$$

with the convention that $f(1)=\infty$. Then, f_* is a generator function of a disjunctive Dombi operator. Let $\nu \in (0,1)$. Since $f(x)f_*(x)=1$ holds for any $x \in [0,1]$, the aggregative operators induced by f and f_* coincide. Let a_ν denote this common aggregative operator. We also know that f and f_* induce a common generalized sigmoid function $\sigma_\nu^{(\lambda)}\colon \mathbb{R} \to [0,1]$, which is given by

$$\sigma_\nu^{(\lambda)}(x) = \frac{1}{1+\frac{1-\nu}{\nu}\mathrm{e}^{-\lambda\frac{1}{\nu(1-\nu)}x}} \tag{28}$$

with the convention stated in Remark 5. Now, let $g\colon [0,1] \to [0,\infty]$ be given by $g(x)=f(x)$ for any $x \in [0,1]$. Next, let $\varepsilon \in \{-1,1\}$ be an indicator parameter, and for $\varepsilon = 1$ ($\varepsilon=-1$, respectively) let the function $t\colon [0,1] \to [0,\nu]$ ($t\colon [0,1] \to [\nu,1]$, respectively) be given by

$$t(x) = \sigma_\nu^{(\lambda\varepsilon)}\left(g^\varepsilon(x)\right),$$

where $\lambda \in \mathbb{R}$, $\lambda < 0$. Then, noting Eq. (28), $t(x)$ can be written as

$$t(x) = \frac{1}{1+\frac{1-\nu}{\nu}\mathrm{e}^{-\lambda\varepsilon\frac{1}{\nu(1-\nu)}\left(\frac{1-x}{x}\right)^\varepsilon}}, \tag{29}$$

where $\lambda \in \mathbb{R}$, $\lambda < 0$ and $\varepsilon \in \{-1,1\}$. Now, exploiting Theorem 6, we find that the function $o_\varepsilon\colon [0,1]^2 \to [0,1]$, which is given by

$$o_\varepsilon(x,y) = t^{-1}\left(a_\nu(t(x),t(y))\right) \tag{30}$$

is a conjunctive operator if $\varepsilon = 1$, and it is a disjunctive operator if $\varepsilon=-1$.

Furthermore, if $\nu = \frac{1}{2}$, which is frequently utilized in practice, and the value of λ is set as $\lambda = \frac{1}{4}$, then the function t in Eq. (29) becomes very simple:

$$t(x) = \begin{cases} \frac{1}{1+\mathrm{e}^{-\frac{1-x}{x}}}, & \text{if } \varepsilon=+1\text{, i.e., the operator } o \text{ is conjunctive} \\ \frac{1}{1+\mathrm{e}^{\frac{x}{1-x}}}, & \text{if } \varepsilon=-1\text{, i.e., the operator } o \text{ is disjunctive.} \end{cases} \tag{31}$$

Remark 8. The advantage of the approach presented in Example 1 lies in the fact that in a Pliant system, we have only one aggregative operator and one generalized sigmoid function, which are both induced by a generator function of the same disjunctive or conjunctive operator. Hence, using Eq. (30), we can derive conjunctive or disjunctive operators from the aggregative operator by changing the value of the ε parameter. Recall that the function t in Eq. (30) depends on the parameters ν, λ and ε, but the operator o_ε depends only on the parameters ν and ε.

5. Generalized sigmoid functions as arguments of operators

In this section, we will present some interesting results related to operations over fuzzy sets that have the generalized sigmoid functions as membership functions. Namely, we will show that, under certain conditions, a set of generalized sigmoid functions is closed under the Pliant negation, the aggregative operation and certain modifying operations. Here, we will use the following interpretation of the generalized sigmoid function.

Let f be a differentiable generator function of a conjunctive or disjunctive operator, and let the generalized sigmoid function $\sigma_{\nu,x_0}^{(\lambda)}\colon \mathbb{R} \to (0,1)$ be given by

$$\sigma_{x_0,\nu}^{(\lambda)}(x) = f^{-1}\left(f(\nu)\,\mathrm{e}^{\lambda\frac{f'(\nu)}{f(\nu)}(x-x_0)}\right), \tag{32}$$

where $\nu \in (0,1)$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $x_0 \in \mathbb{R}$.

