The tau-additive measure and its connection with the lambda-additive measure

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Abstract

In this paper, we study monotone set functions defined as the composition of an additive measure with a strictly increasing function. This function is a unary operator in continuous-valued logic, called the tau function, and it is a generator function-based parametric mapping. We provide a necessary and sufficient condition for the equality of two tau functions that are induced by different generator functions. Using the tau function and its properties, we introduce a new monotone measure that we call the tau-additive measure. This measure is computationally simple and it can be viewed as an upper or lower probability depending on the parameter settings of the tau function. We present the parameter-dependent submodularity and supermodularity of the tau-additive measure and show how this measure can be constructed from a set function on a finite set. This procedure is analogous to how the well-known lambda-additive measure can be constructed, but our method is computationally simpler. We demonstrate that the tau-additive measure can be used to approximate the lambda-additive measure. Lastly, exploiting these theoretical results, we present an application in the area of human resource management.

1. Introduction

It is a well-known fact that fuzzy measures (also known as monotone measures, non-additive measures or capacities) play an important role in many areas of science. This is why there has been a steady interest in them. Without claiming completeness, see, e.g., the publications [1–8].

One of the most widely applied class of monotone measures is the class of λ-additive measures (Sugeno λ-measures) [9]. There are many theoretical and practical articles related to this class of monotone measures, see, e.g.

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[10–14]. In previous articles of ours, we presented the so-called ν-additive measure, which may be viewed as an alternative to lambda-additive measure (see [15,16]). We also provided the general Poincaré-formula for lambda-additive measures and some inequalities related to these measures (see [16–18]).

In our study, we sought to construct a flexible monotone measure that could be applied in solving a wide range of problems and may be treated as a new alternative to lambda-additive measure. For this purpose, we constructed the tau-additive measure, which is a composition of an appropriate T-transformation (i.e., a continuous and strictly increasing mapping on the interval [0, 1] with the range [0, 1]) and an additive measure. In our approach, the T-transformation is a generator function-based parametric transformation called the tau function. This function is also known as a unary operator in continuous-valued logic.

The tau function is generator function-dependent and it has two parameters, ν and ν_0, whose values are in the interval (0, 1). These features of the tau transformation make the tau-additive measure very flexible. On the one hand, it can be adjusted via the parameters of the tau function; and on the other hand, for fixed values of the parameters, various tau-additive measures can be obtained depending on the choice of the generator function of the tau transformation. Furthermore, we sought to find a monotone measure, which is computationally simple and, like the lambda-additive measure, can be treated as a parameter-dependent upper or lower probability measure. Here, we should remark that the tau transformation is mathematically simple and, depending on its parameter values, the corresponding tau-additive measure can be subadditive, superadditive or additive. As we will show, if the generator function of the tau function is that of the Dombi operators in continuous valued-logic (see [19,20]), then the form of the tau function becomes very simple, it is just the fraction of two first order polynomials. Since the tau function is generator function-dependent, it is an interesting question under what conditions two tau functions, which are induced by different generators, coincide. In our study, we will provide a necessary and sufficient condition for the identity of two tau functions that are induced by different generator functions.

Since λ-additive measures can be utilized in many areas, we attempted to construct a monotone measure that, besides having the above-mentioned properties, approximates the λ-additive measure well. We will demonstrate that using the generator function of the Dombi operators as generator of the tau function, with appropriate parameter settings, we can get a particular tau function that approximates the T-function in the additive representation of the λ-additive measure quite well.

As we will see later on, if we fix one of the parameters of a tau function, then we can change its shape by adjusting the value of the other parameter. The value of the λ parameter of a λ-additive measure lies in the interval $(-1, \infty)$, and if $\lambda > 0$ ($\lambda < 0$, respectively), then the λ-additive measure is superadditive (subadditive, respectively). The ν and ν_0 parameters of a tau-additive measure are both in the interval (0, 1), and if $\nu > \nu_0$ ($\nu < \nu_0$, respectively), then the tau-additive measure is superadditive (subadditive, respectively). That is, in the case of the λ-additive measure, zero value of λ does not divide the $(-1, \infty)$ domain into two symmetric subdomains that correspond to the subadditive and superadditive properties of the measure. In the case of the tau-additive measure, if $\nu_0 = \frac{1}{2}$, then the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ are the two domains of ν that correspond to the subadditive and superadditive characteristic of the tau-additive measure, respectively. This also means that the value of parameter ν may be viewed as the 'level of subadditivity or superadditivity' when ν_0 has a fixed value. Moreover, if we need to find the value of the λ parameter of a λ-additive measure numerically, for instance in a curve fitting problem, then we need to find the value of λ in the $(-1, \infty)$ interval. At the same time, in the case of the tau-additive measure, we can fix the value of ν_0, and look for the value of ν in the (0, 1) interval. This can greatly simplify the numerical computations.

It is well known that if we have the values of a set function for n subsets of a finite set X such that these values are all non-negative, less than one, at least two of them are positive and the union of the subsets is X, then there exists a unique $\lambda \in (-1, \infty)$ such that the corresponding λ-additive measure coincides with the set function at each subset in question (see Theorem 4.7 in [4]). We will show that the tau-additive measure has a similar property, which makes it suitable for modeling uncertainty. By means of an example, we will demonstrate how this property and the sub- or superadditivity of the tau-additive measure can be utilized in human resource management.

This paper is structured as follows. In Section 2, we briefly review some basic concepts, constructions and previous results that are connected with our study. The properties of the tau function important from a measure theoretical point of view are discussed in Section 3. The tau-additive measure and its most important features are then introduced in Section 4. Next, in Section 5, we show how the tau-additive measure can be used to approximate the lambda-additive measure. In Section 6, an application of the tau-additive measure in human resource management is presented. Lastly, the key findings and the main conclusions of this study are summarized in Section 7.

2. Preliminaries

In this study, we will use the common notations $\cap$ and $\cup$ for the intersection and union operations over sets, respectively. Also, we will use the notation $\overline{A}$ for the complement of set A. Here, $\mathcal{P}(X)$ will denote the power set of a finite set X. Now, we will briefly review some basic concepts, constructions and previous results, which will be utilized later.

2.1. Set functions and monotone measures

A set function μ on a finite set X is non-negative, extended real-valued and zero at the empty set, i.e.,

$$\mu \colon \mathcal{P}(X) \to [0, \infty], \quad \mu(\emptyset) = 0,$$

see, e.g., Chapter 2 in Denneberg's book [21].

Definition 1. Let $\mu \colon \mathcal{P}(X) \to [0, \infty]$ be a set function on the finite set X. We say that μ is

monotone	if $A, B \in \mathcal{P}(X)$, $A \subseteq B$ implies $\mu(A) \leq \mu(B)$
submodular	if $A, B \in \mathcal{P}(X)$ implies $\mu(A \cup B) \leq \mu(A) + \mu(B) - \mu(A \cap B)$
supermodular	if $A, B \in \mathcal{P}(X)$ implies $\mu(A \cup B) \geq \mu(A) + \mu(B) - \mu(A \cap B)$
subadditive	if $A, B \in \mathcal{P}(X)$, $A \cap B = \emptyset$ implies $\mu(A \cup B) \leq \mu(A) + \mu(B)$
superadditive	if $A, B \in \mathcal{P}(X)$, $A \cap B = \emptyset$ implies $\mu(A \cup B) \geq \mu(A) + \mu(B)$
additive	if μ is sub- and superadditive.

