



PAPER • OPEN ACCESS

# Bi-Hamiltonian structure of Sutherland models coupled to two $u(n)^*$ -valued spins from Poisson reduction

To cite this article: L Fehér 2022 *Nonlinearity* **35** 2971

View the [article online](#) for updates and enhancements.

## You may also like

- [Effects of dipole–dipole interaction between cigar-shaped BECs of cold alkali atoms: towards inverse-squared interactions](#)  
Yue Yu, Zhuxi Luo and Ziqiang Wang
- [Stability of relative equilibria of multidimensional rigid body](#)  
Anton Izosimov
- [Pencils of compatible metrics and integrable systems](#)  
O. I. Mokhov

# Bi-Hamiltonian structure of Sutherland models coupled to two $u(n)^*$ -valued spins from Poisson reduction

L Fehér\*

Department of Theoretical Physics, University of Szeged, Tisza Lajos krt 84-86,  
H-6720 Szeged, Hungary  
Department of Theoretical Physics, WIGNER RCP, RMKI H-1525 Budapest,  
P.O.B. 49, Hungary

E-mail: [lfeher@physx.u-szeged.hu](mailto:lfeher@physx.u-szeged.hu)

Received 17 September 2021, revised 26 March 2022

Accepted for publication 3 May 2022

Published 30 May 2022



CrossMark

## Abstract

We introduce a bi-Hamiltonian hierarchy on the cotangent bundle of the *real* Lie group  $GL(n, \mathbb{C})$ , and study its Poisson reduction with respect to the action of the product group  $U(n) \times U(n)$  arising from left- and right-multiplications. One of the pertinent Poisson structures is the canonical one, while the other is suitably transferred from the real Heisenberg double of  $GL(n, \mathbb{C})$ . When taking the quotient of  $T^*GL(n, \mathbb{C})$  we focus on the dense open subset of  $GL(n, \mathbb{C})$  whose elements have pairwise distinct singular values. We develop a convenient description of the Poisson algebras of the  $U(n) \times U(n)$  invariant functions, and show that one of the Hamiltonians of the reduced bi-Hamiltonian hierarchy yields a hyperbolic Sutherland model coupled to two  $u(n)^*$ -valued spins. Thus we obtain a new bi-Hamiltonian interpretation of this model, which represents a special case of Sutherland models coupled to two spins obtained earlier from reductions of cotangent bundles of reductive Lie groups equipped with their canonical Poisson structure. Upon setting one of the spins to zero, we recover the bi-Hamiltonian structure of the standard hyperbolic spin Sutherland model that was derived recently by a different method.

Keywords: integrable systems, bi-Hamiltonian systems, spin Sutherland model, Poisson reduction

Mathematics Subject Classification numbers: 70H06, 37J15, 37K10.

\*Author to whom any correspondence should be addressed.

Recommended by Professor Beatrice Pelloni.



Original content from this work may be used under the terms of the [Creative Commons Attribution 3.0 licence](https://creativecommons.org/licenses/by/3.0/). Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

## 1. Introduction and background

The investigation of integrable many-body models of Calogero–Moser–Sutherland type has been initiated half a century ago [7, 38, 52], and by now an extensive body of knowledge has been amassed about such systems and their manifold applications in mathematics and physics [3, 11, 12, 47]. Remarkably, we are still witnessing intense research in this area, regarding especially the ‘relativistic’ deformations [48] of the originally studied particle systems and their spin extensions [17, 26, 30, 34]. Recent enquiries on these models are concerned, for example, with their derivation by Hamiltonian reduction [4, 9, 14], relations to moduli spaces of flat connections [2], harmonic analysis and special functions [10, 31, 46], pole dynamics in the matrix KP hierarchy [43] and quiver varieties [8, 13].

Interestingly, several classical integrable systems admit a bi-Hamiltonian structure, which means that the evolution equation can be encoded by two different Hamiltonians and corresponding Poisson brackets that are compatible in the sense that their linear combinations also satisfy the Jacobi identity. This can often be used to generate a family of commuting Hamiltonians through recursion. The first example of a bi-Hamiltonian structure was found by Magri [35] for the KdV equation. Since then the bi-Hamiltonian approach has become one of the central tools in the study of soliton equations [6, 36]. The bi-Hamiltonian structures of Toda lattices also received considerable attention [51], but (except for the simplest rational model [1, 5, 15, 36]) the construction of bi-Hamiltonian structures for Calogero–Moser–Sutherland type models has been left largely unexplored in the literature. The aim of our current research is to take some steps towards filling this gap.

The present report is a continuation of our series of papers [18–20] devoted to the study of bi-Hamiltonian aspects of spin Sutherland models. The papers [18, 19] dealt with bi-Hamiltonian structures of the hyperbolic and trigonometric real spin Sutherland models, and then in [20] we investigated the corresponding complex holomorphic system. Here, we construct a bi-Hamiltonian structure for a more general real Sutherland type model that involves two spin variables belonging to the dual  $\mathfrak{u}(n)^*$  of the Lie algebra of the unitary group  $U(n)$ . The bi-Hamiltonian structure of the hyperbolic spin Sutherland model will be recovered by setting one of the spins to zero. The model of our interest is a special case of models investigated previously in [16, 24, 29, 45], but in those papers bi-Hamiltonian structures were not addressed, while this is the issue that concerns us here.

Our work fits into the Hamiltonian reduction approach to integrable many-body models. We now overview the chain of developments that motivated us and sketch the content of the present report. The first step was taken by Olshanetsky and Perelomov [41] who derived the hyperbolic Sutherland model with Hamiltonian,

$$\mathcal{H}_{\text{Suth}} = \frac{1}{2} \sum_{i=1}^n p_i^2 + \sum_{1 \leq i < j \leq n} \frac{\kappa^2}{\sinh^2(q_i - q_j)}, \quad \kappa \in \mathbb{R}^*, \quad (1.1)$$

by reduction of geodesic motion on the Riemannian symmetric space of positive matrices. One may view the symmetric space in question as the coset space  $G/U(n)$  for  $G := GL(n, \mathbb{C})$  regarded as a real Lie group. The pioneering work [41] focused on solving the equations of motion defined by the Hamiltonian (1.1) via projection of suitably constrained geodesics on  $G/U(n)$ . It is fruitful to interpret this as Hamiltonian reduction of the phase space  $T^*G$  as follows [21]. There is a natural action of the group  $U(n) \times U(n)$  on  $T^*G$  (see equations (3.1) and (3.2) below), which is generated by a moment map

$$(\Phi_L, \Phi_R) : T^*G \rightarrow \mathfrak{u}(n)^* \oplus \mathfrak{u}(n)^*. \quad (1.2)$$

The Sutherland model lives on the reduced phase space

$$\Phi_L^{-1}(0) \cap \Phi_R^{-1}(\mathcal{O}_{\text{KKS}})/(\text{U}(n) \times \text{U}(n)), \tag{1.3}$$

where  $\mathcal{O}_{\text{KKS}}$  (named after [28]) is a coadjoint orbit of  $\text{U}(n)$  of dimension  $2(n - 1)$  (that is, the smallest non-trivial coadjoint orbit). One can also perform the reduction in two consecutive steps, descending first to  $\Phi_L^{-1}(0)/\text{U}(n)$ , which is nothing but the cotangent bundle of the symmetric space. In this construction it is useful to trivialize  $T^*G$ , say by right translations, and also identify  $\mathcal{G} := \mathfrak{gl}(n, \mathbb{C})$  with its own dual space by means of the pairing given by the real part of the trace form, see (2.1). Thus  $T^*G$  gets identified with the manifold

$$\mathfrak{M} := \{(g, J) | g \in G, J \in \mathcal{G}\}, \tag{1.4}$$

and  $\mathcal{H}_{\text{Suth}}$  (1.1) arises from the unreduced Hamiltonian

$$H(g, J) = \frac{1}{2} \langle J, J \rangle. \tag{1.5}$$

Of course, the role of the two  $\text{U}(n)$  factors can be exchanged in (1.3).

A generalization [23] of the construction just outlined is to consider the reduction to

$$\Phi_L^{-1}(0)/(\text{U}(n) \times \text{U}(n)). \tag{1.6}$$

After restriction to a dense open subset, this gives rise to the spin Sutherland model with Hamiltonian

$$\mathcal{H}_{\text{spin-1}} = \frac{1}{2} \sum_{i=1}^n p_i^2 + \sum_{1 \leq i < j \leq n} \frac{|\xi_{ij}|^2}{\sinh^2(q_i - q_j)}, \tag{1.7}$$

where the ‘spin variable’  $\xi \in \mathfrak{u}(n)^* \simeq \mathfrak{u}(n)$  has zero diagonal part. The spin  $\xi$  can be restricted to a coadjoint orbit by imposing a corresponding constraint on  $\Phi_R$ , as was done in [23], but this will not be advantageous for our present purpose.