Here, we will consider the generalized sigmoid function $\sigma_{x_0,\nu}^{(\lambda)}\colon \mathbb{R} \to (0,1)$ as being a fuzzy membership function.

5.1. Building complement using Pliant negation

In the following theorem, which can be proved via direct calculations, we will show that applying the Pliant negation to a generalized sigmoid function results in a generalized sigmoid function. We find that this operation inverts the sign of the λ parameter.

Theorem 7. *Let f be a differentiable generator function of a conjunctive or disjunctive operator, let $\nu \in (0,1)$, $x_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Let the generalized sigmoid function $\sigma_{x_0,\nu}^{(\lambda)}$ be given by Eq. (32) and let the Pliant negation operator η_ν be induced by f according to Eq. (3). Then,*

$$\eta_\nu\left(\sigma^{(\lambda)}_{x_0,\nu}(x)\right)=\sigma^{(-\lambda)}_{x_0,\nu}(x)$$

holds for any $x\in\mathbb{R}$.

5.2. The aggregative operator case

Theorem 8 describes the case where a set of generalized sigmoid functions is closed under the aggregative operation. The proof of this theorem is straightforward.

Theorem 8. *Let* f *be a differentiable generator function of a conjunctive or disjunctive operator, let* $\nu\in(0,1)$*,* $x_{0,i}\in\mathbb{R}$*,* $\lambda_i\in\mathbb{R}\setminus\{0\}$*,* $i=1,2,\ldots,n$ *and* $n\geq 2$*. Let the generalized sigmoid function* $\sigma^{(\lambda_i)}_{x_{0,i},\nu}$ *be given by Eq. (32) and let the aggregative operator* a_ν *be induced by* f *according to Eq. (1).*

If $\sum_{i=1}^n\lambda_i\neq 0$*, then*

$$a_\nu\left(\sigma^{(\lambda_1)}_{x_{0,1},\nu}(x),\sigma^{(\lambda_2)}_{x_{0,2},\nu}(x),\ldots,\sigma^{(\lambda_n)}_{x_{0,n},\nu}(x)\right)=\sigma^{(\lambda)}_{x_0,\nu}(x)$$

holds for any $x\in\mathbb{R}$*, where*

$$\lambda=\sum_{i=1}^n\lambda_i \quad \textit{and} \quad x_0=\frac{\sum_{i=1}^n\lambda_i x_{0,i}}{\sum_{i=1}^n\lambda_i}. \tag{33}$$

5.3. The modifier operator case

In a previous paper of ours (see [34]) we showed that the operator $\tau_{\nu,\nu_0}\colon[0,1]\to[0,1]$ given by

$$\tau_{\nu,\nu_0}(x)=f^{-1}\left(f(\nu_0)\frac{f(x)}{f(\nu)}\right), \tag{34}$$

where $\nu,\nu_0\in(0,1)$ and f is a generator function of a conjunctive or disjunctive operator, may be viewed as the common form of substantiating and weakening modifier operators, modal operators and linguistic hedges.

The following theorem, which can be verified via direct calculation, states that applying the τ_{ν,ν_0} operator to a generalized sigmoid function results in a generalized sigmoid function.

Theorem 9. *Let* f *be a differentiable generator function of a conjunctive or disjunctive operator, let* $\nu,\nu_0\in(0,1)$*,* $x_0\in\mathbb{R}$ *and* $\lambda\in\mathbb{R}\setminus\{0\}$*. Let the generalized sigmoid function* $\sigma^{(\lambda)}_{x_0,\nu}$ *be given by Eq. (32) and let the operator* $\tau_{\nu,\nu_0}\colon[0,1]\to[0,1]$ *be induced by* f *according to Eq. (34). Then,*

$$\tau_{\nu,\nu_0}\left(\sigma^{(\lambda)}_{x_0,\nu}(x)\right)=\sigma^{(\lambda)}_{x_0^*,\nu}(x)$$

holds for any $x\in\mathbb{R}$*, where*

$$x_0^*=x_0-\frac{1}{\lambda}\frac{f(\nu)}{f'(\nu)}\ln\left(\frac{f(\nu_0)}{f(\nu)}\right). \tag{35}$$

Based on Theorem 9, we may conclude that application of the modifier operator in Eq. (34) on a generalized sigmoid function shifts the function along the horizontal axis.