If a set function μ on a finite set X is additive with $\mu(X) = 1$, then μ is a probability measure on X. Hereafter, a monotone measure μ on a finite set X will always be a monotone set function $\mu \colon \mathcal{P}(X) \to [0, 1]$ with $\mu(X) = 1$.

Following Wang and Klir (see Section 4.4 in [4]), the T-function and the additive representability of a monotone measure are defined as follows.

Definition 2. We say that the mapping $\theta \colon [0, 1] \to [0, 1]$ is a T-function if and only if θ is continuous and strictly increasing such that $\theta(0) = 0$ and $\theta(1) = 1$.

Definition 3. Let $\mu \colon \mathcal{P}(X) \to [0, 1]$ be a monotone measure on the finite set X. We say that μ is representable if there exists an additive measure $m \colon \mathcal{P}(X) \to [0, 1]$ and a T-function $\theta \colon [0, 1] \to [0, 1]$ such that

$$\mu = \theta \circ m.$$

In this case, we say that the pair (m, θ) represents μ.

2.2. λ-additive measure

The λ-additive measure, which was proposed by Sugeno in 1974 [9], plays an important role in computer science.

Definition 4. The function $Q_\lambda \colon \mathcal{P}(X) \to [0, 1]$ is a λ-additive measure (Sugeno λ-measure) on the finite set X, if and only if Q_λ satisfies the following requirements:

(1) $Q_\lambda(X) = 1$
(2) For any $A, B \in \mathcal{P}(X)$ and $A \cap B = \emptyset$,

$$Q_\lambda(A \cup B) = Q_\lambda(A) + Q_\lambda(B) + \lambda Q_\lambda(A) Q_\lambda(B), \tag{1}$$

where $\lambda \in (-1, \infty)$.

It is a well-known fact that the lambda additive measure Q_λ is representable (see, e.g., Section 4.4 in [4] or [12]). More precisely, $Q_\lambda = h_\lambda \circ \mu$ for a uniquely determined additive measure $\mu \colon \mathcal{P}(X) \to [0, 1]$, where the T-function $h_\lambda \colon [0, 1] \to [0, 1]$ is given by

$$h_\lambda(x) = \begin{cases} \dfrac{(1+\lambda)^x - 1}{\lambda}, & \text{if } \lambda \neq 0 \\ x, & \text{if } \lambda = 0, \end{cases} \tag{2}$$

and $\lambda \in (-1, \infty)$.

2.3. The tau function

Later, in Section 4, we will construct a monotone measure called the tau-additive measure, which is represented by a T-function called the tau function. This function was first introduced by Dombi as a unary modifier operator in continuous-valued logic (see [22]), and it is defined as follows.

Definition 5 *(Tau function).* Let the continuous function $f \colon [0, 1] \to [0, \infty]$ be either

(a) strictly decreasing with $f(1) = 0$ and $\lim_{x \to 0} f(x) = \infty$ or
(b) strictly increasing with $f(0) = 0$ and $\lim_{x \to 1} f(x) = \infty$.

Let $v, v_0 \in (0, 1)$. We say that the mapping $\tau_{v,v_0} \colon [0, 1] \to [0, 1]$ is a tau function with the parameters v, v_0, induced by function f, if and only if τ_{v,v_0} is given by

$$\tau_{v,v_0}(x) = f^{-1}\left(f(v_0) \frac{f(x)}{f(v)} \right). \tag{3}$$

Here, the function f is called a generator function of τ_{v,v_0}.

Remark 1. Note that in case (a) f is a generator function of a strict t-norm and in case (b) f is a generator function of a strict t-conorm. In both cases, f is uniquely determined up to a positive multiplicative constant (see [23]).

From now on, if f is strictly decreasing, then we will interpret $f(0) = \infty$ and $f^{-1}(\infty) = 0$. And if f is strictly increasing, then we will interpret $f(1) = \infty$ and $f^{-1}(\infty) = 1$.

Remark 2. Here, will make use of the extended real line $[-\infty, \infty]$ and we will employ the following conventions:

$$\frac{1}{0} = \infty \quad \text{and} \quad \frac{1}{\infty} = 0,$$
$$e^{-\infty} = 0, \quad e^{\infty} = \infty, \quad \ln(0) = -\infty, \quad \text{and} \quad \ln(\infty) = \infty.$$

The following proposition states the most important properties of the tau function given in Definition 5.

Proposition 1. *The function τ_{v,v_0} stated in Definition 5 has the following properties:*

(a) *τ_{v,v_0} is continuous in $[0, 1]$, and if f is differentiable, then τ_{v,v_0} is differentiable in $(0, 1)$*
(b) *τ_{v,v_0} is strictly increasing in $[0, 1]$, $\tau_{v,v_0}(0) = 0$ and $\tau_{v,v_0}(1) = 1$*
(c) *$\tau_{v,v_0}(v) = v_0$*
(d) *For any $x \in (0, 1)$,*
    (d1) *if $v = v_0$, then $\tau_{v,v_0}(x) = x$*
    (d2) *if $v < v_0$, then $\tau_{v,v_0}(x) > x$*
    (d3) *if $v > v_0$, then $\tau_{v,v_0}(x) < x$*
(e) *If $v < v_0$ ($v > v_0$, respectively), then τ_{v,v_0} is strictly concave (convex, respectively) in the interval $[0, 1]$.*

Proof. These properties immediately follow from Definition 5. $\quad \square$

Notice that it immediately follows from Proposition 1 that the function τ_{v,v_0} satisfies the requirements for a T-function.

Other properties of the tau function, which are important from a measure theoretical point of view, will be discussed in the next section.

3. The tau function from a measure theoretical point of view

Here, we will demonstrate some important additional properties of the tau function. We will make use of these properties in Section 4, where the tau-additive measure will be presented.

We should emphasize that the tau function is generator function-dependent. That is, for fixed values of the parameters v and v_0, we can obtain various τ_{v,v_0} functions depending on the choice of its generator function f. This means that treating the tau function as a T-function, we can generate flexible monotone measures. Namely, these monotone measures can be adjusted not just through the values of the parameters v and v_0, but via the generator function f as well. The following examples show how a particular tau function can be derived from a given generator function. Note that in both examples, we will use the conventions shown in Remark 2.

Example 1. Let us consider the function $f_\alpha : [0, 1] \to [0, 1]$, which is given by

$$f_\alpha(x) = \left(\frac{1-x}{x} \right)^\alpha \tag{4}$$

with a parameter $\alpha \neq 0$. In continuous-valued logic, this function is known as the generator function of the Dombi operators (see [19,20]). We see that if $\alpha > 0$ ($\alpha < 0$, respectively), then f_α is a generator function of a strict t-norm (strict t-conorm, respectively).

Now, let $v, v_0 \in (0, 1)$. Based on Eq. (4), we have

$$f_\alpha^{-1}(x) = \frac{1}{1 + x^\alpha}$$

and noting the definition for a tau function in Definition 5, after direct calculation, we get that the tau function induced by f_α is

$$\tau_{v,v_0}(x) = f_\alpha^{-1} \left(f_\alpha(v_0) \frac{f_\alpha(x)}{f_\alpha(v)} \right) = \frac{1}{1 + \frac{1-v_0}{v_0} \frac{v}{1-v} \frac{1-x}{x}}, \tag{5}$$

where $x \in [0, 1]$. Notice that here τ_{v,v_0} is independent of α. Fig. 1 shows example plots of tau functions induced by the generator function f_α.

