European Journal of Operational Research 299 (2022) 580–588

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Weighted aggregation systems and an expectation level-based weighting and scoring procedure

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ARTICLE INFO

Article history:
Received 22 December 2020
Accepted 31 August 2021
Available online 27 September 2021

Keywords:
Decision support systems
Weighting transformation
Weighted quasi-arithmetic means
Expectation level
Multi-criteria decision model

ABSTRACT

This paper presents a novel approach to the weighted aggregation and to determination of weights in an aggregation procedure. In our study, we introduce the concept of a weighted aggregation system that consists of two components: (1) a weighting transformation and (2) an aggregation operator, both induced by a common generator function. We provide the necessary and sufficient condition for the form of a generator function-based weighted aggregation system. We show that the weighted quasi-arithmetic means on the non-negative extended real line are none other than the aggregation functions induced by weighted aggregation systems, i.e., these means are compositions of an n-ary aggregation operator and n weighting transformations ($n \in \mathbb{N}$, $n \geq 1$). Next, using weighted quasi-arithmetic means on the unit interval, we introduce a new, expectation level-based weight determination method and a scoring procedure. In this method, the decision-maker's expectation levels for the input variables are directly transformed into weights by making use of the generator function of a weighted quasi-arithmetic mean. We utilize this mean as a scoring function to evaluate the decision alternatives. Lastly, by the means of illustrative numerical examples, we present a novel decision model, in which the expectation levels can be even intervals, i.e., the weights are also intervals. Finally, we get an interval-valued score for each alternative.

1. Introduction

There are many decision making problems that require the incorporation of importance into the final decision. For example, in a fuzzy inference method, the activation levels of the fuzzy rule antecedents may be viewed as importance that determine the weights of the rule consequences in the output. That is, this output may be treated as a result of a weighted aggregation procedure. Calvo, Mesiar, & Yager (2004) described a general method of incorporating quantitative weights into aggregation (see also Calvo & Mesiar, 2001). Their method is founded on the so-called strong idempotent property of the aggregation operators. Qualitative aspects of weighted aggregation arise when importance of a single criterion needs to be aggregated. The qualitative weighted aggregations are discussed, e.g., in Dubois & Prade (1985); Dubois, Prade, Rico, & Teheux (2017); Fodor & Roubens (1994); Yager (2001). Yager (2001) introduced the so-called relevancy transformation (RET operator), which is an operation that obtains the effective rule output from the rule relevancy and the rule consequent in a fuzzy inference system.

The quasi-arithmetic means, treated as extended aggregation functions were introduced by Kolmogorov (1930) and Nagumo (1930). The weighted quasi-arithmetic means may be viewed as generalizations of the quasi-arithmetic means. These aggregation functions are of great importance in various areas (see, e.g., Calvo et al., 2004; Grabisch, Marichal, Mesiar, & Pap, 2011; Matkowski, 2010; Mesiar & Špirková, 2006; Wadbro & Hägg, 2015; Yoshida, 2011). The ordered weighted averaging operator and the quasi-ordered weighted averaging function also play a key role in operational research and decision making (see, e.g., Hou et al., 2021; León-Castro, Espinoza-Audelo, Merigó, Herrera-Viedma, & Herrera, 2020; Maldonado, Merigó, & Miranda, 2018; Mesiar, Stupňanová, & Yager, 2018).

Determining the appropriate weights of criteria or attributes is an important topic in multi-criteria decision making. This is why this topic has been attracting a lot of attention in recent years (see, e.g., de Almeida, de Almeida, Costa, & de Almeida-Filho, 2016; Beliakov, Gmez, James, Montero, & Rodrguez, 2017; Corrente, Figueira, Greco, & Słowiński, 2017; Keshavarz-Ghorabaee, Amiri, Zavadskas, Turskis, & Antucheviciene, 2018; Liu & Wan, 2019; Lolli et al., 2019;

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https://doi.org/10.1016/j.ejor.2021.08.049

Mi & Liao, 2019; Zargini et al., 2020; Žižović & Pamucar, 2019; Zolfani, Yazdani, & Zavadskas, 2018).

1.1. Motivations

Here, we will briefly summarize our research motivations. Based on the studied literature, we can see the following characteristics of the above-mentioned methods and approaches. The weighted quasi-arithmetic means, the ordered weighted averaging operator and the quasi-ordered weighted averaging function are crucial aggregation methods. All these methods are treated as averages or means, in which the weights have a role of expressing importance. However, we may also look at the weights from the perspective of their transformation role. Namely, we may consider these aggregation methods as being two-step procedures. First, the weights applied to the variables transform their values, and then these transformed variable values are aggregated into the final output. Thus, these weighted aggregation functions may be viewed as compositions of an aggregation operation and weighting transformations. In our study, we will look at the weighted aggregation functions from this point of view.

The methods used for determining the weights of criteria or attributes intend to obtain information about the importance of these criteria or attributes. Then, the obtained information is expressed in terms of weights. In our study, we address the question of whether we can treat the weights as 'descriptors' of the decision-maker's expectation levels (expectations in short).

In practice, the weighted version of an aggregation method is often intuitively determined. This approach is based on the presumption that the weighting and the aggregation are independent and so various weighted aggregation functions can be derived from an aggregation operator. Using the following example, we will show that this presumption is generally not valid and it may lead to inconsistencies. Let us consider the

$$o(x_1, x_2) = 1 - (1 - x_1)(1 - x_2) = x_1 + x_2 - x_1 x_2$$

operator, where $x_1, x_2 \in [0, 1]$. Note that this is the product (probabilistic) disjunction operator in continuous-valued logic. Let w_1 and w_2 be two weights, $w_1, w_2 \in [0, 1]$. Then, intuitively, we may link the weighted aggregation function

$$A(w_1, w_2, x_1, x_2) = w_1 x_1 + w_2 x_2 - w_1 w_2 x_1 x_2$$

or

$$A(w_1, w_2, x_1, x_2) = x_1^{w_1} + x_2^{w_2} - x_1^{w_1} x_2^{w_2}$$

or

$$A(w_1, w_2, x_1, x_2) = w_1 x_1 + w_2 x_2 - x_1^{w_1} x_2^{w_2}$$

to the aggregation operator o. This example highlights the fact that it is not always obvious how to identify the appropriate weighted aggregation function for an aggregation operator. In this study, we shall present a consistent approach in which the aggregation operator and the weighting method belong to each other and so the weighted aggregation function can be unambiguously determined. Later, we will see that this can be achieved by defining the weighted aggregation function as a composition of an aggregation operator and weighting transformations; and both the aggregation operator and the weighting transformation operator are induced by the same generator function. In Section 4.2, we will show that the appropriate weighted aggregation function corresponding to the above aggregation operator o is different from all the above-mentioned intuitive suggestions.