The most general reduction of  $\mathfrak{M}$  based on the group  $\text{U}(n) \times \text{U}(n)$  consists in descending to the quotient

$$\mathfrak{M}/(\text{U}(n) \times \text{U}(n)), \tag{1.8}$$

which is naturally a Poisson space. Taking the quotient of a Poisson manifold by a group action, having the key property that the Poisson bracket closes on the invariant functions, is usually called Poisson reduction. The smooth functions on the quotient stem from the smooth invariant functions upstairs. In the paper [24] the closely related method of symplectic reduction was applied, which means that we considered

$$\Phi_L^{-1}(\mathcal{O}_L) \times \Phi_R^{-1}(\mathcal{O}_R)/(\text{U}(n) \times \text{U}(n)) \tag{1.9}$$

with two arbitrary coadjoint orbits of  $\text{U}(n)$ . By general principles [42], these quotients represent Poisson subspaces of the (singular) Poisson space (1.8). They are stable under the projections of the Hamiltonian flows of all the  $\text{U}(n) \times \text{U}(n)$  invariant functions.

The fact that the quotient space (1.8) is not a smooth manifold does not cause any serious difficulty. Still, it is convenient to focus on the dense open subset  $\mathfrak{M}^{\text{reg}} \subset \mathfrak{M}$ , where the singular value decomposition of  $g \in G$  has the form

$$g = Ae^q B^{-1} \quad \text{with} \quad q = \text{diag}(q_1, q_2, \dots, q_n), \quad q_i \in \mathbb{R}, \quad q_1 > q_2 > \dots > q_n, \tag{1.10}$$

and  $A, B \in U(n)$ . Note that  $q$  is uniquely determined by  $g$ , but  $A$  and  $B$  are unique only up to the transformations

$$(A, B) \mapsto (A\tau, B\tau), \quad \forall \tau \in \mathbb{T}^n, \tag{1.11}$$

with  $\mathbb{T}^n < U(n)$  denoting the diagonal subgroup. Introduce the space

$$\mathfrak{M}_0^{\text{reg}} := \{(e^q, \mathcal{J})\}, \tag{1.12}$$

where  $q$  is restricted as in (1.10) and  $\mathcal{J} \in \mathcal{G}$ . Then consider the mapping

$$(g, J) \mapsto (e^q, \mathcal{J}) \quad \text{with} \quad \mathcal{J} := A^{-1}JA \tag{1.13}$$

from  $\mathfrak{M}^{\text{reg}}$  onto  $\mathfrak{M}_0^{\text{reg}}$ . This induces a bijective mapping of the dense subset  $\mathfrak{M}^{\text{reg}}/(U(n) \times U(n))$  of the quotient (1.8) onto  $\mathfrak{M}_0^{\text{reg}}/\mathbb{T}^n$ . The quotient by  $\mathbb{T}^n$  expresses that  $\mathcal{J}$  matters up to the gauge transformations  $\mathcal{J} \mapsto \tau \mathcal{J} \tau^{-1}$ , as inherited from (1.11). According to earlier results (see equation (2.25) in [24], and see also [29, 45]), one can parameterize the matrix  $\mathcal{J}$  in terms of two  $u(n)^*$ -valued spin variables  $\xi^l$  and  $\xi^r$  as follows:

$$\mathcal{J}_{ij} = p_i \delta_{ij} - (1 - \delta_{ij}) (\coth(q_i - q_j) \xi_{ij}^l + \xi_{ij}^r / \sinh(q_i - q_j)) - \xi_{ij}^l. \tag{1.14}$$

The spin variables Poisson commute with the canonical pairs  $q_i, p_i$ . We may decompose any  $n \times n$  complex matrix  $X$  into anti-Hermitian part  $X^+$  and Hermitian part  $X^-$ , and then we have  $\xi^l = -\mathcal{J}^+$  and  $\xi^r = (e^{-q} \mathcal{J} e^q)^+$ . The two spins are coupled by the constraint that the diagonal part of  $(\xi^l + \xi^r)$  vanishes, and the  $\mathbb{T}^n$  gauge transformations act on them by simultaneous conjugations. The reduction of the Hamiltonian (1.5) takes the form

$$\mathcal{H}_{\text{spin-2}} = \frac{1}{2} \sum_{i=1}^n (p_i^2 - |\xi_{ii}^l|^2) + \sum_{1 \leq i < j \leq n} \left( \frac{|\xi_{ij}^l|^2 + |\xi_{ij}^r|^2 - 2\Re(\xi_{ij}^r \xi_{ji}^l)}{\sinh^2(q_i - q_j)} + \frac{\Re(\xi_{ij}^r \xi_{ji}^l)}{\sinh^2((q_i - q_j)/2)} \right). \tag{1.15}$$

Upon setting  $\xi^l = 0$ ,  $\mathcal{J}$  simplifies to the Hermitian Lax matrix of the spin Sutherland model and  $\mathcal{H}_{\text{spin-2}}$  turns into the spin Sutherland Hamiltonian (1.7).

The essential new contribution of the present paper is this. We first exhibit an interesting bi-Hamiltonian structure on the cotangent bundle  $T^*G$ . This consists of a quadratic Poisson bracket (in terms of natural coordinate functions) and the canonical one. The quadratic Poisson bracket comes from the theory of Poisson–Lie groups [49], and its suitable Lie derivative is the canonical Poisson bracket. The Hamiltonians given by the real and imaginary parts of  $\text{tr}(J^k)$ , for  $k \in \mathbb{N}$ , generate a hierarchy of bi-Hamiltonian evolution equations on  $\mathfrak{M}$ . The Poisson reduction with respect to  $U(n) \times U(n)$  gives rise to a bi-Hamiltonian structure on the quotient space (1.8), which we will realize in the form of compatible Poisson brackets on the space of  $\mathbb{T}^n$  invariant functions  $C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ . These functions represent the ring of smooth functions on the regular part of the quotient (1.8). In practice, this means that we present the reduced Poisson brackets in terms of  $\mathbb{T}^n$  invariant functions of the pair  $(q, \mathcal{J})$ . By constraining the anti-Hermitian part of  $\mathcal{J}$  to zero, we reach a joint Poisson subspace of the compatible Poisson brackets, and on this subspace we recover the bi-Hamiltonian structure of the spin Sutherland model (1.7) that was derived originally by a rather ad hoc method [18]. In the general case,

we shall explain that in terms of the variables  $(q, p, \xi^l, \xi^r)$  the first reduced Poisson structure takes the form that underlies the interpretation of the Hamiltonian (1.15) as a Sutherland model coupled to two spins.

The article is organized as follows. In section 2, we describe the compatible Poisson brackets and a family of bi-Hamiltonian evolution equations on  $\mathfrak{M}$ . In section 3, we develop the reduction of these structures. Section 4 is devoted to the spin Sutherland interpretation of the reduced system. Our conclusions are given in section 5. There are also three appendices. Appendix A provides an explanation of the origin of the second Poisson bracket on  $\mathfrak{M}$ , appendix B expounds some technical details, and appendix C contains the proof of the key property of the change of variables behind the spin Sutherland interpretation.

Finally, let us state explicitly what we consider as our new results and specify their location in the text. First, we think that the unreduced bi-Hamiltonian hierarchy that we present in section 2 has not been described before. We studied the analogous holomorphic system in [20]. Our principal results are contained in section 3: theorems 3.4 and 3.5 give the reduced Poisson brackets in terms of  $\mathbb{T}^n$  invariant functions on  $\mathfrak{M}_0^{\text{reg}}$ , propositions 3.6 and 3.8 characterize the reduced bi-Hamiltonian hierarchy of evolution equations. In the general case, the interpretation as a spin Sutherland model with two spins is explained by lemma 4.6 and proposition 4.7, which were motivated by previous results [24]. Theorem 4.3 represents an important new result, since it clarifies the origin of the bi-Hamiltonian structure of the hyperbolic spin Sutherland model having the ‘main Hamiltonian’ (1.7).

**Remark 1.1.** To be more precise about the related literature [16, 24, 29, 45], we note that [16, 45] considered generalized spin Sutherland models on symplectic leaves of quotient spaces of the form  $K_1 \backslash T^*G / K_2$ , where  $K_1$  and  $K_2$  are the fixed point subgroups of two involutions of a semisimple or reductive Lie group  $G$ , while in [24, 29]  $K_1 = K_2$  was taken to be the maximal compact subgroup. Our construction of the bi-Hamiltonian master system on  $T^*G$  is specific to  $G = \text{GL}(n, \mathbb{C})$ , since it uses that  $\text{GL}(n, \mathbb{C})$  is an open subset of its Lie algebra, which contains the unit matrix. One may also apply a similar construction to  $\text{GL}(n, \mathbb{R})$ , but no other generalizations are apparent to us.