Example 2. In a similar way, it can be shown that if $f(x) = -\ln(x)$, $x \in [0, 1]$, then the corresponding tau function is

$$\tau_{v,v_0}(x) = v_0^{\frac{\ln(x)}{\ln(v)}}.$$

The tau function is generator function-dependent. Therefore, it is interesting to ask under what conditions two tau functions induced by different generators coincide. Now, we will provide the necessary and sufficient condition for the identity of two tau functions, which have the same parameter values, but are induced by different generator functions.

Theorem 1. *Let f and g be generator functions of strict t-norms or strict t-conorms. Let the tau functions $\tau_{v,v_0,f}$ and $\tau_{v,v_0,g}$ be induced by the generator functions f and g, respectively, where $v, v_0 \in (0, 1)$ and $v \neq v_0$. Then,*

$$\tau_{v,v_0,f}(x) = \tau_{v,v_0,g}(x) \tag{6}$$

holds for any $x \in (0, 1)$ if and only if there exists a pair (α, β) with $\alpha \neq 0$ and $\beta > 0$ such that

$$f(x) = \beta g^\alpha(x) \tag{7}$$

holds for any $x \in (0, 1)$.
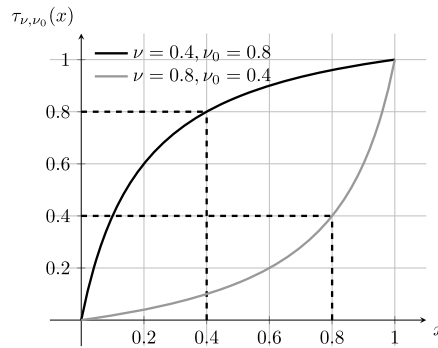
$$\tau_{\nu,\nu_0}(x)$$

Fig. 1. Example plots of tau functions for $f(x) = \frac{1-x}{x}$, $x \in [0, 1]$.

Proof. Let $\nu \neq \nu_0$. If for some $\alpha \neq 0$ and $\beta > 0$, Eq. (7) holds for any arbitrarily fixed $x \in (0, 1)$, then using Definition 5, after direct calculation we arrive at Eq. (6).

Now, we will show that under the conditions of this theorem, if Eq. (6) holds for an arbitrary $x \in (0, 1)$, then there exist $\alpha \neq 0$ and $\beta > 0$ such that Eq. (7) holds. Suppose that Eq. (6) holds for any $x \in (0, 1)$, where $\nu, \nu_0 \in (0, 1)$ and $\nu \neq \nu_0$. Then, by noting the definition for a tau function, we have

$$f^{-1}\left(f(\nu_0)\frac{f(x)}{f(\nu)}\right) = g^{-1}\left(g(\nu_0)\frac{g(x)}{g(\nu)}\right) \tag{8}$$

for any $x \in (0, 1)$. Let the function $F: (0, \infty) \to (0, \infty)$ be given by

$$F(x) = f\left(g^{-1}(x)\right). \tag{9}$$

Then, $f(\nu_0)$, $f(x)$ and $f(\nu)$ can be written as

$$f(\nu_0) = F(g(\nu_0)), \quad f(x) = F(g(x)) \quad \text{and} \quad f(\nu) = F(g(\nu)), \tag{10}$$

where $x \in (0, 1)$. Next, let the variable X and the constants Y_0 and Z_0 be given by

$$X = g(x), \quad Y_0 = g(\nu_0) \quad \text{and} \quad Z_0 = g(\nu). \tag{11}$$

Notice that $X, Y_0, Z_0 \in (0, \infty)$ and $F(X), F(Y_0), F(Z_0) \in (0, \infty)$. Using Eq. (9), Eq. (10) and Eq. (11), Eq. (8) can be written as

$$\frac{F(Y_0)}{F(Z_0)}F(X) = F\left(\frac{Y_0}{Z_0}X\right). \tag{12}$$

Now let

$$a = \frac{Y_0}{Z_0} \quad \text{and} \quad b = \frac{F(Y_0)}{F(Z_0)}.$$

Here, a and b are always positive. Noting that f and g are strictly monotonic functions, F is also strictly monotonic. Therefore, taking into account the condition that $\nu \neq \nu_0$, we get $a \neq 1$ and $b \neq 1$. Using a and b, from Eq. (12) we have the functional equation

$$F(aX) - bF(X) = 0,$$

where $a, b, X > 0$. It is known (see [24]) that the solution of this functional equation is

$$F(X) = \theta(\ln(X))X^{\frac{\ln(b)}{\ln(a)}}, \tag{13}$$

where θ is a periodic function with a period of $\ln(a)$. Here, F is a strictly monotonic function with the range $(0, \infty)$, so the periodic function θ is necessarily a constant function, i.e., $\theta(\ln(x)) = \beta$, where $\beta > 0$ is a constant. Hence, from Eq. (13), we have

$$F(X) = \beta X^\alpha, \tag{14}$$

where $\alpha = \frac{\ln(b)}{\ln(a)} \neq 0$. Using Eq. (14), the definition of F in Eq. (9) and the definition of X in Eq. (11), we have

$$F(X) = F(g(x)) = f\left(g^{-1}\left(g(x)\right)\right) = f(x)$$

and

$$F(X) = \beta X^\alpha = \beta(g(x))^\alpha.$$

That is,

$$f(x) = \beta g^\alpha(x). \quad \Box \tag{15}$$

Now, we will show that the inverse of a tau function is a tau function as well. More precisely, a tau function can be inverted by swapping its parameter values (ν and ν_0). This property of the tau function allows us to define the tau-additive measure in a different way as we will show later in Remark 3 in Section 4. We will also see that the tau-additive measure is a composition of a tau function and an additive measure. Thus, using the result of the following proposition makes it very simple to express the additive measure, which along with a tau function represents a tau-additive measure, in terms of the tau-additive measure itself.

Proposition 2. *Let ν, $\nu_0 \in (0, 1)$ and let τ_{ν, ν_0} be a tau function induced by a generator function f. The inverse of τ_{ν, ν_0} is the tau function $\tau_{\nu_0, \nu}$. That is*

$$\tau_{\nu, \nu_0}^{-1}(x) = \tau_{\nu_0, \nu}(x) \tag{16}$$

for any $x \in [0, 1]$.

Proof. Using the definition for a tau function in Definition 5, inverting τ_{ν, ν_0} leads to

$$\tau_{\nu, \nu_0}^{-1}(x) = f^{-1}\left(f(\nu)\frac{f(x)}{f(\nu_0)}\right) \tag{17}$$

for any $x \in [0, 1]$. Noting again Definition 5, Eq. (17) means that Eq. (16) holds for any $x \in [0, 1]$. $\quad \Box$

Later, in Theorem 4, we will show how a tau-additive measure can be constructed using the values of a set function at subsets of a finite set. In the proof of Theorem 4, we will utilize the results of the following proposition.

Proposition 3. *Let ν, $\nu_0 \in (0, 1)$ and let τ_{ν, ν_0} be a tau function induced by a generator function f. For any arbitrary fixed $x \in (0, 1)$ and also fixed $\nu_0 \in (0, 1)$, the inverse function of τ_{ν, ν_0} is a strictly increasing function of ν.*

Proof. Let $x \in (0, 1)$ and $\nu_0 \in (0, 1)$ have arbitrarily fixed values. Here, we will distinguish two cases: (a) f is a strictly increasing function; (b) f is a strictly decreasing function.