6. The generalized sigmoid function and artificial neural networks

Now, we will demonstrate an important connection between the weighted aggregative operator and the generalized sigmoid function. Based on this connection, we can provide a new interpretation of neurons in an artificial neural network.

Here, we consider the following weighted form of the aggregative operator, which was introduced by Dombi (see [25]).

Definition 9. The function $a_\nu\colon[0,1]^n\times[0,1]^n\to[0,1]$ is a weighted aggregative operator if and only if a_ν is given by

$$a_\nu(\mathbf{w};\mathbf{x})=f^{-1}\left(f(\nu)\prod_{i=1}^n\left(\frac{f(x_i)}{f(\nu)}\right)^{w_i}\right), \tag{36}$$

where $\nu\in(0,1)$, $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in[0,1]^n$, $\mathbf{w}=(w_1,w_2,\ldots,w_n)\in[0,1]^n$, $\sum_{i=1}^n w_i=1$ and f is a generator function of a conjunctive or disjunctive operator.

Theorem 10. *Let $\mu\colon \mathbb{R} \to (0,1)$ be a strictly monotonic mapping. Let f be a differentiable generator function of a conjunctive or disjunctive operator, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$, $i = 1, 2, \ldots, n$ and $n \geq 2$. Let $\nu \in (0,1)$ and let the weighted aggregative operator a_ν be induced by f according to Eq. (36).*

(a) *If for a $\lambda \in \mathbb{R} \setminus \{0\}$,*

$$\mu(x) = \sigma_\nu^{(\lambda)}(x) \tag{37}$$

holds for any $x \in \mathbb{R}$, then

$$a_\nu\left((w_1, w_2, \ldots, w_n); (\mu(x_1), \mu(x_2), \ldots, \mu(x_n))\right) = \mu\left(\sum_{i=1}^{n} w_i x_i\right) \tag{38}$$

holds for any $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where $\sigma_\nu^{(\lambda)}$ is the generalized sigmoid function defined in Definition 8.

(b) *If Eq. (38) holds for any $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then there exists a $\lambda \in \mathbb{R} \setminus \{0\}$ and an $a \in \mathbb{R}$ such that*

$$\mu(x) = \sigma_\nu^{(\lambda)}(x - a) \tag{39}$$

holds for any $x \in \mathbb{R}$.

Proof. Proof of (a). Let $\lambda \in \mathbb{R} \setminus \{0\}$ and let us suppose that Eq. (37) holds for any $x \in \mathbb{R}$. Noting Eq. (37), the definition of the generalized sigmoid function in Definition 8 and the definition of the weighted aggregative operator in Definition 9, by direct calculation we get

$$\begin{aligned} & a_\nu\left((w_1, w_2, \ldots, w_n); (\mu(x_1), \mu(x_2), \ldots, \mu(x_n))\right) = \\ = f^{-1}&\left(f(\nu)\prod_{i=1}^{n}\left(\frac{f(\mu(x_i))}{f(\nu)}\right)^{w_i}\right) = f^{-1}\left(f(\nu)\prod_{i=1}^{n} \mathrm{e}^{\lambda \frac{f'(\nu)}{f(\nu)} w_i x_i}\right) = \\ & = f^{-1}\left(f(\nu)\,\mathrm{e}^{\lambda \frac{f'(\nu)}{f(\nu)} \sum_{i=1}^{n} w_i x_i}\right) = \mu\left(\sum_{i=1}^{n} w_i x_i\right) \end{aligned}$$

for any $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Proof of (b). Suppose that Eq. (38) holds for any $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Using the definition of the weighted aggregative operator in Definition 9, Eq. (38) can be written as

$$f^{-1}\left(f(\nu)\prod_{i=1}^{n}\left(\frac{f(\mu(x_i))}{f(\nu)}\right)^{w_i}\right) = \mu\left(\sum_{i=1}^{n} w_i x_i\right) \tag{40}$$

for any $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. After applying f to both sides of Eq. (40), we get

$$\prod_{i=1}^{n}\left(\frac{f(\mu(x_i))}{f(\nu)}\right)^{w_i} = \frac{f\left(\mu\left(\sum_{i=1}^{n} w_i x_i\right)\right)}{f(\nu)}. \tag{41}$$