## 2. Master system on the cotangent bundle of $\text{GL}(n, \mathbb{C})$

We first fix some notations. Let us consider  $G := \text{GL}(n, \mathbb{C})$ , regarded as a *real* Lie group, and endow its Lie algebra  $\mathcal{G} := \mathfrak{gl}(n, \mathbb{C})$  with the trace form

$$\langle X, Y \rangle := \Re \text{tr}(XY), \quad \forall X, Y \in \mathcal{G}. \quad (2.1)$$

Any  $X \in \mathcal{G}$  admits the unique decomposition

$$X = X_{>} + X_0 + X_{<} \quad (2.2)$$

into strictly upper triangular part  $X_{>}$ , diagonal part  $X_0$ , and strictly lower triangular part  $X_{<}$ . Correspondingly, one obtains the vector space direct sum

$$\mathcal{G} = \mathcal{G}_{>} + \mathcal{G}_0 + \mathcal{G}_{<}. \quad (2.3)$$

This decomposition gives rise to the standard solution of the modified classical Yang–Baxter equation on  $\mathcal{G}$ ,  $r \in \text{End}(\mathcal{G})$ . Specifically, we define

$$r(X) := \frac{1}{2}(X_{>} - X_{<}), \quad \forall X \in \mathcal{G}, \quad (2.4)$$

and also introduce

$$r_{\pm} := r \pm \frac{1}{2} \text{id}, \tag{2.5}$$

where  $\text{id}(X) := X$ . Note that

$$\langle r(X), Y \rangle = -\langle X, r(Y) \rangle, \quad \forall X, Y \in \mathcal{G}. \tag{2.6}$$

Let  $\mathcal{G}^+$  denote the subalgebra of anti-Hermitian matrices and  $\mathcal{G}^-$  the subspace of Hermitian matrices,

$$\mathcal{G}^+ := \mathfrak{u}(n) = \{X \in \mathcal{G} \mid X^* = -X\}, \quad \mathcal{G}^- := \mathfrak{i}\mathfrak{u}(n). \tag{2.7}$$

They yield another direct sum decomposition

$$\mathcal{G} = \mathcal{G}^+ + \mathcal{G}^-, \tag{2.8}$$

where the two subspaces are orthogonal to each other with respect to the bilinear form (2.1). Thus we may decompose any  $X \in \mathcal{G}$  as  $X = X^+ + X^-$  with  $X^{\pm} \in \mathcal{G}^{\pm}$ . We shall use that

$$[\mathcal{G}^+, \mathcal{G}^+] \subset \mathcal{G}^+, \quad [\mathcal{G}^-, \mathcal{G}^-] \subset \mathcal{G}^+, \quad [\mathcal{G}^+, \mathcal{G}^-] \subset \mathcal{G}^-, \tag{2.9}$$

and, in consequence of (2.4),

$$r(\mathcal{G}^+) \subset \mathcal{G}^- \quad \text{and} \quad r(\mathcal{G}^-) \subset \mathcal{G}^+. \tag{2.10}$$

Our immediate aim is to present two Poisson structures on the smooth, real manifold

$$\mathfrak{M} := G \times \mathcal{G} = \{(g, J) \mid g \in G, J \in \mathcal{G}\}. \tag{2.11}$$

We view  $\mathfrak{M}$  as a model of the cotangent bundle of  $T^*G$ . Here, the cotangent bundle is trivialized by means of right translations, and  $\mathcal{G}^*$  is identified with  $\mathcal{G}$  using the trace form.

For any  $F \in C^\infty(\mathfrak{M}, \mathbb{R})$ , introduce the  $\mathcal{G}$ -valued derivatives  $\nabla_1 F$ ,  $\nabla'_1 F$  and  $d_2 F$  by the defining relations

$$\langle \nabla_1 F(g, J), X \rangle = \left. \frac{d}{dt} \right|_{t=0} F(e^{tX} g, J), \quad \langle \nabla'_1 F(g, J), X \rangle = \left. \frac{d}{dt} \right|_{t=0} F(g e^{tX}, J), \tag{2.12}$$

and

$$\langle d_2 F(g, J), X \rangle = \left. \frac{d}{dt} \right|_{t=0} F(g, J + tX), \tag{2.13}$$

where  $t$  is a real parameter and  $X \in \mathcal{G}$  is arbitrary. In addition, it will be convenient to define the  $\mathcal{G}$ -valued functions  $\nabla_2 F$  and  $\nabla'_2 F$  by

$$\nabla_2 F(g, J) := J(d_2 F(g, J)), \quad \nabla'_2 F(g, J) := (d_2 F(g, J))J. \tag{2.14}$$

It is worth remarking that  $\nabla_1 F(g, J) = g (\nabla'_1 F(g, J)) g^{-1}$ .

**Theorem 2.1.** For  $F, H \in C^\infty(\mathfrak{M}, \mathbb{R})$ , the following formulae define two Poisson brackets:

$$\{F, H\}_1(g, J) = \langle \nabla_1 F, d_2 H \rangle - \langle \nabla_1 H, d_2 F \rangle + \langle J, [d_2 F, d_2 H] \rangle, \tag{2.15}$$

and

$$\begin{aligned} \{F, H\}_2(g, J) &= \langle r\nabla_1 F, \nabla_1 H \rangle - \langle r\nabla'_1 F, \nabla'_1 H \rangle + \langle \nabla_2 F - \nabla'_2 F, r_+ \nabla'_2 H - r_- \nabla_2 H \rangle \\ &\quad + \langle \nabla_1 F, r_+ \nabla'_2 H - r_- \nabla_2 H \rangle - \langle \nabla_1 H, r_+ \nabla'_2 F - r_- \nabla_2 F \rangle, \end{aligned} \tag{2.16}$$

where the derivatives are evaluated at  $(g, J)$ , and we put  $rX$  for  $r(X)$ .

**Proof.** The first Poisson bracket is easily recognized to be the canonical one carried by the cotangent bundle  $T^*G$ , presented by means of right trivialization. The second one is the real analogue of the holomorphic Poisson bracket of  $\mathfrak{M}$ , then regarded as a complex manifold, which has been constructed in [20]. The formula (2.16) has the same form as the second Poisson bracket in [20], except that now we use smooth real functions and the real part of the trace for the pairing  $\langle, \rangle$ . We give a terse outline of the origin of this Poisson bracket in appendix A. Of course, its Jacobi identity can be verified directly as well.  $\square$

Let us recall [35, 36] that two Poisson brackets are called *compatible* if their arbitrary linear combination is also a Poisson bracket. A *bi-Hamiltonian manifold* is a manifold equipped with a pair of compatible Poisson brackets. The non-trivial condition is posed by the Jacobi identity for the linear combination of the Poisson brackets. In several examples it is guaranteed by the following mechanism. Suppose that  $\mathcal{D}$  is a derivation of the functions, and  $\{, \}$  is a Poisson bracket. Then the formula

$$\{F, H\}^{\mathcal{D}} := \mathcal{D}[\{F, H\}] - \{\mathcal{D}[F], H\} - \{F, \mathcal{D}[H]\}, \tag{2.17}$$

where  $\mathcal{D}[F]$  is the derivative of the function  $F$ , provides an anti-symmetric bi-derivation. This is called the Lie derivative bracket (since its underlying bi-vector is the Lie derivative of the Poisson tensor of  $\{, \}$  along the corresponding vector field). It is well-known [50] that if  $\{, \}^{\mathcal{D}}$  satisfies the Jacobi identity, then this holds also for arbitrary linear combinations of  $\{, \}$  and  $\{, \}^{\mathcal{D}}$ .

Now consider the vector field on  $\mathfrak{M}$  whose flow through  $(g, J)$  is

$$(g, J + t1_n), \quad t \in \mathbb{R}, \tag{2.18}$$

where  $1_n$  denotes the  $n \times n$  unit matrix, and let  $\mathcal{D}$  be the corresponding derivation of  $C^\infty(\mathfrak{M}, \mathbb{R})$ . As coordinate functions on  $\mathfrak{M}$ , we may employ ‘evaluation functions’ that on  $(g, J)$  return the real and imaginary parts of the matrix elements of  $g$ ,

$$g_{\alpha\beta}^r := \Re g_{\alpha\beta}, \quad g_{\alpha\beta}^i := \Im g_{\alpha\beta}, \tag{2.19}$$

and the components  $J_k := \langle T_k, J \rangle$  with respect to an arbitrary basis  $T_k$  ( $k = 1, \dots, 2n^2$ ) of the real vector space  $\mathcal{G}$ . Their derivatives are

$$\mathcal{D}[g_{\alpha\beta}] = 0, \quad \mathcal{D}[J_k] = \langle T_k, 1_n \rangle, \tag{2.20}$$

where we extended the derivation to complex functions.