(a) In this case, f is strictly increasing. Therefore, if $\nu_1 < \nu_2$, then

$$f^{-1}\left(f(\nu_1)\frac{f(x)}{f(\nu_0)}\right) < f^{-1}\left(f(\nu_2)\frac{f(x)}{f(\nu_0)}\right),$$

which, based on Eq. (17), means that $\tau_{\nu_1, \nu_0}^{-1}(x) < \tau_{\nu_2, \nu_0}^{-1}(x)$.

(b) Now, f is strictly decreasing. Hence, if $\nu_1 < \nu_2$, then

$$f(\nu_1)\frac{f(x)}{f(\nu_0)} > f(\nu_2)\frac{f(x)}{f(\nu_0)}.$$

Next, applying f^{-1} to both sides of the last inequality, we get

$$f^{-1}\left(f(\nu_1)\frac{f(x)}{f(\nu_0)}\right) < f^{-1}\left(f(\nu_2)\frac{f(x)}{f(\nu_0)}\right),$$

which, noting Eq. (17), means that $\tau_{\nu_1, \nu_0}^{-1}(x) < \tau_{\nu_2, \nu_0}^{-1}(x)$. $\quad \Box$

The result of the following theorem is important as it can be used to show that the tau-additive measure can be submodular or supermodular depending on its parameter settings.

Theorem 2. *Let $\nu, \nu_0 \in (0, 1)$ and let τ_{ν,ν_0} be a tau function according to Definition 5. For any $x, y, z \in [0, 1]$, if $z \leq \min(x, y)$ and $x + y - z \leq 1$, then*

(a) *if $\nu > \nu_0$, then*

$$\tau_{\nu,\nu_0}(x + y - z) \geq \tau_{\nu,\nu_0}(x) + \tau_{\nu,\nu_0}(y) - \tau_{\nu,\nu_0}(z) \tag{18}$$

(b) *if $\nu < \nu_0$, then*

$$\tau_{\nu,\nu_0}(x + y - z) \leq \tau_{\nu,\nu_0}(x) + \tau_{\nu,\nu_0}(y) - \tau_{\nu,\nu_0}(z).$$

Proof. Here, we will prove case (a), and the proof of case (b) can be given in a similar way.

Without loss of generality, we may assume that $x \leq y$. Since $0 \leq z \leq \min(x, y)$, there exists an $a \geq 0$ such that $z = x - a \geq 0$. Noting that $x + y - z \leq 1$, we also have $x + y - (x - a) \leq 1$, and so $y + a \leq 1$. That is, using the condition that $x, y, z \in [0, 1]$, we have that $x - a, x, y, y + a \in [0, 1]$. Based on Proposition 1, τ_{ν,ν_0} is strictly increasing, $\tau_{\nu,\nu_0}(0) = 0$, $\tau_{\nu,\nu_0}(1) = 1$ and for a $\nu > \nu_0$, τ_{ν,ν_0} is strictly convex in $[0, 1]$. Therefore, we have

$$\tau_{\nu,\nu_0}(y + a) - \tau_{\nu,\nu_0}(y) \geq \tau_{\nu,\nu_0}(x) - \tau_{\nu,\nu_0}(x - a),$$

from which we get

$$\tau_{\nu,\nu_0}(x + y - (x - a)) \geq \tau_{\nu,\nu_0}(x) + \tau_{\nu,\nu_0}(y) - \tau_{\nu,\nu_0}(x - a). \tag{19}$$

Noting the fact that $z = x - a$, from Eq. (19), we get Eq. (18). $\quad\square$

The following corollary, which follows from Theorem 2, can be used to show that a tau-additive measure can be subadditive or superadditive depending on its parameter values.

Corollary 1. *Let $\nu \in (0, 1)$ and let τ_{ν,ν_0} be a tau function according to Definition 5. For any $x, y \in [0, 1]$, if $x + y \leq 1$, then*

(a) *if $\nu > \nu_0$, then*

$$\tau_{\nu,\nu_0}(x + y) \geq \tau_{\nu,\nu_0}(x) + \tau_{\nu,\nu_0}(y)$$

(b) *if $\nu < \nu_0$, then*

$$\tau_{\nu,\nu_0}(x + y) \leq \tau_{\nu,\nu_0}(x) + \tau_{\nu,\nu_0}(y).$$

Proof. We get the statement of this corollary by applying Theorem 2 with $z = 0$ and noting that $\tau_{\nu,\nu_0}(0) = 0$. $\quad\square$

4. The tau-additive measure

Now, we will introduce the tau-additive measure and show that it can be represented by the tau function.

Definition 6. We say that the set function $\mu_{\tau_{\nu,\nu_0}} : \mathcal{P}(X) \to [0, 1]$ is a tau-additive measure on the finite set X if and only if $\mu_{\tau_{\nu,\nu_0}}$ satisfies the following requirements:

(1) $\mu_{\tau_{\nu,\nu_0}}(X) = 1$
(2) For any $A, B \in \mathcal{P}(X)$ and $A \cap B = \emptyset$,

$$\mu_{\tau_{\nu,\nu_0}}(A \cup B) = \tau_{\nu,\nu_0}\left(\tau_{\nu,\nu_0}^{-1}(\mu_{\tau_{\nu,\nu_0}}(A)) + \tau_{\nu,\nu_0}^{-1}(\mu_{\tau_{\nu,\nu_0}}(B))\right), \tag{20}$$

where τ_{ν,ν_0} is a tau function with parameters $\nu, \nu_0 \in (0, 1)$.

Remark 3. Noting Proposition 2, Eq. (20) can alternatively be written as

$$\mu_{\tau_{\nu,\nu_0}}(A \cup B) = \tau_{\nu,\nu_0}\left(\tau_{\nu_0,\nu}(\mu_{\tau_{\nu,\nu_0}}(A)) + \tau_{\nu_0,\nu}(\mu_{\tau_{\nu,\nu_0}}(B))\right).$$

The following lemma demonstrates that the tau-additive measure can be represented by the tau-function τ_{ν,ν_0}.

Lemma 1. *The tau-additive measure* $\mu_{\tau_{\nu,\nu_0}} : \mathcal{P}(X) \to [0, 1]$ *on a finite set* X, *where* $\nu, \nu_0 \in (0, 1)$, *is represented by the tau-function and an additive measure.*

Proof. Here, $\mu_{\tau_{\nu,\nu_0}}$ is a tau-additive measure, and so, applying τ_{ν,ν_0}^{-1} to both sides of Eq. (20), we have

$$\tau_{\nu,\nu_0}^{-1}\left(\mu_{\tau_{\nu,\nu_0}}(A \cup B)\right) = \tau_{\nu,\nu_0}^{-1}(\mu_{\tau_{\nu,\nu_0}}(A)) + \tau_{\nu,\nu_0}^{-1}(\mu_{\tau_{\nu,\nu_0}}(B)) \tag{21}$$

for any $A, B \in \mathcal{P}(X)$. Now, let

$$m = \tau_{\nu,\nu_0}^{-1} \circ \mu_{\tau_{\nu,\nu_0}}. \tag{22}$$

Based on Eq. (21), m is an additive measure on X. Moreover, $\tau_{\nu,\nu_0}^{-1}(1) = 1$ and $\mu_{\tau_{\nu,\nu_0}}(X) = 1$ imply $m(X) = 1$. Therefore, m is a probability measure on X. We know that τ_{ν,ν_0} is a T-function, and from Eq. (22) we have

$$\mu_{\tau_{\nu,\nu_0}} = \tau_{\nu,\nu_0} \circ m,$$

which means that $\mu_{\tau_{\nu,\nu_0}}$ can be represented by the probability measure m and the T-function τ_{ν,ν_0}. $\quad\square$

Remark 4. Based on Lemma 1, a tau-additive measure $\mu_{\tau_{\nu,\nu_0}}$ can always be represented by the pair (Pr, τ_{ν,ν_0}), where Pr is a probability measure and τ_{ν,ν_0} is the same tau function as that in $\mu_{\tau_{\nu,\nu_0}}$. That is, we have

$$\mu_{\tau_{\nu,\nu_0}} = \tau_{\nu,\nu_0} \circ Pr \tag{23}$$

and

$$Pr = \tau_{\nu,\nu_0}^{-1} \circ \mu_{\tau_{\nu,\nu_0}}. \tag{24}$$

This means that we can always create a tau-additive measure from a probability measure, and a probability measure from a tau-additive measure.