In a practical decision-making procedure, the expectations of a decision-maker play an important role. On the one hand, the expectations may be viewed as importance weights, on the other hand, the levels of expectations also represent the decision-maker's preferences. In our study, we seek to find a method that can establish a connection between the expectation levels and the weights. After identifying this connection, we can treat the weights and the expectations as being interchangeable. We should add that, to the best of our knowledge, the literature lacks methods that can directly translate decision-maker's expectation levels into quantitative weights. Since in many situations, the decision-maker's expectations are represented by intervals, we aim to present a procedure that can treat the expectation intervals as inputs and results in an interval-valued score for each assessed decision alternative.

1.2. Contribution of this study

In our study, we present a novel approach to weighted aggregation and to the determination of weights in an aggregation procedure.

It is well known that the weighted quasi-arithmetic means are characterized by the continuity, strictly increasing monotonicty, idempotency and bisymmetry properties (see, e.g., Theorem 4 in Grabisch et al., 2011). In our approach, we introduce the concept of a weighted aggregation system which consists of two elements: (1) a weighting transformation and (2) an aggregation operator. These two elements are both induced by a common generator function. Then, we show the necessary and sufficient condition for the forms of generator function-based weighting transformation and aggregation operator in a weighted aggregation system. Setting appropriate requirements for a weighted aggregation system allows us to have a new representation of weighted quasi-arithmetic means. Namely, we show that the weighted quasi-arithmetic means on the non-negative extended real line are none other than compositions of an n-ary aggregation operator and n weighting transformations ($n \in \mathbb{N}$, $n \geq 1$). We should add that our weighting transformation is similar to Yager's relevancy transformation (RET) (see Yager, 2001), which is used to obtain the effective rule output from the rule relevancy and the rule consequent in a fuzzy system. In our approach, we consider the weighting transformation – as well as the aggregation operator – being induced by a common generator. We also demonstrate that the generator functions of strict triangular norms or strict triangular conorms induce weighted aggregation systems on the unit interval. Here, we show that the weighted aggregation function of such a system is a weighted quasi-arithmetic mean on [0,1] with a single annihilator element 0 or 1.

In this paper, we present an expectation level-based weight determination and scoring procedure that utilizes the weighted quasi-arithmetic means on the unit interval. In this method, we obtain the decision-makers expectation level for each input variable of a decision making procedure. Next, we translate these expectation levels to weights using the generator function of a weighted quasi-arithmetic mean and then we use this mean as a scoring function to evaluate the decision alternatives. It is worth mentioning that, to the best of our knowledge, our method of weight determination is unique. Furthermore, we present two numerical examples of how our method can be used in practice. In the first example, the expectations levels are real-valued scalars, while in the second one, the decision-maker's expectations are interval-valued quantities. In the latter case, we obtain an interval-valued score for each decision alternative. The proposed techniques may also be viewed as novel methods in multi-criteria decision making.

This study is structured as follows. After some preliminary topics in Section 2, a general approach to weighted aggregation systems is presented in Section 3. Next, in Section 4, we present the connections between the weighted quasi-arithmetic means and the aggregation functions induced by weighted aggregation systems on the non-negative extended real line. Here, we also describe the

Table 1
Values and weights.

Patient	Symptoms		
	s_1	...	s_n
p_1	$x_{1,1}$	...	$x_{1,n}$
p_2	$x_{2,1}$	...	$x_{2,n}$
$\vdots$	$\vdots$		$\vdots$
p_m	$x_{m,1}$	...	$x_{m,n}$
Diagnosis	Symptoms		
	s_1	...	s_n
d_1	$w_{1,1}$	...	$w_{1,n}$
d_2	$w_{2,1}$	...	$w_{2,n}$
$\vdots$	$\vdots$		$\vdots$
d_l	$w_{l,1}$	...	$w_{l,n}$

weighted aggregation systems induced by strict logical operators. In Section 5, we introduce an expectation level-based weighting and scoring procedure that utilizes the weighted quasi-arithmetic means on the unit interval. In this section, we present two illustrative numerical examples of how our method can be applied in the practice of multi-criteria decision making. Lastly, the main conclusions, the limitations of our method and our future research ideas are summarized in Section 6.

2. Preliminaries

First we would like to clarify what the weighting of criteria means. In real-life decision making problems, not just the magnitude of each attribute needs to be considered, but also its degree of importance (i.e., its weight). Let us assume that we measure various symptoms of patients, and based on the measured values, we would like to give diagnoses. We should take into account the fact that the symptoms may have a different significance (weights, degrees of importance) for various types of health problems.

The left hand side part of Table 1 shows the patients from p_1 to p_m and the measured values of their symptoms $x_{i,j}$, ($i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$). The right hand side part of this table shows the degree of importance of the jth symptom in the kth diagnosis, i.e., the weight $w_{k,j}$ ($k = 1, 2, \ldots, l$). In practice, it is quite normal that the same symptom has different significance in different diagnoses, which means that there is no unique set of weights for the symptoms.

Another example is the buying of a car. In this case, the parameters of a car may also have various degrees of importance depending on whether we wish to buy a sports car or a family car. That is, in this manner, there is no so-called best car. The best car exists only for a given set of weights.

In decision procedures, one of the main tasks is to find proper weights. It is customarily supposed that there exists a unique set of weights. The previous examples suggest that this presumption is generally not valid.

Now, we will give a mathematical description of the above concept. Suppose that we have the attributes $a_1, a_2, \ldots, a_n$, which characterize an entity. Let the variables $x_1, x_2, \ldots, x_n$ and the weights $w_1, w_2, \ldots, w_n$ be the inputs of a decision procedure. Here, we interpret the value of variable x_i and the value of weight w_i as the utility value and the importance value of the ith attribute, respectively, in the preference system of a decision-maker. In practice, there are two well-known weighed aggregation methods, called the weighted arithmetic mean and the weighted geometric mean, which are:

$$\sum_{i=1}^{n} w_i x_i \quad \text{(weighted arithmetic mean)} \tag{1}$$

$$\prod_{i=1}^{n} x_i^{w_i} \quad \text{(weighted geometric mean).} \tag{2}$$

We will use the common notation $\mathbb{R}$ for the real line and $\overline{\mathbb{R}}$ for the extended real line, i.e., $\overline{\mathbb{R}} = [-\infty, \infty]$. Here, we suppose that $x_i \in \overline{\mathbb{R}}_+$, where $\overline{\mathbb{R}}_+$ is the non-negative extended real line, i.e., $\overline{\mathbb{R}}_+ = [0, \infty]$, $i \in \{1, 2, \ldots, n\}$. Note that based on Klement, Mesiar, & Pap (2013) and Grabisch et al. (2011), here we adopt the following conventions:

$$\frac{1}{0} = \infty, \quad \frac{1}{\infty} = 0, \quad +\infty + (-\infty) = -\infty \quad \text{and} \quad 0 \cdot (\pm\infty) = 0,$$

for any $x \in \mathbb{R}$:

$$x + \infty = \infty \quad \text{and} \quad x - \infty = -\infty$$

for any $x \in \mathbb{R} \setminus \{0\}$:

$$x \cdot \infty = \begin{cases} \infty, & \text{if } x > 0 \\ -\infty, & \text{if } x < 0 \end{cases} \quad \text{and} \quad \infty^x = \begin{cases} \infty, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}$$

$$e^{-\infty} = 0, \quad e^{\infty} = \infty, \quad \ln(0) = -\infty, \quad \text{and} \quad \ln(\infty) = \infty.$$