**Proposition 2.2.** *The first Poisson bracket (2.15) on  $\mathfrak{M}$  is the Lie derivative of the second Poisson bracket (2.16) with respect to the derivation  $\mathcal{D}$ , i.e. we have*

$$\{F, H\}_1 = \{F, H\}_2^{\mathcal{D}}, \quad \forall F, H \in C^\infty(\mathfrak{M}, \mathbb{R}). \tag{2.21}$$

Consequently, the two Poisson brackets of theorem 2.1 are compatible.

**Proof.** Since both  $\{\cdot, \cdot\}_1$  and  $\{\cdot, \cdot\}_2^{\mathcal{D}}$  are bi-derivations, it is enough to verify the equality (2.21) for coordinate functions. Hence, we inspect the cases for which  $F$  and  $H$  are taken from the functions  $J_k$  and  $g_{\alpha\beta}^i, g_{\alpha\beta}^r$ . Since  $\mathcal{D}[g_{\alpha\beta}] = 0$  and the second Poisson bracket closes on the functions that depend only on  $g$ , we see that the equality (2.21) holds if both  $F$  and  $H$  depend only on  $g$ , for in these cases both sides of (2.21) vanish. Next, we note that

$$\{J_k, J_1\}_2 = \langle [J, T_k], r[T_1, J] + \frac{1}{2}(T_1 J + J T_1) \rangle, \tag{2.22}$$

and this leads to

$$\{J_k, J_1\}_2^{\mathcal{D}} = \langle J, [T_k, T_1] \rangle, \tag{2.23}$$

which equals  $\{J_k, J_1\}_1$ . To continue, we introduce the  $n \times n$  matrix  $\{g, J_k\}_a$ , for  $a = 1, 2$ , by defining its  $\alpha\beta$  entry to be

$$\{g_{\alpha\beta}^r, J_k\}_a + i\{g_{\alpha\beta}^i, J_k\}_a, \quad 1 \leq \alpha, \beta \leq n. \tag{2.24}$$

The formulae of theorem 2.1 give

$$\{g, J_k\}_1 = T_k g \quad \text{and} \quad \{g, J_k\}_2 = \left( r[T_k, J] + \frac{1}{2}(J T_k + T_k J) \right) g \tag{2.25}$$

From here it is easily checked that

$$\{g_{\alpha\beta}^r, J_k\}_1 + i\{g_{\alpha\beta}^i, J_k\}_1 = \{g_{\alpha\beta}^r, J_k\}_2^{\mathcal{D}} + i\{g_{\alpha\beta}^i, J_k\}_2^{\mathcal{D}}, \tag{2.26}$$

whereby the proof is complete. □

The evolution equation of a *bi-Hamiltonian system* can be written in Hamiltonian form with respect to two compatible, linearly independent, Poisson brackets, and respective Hamiltonians. We now present interesting bi-Hamiltonian systems on  $\mathfrak{M}$ .

**Proposition 2.3.** *For any  $k \in \mathbb{N}$ , define the Hamiltonians  $H_k$  and  $\tilde{H}_k$  on  $\mathfrak{M}$  by*

$$H_k(g, J) := \frac{1}{k} \Re \text{tr}(J^k), \quad \tilde{H}_k(g, J) := \frac{1}{k} \Im \text{tr}(J^k). \tag{2.27}$$

*All these Hamiltonians are in involution, and they define bi-Hamiltonian systems according to the relations, and corresponding flows, listed as follows:*

$$\{\cdot, H_k\}_2 = \{\cdot, H_{k+1}\}_1, \quad (g(t), J(t)) = (\exp(J(0)^k t) g(0), J(0)) \tag{2.28}$$

and

$$\{\cdot, \tilde{H}_k\}_2 = \{\cdot, \tilde{H}_{k+1}\}_1, \quad (g(t), J(t)) = (\exp(-iJ(0)^k t) g(0), J(0)). \tag{2.29}$$

**Proof.** We can calculate that

$$d_2 H_k = J^{k-1}, \quad d_2 \tilde{H}_k = -iJ^{k-1}. \tag{2.30}$$

Therefore  $(\nabla'_2 H_k - \nabla_2 H_k)$  and  $(\nabla'_2 \tilde{H}_k - \nabla_2 \tilde{H}_k)$  both vanish, and we obtain

$$\begin{aligned} r_+ \nabla'_2 H_k - r_- \nabla_2 H_k &= d_2 H_{k+1} = J^k, \\ r_+ \nabla'_2 \tilde{H}_k - r_- \nabla_2 \tilde{H}_k &= d_2 \tilde{H}_{k+1} = -iJ^k. \end{aligned} \tag{2.31}$$

Taking an arbitrary  $F \in C^\infty(\mathfrak{M}, \mathbb{R})$ , the formulae of theorem 2.1 then yield

$$\{F, H_k\}_2(g, J) = \{F, H_{k+1}\}_1(g, J) = \langle \nabla_1 F(g, J), J^k \rangle, \tag{2.32}$$

and

$$\{F, \tilde{H}_k\}_2(g, J) = \{F, \tilde{H}_{k+1}\}_1(g, J) = \langle \nabla_1 F(g, J), -iJ^k \rangle. \tag{2.33}$$

This means that the Poisson bracket relations (2.28) and (2.29) hold. It follows also immediately that the corresponding Hamiltonian flows through the initial value  $(g(0), J(0))$  have the simple form as claimed. In particular,  $J$  and all its functions are constant along these flows, which implies the involutivity of the underlying Hamiltonians.  $\square$

**Remark 2.4.** Let us define the  $\mathcal{G}$ -valued function  $\tilde{J}$  on  $\mathfrak{M}$  by

$$\tilde{J}(g, J) := -g^{-1}Jg, \tag{2.34}$$

and notice that  $\tilde{J}$  is constant along all the bi-Hamiltonian flows of proposition 2.3. The functions of  $J$  and  $\tilde{J}$  are related by the  $2n$  functional relations

$$\mathfrak{R}\text{tr}(J^k) = (-1)^k \mathfrak{R}\text{tr}(\tilde{J}^k), \quad \mathfrak{I}\text{tr}(J^k) = (-1)^k \mathfrak{I}\text{tr}(\tilde{J}^k), \quad \forall k = 1, \dots, n. \tag{2.35}$$

Thus the functional dimension of the ring of functions,  $\mathfrak{A}$ , depending on  $J$  and  $\tilde{J}$  equals  $4n^2 - 2n$ . This ring is closed under both Poisson brackets, since in addition to (2.22) one has  $\{J_k, \tilde{J}_1\}_i = 0$  ( $i = 1, 2$ ) and

$$\{\tilde{J}_k, \tilde{J}_1\}_2 = -\langle [\tilde{J}, T_k], r[T_1, \tilde{J}] + \frac{1}{2}(T_1 \tilde{J} + \tilde{J} T_1) \rangle. \tag{2.36}$$

The elements of  $\mathfrak{A}$  Poisson commute with the elements of the ring of functions,  $\mathfrak{F} \subset \mathfrak{A}$ , generated by the Hamiltonians (2.27). It is clear that the functional dimension of  $\mathfrak{F}$  is  $2n$ . Let us recall [39] that a *degenerate integrable system* on a symplectic manifold of dimension  $N$  is given by an abelian Poisson algebra of functional dimension  $N_1 < N/2$ , whose element admit  $N - N_1$  functionally independent joint constants of motion<sup>1</sup>. We can directly apply this definition to the first Poisson structure, which is symplectic. Therefore we see that the Hamiltonians (2.27) define a degenerate integrable system, at least with respect to the first Poisson bracket. The origin of the second Poisson bracket from Poisson–Lie groups (see appendix A) shows that it is also symplectic on a dense open subset of  $\mathfrak{M}$ , and therefore we may say that the Hamiltonians of the ‘bi-Hamiltonian hierarchy’ of proposition 2.3 form a degenerate integrable system with respect to both Poisson brackets.

<sup>1</sup> Degenerate integrable systems are also called *superintegrable* or *integrable in the non-commutative sense* [32, 37, 44]. Liouville integrability corresponds to the limiting case  $N_1 = N/2$ .

### 3. Poisson reduction with respect to $U(n) \times U(n)$

We are going to reduce with respect to an action of  $U(n) \times U(n)$ , which is engendered by two commuting actions of  $U(n)$ , called right-handed and left-handed actions. Regarding the first Poisson structure, it will be obvious that the Poisson bracket closes on the invariant functions. For the second Poisson bracket, this needs a proof, and we start by providing it.