4.1. The properties of the tau-additive measure

Here, we will describe the main properties of the tau-additive measure. Noting Proposition 1 and the definition for a tau-additive measure in Definition 6, we immediately get the following proposition.

Proposition 4. *Let* $\mu_{\tau_{\nu,\nu_0}}$ *be a tau-additive measure on a finite set* X, *and let* $\mu_{\tau_{\nu,\nu_0}}$ *be represented by the probability measure* Pr. *Then, for any* $\nu, \nu_0 \in (0, 1)$, $\mu_{\tau_{\nu,\nu_0}}$ *is a monotone measure on* X. *Also, for any* $A \in \mathcal{P}(X)$,

(a) *if* $\nu = \nu_0$, *then* $\mu_{\tau_{\nu,\nu_0}}(A) = Pr(A)$
(b) *if* $\nu < \nu_0$, *then* $\mu_{\tau_{\nu,\nu_0}}(A) \geq Pr(A)$
(c) *if* $\nu > \nu_0$, *then* $\mu_{\tau_{\nu,\nu_0}}(A) \leq Pr(A)$.

We should mention that, based on Proposition 4, the tau-additive measure $\mu_{\tau_{\nu,\nu_0}}$ may be viewed as an upper or a lower probability measure depending on the values of its parameters.

Now, we will show how to compute the tau-additive measure of the union of n pairwise disjoints sets.

Proposition 5. *Let* $\mu_{\tau_{\nu,\nu_0}}$ *be a tau-additive measure on a finite set* X. *For any* $A_1, A_2, \ldots, A_n \in \mathcal{P}(X)$ *pairwise disjoint sets,*

$$\mu_{\tau_{\nu,\nu_0}}\left(\bigcup_{i=1}^{n} A_i\right) = \tau_{\nu,\nu_0}\left(\sum_{i=1}^{n} \tau_{\nu,\nu_0}^{-1}(\mu_{\tau_{\nu,\nu_0}}(A_i))\right). \tag{25}$$

Proof. We will prove Eq. (25) by induction. Taking into account the definition for a tau-additive measure in Definition 6, we immediately have that Eq. (25) holds for $n = 2$. Now, let us assume that Eq. (25) holds for any $n > 2$. Then, noting the associativity of the set union operation, Definition 6 and the inductive condition that Eq. (25) holds for n, we can write

$$\mu_{\tau_{v,v_0}}\left(\bigcup_{i=1}^{n+1} A_i\right) = \mu_{\tau_{v,v_0}}\left(\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right) =$$

$$= \tau_{v,v_0}\left(\tau_{v,v_0}^{-1}\left(\mu_{\tau_{v,v_0}}\left(\bigcup_{i=1}^{n} A_i\right)\right) + \tau_{v,v_0}^{-1}(\mu_{\tau_{v,v_0}}(A_{n+1}))\right) =$$

$$= \tau_{v,v_0}\left(\tau_{v,v_0}^{-1}\left(\tau_{v,v_0}\left(\sum_{i=1}^{n} \tau_{v,v_0}^{-1}(\mu_{\tau_{v,v_0}}(A_i))\right)\right) + \tau_{v,v_0}^{-1}(\mu_{\tau_{v,v_0}}(A_{n+1}))\right) =$$

$$= \tau_{v,v_0}\left(\sum_{i=1}^{n+1} \tau_{v,v_0}^{-1}(\mu_{\tau_{v,v_0}}(A_i))\right),$$

which is Eq. (25) for $n + 1$. $\square$

The following proposition concerns the tau-additive measure of the difference of two sets.

Proposition 6. *Let $\mu_{\tau_{v,v_0}}$ be a tau-additive measure on a finite set X. For any $A, B \in \mathcal{P}(X)$, if $A \subseteq B$, then*

$$\mu_{\tau_{v,v_0}}(B \setminus A) = \tau_{v,v_0}\left(\tau_{v,v_0}^{-1}(\mu_{\tau_{v,v_0}}(B)) - \tau_{v,v_0}^{-1}(\mu_{\tau_{v,v_0}}(A))\right). \tag{26}$$

Proof. Let Pr be a probability measure on the set X. Then, based on Lemma 1, Eq. (24) holds. Since Pr is a probability measure, for any $A, B \in \mathcal{P}(X)$, $A \subseteq B$, we have

$$Pr(B \setminus A) = Pr(B) - Pr(A). \tag{27}$$

Using Eq. (24), Eq. (27) can be written as

$$\tau_{v,v_0}^{-1}(\mu_{\tau_{v,v_0}}(B \setminus A)) = \tau_{v,v_0}^{-1}(\mu_{\tau_{v,v_0}}(B)) - \tau_{v,v_0}^{-1}(\mu_{\tau_{v,v_0}}(A)).$$

Since both sides of this equation are non-negative, applying τ_{v,v_0} to both sides leads to Eq. (26). $\square$

The following proposition describes how the tau-additive measure of a complement set can be computed.

Proposition 7. *Let $\mu_{\tau_{v,v_0}}$ be a tau-additive measure on a finite set X. For any $A \in \mathcal{P}(X)$*

$$\mu_{\tau_{v,v_0}}(\overline{A}) = \tau_{v,v_0}\left(1 - \tau_{v,v_0}^{-1}(\mu_{\tau_{v,v_0}}(A))\right), \tag{28}$$

where $\overline{A}$ is the complement set of A (i.e. $\overline{A} = X \setminus A$).

Proof. Noting the fact that $\tau_{v,v_0}^{-1}(\mu_{\tau_{v,v_0}}(X)) = 1$ and applying Proposition 6 with $B = X$, we get Eq. (28). $\square$

The following proposition may be viewed as the law of total tau-additive measure.

Proposition 8. *Let $\mu_{\tau_{v,v_0}}$ be a tau-additive measure on a finite set X. If $B_1, B_2, \ldots, B_n$ are pairwise disjoint subsets of X such that $\bigcup_{i=1}^{n} B_i = X$, then for any $A \in \mathcal{P}(X)$,*

$$\mu_{\tau_{v,v_0}}(A) = \tau_{v,v_0}\left(\sum_{i=1}^{n} \tau_{v,v_0}^{-1}(\mu_{\tau_{v,v_0}}(A \cap B_i))\right). \tag{29}$$

Proof. Let $\mu_{\tau_{\nu,\nu_0}}$ be represented by the probability measure Pr, i.e., Eq. (24) holds. Pr is a probability measure, and so, using the law of total probability, we have

$$Pr(A) = \sum_{i=1}^{n} Pr(A \cap B_i). \tag{30}$$

With Eq. (24), Eq. (30) can be written as

$$\tau_{\nu,\nu_0}^{-1}(\mu_{\tau_{\nu,\nu_0}}(A)) = \sum_{i=1}^{n} \tau_{\nu,\nu_0}^{-1}(\mu_{\tau_{\nu,\nu_0}}(A \cap B_i)).$$

Now, applying τ_{ν,ν_0} to both sides of the last equation, we get Eq. (29). □

Here, we will demonstrate that the tau-additive measure can be submodular or supermodular, depending on the value of parameter ν. This characteristic of the tau-additive measure is important from an uncertainty modeling point of view.