We should add that these conventions can be overwritten by results of particular limits. Also, we shall assume that $w_i \in [0, 1]$ for any $i \in \{1, 2, \ldots, n\}$ and $\sum_{i=1}^{n} w_i = 1$. We observe that there are two types of operations underlying both the weighted arithmetic and the weighted geometric means given by Eqs. (1) and (2), respectively. One of these operations transforms the x_i value to an $x'_i \in \overline{\mathbb{R}}_+$ value, while the other aggregates the transformed x'_i values into a value in $\overline{\mathbb{R}}_+$. We will call the operation that transforms x_i to x'_i the weighting transformation. And, we will call the operation that aggregates the x'_i values the aggregation operation. The weighting transformation $\omega : [0, 1] \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is a continuous function of a weight w and a variable x. That is, x' is given by

$$x' = \omega(w, x).$$

The aggregation operator o is an $o : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ mapping. For the weighted arithmetic and the weighted geometric means, the weighting transformations are given by

$$x'_i = \omega(w_i, x_i) = w_i x_i \tag{3}$$

and

$$x'_i = \omega(w_i, x_i) = x_i^{w_i}, \tag{4}$$

respectively, where $w_i \in [0, 1]$ and $x_i \in \overline{\mathbb{R}}_+$. The aggregation operations o for the weighted arithmetic mean and the weighted geometric mean are given by

$$o(x'_1, x'_2, \ldots, x'_n) = \sum_{i=1}^{n} x'_i \tag{5}$$

$$o(x'_1, x'_2, \ldots, x'_n) = \prod_{i=1}^{n} x'_i, \tag{6}$$

respectively, $x'_i \in \overline{\mathbb{R}}_+$.

Based on this line of thinking, we can state that the operator pair (ω, o) form a weighted aggregation system. Here, we seek to generalize the concept of a weighted aggregation system. In order to find the general requirements for a weighted aggregation system (ω, o), first we study the weighting transformation operation and the aggregation operation of the weighted arithmetic and the weighted geometric means. The weighting transformations of these two aggregations, given by Eqs. (3) and (4), have the following properties:

	Weighted arithmetic mean	Weighted geometric mean
(a)	for any $w \in (0,1)$, $x_1 < x_2$ implies $wx_1 < wx_2$	for any $w \in (0,1)$, $x_1 < x_2$ implies $x_1^w < x_2^w$
(b)	$1x = x, \quad 0x = 0$	$x^1 = x, \quad x^0 = 1$
(c)	$w(x_1 + x_2) = wx_1 + wx_2$	$(x_1x_2)^w = x_1^w x_2^w$
(d)	$(w_1 + w_2)x = w_1x + w_2x$	$x^{w_1+w_2} = x^{w_1}x^{w_2}$
(e)	$(w_1w_2)x = w_1(w_2x)$	$x^{w_1w_2} = (x^{w_2})^{w_1}$,

where $x, x_1, x_2 \in \overline{\mathbb{R}}_+$, $w, w_1, w_2 \in [0,1]$ and $w_1 + w_2 \leq 1$. The aggregation operator o for these two methods, given by Eqs. (5) and (6), have the following properties:

(a) o is non-decreasing in each variable
(b) If the arity of o is one, then $o(x') = x'$ for any $x' \in \overline{\mathbb{R}}_+$.

3. A general approach to weighted aggregation systems

From now on, we will suppose that the domain of the weighted aggregation is the non-negative extended real line $\overline{\mathbb{R}}_+$. Following the properties of the weighted arithmetic mean and weighted geometric mean, the general requirements for a weighted aggregation system are as follows.

Definition 1. Let $n \in \mathbb{N}$, $n \geq 1$. We say that the mapping $o : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ is an aggregation operator if and only if o satisfies the following requirements:

(o1) Continuity
o is continuous for every bounded input vector of $\overline{\mathbb{R}}_+^n$.
(o2) Monotonicity
o is strictly increasing in each of its arguments for every bounded input vector of $\overline{\mathbb{R}}_+^n$.
(o3) Identity when the arity of o is one.
For any $x \in \overline{\mathbb{R}}_+$:

$$o(x) = x.$$

Note that the aggregation operator o is defined for any $n \in \mathbb{N}$, $n \geq 1$ arity.

Now, we will generalize requirements (a)–(e), that we identified for the weighted arithmetic and geometric means.

Definition 2 (weighted aggregation system)**.** Let $n \in \mathbb{N}$, $n \geq 1$ and let $o : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ be an aggregation operator given by Definition 1. We say that the mapping $\omega : [0,1] \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$, called the weighting transformation, and the aggregation operator o form a weighted aggregation system (ω, o) if and only if ω satisfies the following requirements:

(ω1) Continuity
$\omega(w, x)$ is continuous in w and x, where $w \in [0,1]$, $x \in \overline{\mathbb{R}}_+$ and $x < \infty$.
(ω2) Monotonicity in x
For any $w \in (0,1)$ and $x_1, x_2 \in \overline{\mathbb{R}}_+$:

$$x_1 < x_2 \quad \text{implies} \quad \omega(w, x_1) < \omega(w, x_2)$$

(ω3) Neutrality
For any $x \in \overline{\mathbb{R}}_+$:

$$\omega(1, x) = x$$

(ω4) Multiplicative property
For any $w_1, w_2 \in [0,1]$, $w_1 + w_2 \leq 1$ and $x \in \overline{\mathbb{R}}_+$:

$$\omega(w_1 w_2, x) = \omega(w_1, \omega(w_2, x))$$

(ωo1) Distributivity of ω over o
For any $w \in [0,1]$ and $x_1, x_2, \ldots, x_n \in \overline{\mathbb{R}}_+$:

$$\omega(w, o(x_1, x_2, \ldots, x_n)) = o(\omega(w, x_1), \omega(w, x_2), \ldots, \omega(w, x_2))$$

(ωo2) Additive property
For any $w_1, w_2, \ldots, w_n \in [0,1]$, $\sum_{i=1}^n w_i \leq 1$ and $x \in \overline{\mathbb{R}}_+$:

$$\omega\left(\sum_{i=1}^n w_i, x\right) = o(\omega(w_1, x), \omega(w_2, x), \ldots, \omega(w_n, x)).$$

Using the concept of a weighted aggregation system, we interpret the weighted aggregation function induced by a weighted aggregation system as follows.