We define the right-handed action of  $U(n)$  on  $\mathfrak{M}$  by associating with any  $\eta \in U(n)$  the diffeomorphism  $\mathcal{R}_\eta$  of  $\mathfrak{M}$  that acts according to

$$\mathcal{R}_\eta : (g, J) \mapsto (g\eta^{-1}, J). \tag{3.1}$$

The left-handed action is given by  $\mathcal{L}_\eta$ ,

$$\mathcal{L}_\eta : (g, J) \mapsto (\eta g, \eta J \eta^{-1}). \tag{3.2}$$

These are commuting  $U(n)$  actions on  $\mathfrak{M}$ , for we have

$$\begin{aligned} \mathcal{R}_{\eta_1} \circ \mathcal{R}_{\eta_2} &= \mathcal{R}_{\eta_1 \eta_2}, \\ \mathcal{L}_{\eta_1} \circ \mathcal{L}_{\eta_2} &= \mathcal{L}_{\eta_1 \eta_2}, \\ \mathcal{L}_{\eta_1} \circ \mathcal{R}_{\eta_2} &= \mathcal{R}_{\eta_2} \circ \mathcal{L}_{\eta_1}, \forall \eta_1, \eta_2 \in U(n). \end{aligned} \tag{3.3}$$

We shall consider the respective sets of invariant functions,

$$C^\infty(\mathfrak{M})_{\mathcal{R}}^{U(n)} := \{F \in C^\infty(\mathfrak{M}, \mathbb{R}) \mid F \circ \mathcal{R}_\eta = F, \quad \forall \eta \in U(n)\}, \tag{3.4}$$

$$C^\infty(\mathfrak{M})_{\mathcal{L}}^{U(n)} := \{F \in C^\infty(\mathfrak{M}, \mathbb{R}) \mid F \circ \mathcal{L}_\eta = F, \quad \forall \eta \in U(n)\}, \tag{3.5}$$

and

$$C^\infty(\mathfrak{M})^{U(n) \times U(n)} = C^\infty(\mathfrak{M})_{\mathcal{R}}^{U(n)} \cap C^\infty(\mathfrak{M})_{\mathcal{L}}^{U(n)}. \tag{3.6}$$

**Lemma 3.1.** For  $F, H \in C^\infty(\mathfrak{M})_{\mathcal{R}}^{U(n)}$ ,  $\{F, H\}_2$  (2.16) takes the form

$$\begin{aligned} \{F, H\}_2 &= \langle r\nabla_1 F, \nabla_1 H \rangle + \langle \nabla_2 F - \nabla_2' F, r_+ \nabla_2' H - r_- \nabla_2 H \rangle \\ &\quad + \langle \nabla_1 F, r_+ \nabla_2' H - r_- \nabla_2 H \rangle - \langle \nabla_1 H, r_+ \nabla_2' F - r_- \nabla_2 F \rangle, \end{aligned} \tag{3.7}$$

and it belongs to  $C^\infty(\mathfrak{M})_{\mathcal{R}}^{U(n)}$ .

**Proof.** If  $F \in C^\infty(\mathfrak{M})_{\mathcal{R}}^{U(n)}$ , then

$$F(ge^{tX}, J) = F(g, J), \quad \forall X \in \mathcal{G}^+, \tag{3.8}$$

and therefore

$$\nabla_1' F(g, J) \in \mathcal{G}^-. \tag{3.9}$$

On account of (2.10) and the orthogonality of the subspaces (2.8), we see that for right-invariant functions  $F, H$

$$\langle r\nabla_1' F, \nabla_1' H \rangle = 0. \tag{3.10}$$

Thus we obtain (3.7) from (2.16). The closure of the Poisson bracket on  $C^\infty(\mathfrak{M})_{\mathcal{R}}^{U(n)}$  is then a consequence of the relations

$$(\nabla_1 F) \circ \mathcal{R}_\eta = \nabla_1 F, \quad (\nabla_2 F) \circ \mathcal{R}_\eta = \nabla_2 F, \quad (\nabla'_2 F) \circ \mathcal{R}_\eta = \nabla'_2 F, \tag{3.11}$$

which follow directly from the definitions.  $\square$

**Lemma 3.2.** For  $F, H \in C^\infty(\mathfrak{M})_{\mathcal{L}}^{U(n)}$ ,  $\{F, H\}_2$  (2.16) takes the form

$$\begin{aligned} \{F, H\}_2 &= \langle \nabla'_1 F, r\nabla'_1 H \rangle + \frac{1}{2} \langle \nabla_2 F, \nabla'_2 H \rangle - \frac{1}{2} \langle \nabla_2 H, \nabla'_2 F \rangle \\ &\quad + \frac{1}{2} \langle \nabla_1 F, \nabla'_2 H + \nabla_2 H \rangle - \frac{1}{2} \langle \nabla_1 H, \nabla'_2 F + \nabla_2 F \rangle, \end{aligned} \tag{3.12}$$

and it belongs to  $C^\infty(\mathfrak{M})_{\mathcal{L}}^{U(n)}$ .

**Proof.** The function  $F$  (and similarly  $H$ ) enjoys the invariance property

$$F(e^{tX}g, e^{tX}Je^{-tX}) = F(g, J), \quad \forall X \in \mathcal{G}^+. \tag{3.13}$$

Taking the derivative at  $t = 0$  we obtain

$$(\nabla_1 F)^+ = (\nabla'_2 F - \nabla_2 F)^+. \tag{3.14}$$

Here and below, we apply the decomposition  $Y = Y^+ + Y^-$  for any  $Y \in \mathcal{G}$ , as defined by the direct sum (2.8). We now look at the terms that appear in the formula (2.16). By using that  $r$  maps  $\mathcal{G}^\pm$  into  $\mathcal{G}^\mp$  (2.10) and taking (3.14) into account, we derive the identity

$$\langle r\nabla_1 F, \nabla_1 H \rangle = \langle r(\nabla_1 H)^-, (\nabla_2 F - \nabla'_2 F)^+ \rangle - \langle r(\nabla_1 F)^-, (\nabla_2 H - \nabla'_2 H)^+ \rangle. \tag{3.15}$$

In a similar manner, we obtain

$$\begin{aligned} &\langle \nabla_1 F, r\nabla'_2 H - r\nabla_2 H \rangle - \langle \nabla_1 H, r\nabla'_2 F - r\nabla_2 F \rangle \\ &= \langle r(\nabla_1 F)^-, (\nabla_2 H - \nabla'_2 H)^+ \rangle - \langle r(\nabla_1 H)^-, (\nabla_2 F - \nabla'_2 F)^+ \rangle \\ &\quad + \langle r(\nabla'_2 F - \nabla_2 F)^+, (\nabla_2 H - \nabla'_2 H)^- \rangle \\ &\quad - \langle r(\nabla'_2 H - \nabla_2 H)^+, (\nabla_2 F - \nabla'_2 F)^- \rangle. \end{aligned} \tag{3.16}$$

Next, we can write

$$\begin{aligned} \langle \nabla_2 F - \nabla'_2 F, r\nabla'_2 H - r\nabla_2 H \rangle &= \langle r(\nabla'_2 F - \nabla_2 F)^+, (\nabla'_2 H - \nabla_2 H)^- \rangle \\ &\quad - \langle r(\nabla'_2 H - \nabla_2 H)^+, (\nabla'_2 F - \nabla_2 F)^- \rangle. \end{aligned} \tag{3.17}$$

In these derivations we used the anti-symmetry of  $r$  (2.6). Observe that the terms given by the last three equations cancel altogether. In other words, the terms that contain  $r$  in  $r_\pm$  (2.5) all cancel from (2.16). Noticing the elementary identity

$$\langle \nabla_2 F - \nabla'_2 F, \nabla'_2 H + \nabla_2 H \rangle = \langle \nabla_2 F, \nabla'_2 H \rangle - \langle \nabla'_2 F, \nabla_2 H \rangle, \tag{3.18}$$

we then obtain (3.12) from (2.16). The closure of the Poisson bracket on the left  $U(n)$  invariant functions follows from (3.12) and the transformation rules of the derivatives,

$$\begin{aligned} (\nabla_1 F) \circ \mathcal{L}_\eta &= \eta(\nabla_1 F)\eta^{-1}, \\ (\nabla'_1 F) \circ \mathcal{L}_\eta &= \nabla'_1 F, \\ (\nabla_2 F) \circ \mathcal{L}_\eta &= \eta(\nabla_2 F)\eta^{-1}, \\ (\nabla'_2 F) \circ \mathcal{L}_\eta &= \eta(\nabla'_2 F)\eta^{-1}, \end{aligned} \tag{3.19}$$

which hold for all  $\eta \in U(n)$ . □

We have seen that the  $U(n)$  invariant functions form Poisson subalgebras with respect to the second Poisson bracket (2.16). Of course, the same is true regarding the first Poisson bracket (2.15). This can be seen directly from (2.15), and also follows from well known results about cotangent lifts of actions on a configuration space.