Theorem 3. *Let $\mu_{\tau_{\nu,\nu_0}}$ be a tau-additive measure on a finite set X.*

(a) *If $\nu > \nu_0$, then $\mu_{\tau_{\nu,\nu_0}}$ is supermodular.*
(b) *If $\nu < \nu_0$, then $\mu_{\tau_{\nu,\nu_0}}$ is submodular.*

Proof. Here, we will prove case (a), the proof of case (b) being similar to that of case (a).

The supermodularity of $\mu_{\tau_{\nu,\nu_0}}$ means that for any $A, B \in \mathcal{P}(X)$,

$$\mu_{\tau_{\nu,\nu_0}}(A \cup B) \geq \mu_{\tau_{\nu,\nu_0}}(A) + \mu_{\tau_{\nu,\nu_0}}(B) - \mu_{\tau_{\nu,\nu_0}}(A \cap B). \tag{31}$$

Let $\mu_{\tau_{\nu,\nu_0}}$ be represented by the probability measure Pr, i.e., Eq. (23) holds. Then Eq. (31) is equivalent to

$$\tau_{\nu,\nu_0}(Pr(A \cup B)) \geq \tau_{\nu,\nu_0}(Pr(A)) + \tau_{\nu,\nu_0}(Pr(B)) - \tau_{\nu,\nu_0}(Pr(A \cap B)). \tag{32}$$

Noting the Poincaré formula of probability theory, Eq. (32) is equivalent to

$$\begin{aligned}\tau_{\nu,\nu_0}(Pr(A) + Pr(B) - Pr(A \cap B)) \geq \\ \geq \tau_{\nu,\nu_0}(Pr(A)) + \tau_{\nu,\nu_0}(Pr(B)) - \tau_{\nu,\nu_0}(Pr(A \cap B)).\end{aligned} \tag{33}$$

Since Pr is a monotone measure, we have $Pr(A \cap B) \leq \min(Pr(A), Pr(B))$. Now, let

$$x = Pr(A), \quad y = Pr(B), \quad z = Pr(A \cap B).$$

Then $x, y, z \in [0, 1]$, $z \leq \min(x, y)$ and $x + y - z \leq 1$. Therefore, noting Theorem 2, $\nu > \nu_0$ implies

$$\tau_{\nu,\nu_0}(x + y - z) \geq \tau_{\nu,\nu_0}(x) + \tau_{\nu,\nu_0}(y) - \tau_{\nu,\nu_0}(z).$$

This also means that if $\nu > \nu_0$, then Eq. (33) holds. Since Eq. (33) and Eq. (31) are equivalent, if $\nu > \nu_0$, then Eq. (31) holds as well. □

It is well known that every submodular (supermodular, respectively) measure is subadditive (superadditive, respectively) (see, e.g., [21]). Therefore, it readily follows from Theorem 3 that a tau-additive measure $\mu_{\tau_{\nu,\nu_0}}$ is superadditive if $\nu > \nu_0$, and it is subadditive if $\nu < \nu_0$. This property can also be proved by exploiting the result of Corollary 1. And the following is an immediate consequence of the subadditivity or superadditivity of the tau-additive measure.

Corollary 2. *Let $\mu_{\tau_{\nu,\nu_0}}$ be a tau-additive measure on a finite set X. Then for any $A \in \mathcal{P}(X)$,*

(a) *if $\nu > \nu_0$, then $1 \geq \mu_{\tau_{\nu,\nu_0}}(A) + \mu_{\tau_{\nu,\nu_0}}(\overline{A})$.*
(b) *If $\nu < \nu_0$, then $1 \leq \mu_{\tau_{\nu,\nu_0}}(A) + \mu_{\tau_{\nu,\nu_0}}(\overline{A})$.*

Proof. Noting that $A \cap \overline{A} = \emptyset$ and $\mu_{\tau_{\nu,\nu_0}}(A \cup \overline{A}) = 1$, by applying Theorem 3 with $B = \overline{A}$, we immediately get (a) and (b) of this corollary. $\quad \square$

The result of the following proposition will be utilized in Section 6, where we will present an application of the tau-additive measure in human resource management. Namely, we will show how the tau-additive measure can be utilized to model the effect of team merging on team performance.

Proposition 9. *Let $\mu_{\tau_{\nu,\nu_0}}$ be a tau-additive measure on a finite set X. For any $A_1, A_2, \ldots, A_n \in \mathcal{P}(X)$ pairwise disjoint sets, the following hold:*

(a) *If $\nu > \nu_0$, then*

$$\mu_{\tau_{\nu,\nu_0}}\left(\bigcup_{i=1}^n A_i\right) \geq \sum_{i=1}^n \mu_{\tau_{\nu,\nu_0}}(A_i) \tag{34}$$

(b) *If $\nu < \nu_0$, then*

$$\mu_{\tau_{\nu,\nu_0}}\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \mu_{\tau_{\nu,\nu_0}}(A_i)$$

Proof. Here, we will prove case (a) by induction. The proof of case (b) can be provided in a similar way. Exploiting the result of Theorem 3, we know that if $\nu > \nu_0$, then Eq. (34) holds for $n = 2$. Now, suppose that Eq. (34) holds for any $n > 2$. Then, using the case for $n = 2$, the associativity of the set union operation and the inductive condition, we can write

$$\begin{aligned} \mu_{\tau_{\nu,\nu_0}}(A_1 \cup A_2 \cup \cdots \cup A_{n+1}) = \mu_{\tau_{\nu,\nu_0}}((A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1}) \geq \\ \geq \mu_{\tau_{\nu,\nu_0}}(A_1 \cup A_2 \cup \cdots \cup A_n) + \mu_{\tau_{\nu,\nu_0}}(A_{n+1}) \geq \\ \geq \mu_{\tau_{\nu,\nu_0}}(A_1) + \mu_{\tau_{\nu,\nu_0}}(A_2) + \cdots + \mu_{\tau_{\nu,\nu_0}}(A_{n+1}). \quad \square \end{aligned} \tag{35}$$

4.2. Construction of a tau-additive measure from a set function

Constructing a tau-additive measure is an interesting problem. In this part, we will describe how a tau-additive measure can be constructed on a finite set. Later, in Section 6, we will present an application connected with the results presented here.

It is well known that the λ-additive measure has the following property (see Theorem 4.7 in [4]). If X is a non-empty finite set, $\mu: \mathcal{P}(X) \to [0, 1]$ is a set function on X, $A_1, A_2, \ldots, A_n$ are pairwise disjoint subsets of X such that $\bigcup_{i=1}^n A_i = X$, $\mu(A_i) < 1$ for $i = 1, 2, \ldots, n$ and there are at least two subsets, A_i and A_j, satisfying $\mu(A_i), \mu(A_j) > 0$, $i, j \in \{1, 2, \ldots, n\}$, then μ is a λ-additive measure on X and the value of the parameter $\lambda \in (-1, \infty)$ is uniquely determined by the equation

$$1 + \lambda = \prod_{i=1}^n (1 + \lambda \mu(A_i)). \tag{36}$$

This is an important property of the λ-additive measure as it tells us how it can be constructed on finite set. Now, we will show that the tau-additive measure has a similar property, which allows us to construct a particular tau-additive measure on a finite set.