Definition 3. Let $n \in \mathbb{N}$, $n \geq 1$ and let (ω, o) be a weighted aggregation system with the weighting transformation $\omega : [0,1] \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ and the aggregation operator $o : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$. We say that the function $A_{\omega,o} : [0,1]^n \times \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$, which is given by

$$A_{\omega,o}(\mathbf{w}, \mathbf{x}) = o(\omega(w_1, x_1), \omega(w_2, x_2), \ldots, \omega(w_n, x_n)), \tag{7}$$

is the aggregation function induced by the weighted aggregation system (ω, o), where $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, $w_i \geq 0$, $\sum_{i=1}^n w_i = 1$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

4. Weighted quasi-arithmetic means as aggregation functions induced by weighted aggregation systems

Following Grabisch et al. (2011), we define the weighted quasi-arithmetic mean on $\overline{\mathbb{R}}_+$ as follows (see also Daróczy & Páles, 2003).

Definition 4. Let $n \in \mathbb{N}$, $n \geq 1$. We say that the function $M_{f,\mathbf{w}} : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ is a weighted quasi-arithmetic mean on $\overline{\mathbb{R}}_+$ if there exists a function $f : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}$, which is continuous and strictly monotonic on $\mathbb{R}$, and a real valued vector $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in (0,1)$ satisfying $\sum_{i=1}^n w_i = 1$ such that

$$M_{f,\mathbf{w}}(\mathbf{x}) = f^{-1}\left(\sum_{i=1}^n w_i f(x_i)\right), \tag{8}$$

holds for any $x_1, x_2, \ldots, x_n \in \overline{\mathbb{R}}_+$.

We will say that the function f in Eq. (8) is a generator function of the weighted quasi-arithmetic mean $M_{f,\mathbf{w}}$. We can see by direct calculation that $M_{f,\mathbf{w}}$ is uniquely determined up to a positive multiplier of f.

4.1. Representing weighted quasi-arithmetic means on the non-negative extended real line

Here, we will show that any weighted quasi-arithmetic mean on $\overline{\mathbb{R}}_+$ can be represented by a weighted aggregation system.

Proposition 1. *Let $n \in \mathbb{N}$, $n \geq 1$ and let $f : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}$ be a continuous and strictly monotonic function on $\mathbb{R}$, and let the function $o_f : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ be given by*

$$o_f(x_1, x_2, \ldots, x_n) = f^{-1}\left(\sum_{i=1}^n f(x_i)\right). \tag{9}$$

Then, o_f satisfies the requirements for an aggregation operator given by Definition 1.

Proof. With Eq. (9) and Definition 1, the proof is straightforward. □

Remark 1. Note that Eq. (9) is a characterization of associative functions. The necessary and sufficient conditions for characterization of associative functions can be found in Section 6.2 in Aczél (1966). The function f in Proposition 1 is called a generator function of o_f, and f is determined up to a multiplicative constant.

The following theorem gives a sufficient condition for the form of a generator function-based weighting transformation ω_f such that the (ω_f, o_f) pair satisfies the definition for a weighted aggregation system.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geq 1$ and let the function $o_f : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ be given by Eq. (9), where f is a generator function of o_f. If the function $\omega_f : [0,1] \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is given by*

$$\omega_f(w, x) = f^{-1}(wf(x)), \tag{10}$$

then the pair (ω_f, o_f) satisfies the requirements for a weighted aggregation system given in Definition 2.

Proof. Based on Proposition 1, the function o_f satisfies the requirements for an aggregation operator given by Definition 1. Hence, to show that (ω_f, o_f) is a weighted aggregation system, we need to demonstrate that ω_f satisfies the requirements from $\omega 1$ to $\omega 4$ and $\omega o1$ and $\omega o2$ in Definition 2. Recall that $f : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}$ is a continuous and strictly monotonic function on $\mathbb{R}$.

(ω1) Since f is a continuous function on $\mathbb{R}$, ω_f is continuous in w and in every $x \in \overline{\mathbb{R}}_+$, $x < \infty$.

(ω2) As f is a strictly monotonic function, ω_f is strictly increasing in x.

(ω3) For any $x \in \overline{\mathbb{R}}_+$, we have $\omega(1, x) = f^{-1}(1f(x)) = x$; that is, ω satisfies (ω3).

(ω4) Using Eq. (10), for any $w_1, w_2 \in [0,1]$, $w_1 + w_2 \leq 1$ and $x \in \overline{\mathbb{R}}_+$, we can write

$$\begin{aligned}&\omega_f(w_1 w_2, x)\\ &= f^{-1}(w_1 w_2 f(x)) = f^{-1}(w_1 f(f^{-1}(w_2 f(x)))) =\\ &= f^{-1}(w_1 f(\omega_f(w_2, x))) = \omega_f(w_1, \omega_f(w_2, x)).\end{aligned}$$

(ωo1) With Eqs. (9) and (10), for any $w \in [0,1]$ and $x_1, x_2, \ldots, x_n \in \overline{\mathbb{R}}_+$, we can write

$$\begin{aligned}\omega_f(w, o_f(x_1, x_2, \ldots, x_n)) &= f^{-1}\left(wf\left(f^{-1}\left(\sum_{i=1}^n f(x_i)\right)\right)\right) =\\ &= f^{-1}\left(\sum_{i=1}^n wf(x_i)\right) = f^{-1}\left(\sum_{i=1}^n f\left(f^{-1}(wf(x_i))\right)\right) =\\ &= f^{-1}\left(\sum_{i=1}^n f\left(\omega_f(w, x_i)\right)\right)\\ &= o_f(\omega_f(w, x_1), \omega_f(w, x_2), \ldots, \omega_f(w, x_n)).\end{aligned}$$