Now, let the function $F\colon \mathbb{R} \to (0, \infty)$ be given by

$$F(x) = \frac{f(\mu(x))}{f(\nu)}. \tag{42}$$

Then, Eq. (41) can be written as

$$\prod_{i=1}^{n} F^{w_i}(x_i) = F\left(\left(\sum_{i=1}^{n} w_i x_i\right)\right). \tag{43}$$

Since both sides of Eq. (43) are positive, by taking the logarithm of both sides of this equation and introducing

$$G(x) = \ln\left(F(x)\right), \tag{44}$$

we can write Eq. (43) as

$$\sum_{i=1}^{n} w_i G(x_i) = G\left(\sum_{i=1}^{n} w_i x_i\right). \tag{45}$$

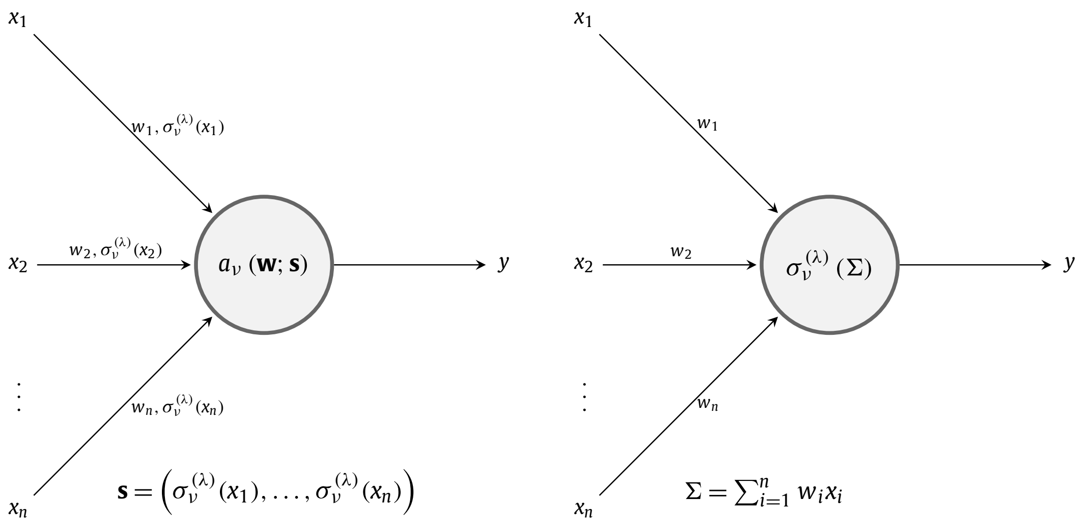

Fig. 3. Two equivalent models of a perceptron.

The functional equation in Eq. (45) is a Cauchy functional equation with $\sum_{i=1}^n w_i = 1$. It is well known that the solution of Eq. (45) has the form

$$G(x) = cx + d, \tag{46}$$

where $c, d \in \mathbb{R}$ are constants (see Section 2.2.6. in [23]). Noting Eq. (42), Eq. (44) and Eq. (46), we have

$$e^{cx+d} = \frac{f(\mu(x))}{f(\nu)},$$

from which

$$\mu(x) = f^{-1}\left(f(\nu)\, e^{cx+d}\right)$$

follows for any $x \in \mathbb{R}$. Noting that F is a strictly monotonic function, G is a strictly monotonic function as well. This means that $c \neq 0$ necessarily holds. Therefore, there exists a $\lambda \neq 0$ such that

$$c = \lambda \frac{f'(\nu)}{f(\nu)},$$

which means that $\mu(x)$ can be written as

$$\mu(x) = f^{-1}\left(f(\nu)\, e^{\lambda \frac{f'(\nu)}{f(\nu)}(x-a)}\right) = \sigma_\nu^{(\lambda)}(x-a).$$

for any $x \in \mathbb{R}$, where $a = -\frac{d}{c}$. □

Theorem 10 (a) allows us to provide two equivalent representations of a perceptron in an artificial neural network. Fig. 3 shows these two representations.