Theorem 4. *Let X be a non-empty finite set and let $A_1, A_2, \ldots, A_n$ be pairwise disjoint subsets of X such that $\bigcup_{i=1}^n A_i = X$. Let $\mu: \mathcal{P}(X) \to [0, 1]$ be a set function that satisfies the following requirements:*

(a) $\mu(X) = 1$
(b) $0 \leq \mu(A_i) < 1$ *for any $i \in \{1, 2, \ldots, n\}$*
(c) *there are at least two sets, A_{i_1} and A_{i_2}, $i_1, i_2 \in \{1, 2, \ldots, n\}$, $i_1 \neq i_2$, such that $\mu(A_{i_1}), \mu(A_{i_2}) > 0$.*

Then, for any arbitrary fixed $v_0 \in (0, 1)$, μ is a tau-additive measure with some parameter $v \in (0, 1)$, and the value of v is uniquely determined by the equation

$$1 = \tau_{v,v_0}\left(\sum_{i=1}^{n} \tau_{v,v_0}^{-1}(\mu(A_i))\right). \tag{37}$$

Proof. Since $\mu(X) = 1$, Eq. (37) can be written as

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \tau_{v,v_0}\left(\sum_{i=1}^{n} \tau_{v,v_0}^{-1}(\mu(A_i))\right). \tag{38}$$

Therefore, for an arbitrary fixed $v_0 \in (0, 1)$, μ is a tau-additive measure with some parameter v. Now, we will show that under the condition of this theorem, Eq. (37) uniquely determines the value of v; that is, Eq. (37) has exactly one solution for v in $(0, 1)$ when the value of $v_0 \in (0, 1)$ is fixed. Applying τ_{v,v_0}^{-1} to both sides of Eq. (37), we get

$$\tau_{v,v_0}^{-1}(1) = \sum_{i=1}^{n} \tau_{v,v_0}^{-1}(\mu(A_i)), \tag{39}$$

which is equivalent to Eq. (37). Noting the fact that $\tau_{v,v_0}^{-1}(1) = 1$, the equation

$$1 = \sum_{i=1}^{n} \tau_{v,v_0}^{-1}(\mu(A_i)) \tag{40}$$

is equivalent to Eq. (38) and Eq. (37). Now, we will show that under the condition of this theorem, Eq. (40) has exactly one solution. Let

$$g_i(v) = \tau_{v,v_0}^{-1}(\mu(A_i)) \tag{41}$$

for $i = 1, 2, \ldots, n$, and let

$$g(v) = \sum_{i=1}^{n} g_i(v). \tag{42}$$

Then, Eq. (40) can be written as

$$1 = g(v).$$

Noting the condition that $0 \le \mu(A_i) < 1$, the properties of the tau function, Proposition 2 and Proposition 3, g_i in Eq. (41) is a continuous function of v and

(a) if $\mu(A_i) = 0$, then $g_i(v) = 0$ for any $v \in (0, 1)$
(b) if $\mu(A_i) > 0$, then g_i is a strictly increasing function of v with the range $(0, 1)$, where $v \in (0, 1)$.

Since function g in Eq. (42) is a sum of continuous functions, it is continuous as well. Taking into account the fact that there are at least two sets, A_{i_1} and A_{i_2}, $i_1, i_2 \in \{1, 2, \ldots, n\}$, $i_1 \ne i_2$, such that $\mu(A_{i_1}), \mu(A_{i_2}) > 0$, based on (a) and (b), the function $g(v) = \sum_{i=1}^{n} g_i(v)$ is a strictly increasing, continuous function with the range $(0, r)$, where $r \ge 2$. Therefore, there exists exactly one $v \in (0, 1)$ such that $g(v) = 1$. $\quad\square$

As we pointed out above, when constructing a λ-additive measure, we need to solve Eq. (36) for λ. This equation can be solved numerically. In this case, we need to find the value of λ in the $(-1, \infty)$ interval. At the same time, based on Theorem 4, we can construct a tau-additive measure by fixing the value of v_0 and solving Eq. (40) for v. Since $v \in (0, 1)$, the numerical solution of Eq. (40) for v is simpler than that of Eq. (36) for λ. Furthermore, exploiting Theorem 4 and the results that will be described in Section 5, we can easily construct an approximate λ-additive measure on a finite set.

5. The tau-additive measure as an approximation to lambda-additive measure

Here, we will show how the lambda-additive measure can be approximated by the tau-additive measure.

Now, let us consider the tau function in Eq. (5) with the parameter $\nu_0 = \frac{1}{2}$; that is,

$$\tau_\nu(x) = \tau_{\nu,\nu_0}(x)|_{\nu_0 = \frac{1}{2}} = \frac{1}{1 + \frac{\nu}{1-\nu}\frac{1-x}{x}}, \tag{43}$$

where $x \in [0, 1]$. Notice that in this case, ν_0 drops out from Eq. (43), and so, we can use the simplified τ_ν notation instead of τ_{ν,ν_0}. Next, let the tau-additive measure be represented by the tau function in Eq. (43) and the additive measure μ; that is, $\mu_{\tau_\nu} = \tau_\nu \circ \mu$.

Now, consider the

$$\lambda = \left(\frac{\nu}{1-\nu}\right)^2 - 1 \tag{44}$$

bijective mapping, or alternatively, its inverse transformation

$$\nu = \frac{\sqrt{1+\lambda}}{1 + \sqrt{1+\lambda}}, \tag{45}$$

where $\lambda \in (-1, \infty)$ and $\nu \in (0, 1)$. This also means, that $\lambda = 0$ corresponds to $\nu = \frac{1}{2}$. After direct calculation, we can see that if Eq. (44) or Eq. (45) holds, then

$$h_\lambda\left(\frac{1}{2}\right) = \tau_\nu\left(\frac{1}{2}\right), \tag{46}$$

where h_λ is the T-function in the additive representation of the λ-additive measure (see Eq. (2)). Also, the first derivatives of h_λ and τ_ν at $x = \frac{1}{2}$ are

$$\frac{\mathrm{d}h_\lambda(x)}{\mathrm{d}x}\bigg|_{x=\frac{1}{2}} = \frac{(1+\lambda)^x \ln(1+\lambda)}{\lambda}\bigg|_{x=\frac{1}{2}} = \frac{\sqrt{1+\lambda}\ln(1+\lambda)}{\lambda}$$

and

$$\frac{\mathrm{d}\tau_\nu(x)}{\mathrm{d}x}\bigg|_{x=\frac{1}{2}} = \frac{\nu(1-\nu)}{((2\nu-1)x-\nu)^2}\bigg|_{x=\frac{1}{2}} = 4\nu(1-\nu).$$

Since h_λ and τ_ν are continuous functions in $(0, 1)$, $\lambda = 0$ corresponds to $\nu = \frac{1}{2}$, and

$$\lim_{\lambda \to 0} \frac{\mathrm{d}h_\lambda(x)}{\mathrm{d}x}\bigg|_{x=\frac{1}{2}} = \lim_{\lambda \to 0} \frac{\sqrt{1+\lambda}\ln(1+\lambda)}{\lambda} = 1$$

and

$$\lim_{\nu \to \frac{1}{2}} \frac{\mathrm{d}\tau_\nu(x)}{\mathrm{d}x}\bigg|_{x=\frac{1}{2}} = \lim_{\nu \to \frac{1}{2}} 4\nu(1-\nu) = 1,$$

we get that if Eq. (44) (or equivalently, Eq. (45)) holds, and $\lambda \approx 0$, (or equivalently $\nu \approx \frac{1}{2}$), then the first derivatives of h_λ and τ_ν at $x = \frac{1}{2}$ are approximately equal. Therefore, using this observation and Eq. (46), we may conclude that if Eq. (44) (or equivalently, Eq. (45)) holds, and $\lambda \approx 0$, (or equivalently $\nu \approx \frac{1}{2}$), then τ_ν approximates h_λ quite well around $x = \frac{1}{2}$. Moreover, it can be shown numerically that if Eq. (44) (or equivalently, Eq. (45)) holds, then for $\nu \in (0.25, 0.75)$ (or equivalently, $\lambda \in (-8/9, 8)$),

$$\max_{x \in (0,1)} |h_\lambda(x) - \tau_\nu(x)| \leq 0.0304.$$

Fig. 2 shows examples of how a tau-additive measure can approximate a lambda-additive measure. In practical applications of the lambda-additive measure, the value of parameter λ is typically not much greater and not much less than zero. In such cases, the tau-additive measure can be used as an alternative to the lambda-additive measure.