(ωo2) With Eqs. (9) and (10), for any $w_1, w_2, \ldots, w_n \in [0,1]$, $\sum_{i=1}^n w_i \leq 1$ and $x \in \overline{\mathbb{R}}_+$, we can write

$$\begin{aligned}\omega_f\left(\sum_{i=1}^n w_i, x\right) &= f^{-1}\left(\left(\sum_{i=1}^n w_i\right) f(x)\right)\\ &= f^{-1}\left(\sum_{i=1}^n w_i f(x)\right) =\\ &= f^{-1}\left(\sum_{i=1}^n f\left(f^{-1}(w_i f(x))\right)\right)\\ &= f^{-1}\left(\sum_{i=1}^n f\left(\omega_f(w_i, x)\right)\right) =\\ &= o_f(\omega_f(w_1, x), \omega_f(w_2, x), \ldots, \omega_f(w_n, x)).\end{aligned}$$

□

We should add that the weighting transformation ω_f is similar to Yager's RET operator which can be used in rule-based fuzzy systems to obtain the output of a rule by considering its relevancy and its consequent (see Yager, 2001). The two main differences between the RET operator and the weighting transformation are as follows. (1) Yager's RET operator is a binary operator taking values from I^2 to I, while the weighting transformation ω_f is a $[0,1] \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ binary operator. That is, if ω_f were a $[0,1] \times [0,1] \to [0,1]$ mapping, then it would be a RET operator with $I = [0,1]$. (2) The weighting transformation ω_f is generator-dependent, while in Yager's paper, the RET operator is interpreted as a function that fulfills certain requirements, but not induced by a generator.

Remark 2. It should be emphasized that the elements of a weighted aggregation system (ω_f, o_f) induced by the generator function f belong to each other. That is, both the aggregation operator o_f and the weighting transformation ω_f depend on a common generator function.

Here, we shall give the necessary and sufficient condition for the form of a generator function-based weighting transformation ω_f such that the (ω_f, o_f) pair satisfies the definition for a weighted aggregation system. We will show that from the requirements in Definition 2, only two, namely, (ω3) and (ωo2), are the necessary conditions for a weighted aggregation system.

Theorem 2. *Let $n \in \mathbb{N}$, $n \geq 1$ and let $o_f : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ be an aggregation operator given by Eq. (9), where f is a generator function of o_f. The pair (ω_f, o_f) satisfies the requirements (ω3) and (ωo2) for a weighted aggregation system given in Definition 2 if and only if the function $\omega_f : [0,1] \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ has the form given by Eq. (10).*

Proof. Based on Theorem 1, if ω_f has the form given by Eq. (10), then the pair (ω_f, o_f) satisfies all the requirements for a weighted aggregation system given in Definition 2. Therefore, (ω_f, o_f) satisfies the requirements (ω3) and (ωo2) for such a system as well.

Conversely, let us assume that (ω_f, o_f) satisfies the requirements (ω3) and (ωo2) for a weighted aggregation system given in Definition 2. Let $x \in \overline{\mathbb{R}}_+$ have an arbitrary fixed value. Then, noting the definition of o_f in Eq. (9) and the assumption that (ω_f, o_f) satisfies the requirement ($o\omega$2) in Definition 2, we have

$$\omega_f\left(\sum_{i=1}^n w_i, x\right) = f^{-1}\left(\sum_{i=1}^n f\left(\omega_f(w_i, x)\right)\right). \tag{11}$$

Now, let the function $g : [0,1] \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}$ be given by

$$g(w, x) = f(\omega_f(w, x)). \tag{12}$$

Then, Eq. (11) can be written as

$$g\left(\sum_{i=1}^n w_i, x\right) = \sum_{i=1}^n g(w_i, x). \tag{13}$$

Since $x \in \overline{\mathbb{R}}_+$ has a fixed value, the variables in Eq. (13) are $w_1, w_2, \ldots, w_n$. Noting the Cauchy functional equation, the solution of Eq. (13) is

$$g(w, x) = c_x w, \tag{14}$$

where c_x is a constant, which depends on the fixed value of x. Next, from Eq. (12) and Eq. (14), we have

$$f(\omega_f(w, x)) = c_x w. \tag{15}$$

Since the pair (ω_f, o_f) satisfies the requirement (ω3) for a weighted aggregation system given in Definition 2, from Eq. (15), we have

$$f(\omega_f(1, x)) = f(x) = c_x. \tag{16}$$

Using Eqs. (15) and (16), we can write

$$f(\omega_f(w, x)) = wf(x), \tag{17}$$

which means that ω_f has the form given by Eq. (10). □

Table 2
Weighting transformations, aggregation operators and weighted aggregation functions induced by some generator functions.

$f(x)$	$\omega_f(w,x)$	$o_f(\mathbf{x})$	$A_{\omega_f,o_f}(\mathbf{w},\mathbf{x})$
x	wx	$\sum_{i=1}^{n} x_i$	$\sum_{i=1}^{n} w_i x_i$
$\ln(x)$	x^w	$\prod_{i=1}^{n} x_i$	$\prod_{i=1}^{n} x_i^{w_i}$
x^2	$\sqrt{w}x$	$\sum_{i=1}^{n} x_i^2$	$\sum_{i=1}^{n} w_i x_i^2$
x^{-1}	$\frac{x}{w}$	$\frac{1}{\sum_{i=1}^{n}\frac{1}{x_i}}$	$\frac{1}{\sum_{i=1}^{n}\frac{w_i}{x_i}}$
$x^\alpha (\alpha \neq 0)$	$w^{\frac{1}{\alpha}}x$	$\left(\sum_{i=1}^{n} x_i^\alpha\right)^{\frac{1}{\alpha}}$	$\left(\sum_{i=1}^{n} w_i x_i^\alpha\right)^{\frac{1}{\alpha}}$
$e^{\alpha x} (\alpha \neq 0)$	$\frac{1}{\alpha}\ln(we^{\alpha x})$	$\frac{1}{\alpha}\ln\left(\sum_{i=1}^{n} e^{\alpha x_i}\right)$	$\frac{1}{\alpha}\ln\left(\sum_{i=1}^{n} w_i e^{\alpha x_i}\right)$

Based on the above results, we will introduce the concept of the weighting aggregation system induced by a generator function.

Definition 5. Let $n \in \mathbb{N}$, $n \geq 1$ and let $f : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}$ be a continuous and strictly monotonic function on $\mathbb{R}$, let the aggregation operator $o_f : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ be given by Eq. (9) and let the weighting transformation $\omega_f : [0,1] \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ be given by Eq. (10). We say that the weighted aggregation system (ω_f, o_f) is induced by the generator function f.

Suppose we have the weighted aggregation system (ω_f, o_f) induced by a generator function f. Then, using Definition 3, the weighted aggregation function induced by (ω_f, o_f) is

$$A_{\omega_f,o_f}(\mathbf{w},\mathbf{x}) = o_f(\omega_f(w_1,x_1), \omega_f(w_2,x_2), \ldots, \omega_f(w_n,x_n)) = \\ = f^{-1}\left(\sum_{i=1}^{n} w_i f(x_i)\right), \tag{18}$$

where $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$.