In the left hand side model in Fig. 3, the perceptron is based on an aggregative operator. This operator receives the $w_1, w_2, \dots, w_n$ weight values for the $x_1, x_2, \dots, x_n$ inputs, respectively, and the values of the generalized sigmoid function computed at $x_1, x_2, \dots, x_n$. That is, the inputs of the aggregative operator are the vectors $\mathbf{w} = (w_1, w_2, \dots, w_n)$ and $\mathbf{s} = \left(\sigma_\nu^{(\lambda)}(x_1), \dots, \sigma_\nu^{(\lambda)}(x_n)\right)$. The perceptron computes the $y = a_\nu(\mathbf{w}; \mathbf{s})$ output value.

The right hand side graph in Fig. 3 shows the classical model of a perceptron. In this case, the perceptron is based on a generalized sigmoid function, which receives the $w_1, w_2, \dots, w_n$ weight values and the $x_1, x_2, \dots, x_n$ input variable values, and it computes the $y = \sigma_\nu^{(\lambda)}\left(\sum_{i=1}^n w_i x_i\right)$ output value.

Based on Theorem 10 (a), $a_\nu(\mathbf{w}; \mathbf{s}) = \sigma_\nu^{(\lambda)}\left(\sum_{i=1}^n w_i x_i\right)$. Therefore, the two representations of a perceptron in Fig. 3 are equivalent.

Remark 9. In the light of the above results, the classical artificial neural networks (perceptron back-propagation, etc.) may be viewed as constructions that are based on the aggregative operator and the generalized sigmoid function.

Table 1
Excerpts of Theorem 3 in two contexts.

Theorem 3 in aggregative operator context: $t^{-1}\left(f^{-1}\left(f(\nu)\frac{f(t(x))}{f(\nu)}\frac{f(t(y))}{f(\nu)}\right)\right)=g^{-1}(g(x)+g(y))$ iff $t(x)=\sigma_\nu^{(\lambda)}(g(x))=f^{-1}\left(f(\nu)\,\mathrm{e}^{\lambda\frac{f'(\nu)}{f(\nu)}g(x)}\right)$, where g is an additive generator of a conjunctive (disjunctive, respectively) operator and $\lambda<0$ ($\lambda>0$, respectively).
Theorem 3 in representable uninorm context: $t^{-1}\left(g_u^{-1}(g_u(t(x))+g_u(t(y)))\right)=g^{-1}(g(x)+g(y))$ iff $t(x)=\sigma_\nu^{(\lambda)}(g(x))=f^{-1}\left(f(\nu)\,\mathrm{e}^{\lambda\frac{f'(\nu)}{f(\nu)}g(x)}\right)=g_u^{-1}(g(x))$, where g is an additive generator of a conjunctive (disjunctive, respectively) operator and $\lambda<0$ ($\lambda>0$, respectively).

7. Conclusions and future research

7.1. Main findings

The main findings of our study can be summarized as follows.

(a) In this paper, we introduced the concept of the so-called operator-dependent generalized sigmoid function, which can be derived from the unary operator called the kappa function.
(b) We studied how the generalized sigmoid function is connected to the aggregative operator. Namely, we showed that the aggregative operator is a uninorm which can be represented by the inverse of a generalized sigmoid function. Then we demonstrated that in certain domains the aggregative operator is a conjunctive or a disjunctive operator.
(c) We provided the necessary and sufficient conditions for the form of the transformation that turns the aggregative operator into a logical operator. Here, we proved that this transformation is based on the generalized sigmoid function.
(d) Next, we described the advantages of using the above results in a Pliant logical system. Here, we showed how conjunctive and disjunctive operators, which form a De Morgan system with a Pliant negation, can be derived from an aggregative operator (see Eq. (29)).
(e) It was also shown that, under certain conditions, a set of generalized sigmoid functions is closed under the Pliant negation, the aggregative operation and certain modifying operations.
(f) Lastly, we demonstrated an important connection between the weighted aggregative operator and the generalized sigmoid function. Based on this, we provided a new interpretation of neurons in an artificial neural network. Namely, we showed that a feed-forward neural network may be treated as a construction that is based on the aggregative operator and the generalized sigmoid function.
(g) We showed that the aggregative operator is a representable uninorm. This means that for any $x_1, x_2, \ldots, x_n \in [0,1]$,