According to the singular value decomposition, also called Cartan (KAK) decomposition, every element  $g \in GL(n, \mathbb{C})$  can be decomposed as

$$\begin{aligned} g &= \eta_L e^q \eta_R^{-1}, \quad \eta_L, \eta_R \in U(n), \\ q &= \text{diag}(q_1, q_2, \dots, q_n), \quad q_i \in \mathbb{R}, \quad q_1 \geq q_2 \geq \dots \geq q_n. \end{aligned} \tag{3.20}$$

Here,  $q$  is uniquely determined by  $g$ , and if  $q_1 > q_2 > \dots > q_n$  then  $\eta_L$  and  $\eta_R$  are unique up to the freedom

$$(\eta_L, \eta_R) \mapsto (\eta_L \tau, \eta_R \tau), \quad \forall \tau \in \mathbb{T}^n, \tag{3.21}$$

with the maximal torus  $\mathbb{T}^n < U(n)$ . In this paper, we call  $GL(n, \mathbb{C})^{\text{reg}}$  the dense open subset of  $GL(n, \mathbb{C})$  whose elements obey the strict inequalities, and we also introduce

$$\mathfrak{M}^{\text{reg}} := GL(n, \mathbb{C})^{\text{reg}} \times \mathcal{G}. \tag{3.22}$$

It is easily seen that every invariant function  $F \in C^\infty(\mathfrak{M})^{U(n) \times U(n)}$  can be recovered from its restriction to the following submanifold of  $\mathfrak{M}$ :

$$\mathfrak{M}_0^{\text{reg}} := \{(e^q, J) \mid J \in \mathcal{G}, \quad q = \text{diag}(q_1, q_2, \dots, q_n), \quad q_1 > q_2 > \dots > q_n\}. \tag{3.23}$$

With the tautological embedding  $\iota : \mathfrak{M}_0^{\text{reg}} \rightarrow \mathfrak{M}$ , the restriction of  $F$  reads

$$f = F \circ \iota. \tag{3.24}$$

Obviously,  $f$  is invariant under the  $\mathbb{T}^n$  action on  $\mathfrak{M}_0^{\text{reg}}$  given by the diffeomorphisms  $A_\tau$ ,

$$A_\tau : (e^q, J) \mapsto (e^q, \tau J \tau^{-1}), \quad \forall \tau \in \mathbb{T}^n. \tag{3.25}$$

That is to say,  $f = F \circ \iota$  belongs to  $C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ . Next, we introduce the reduced ring of functions,

$$C^\infty(\mathfrak{M})_{\text{red}} := \{f \in C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n} \mid f = F \circ \iota, \quad F \in C^\infty(\mathfrak{M})^{U(n) \times U(n)}\}. \tag{3.26}$$

This space of functions naturally inherits a pair of compatible Poisson brackets, called *reduced Poisson brackets*, which are defined as follows:

$$\{F \circ \iota, H \circ \iota\}_i^{\text{red}} := \{F, H\}_i \circ \iota, \quad \forall F, H \in C^\infty(\mathfrak{M})^{\text{U}(n) \times \text{U}(n)}, \quad i = 1, 2. \tag{3.27}$$

The reduced Poisson brackets are well-defined, since the original Poisson brackets close on the  $\text{U}(n) \times \text{U}(n)$  invariant functions. They contain all information about the Poisson algebra carried by the (singular) quotient space

$$\mathfrak{M}_{\text{red}} := \mathfrak{M}/(\text{U}(n) \times \text{U}(n)), \tag{3.28}$$

whose space of smooth functions is  $C^\infty(\mathfrak{M})^{\text{U}(n) \times \text{U}(n)}$ .

Now our goal is to establish intrinsic formulae of the reduced Poisson brackets, which contain only derivatives with respect to the variables on  $\mathfrak{M}_0^{\text{reg}}$ . To this end, we need to express the derivatives of  $F$  at  $(e^q, J) \in \mathfrak{M}_0^{\text{reg}}$  in terms of the derivatives of the restricted function  $f = F \circ \iota$ . We shall use the decompositions

$$\mathcal{G}^\pm = \mathcal{G}_0^\pm + \mathcal{G}_\perp^\pm, \tag{3.29}$$

where  $\mathcal{G}_0^\pm \subset \mathcal{G}^\pm$  contain the respective diagonal matrices, and  $\mathcal{G}_\perp^\pm$  contain the off-diagonal ones. Thus we can write  $X^\pm = X_0^\pm + X_\perp^\pm$  for any  $X \in \mathcal{G}^\pm$ . Any function  $f \in C^\infty(\mathfrak{M}_0^{\text{reg}})$  has the  $\mathcal{G}_0^-$ -valued derivative  $\nabla_1 f$  and the  $\mathcal{G}$ -valued derivative  $d_2 f$ , determined by

$$\langle \nabla_1 f(e^q, J), X_0 \rangle = \left. \frac{d}{dt} \right|_{t=0} f(e^{tX_0} e^q, J) = \left. \frac{d}{dt} \right|_{t=0} f(e^{q+tX_0}, J), \quad \forall X_0 \in \mathcal{G}_0^-, \tag{3.30}$$

$$\langle d_2 f(e^q, J), X \rangle = \left. \frac{d}{dt} \right|_{t=0} f(e^q, J + tX), \quad \forall X \in \mathcal{G}, \tag{3.31}$$

and we define  $\nabla_2 f$  and  $\nabla_2' f$  similarly to (2.14). The definition (3.30) makes sense since for small enough  $t$  the components of  $(q + tX_0)$  satisfy the same ordering condition as those of  $q$ . Plainly, we have

$$d_2 F(e^q, J) = d_2 f(e^q, J), \tag{3.32}$$

and as a result of the  $\mathbb{T}^n$  invariance

$$[J, d_2 f(e^q, J)]_0^+ = 0. \tag{3.33}$$

Let us introduce the linear operator  $R(q) \in \text{End}(\mathcal{G})$  by letting it act on an arbitrary matrix  $X \in \mathcal{G}$  according to

$$(R(q)X)_{ii} = 0, \quad (R(q)X)_{ij} = X_{ij} \coth(q_i - q_j), \quad 1 \leq i \neq j \leq n. \tag{3.34}$$

Notice that  $R(q)$  maps  $\mathcal{G}_\perp^\pm$  onto  $\mathcal{G}_\perp^\mp$  (3.29), respectively, in an invertible manner, and it satisfies

$$\langle R(q)X, Y \rangle = -\langle X, R(q)Y \rangle, \quad \forall X, Y \in \mathcal{G}. \tag{3.35}$$

Due to the following result,  $R(q)$  will appear in the expressions of the reduced Poisson brackets.

**Lemma 3.3.** *If  $F \in C^\infty(\mathfrak{M})^{\mathbb{U}(n) \times \mathbb{U}(n)}$  and  $f = F \circ \iota$ , then we have*

$$\nabla_1 F(e^q, J) = [d_2 f(e^q, J), J]^+ + \nabla_1 f(e^q, J) + R(q)[d_2 f(e^q, J), J]^+, \quad (3.36)$$

where the superscripts refer to the decomposition (2.7).

**Proof.** We know from (3.9) that  $(\nabla'_1 F)^+ = 0$  holds, because  $F \in C^\infty(\mathfrak{M})_{\mathcal{R}}^{\mathbb{U}(n)}$ . On the other hand,  $(\nabla_1 F)^+ = (\nabla'_2 F - \nabla_2 F)^+$  holds (3.14), because  $F \in C^\infty(\mathfrak{M})_{\mathcal{L}}^{\mathbb{U}(n)}$ . By using that

$$\nabla'_1 F(e^q, J) = e^{-q}(\nabla_1 F(e^q, J))e^q, \quad (3.37)$$

we can write

$$0 = (\nabla'_1 F)^+ = (\cosh \operatorname{ad}_q)((\nabla_1 F)^+) - (\sinh \operatorname{ad}_q)((\nabla_1 F)^-), \quad (3.38)$$

at any  $(e^q, J)$ , where  $\operatorname{ad}_q(X) := [q, X]$ . This implies that

$$(\nabla_1 F)^-_{\perp} = R(q)(\nabla_1 F)^+. \quad (3.39)$$

Finally, since

$$\left. \frac{d}{dt} \right|_{t=0} F(e^{tX_0} e^q, J) = \left. \frac{d}{dt} \right|_{t=0} f(e^{tX_0} e^q, J), \quad \forall X_0 \in \mathcal{G}_0^-, \quad (3.40)$$

we get

$$(\nabla_1 F(e^q, J))_0^- = \nabla_1 f(e^q, J). \quad (3.41)$$

The proof is completed by noting that

$$\nabla'_2 F - \nabla_2 F = \nabla'_2 f - \nabla_2 f = [d_2 f, J] \quad \text{at any } (e^q, J) \in \mathfrak{M}_0^{\text{red}}. \quad (3.42)$$