Theorem 3. *The function $M_{f,\mathbf{w}} : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ is a weighted quasi-arithmetic mean on $\overline{\mathbb{R}}_+$ if and only if there exists a function $f : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}$, which is continuous and strictly monotonic on $\mathbb{R}$, and a real valued vector $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in (0,1)$ satisfying $\sum_{i=1}^{n} w_i = 1$ such that*

$$M_{f,\mathbf{w}}(\mathbf{x}) = A_{\omega_f,o_f}(\mathbf{w},\mathbf{x}), \tag{19}$$

holds for any $x_1, x_2, \ldots, x_n \in \overline{\mathbb{R}}_+$, where $A_{\omega_f,o_f} : [0,1]^n \times \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ is the weighted aggregation function induced by the weighted aggregation system (ω_f, o_f) according to Eq. (18).

Proof. Taking into account Eq. (18) and the definition for a weighted quasi-arithmetic mean in Definition 4, the proof is straightforward. □

Table 2 contains some well-known weighted quasi-arithmetic mean operators represented by weighted aggregation systems.

4.2. Weighted aggregation systems induced by strict logical operators

Now, we will briefly describe those weighted aggregation systems that are induced by generator functions of strict triangular norms (t-norms for short) and strict triangular conorms (t-conorms for short). First, we will show that such systems indeed exist.

Let the function $f : [0,1] \to \overline{\mathbb{R}}_+$ be continuous and either

(a) strictly decreasing with $f(1) = 0$ and $f(0) = \infty$ or
(b) strictly increasing with $f(0) = 0$ and $f(1) = \infty$.

Note that f is uniquely determined up to a positive multiplicative constant, and in case (a) f is a generator function of a strict t-norm and in case (b) f is a generator function of a strict t-conorm (see Klement et al., 2013). Let $n \in \mathbb{N}$, $n \geq 1$. It can be easily verified that the operator $o : [0,1]^n \to [0,1]$, which is given by

$$o_f(x_1, x_2, \ldots, x_n) = f^{-1}\left(\sum_{i=1}^{n} f(x_i)\right), \tag{20}$$

satisfies the requirements for an aggregation operator given by Definition 1. Also note that o_f is an n-ary strict t-norm (case (a)) or strict t-conorm (case (b)).

Next, let the function $\omega_f : [0,1] \times [0,1] \to [0,1]$ be given by

$$\omega_f(w,x) = f^{-1}(wf(x)). \tag{21}$$

Then, similar to the proof of Theorem 1, it can be verified that the pair (ω_f, o_f) with ω_f and o_f given by Eqs. (20) and (21), respectively, satisfies the requirements for a weighted aggregation system given in Definition 2.

It can be readily verified that if f is a generator function of a strict t-norm (strict t-conorm, respectively), then the weighted aggregation function A_{ω_f,o_f} induced by (ω_f, o_f) is a weighted quasi-arithmetic mean on [0,1] with a single annihilator element 0 (1, respectively).

Example 1. If the generator function $f : [0,1] \to \overline{\mathbb{R}}_+$ is $f(x) = -\ln(1-x)$, then

$$o_f(x_1, x_2) = 1 - (1-x_1)(1-x_2) = x_1 + x_2 - x_1 x_2,$$

where $(x_1, x_2) \in [0,1]^2$, and

$$\omega_f(w,x) = 1 - (1-x)^w,$$

where $x \in [0,1]$. Therefore, the weighted aggregation function, which is a weighted quasi-arithmetic mean as well, induced by (ω_f, o_f) is

$$A_{\omega_f,o_f}(w_1, w_2, x_1, x_2) = 1 - (1-x_1)^{w_1}(1-x_2)^{w_2},$$

where $x_1, x_2, w_1, w_2 \in [0,1]$. Notice that o_f is the same aggregation operator as the aggregation operator o in Section 1.1., but the appropriate weighted aggregation function A_{ω_f,o_f} is different from the three intuitively suggested weighted aggregation functions in Section 1.1. In our approach, the aggregation operator and the weighting transformation are both induced by a common generator function. Therefore, this approach results in a consistent way of weighting.

5. Expectation levels, weights and the weighted quasi-arithmetic means as scoring functions on the unit interval

Now, let $f : [0,1] \to \overline{\mathbb{R}}_+$ be a continuous and strictly monotonic function. Here, we assume that the image of f is $\overline{\mathbb{R}}_+$ or a subinterval of $\overline{\mathbb{R}}_+$, i.e., f is not necessarily a generator of a strict t-norm or strict t-conorm. For example, f can be given by $f(x) = x$ or $f(x) = x^2$, where $x \in [0,1]$.

Suppose that the decision-maker has an expectation level $\nu_i \in (0,1)$ for the value of variable $x_i \in (0,1)$, $i = 1, 2, \ldots, n$, $n \in \mathbb{N}$, $n \geq 1$. Here, ν_i expresses the importance of x_i based on the decision-maker's preferences, i.e., the greater the value of ν_i is, the higher the importance of x_i is. Note that a higher importance value does not necessarily imply a higher weight value. For example, in the weighted geometric mean, a lower weight value corresponds to a higher importance. In this case, a greater value of expectation level should correspond to a lower weight value.

Let $t_f : (0,1) \to (0,1)$ be a function that transforms an expectation level ν_i to a weight w_i. Then, t_f needs to satisfy the following requirements:

(R1) $\sum_{i=1}^{n} t_f(\nu_i) = 1$

Table 3
Expectations of a decision maker for the values of car attributes.

Attribute:	Engine power	Max. speed	Fuel consumption	Trunk capacity
Unit of measure:	HP	kilometers/hour	kilometers/liter	liter
Range:	[50,300]	[140,240]	[5,25]	[100,600]
Expected value:	150	210	20	400

(R2) If f is strictly decreasing, then for any $\nu_i > \nu_j$, $t_f(\nu_i) < t_f(\nu_j)$
If f is strictly increasing, then for any $\nu_i > \nu_j$, $t_f(\nu_i) > t_f(\nu_j)$,

where $i, j \in \{1, 2, \ldots, n\}$.

It can be easily verified that the function $t_f : (0,1) \to (0,1)$ given by

$$t_f(\nu_i) = \frac{f(\nu_i)}{\sum_{j=1}^{n} f(\nu_j)}, \tag{22}$$

satisfies the requirements (R1) and (R2). Then, using $w_i = t_f(\nu_i)$ and Eq. (8), the weighted quasi-arithmetic mean on [0,1] induced by f with the weights $w_1, w_2, \ldots, w_n$ can be written as

$$M_{f,\mathbf{w}}(\mathbf{x}) = f^{-1}\left(\sum_{i=1}^{n} w_i f(x_i)\right) = f^{-1}\left(\frac{\sum_{i=1}^{n} f(\nu_i) f(x_i)}{\sum_{i=1}^{n} f(\nu_i)}\right). \tag{23}$$

Here, $M_{f,\mathbf{w}}(\mathbf{x})$ may be treated as a scoring function that assigns a scoring value to the particular input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in (0,1)^n$ using the weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in (0,1)^n$. Notice that in our approach, the weights are directly derived from the decision-makers expectation levels.