$$a_{\nu,f}(x_1,x_2,\ldots,x_n)=U_{\nu,g_u}(x_1,x_2,\ldots,x_n),$$

where

$$a_{\nu,f}(x_1,x_2,\ldots,x_n)=f^{-1}\left(f(\nu)\prod_{i=1}^{n}\frac{f(x_i)}{f(\nu)}\right),$$
$$U_{\nu,g_u}(x_1,x_2,\ldots,x_n)=g_u^{-1}\left(\sum_{i=1}^{n}g_u(x_i)\right),$$
$$g_u(x)=\left(\sigma_\nu^{(\lambda)}\right)^{-1}(x)=\frac{1}{\lambda}\frac{f(\nu)}{f'(\nu)}\ln\left(\frac{f(x)}{f(\nu)}\right),\quad \lambda>0$$

and f is a differentiable additive generator of either a strict t-norm or a strict t-conorm. Using this connection between the aggregative operator and the representable uninorm, most of the results in our study can be provided in two different contexts, namely, in an aggregative operator context and in a representable uninorm context. For example, Table 1 and Table 2 show the excerpts of Theorem 3 and Theorem 6 (c), respectively, in these two contexts.

7.2. Future research plans

One possible direction of future research is the practical application of the aggregative operator and the generalized sigmoid function in neural networks. We pointed out that by using the generalized sigmoid function, we can use the aggregative operator as a conjunctive or disjunctive operator. This also allows us to build neural networks that become interpretable after the training process.

Table 2
Excerpts of Theorem 6 (c) in two contexts.

Let $\nu \in (0,1)$. Let f be a differentiable generator function of a conjunctive (disjunctive, resp.) operator, let g be a generator function of a conjunctive (disjunctive, resp.) operator and let η_ν be a Pliant negation operator induced by f. If f is a conjunctive (disjunctive, resp.) generator function, $\lambda < 0$ ($\lambda > 0$, resp.) and the functions t_1 and t_2 are given by
aggregative operator context: $t_1(x) = \sigma_\nu^{(\lambda)}(g(x))$ and $t_2(x) = \sigma_\nu^{(-\lambda)}\left(\frac{1}{g(x)}\right)$, then $o_1(x,y) = t_1^{-1}\left(f^{-1}\left(f(\nu)\frac{f(t_1(x))}{f(\nu)}\frac{f(t_1(y))}{f(\nu)}\right)\right)$ is a conjunctive (disjunctive, resp.) operator, $o_2(x,y) = t_2^{-1}\left(f^{-1}\left(f(\nu)\frac{f(t_2(x))}{f(\nu)}\frac{f(t_2(y))}{f(\nu)}\right)\right)$ is a disjunctive (conjunctive, resp.) operator; and o_1, o_2 and η_ν form a De Morgan system.
representable uninorm context: $t_1(x) = \sigma_\nu^{(\lambda)}(g(x))$ and $t_2(x) = \sigma_\nu^{(-\lambda)}\left(\frac{1}{g(x)}\right)$, then $o_1(x,y) = t_1^{-1}\left(g_u^{-1}\left(g_u(t_1(x)) + g_u(t_1(y))\right)\right)$ is a conjunctive (disjunctive, resp.) operator, $o_2(x,y) = t_2^{-1}\left(g_u^{-1}\left(g_u(t_2(x)) + g_u(t_2(y))\right)\right)$ is a disjunctive (conjunctive, resp.) operator; and o_1, o_2 and η_ν form a De Morgan system.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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