It should be added that h_λ is an exponential function, while τ_ν is a fraction of first order polynomials, which is an advantage when numerical computations need to be performed.
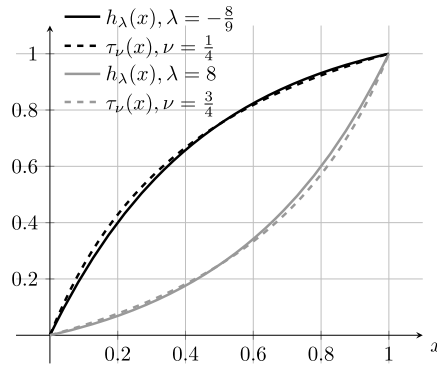
Fig. 2. Lambda-additive measures approximated by tau-additive measures.

6. A demonstrative example in human resource management

Here, we will present an example of the application of the tau-additive measure in the area of human resource management. Namely, we will show how the tau-additive measure can be utilized to model the effect of team merging on team performance. This example is an application of Theorem 4, which tells us how to construct a tau-additive measure on a finite set, and of Proposition 9, which is about the sub- and superadditivity of the tau-additive measure.

Let $A_1, \ldots, A_n$ be n pairwise disjoint groups of people (teams) and let $X = \bigcup_{i=1}^{n} A_i$ be the universe of groups. Suppose that we have the value of $\mu(A_i)$ for all $i = 1, \ldots, n$, where $\mu \colon \mathcal{P}(X) \to [0, 1]$ is a set function. Here, we interpret μ as a performance measure; that is, $\mu(A_i)$ expresses the performance of team A_i. Let us assume that there are at least two teams whose performance is positive. This means that ν satisfies the conditions (a), (b) and (c) of Theorem 4. Therefore, exploiting the results of Theorem 4, for any arbitrarily fixed $\nu_0 \in (0, 1)$, μ is a tau-additive measure with some parameter ν, and the value of ν is uniquely determined by Eq. (40). This equation can be solved numerically, e.g., by using the bisection method. Once we have the value of ν, we can compare it with ν_0. So, based on Proposition 9, for any arbitrarily fixed index set $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$,

(a) if $\nu > \nu_0$, then

$$\mu(A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}) \geq \mu(A_{i_1}) + \mu(A_{i_2}) + \cdots + \mu(A_{i_k}), \tag{47}$$

(b) if $\nu < \nu_0$, then

$$\mu(A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}) \leq \mu(A_{i_1}) + \mu(A_{i_2}) + \cdots + \mu(A_{i_k}). \tag{48}$$

This means that if $\nu > \nu_0$, then uniting some teams into one, the performance of the united team is at least as good as the performance sum of the individual teams. Also, $\nu < \nu_0$ implies that merging some teams into one, the performance of the merged team is at most as good as the performance sum of the individual teams.

Now, let $\nu_0 = \frac{1}{2}$ and let the generator function of the tau function be that of the Dombi operators. In this case, we get the one-parameter tau function τ_ν given by Eq. (43). If we use the above method with τ_ν, then, noting Eq. (40) and Proposition 2, we need to solve the equation

$$1 = \sum_{i=1}^{n} \tau_{\nu,\nu_0}^{-1}(\mu(A_i)) = \sum_{i=1}^{n} \frac{1}{1 + \frac{1-\nu}{\nu} \frac{1-\mu(A_i)}{\mu(A_i)}} \tag{49}$$

for $\nu \in (0, 1)$. Based on the value of ν, we know that if $\nu > \frac{1}{2}$ ($\nu < \frac{1}{2}$, respectively), then μ is superadditive (subadditive, respectively) and Eq. (47) (Eq. (48), respectively) holds for any $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$. Since $\nu = \frac{1}{2}$ symmetrically separates the domains $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$, which correspond to the subadditive and superadditive tau-additive measures, respectively, the value of ν may be viewed as an indicator of sub- or superadditivity of μ. Furthermore, if $\nu \in (0.25, 0.75)$, then, based on the results described in Section 5, the set function μ is approximately a λ-additive measure with the λ parameter value found using Eq. (44). Recall that the numerical solution of Eq. (49)

means that we need to find a particular value of the parameter ν in the interval $(0, 1)$. This task is computationally simpler than finding the solution of Eq. (36) for $\lambda \in (-1, \infty)$. Therefore, we can first find the value of ν and then, using Eq. (44), we can find the corresponding λ.

Remark 5. It is worth noting that the so-called ν-additive measure can be also used for modeling the effect of team merging on team performance. For more details, see [16] or Chapter 2 in [15].

7. Conclusions

In our study, we introduced a new monotone measure called the tau-additive measure, as the composition of an appropriate T-transformation and an additive measure. In our approach, the T-transformation is a generator function-based parametric function, called the tau function, which is also known as a unary operator in continuous-valued logic. Owing to the properties of the tau function, the tau-additive measure is very flexible. It can be adjusted via the parameters of the tau function and various tau-additive measures can be obtained depending on the choice of the generator function of the tau transformation. We should add that the tau-additive measure can be viewed as an upper or lower probability depending on the parameter values of the tau function.

Since the tau function is generator function-dependent, we provided a necessary and sufficient condition for the identity of two tau functions that are induced by different generator functions (see Theorem 1).

We demonstrated that using the generator function of the Dombi operators as generator of the tau function, we can get a particular tau function that approximates the T-function in the additive representation of the λ-additive measure quite well. In this case, the tau-function is a fraction of two first order polynomials and its parameters lie in the interval $(0, 1)$, while the T-function of a lambda-additive measure is an exponential function with $\lambda \in (-1, \infty)$. This means that the tau-additive measure is computationally less complex and it is easier to apply in practice than the λ-additive measure. Therefore, the tau-additive measure may be viewed as a viable alternative to the lambda additive measure.

As a key result of our study (see Theorem 4), we described how a tau-additive measure can be constructed on a finite set. We pointed out that this procedure is similar to how a lambda-additive measure can be constructed, but in the case of the tau-additive measure, the procedure is computationally much simpler than that for the lambda-additive measure. This property of the tau-additive measure and the fact that it can be sub- or supermodular depending on the parameter values of the tau function (see Theorem 2 and 3) together allow us to effectively utilize the tau-additive measure in uncertainty modeling. We demonstrated this capacity of the tau-additive measure by the means of an example in the area of human resource management. And of course our results can be utilized for uncertainty modeling in other areas as well.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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