5.1. Illustrative examples

In this section, by the means of two numerical examples, we will show how our method can be applied in practice. In the first example, we will demonstrate how the assessment of the alternatives can be performed using the decision-maker's expectation levels for the attributes of the assessed entities. Here, the expectation levels are simple real-valued scalars. In the second example, we show how our method can be adapted to the case where the decision-maker's expectations are interval-valued quantities.

Example 2. Suppose that cars are characterized by the following four attributes: (1) engine power, (2) max. speed, (3) fuel consumption and (4) trunk capacity. Table 3 shows the unit of measure and the range for each of these attributes. This table also includes the expectation of a decision-maker for the value of each attribute.

Let a_1, a_2, a_3 and a_4 denote the attributes (1) Engine power, (2) Max. speed, (3) Fuel consumption and (4) Trunk capacity, respectively, and let ν_i be the normalized expected value of attribute a_i:

$$\nu_1 = \frac{150-50}{250} = 0.4, \qquad \nu_2 = \frac{210-140}{100} = 0.7$$
$$\nu_3 = \frac{20-5}{20} = 0.75, \qquad \nu_4 = \frac{400-100}{500} = 0.6.$$

Let the generator function $f : [0,1] \to \overline{\mathbb{R}}_+$ be given by

$$f(x) = -\ln(x),$$

i.e., f is a generator function of the product t-norm. The inverse function of f is

$$f^{-1}(x) = \mathrm{e}^{-x}.$$

Using Eq. (22), we can compute the value of the weight w_i:

$$w_i = \frac{f(\nu_i)}{\sum_{j=1}^{4} f(\nu_j)} = \frac{\ln(\nu_i)}{\sum_{j=1}^{4} \ln(\nu_j)}$$

Table 4
Computation results.

i:	1	2	3	4
ν_i:	0.4	0.7	0.75	0.6
$f(\nu_i)$:	0.9163	0.3567	0.2877	0.5108
$w_i = \frac{f(\nu_i)}{\sum_{j=1}^{4} f(\nu_j)}$:	0.4423	0.1722	0.1389	0.2466

for $i = 1, 2, 3, 4$ (see Table 4). Notice that the greater the expected value is, the smaller the corresponding weight is. This finding is in line with the fact that, in the current case, the weighted quasi-arithmetic mean is

$$M_{f,\mathbf{w}}(\mathbf{x}) = \prod_{i=1}^{4} x_i^{w_i}, \tag{24}$$

where $x_i \in (0,1)$, and so weights for the variables with higher importance (i.e., with higher expected values) are smaller than those for the variables with lower importance (i.e., with lower expected values).

We can easily verify that $\sum_{i=1}^{4} w_i = 1$.

Now, suppose that five cars with the attributes in Table 5 are offered to the decision-maker.

Since we have found the weight vector $\mathbf{w} = (w_1, w_2, w_3, w_4)$, we can compute the value of the weighted quasi-arithmetic mean $M_{f,\mathbf{w}}(\mathbf{x})$ for each alternative in Table 5 by utilizing Eq. (24) and the corresponding variable vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$.

Using the values of the weighted quasi-arithmetic means as scoring values, the preference order of the offered cars (from the most preferred to the least preferred) is:

Car 1, Car 3, Car 2, Car 5, Car 4.

Example 3. Let us assume that for the value of each car attribute in Example 2, a decision-maker can provide an interval that expresses his or her expectation for the attribute value. Assume that the decision-maker's expected intervals are given in Table 6.

That is, in this case, instead of the normalized expected value ν_i, we have the interval $[\nu_{i,l}, \nu_{i,u}]$, $i = 1, 2, \ldots, n$. Here, $\nu_{i,l}$ and $\nu_{i,u}$ stand for a lower and for an upper normalized expected value for the attribute a_i, respectively, where $\nu_{i,l} \le \nu_{i,u}$. Then, following the procedure described in Example 2 and noting that $f(x) = -\ln(x)$, the lower weight value $w_{i,l}$ and the upper weight value $w_{i,u}$ for each attribute can be computed as follows:

$$w_{i,l} = \frac{f(\nu_{i,l})}{\sum_{j=1}^{4} f(\nu_{j,l})}, \quad w_{i,u} = \frac{f(\nu_{i,u})}{\sum_{j=1}^{4} f(\nu_{j,u})}.$$

The corresponding lower and upper normalized expected values $\nu_{i,l}$ and $\nu_{i,u}$, respectively, and the computed lower and upper weight values $w_{i,l}$ and $w_{i,u}$, respectively, are shown for each attribute in Table 7.

Notice that here the aggregation function is strictly decreasing ($f(x) = -\ln(x)$), therefore, a greater expectation level implies a lower weight both in the lower and upper cases. Recall that the weights are derived from expectation levels using a generator function. That is, a weight represents a generator function-dependent importance. Here, we have two sets of weights, namely the lower and upper weights. The value of a weight tell us the importance of the corresponding attribute after considering the set of weights

Table 5
Alternatives and their scoring values.

Car	a_1	a_2	a_3	a_4	x_1	x_2	x_3	x_4	$M_{f,\mathbf{w}}(\mathbf{x})$
Car 1	170	220	23	450	0.48	0.80	0.90	0.70	0.6277
Car 2	130	190	18	350	0.32	0.50	0.65	0.50	0.4257
Car 3	280	230	7	300	0.92	0.90	0.10	0.40	0.5484
Car 4	80	170	24	580	0.12	0.30	0.95	0.96	0.3127
Car 5	130	150	18	400	0.32	0.10	0.65	0.60	0.3375

Table 6
Expectation intervals of a decision maker for the values of car attributes.

Attribute:	Engine power	Max. speed	Fuel consumption	Trunk capacity
Unit of measure:	HP	kilometers/hour	kilometers/liter	liter
Range:	[50,300]	[140,240]	[5,25]	[100,600]
Expectation:	[140,160]	[180,230]	[15,22]	[350,450]

Table 7
Computation results.

i:	1	2	3	4
$\nu_{i,l}$:	0.36	0.4	0.5	0.5
$f(\nu_{i,l})$:	1.0217	0.9163	0.6931	0.6931
$w_{i,l} = \frac{f(\nu_{i,l})}{\sum_{j=1}^{4} f(\nu_{j,l})}$:	0.3073	0.2756	0.2085	0.2085
$\nu_{i,u}$:	0.44	0.9	0.85	0.7
$f(\nu_{i,u})$:	0.8210	0.1054	0.1625	0.3567
$w_{i,u} = \frac{f(\nu_{i,u})}{\sum_{j=1}^{4} f(\nu_{j,u})}$:	0.5679	0.0729	0.1124	0.2467

that the weight in question belongs to. Therefore, an upper weight value may be less than the corresponding lower weight value. For example, in the case of the second attribute, the lower normalized expected value is 0.4. This is not much different from the other lower normalized expected values and the corresponding weight value is 0.2756. Nevertheless, the upper normalized expected value for this attribute is 0.9, which is the highest among the upper normalized expected values, and so, because a strictly decreasing generator function is utilized, this attribute has the lowest weight value (0.0729) among all the upper weight values. In this case, the upper weight value is less than the lower weight value. However, here we should not compare them with each other as they belong to two different sets of weights.

Once the lower and upper weights for each attribute have been identified, we can use them to evaluate the alternatives in Table 5. For each alternative in Table 5, we compute the lower value of the weighted quasi-arithmetic mean function $M_l = M_{f,\mathbf{w}_l}(\mathbf{x})$ using the vector of lower weights $\mathbf{w}_l = (w_{1,l}, w_{2,l}, w_{3,l}, w_{4,l})$ and the upper value of the weighted quasi-arithmetic mean function $M_u = M_{f,\mathbf{w}_u}(\mathbf{x})$ using the vector of upper weights $\mathbf{w}_u = (w_{1,u}, w_{2,u}, w_{3,u}, w_{4,u})$:

$$M_l = M_{f,\mathbf{w}_l}(\mathbf{x}) = \prod_{i=1}^{4} x_i^{w_{i,l}}, \quad M_u = M_{f,\mathbf{w}_u}(\mathbf{x}) = \prod_{i=1}^{4} x_i^{w_{i,u}},$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$ is the variable vector corresponding to the alternative. Note that based on the above comments on the values of weights, the value of M_l may be greater than the value of M_u. Therefore, we shall treat M_l and M_u as being the two endpoints of the scoring interval. That is, we have the $[M_l, M_u]$ scoring interval if $M_l \le M_u$, and we have the $[M_u, M_l]$ scoring interval if $M_u \le M_l$. These scoring intervals are numerically given in Table 8. Fig. 1 shows the plots of the scoring intervals.

In this case, the scores are interval-valued quantities, and so the preference order of the offered cars can be obtained by ranking intervals. There are effective methods that utilize probability-based preference measures to rank intervals (see, e.g. Dombi & Jónás, 2020; Huynh, Nakamori, & Lawry, 2008; Kundu, 1997; Yue, 2016).

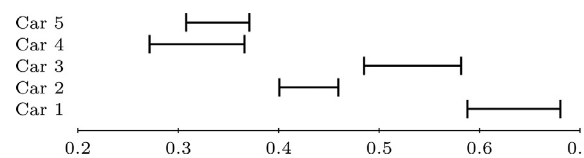

Fig. 1. Scoring intervals.

The main implications of our weight determination and scoring method can be summarized as follows. In the presented procedure, the attribute weights are obtained directly from the information on the decision-maker's expectation levels. This feature makes the method easy-to-use in practice. It is worth adding that, as we demonstrated in Example 3, our method can be easily adapted to the case where the decision-maker's expectations are interval-valued quantities. An advantage of our method is that it preserves its simplicity in this case as well.

6. Conclusions, limitations and future research plans

The chief results of this study can be summarized as follows.

- We presented the concept of weighted aggregation system which consists of a weighting transformation and an aggregation operator.
- We proved that the weighted aggregations functions induced by generator function-based weighted aggregation systems are the weighted quasi-arithmetic means on the non-negative extended real line. This means that we presented a new representation of the weighted quasi-arithmetic means on the non-negative extended real line.
- Next, we provided the necessary and sufficient condition for the form of a generator function-based weighting transformation such that this transformation along with the corresponding generator function-based aggregation operator satisfy the definition for a weighted aggregation system.
- We demonstrated that the generator functions of strict triangular norms or strict triangular conorms induce weighted aggregation systems on the unit interval. We also pointed out that the weighted aggregation functions of these system are weighted quasi-arithmetic means on [0,1] with a single annihilator element 0 or 1.
- Utilizing the weighted quasi-arithmetic means on [0,1], we presented a novel, expectation level-based weight determination and scoring method in which the decision-makers expectation levels are directly transformed into weights.
- By the means of illustrative numerical examples, we showed how our method can be applied in the practice of multi-criteria decision making.

Table 8
Alternatives and their scoring intervals.

Car	a_1	a_2	a_3	a_4	x_1	x_2	x_3	x_4	Scoring interval
Car 1	170	220	23	450	0.48	0.80	0.90	0.70	[0.5869,0.6815]
Car 2	130	190	18	350	0.32	0.50	0.65	0.50	[0.3997,0.4604]
Car 3	280	230	7	300	0.92	0.90	0.10	0.40	[0.4839,0.5827]
Car 4	80	170	24	580	0.12	0.30	0.95	0.96	[0.2704,0.3669]
Car 5	130	150	18	400	0.32	0.10	0.65	0.60	[0.3069,0.3718]

6.1. Limitations

We should mention that our approach has certain limitations. Since our procedure is generator function-dependent, the weights, the weighted aggregation functions and the scoring results all depend on the choice of the generator function. Therefore, it is an unanswered question how the most suitable generator function for a decision making problem can be determined. Also note that the presented weight determination procedure is limited to those cases where the domain of each variable is a bounded interval and the variables can be normalized to the unit interval.

We showed that if a weighting transformation is induced by the same generator as the aggregation operator, and this generator is a continuous and strictly monotonic function $f : \overline{\mathbb{R}}_+ \to [-\infty, \infty]$, then the corresponding weighted aggregation function is a weighted quasi-arithmetic mean on the non-negative extended real line. We did not study whether the aggregation functions other than the weighted quasi-arithmetic means can be represented by utilizing a similar approach.

6.2. Future research plans

A future research direction could be the investigation of how the most suitable generator function for a decision making problem can be determined. We should study if certain modifications to our weighted aggregation systems can result in aggregation functions other than the weighted quasi-arithmetic means. Also, we should note that there is a wide range of existing aggregation functions. A comparative analysis of the deficiencies of the existing methods in the light of our method is taken into account as a future research avenue.

Declaration of Competing Interest

Authors declare that they have no conflict of interest.

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