□

**Theorem 3.4.** *For  $f, h \in C^\infty(\mathfrak{M})_{\text{red}}$  (3.26), the first reduced Poisson bracket (3.27) is given by*

$$\begin{aligned} \{f, h\}_1^{\text{red}}(e^q, J) &= \langle \nabla_1 f, (d_2 h)_0^- \rangle - \langle \nabla_1 h, (d_2 f)_0^- \rangle \\ &\quad + \langle R(q)[d_2 f, J]^+, (d_2 h)^- \rangle - \langle R(q)[d_2 h, J]^+, (d_2 f)^- \rangle \\ &\quad + \langle J^+, [(d_2 f)^-, (d_2 h)^-] - [(d_2 f)^+, (d_2 h)^+] \rangle, \end{aligned} \quad (3.43)$$

where the derivatives are taken at  $(e^q, J)$  and we use  $R(q)$  (3.34). If  $h$  is the restriction of any of the Hamiltonians  $H_k$  or  $\tilde{H}_k$  (2.27), then this formula can be written as

$$\{f, h\}_1^{\text{red}}(e^q, J) = \langle \nabla_1 f, (d_2 h)_0^- \rangle + \langle d_2 f, [R(q)(d_2 h)^- - (d_2 h)^+, J] \rangle. \quad (3.44)$$

**Proof.** The first and second lines of (3.43) represent the contributions of the second and third terms of (3.36) obtained upon substitution in the formula (2.15). Regarding the third line, it arises from the first term of (3.36) by taking the sum of

$$\begin{aligned} \langle [d_2 f, J]^+, d_2 h \rangle - \langle [d_2 h, J]^+, d_2 f \rangle &= -2 \langle J^+, [(d_2 f)^+, (d_2 h)^+] \rangle \\ &\quad - \langle J^-, [d_2 f, d_2 h]^- \rangle \end{aligned} \quad (3.45)$$

and

$$\langle J, [d_2f, d_2h] \rangle = \langle J^+, [d_2f, d_2f]^+ \rangle + \langle J^-, [d_2f, d_2h]^- \rangle. \tag{3.46}$$

To get (3.44) from (3.43), we use that for any Hamiltonian  $h$  proportional with the real or imaginary parts of  $\text{tr}(J^k)$  one has  $[d_2h, J] = 0$ . In this case, the second line of (3.43) gives

$$\langle R(q)[d_2f, J]^+, (d_2h)^- \rangle = \langle R(q)[d_2f, J], (d_2h)^- \rangle = \langle d_2f, [R(q)(d_2h)^-, J] \rangle, \tag{3.47}$$

and the third line becomes

$$\begin{aligned} & \langle J^+, [(d_2f)^-, (d_2h)^-] - [(d_2f)^+, (d_2h)^+] \rangle \\ &= \langle J, [(d_2f)^-, (d_2h)^-] - [(d_2f)^+, (d_2h)^+] \rangle \\ &= \langle J, [(d_2h)^+, (d_2f)^-] + [(d_2h)^+, (d_2f)^+] \rangle = -\langle d_2f, [(d_2h)^+, J] \rangle, \end{aligned} \tag{3.48}$$

which confirm the claim (3.44). □

**Theorem 3.5.** For  $f, h \in C^\infty(\mathfrak{M})_{\text{red}}$  (3.26), the second reduced Poisson bracket (3.27) can be written as

$$\begin{aligned} 2\{f, h\}_2^{\text{red}}(e^q, J) &= \langle \nabla_1 f, (\nabla_2 h + \nabla_2' h)_0^- \rangle - \langle \nabla_1 h, (\nabla_2 f + \nabla_2' f)_0^- \rangle \\ &+ \langle R(q)[d_2f, J]^+, (\nabla_2 h + \nabla_2' h)^- \rangle - \langle R(q)[d_2h, J]^+, (\nabla_2 f + \nabla_2' f)^- \rangle \\ &+ \langle (\nabla_2 f)^-, (\nabla_2' h)^- \rangle + \langle (\nabla_2' f)^+, (\nabla_2 h)^+ \rangle - \langle (\nabla_2' f)^-, (\nabla_2 h)^- \rangle \\ &- \langle (\nabla_2 f)^+, (\nabla_2' h)^+ \rangle, \end{aligned} \tag{3.49}$$

where the derivatives are taken at  $(e^q, J)$  and  $R(q)$  is given by (3.34). If  $h$  is the restriction of any of the Hamiltonians  $H_k$  or  $\tilde{H}_k$  (2.27), then this formula can be recast in the form

$$\{f, h\}_1^{\text{red}}(e^q, J) = \langle \nabla_1 f, (\nabla_2 h)_0^- \rangle + \langle d_2f, [R(q)(\nabla_2 h)^- - (\nabla_2 h)^+, J] \rangle. \tag{3.50}$$

**Proof.** We see by combining (3.10) (obtained in the proof of lemma 3.1) and lemma 3.2 that for  $F, H \in C^\infty(\mathfrak{M})^{\text{U}(n) \times \text{U}(n)}$

$$\begin{aligned} 2\{F, H\}_2 &= \langle \nabla_2 F, \nabla_2' H \rangle - \langle \nabla_2 H, \nabla_2' F \rangle + \langle \nabla_1 F, \nabla_2' H + \nabla_2 H \rangle \\ &- \langle \nabla_1 H, \nabla_2' F + \nabla_2 F \rangle. \end{aligned} \tag{3.51}$$

Putting  $f = F \circ \iota$  and  $h = H \circ \iota$ , we substitute the identity (3.36) both for  $\nabla_1 F$  and  $\nabla_1 H$ . After that, we spell out the first two terms of (3.51) and the contributions coming from the first terms of  $\nabla_1 F$  and  $\nabla_1 H$ . The formula (3.49) is then obtained by collecting terms. Turning to the proof of (3.50), instead of (3.49) it is shorter to go back to (3.51). We observe from (2.30) that in this case  $[d_2H, J] = 0$ . Thus  $\nabla_2 H = \nabla_2' H$  and

$$\langle \nabla_2 F, \nabla_2' H \rangle - \langle \nabla_2 H, \nabla_2' F \rangle = \langle d_2F, [d_2H, J^2] \rangle = 0. \tag{3.52}$$

Then

$$\{f, h\}_2^{\text{red}}(e^q, J) = \langle \nabla_1 F(e^q, J), \nabla_2 H(e^q, J) \rangle$$



$$= \langle [d_2f(e^q, J), J]^+ + \nabla_1 f(e^q, J) + R(q)[d_2f(e^q, J), J]^+, \nabla_2 h(e^q, J) \rangle. \tag{3.53}$$

The verification is finished by noting that

$$\langle R(q)[d_2f, J]^+, \nabla_2 h \rangle = -\langle [d_2f, J], R(q)(\nabla_2 h)^- \rangle = \langle d_2f, [R(q)(\nabla_2 h)^-, J] \rangle, \tag{3.54}$$

and

$$\langle [d_2f, J]^+, \nabla_2 h \rangle = \langle d_2f, [J, (\nabla_2 h)^+] \rangle. \tag{3.55}$$

□

The following statement summarizes the outcome of our construction.

**Proposition 3.6.** *The formulae (3.43) and (3.49) yield two compatible Poisson brackets on  $C^\infty(\mathfrak{M})_{\text{red}}$  (3.26). The commuting Hamiltonians*

$$h_k := \frac{1}{k} \mathfrak{R} \text{tr}(J^k) \quad \text{and} \quad \tilde{h}_k := \frac{1}{k} \mathfrak{I} \text{tr}(J^k), \quad k \in \mathbb{N}, \tag{3.56}$$

give rise to bi-Hamiltonian evolution equations since they satisfy

$$\begin{aligned} \{f, h_k\}_2^{\text{red}} &= \{f, h_{k+1}\}_1^{\text{red}} \quad \text{and} \\ \{f, \tilde{h}_k\}_2^{\text{red}} &= \{f, \tilde{h}_{k+1}\}_1^{\text{red}}, \\ \forall f &\in C^\infty(\mathfrak{M})_{\text{red}}, \quad k \in \mathbb{N}. \end{aligned} \tag{3.57}$$

**Proof.** This is obvious from our construction. In particular, the compatibility of the reduced Poisson brackets follows from the compatibility of the original Poisson brackets on  $C^\infty(\mathfrak{M}, \mathbb{R})$  by applying the definition (3.27). The properties (3.57) are consequences of (3.44) and (3.50) taking into account that

$$\nabla_2 h_k = J^k = d_2 h_{k+1} \quad \text{and} \quad \nabla_2 \tilde{h}_k = -iJ^k = d_2 \tilde{h}_{k+1}, \tag{3.58}$$

which are implied by (2.30). □

In the next remark, we explain that the compatible Poisson brackets can be defined also on the ring of functions  $C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ .

**Remark 3.7.** The ring of functions  $C^\infty(\mathfrak{M})_{\text{red}}$  (3.26) is contained in  $C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ , but is not equal to it. For example, the components of  $q$  give elements of  $C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ , and can be extended to unique,  $U(n) \times U(n)$  invariant continuous functions on  $\mathfrak{M}$ , but these functions lose their differentiability at the locus where  $q_i = q_{i+1}$  for some  $i$ . This holds since the  $e^{2q_i}$  (3.20) are the ordered eigenvalues of  $gg^\dagger$  and, as is well known, the differentiability of eigenvalues is in general lost where they coincide. Nevertheless, the formulae (3.43) and (3.49) define compatible Poisson brackets on the whole of  $C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ . In order to see this, consider  $\mathfrak{M}^{\text{red}}$  (3.22), which is the set of elements of  $\mathfrak{M}$  that can be transformed into  $\mathfrak{M}_0^{\text{reg}}$  by the action of  $U(n) \times U(n)$ . This is a dense open subset and the pull-back by  $\iota : \mathfrak{M}_0^{\text{reg}} \rightarrow \mathfrak{M}^{\text{reg}}$  yields an injective and surjective map from  $C^\infty(\mathfrak{M}^{\text{reg}})^{U(n) \times U(n)}$  onto  $C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ , that is,

$$C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n} = \{f \in C^\infty(\mathfrak{M}_0^{\text{reg}}) \mid f = F \circ \iota, \quad F \in C^\infty(\mathfrak{M}^{\text{reg}})^{U(n) \times U(n)}\}. \tag{3.59}$$

Then the application of (3.27) to  $F, H \in C^\infty(\mathfrak{M}_0^{\text{reg}})^{\text{U}(n) \times \text{U}(n)}$  gives rise to compatible Poisson brackets on  $C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ . They are described by the formulae (3.43) and (3.49) for any  $f, h \in C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ , since they are determined by calculations identical to those presented above.

For a vector field  $\mathcal{E}$  on  $\mathfrak{M}_0^{\text{reg}}$ , we denote the derivative of  $f \in C^\infty(\mathfrak{M}_0^{\text{reg}})$  by  $\mathcal{E}[f]$ . The vector field  $\mathcal{E}$  is encoded by the matrix valued functions  $\mathcal{E}[q]$  and  $\mathcal{E}[J]$ , i.e. by the derivatives of  $q$  and  $J$  regarded as evaluation functions that return  $q$  and  $J$  when applied to  $(q, J) \in \mathfrak{M}_0^{\text{reg}}$ . Then the chain rule reads

$$\mathcal{E}[f] = \langle \nabla_1 f, \mathcal{E}[q] \rangle + \langle d_2 f, \mathcal{E}[J] \rangle. \tag{3.60}$$

For any fixed  $h \in C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ , the two Poisson brackets with  $h$  determine two derivations of  $C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ . These correspond to vector fields  $\mathcal{E}_h^i$  ( $i = 1, 2$ ) that are unique only up to the addition of infinitesimal gauge transformations. The term *infinitesimal gauge transformation* refers to any vector field  $\mathcal{Z}$  for which

$$\mathcal{Z}[q] = 0 \quad \text{and} \quad \mathcal{Z}[J] = [\mathcal{T}, J] \tag{3.61}$$

with some function  $\mathcal{T} : \mathfrak{M}_0^{\text{reg}} \rightarrow \mathcal{G}_0^+$ . Note that  $\mathcal{G}_0^+$  is just the Lie algebra of  $\mathbb{T}^n$ . This ambiguity drops out after projection to the quotient space  $\mathfrak{M}_0^{\text{reg}}/\mathbb{T}^n$ . For definiteness, we shall fix this ambiguity of the vector field  $\mathcal{E}_h^i$  by imposing the condition

$$\mathcal{E}_h^i[f] = \{f, h\}_i^{\text{red}}, \quad \forall f \in C^\infty(\mathfrak{M}_0^{\text{reg}}), \tag{3.62}$$

where  $\{f, h\}_i^{\text{red}}$  is understood to be given by the formulae (3.43) and (3.49). These formulae define anti-symmetric bi-derivations on  $C^\infty(\mathfrak{M}_0^{\text{reg}})$ , but the Jacobi identity holds only for the  $\mathbb{T}^n$  invariant functions. By some abuse of terminology, we call the vector field  $\mathcal{E}_h^i$  the *Hamiltonian vector field* associated with  $h$  by means of the bracket  $\{, \}_i^{\text{red}}$ .

**Proposition 3.8.** *Consider the vector fields  $\mathcal{E}_k^i$  and  $\tilde{\mathcal{E}}_k^i$  defined by*

$$\mathcal{E}_k^i[f] = \{f, h_k\}_i^{\text{red}} \quad \text{and} \quad \tilde{\mathcal{E}}_k^i[f] = \{f, \tilde{h}_k\}_i^{\text{red}}, \quad k \in \mathbb{N}, \quad i = 1, 2, \tag{3.63}$$

with the Hamiltonians (3.56). Using  $R(q)$  (3.34), these vector fields have the explicit form

$$\begin{aligned} \mathcal{E}_k^2[q_j] &= \mathcal{E}_{k+1}^1[q_j] = \Re(J^k)_{jj}, \\ \mathcal{E}_k^2[J] &= \mathcal{E}_{k+1}^1[J] = \frac{1}{2}[R(q)(J^k + (J^k)^*) + ((J^k)^* - J^k), J], \end{aligned} \tag{3.64}$$

and

$$\begin{aligned} \tilde{\mathcal{E}}_k^2[q_j] &= \tilde{\mathcal{E}}_{k+1}^1[q_j] = \Re(-iJ^k)_{jj}, \\ \tilde{\mathcal{E}}_k^2[J] &= \tilde{\mathcal{E}}_{k+1}^1[J] = \frac{1}{2}[iR(q)((J^k)^* - J^k) + i(J^k + (J^k)^*), J]. \end{aligned} \tag{3.65}$$

All these vector fields are tangent to the submanifold

$$\mathfrak{M}_{0,-}^{\text{reg}} = \{(q, J^-) \in \mathfrak{M}_0^{\text{reg}} \mid J^- \in \mathcal{G}^-\} \subset \mathfrak{M}_0^{\text{reg}} \tag{3.66}$$

defined by imposing the constraint  $J^+ = 0$ , as well as to the submanifold

$$\mathfrak{M}_{0,+}^{\text{reg}} = \{(q, J^+) \in \mathfrak{M}_0^{\text{reg}} \mid J^+ \in \mathcal{G}^+\} \subset \mathfrak{M}_0^{\text{reg}} \tag{3.67}$$

defined by imposing the constraint  $J^- = 0$ . The restriction  $\mathcal{V}_k^i$  of  $\mathcal{E}_k^i$  on  $\mathfrak{M}_{0,-}^{\text{reg}}$  gives

$$\mathcal{V}_k^2[q_j] = \mathcal{V}_{k+1}^1[q_j] = (J^-)_{jj}^k, \quad \mathcal{V}_k^2[J^-] = \mathcal{V}_{k+1}^1[J^-] = [R(q)(J^-)^k, J^-], \tag{3.68}$$

while the restriction of  $\tilde{\mathcal{E}}_k^i$  vanishes identically on  $\mathfrak{M}_{0,-}^{\text{reg}}$  for all  $k$  and  $i$ . The vector fields  $\mathcal{V}_k^i$  reproduce the evolutionary vector fields of the spin Sutherland hierarchy described in [18].

Denoting the restrictions of  $\mathcal{E}_k^i$  and  $\tilde{\mathcal{E}}_k^i$  on  $\mathfrak{M}_{0,+}^{\text{reg}}$  by  $\mathcal{U}_k^i$  and  $\tilde{\mathcal{U}}_k^i$ , we obtain, for all  $l \in \mathbb{N}$ ,

$$\mathcal{U}_{2l-1}^2 = \mathcal{U}_{2l}^1 = 0, \quad \tilde{\mathcal{U}}_{2l}^2 = \tilde{\mathcal{U}}_{2l+1}^1 = 0, \tag{3.69}$$

and

$$\begin{aligned} \mathcal{U}_{2l}^2[q_j] &= \mathcal{U}_{2l+1}^1[q_j] = \Re((J^+)^{2l})_{jj}, \\ \mathcal{U}_{2l}^2[J^+] &= \mathcal{U}_{2l+1}^1[J^+] = [R(q)(J^+)^{2l}, J^+], \\ \tilde{\mathcal{U}}_{2l-1}^2[q_j] &= \tilde{\mathcal{U}}_{2l}^1[q_j] = \Re(-i(J^+)^{2l-1})_{jj}, \\ \tilde{\mathcal{U}}_{2l-1}^2[J^+] &= \tilde{\mathcal{U}}_{2l}^1[J^+] = [-iR(q)(J^+)^{2l-1}, J^+]. \end{aligned} \tag{3.70}$$

By making the substitution  $J^+ = iJ^-$ , the vector fields shown in (3.70) get transformed into those that appear in (3.68), up some irrelevant overall signs.

**Proof.** The formulae (3.64) and (3.65) follow directly by applying the formulae (3.44), (3.50) and (3.58). The tangency to  $\mathfrak{M}_{0,\pm}^{\text{reg}}$  is a consequence of the fact that  $[\mathcal{G}^+, \mathcal{G}^\pm] \subset \mathcal{G}^\pm$ . The statements about the form of the restricted vector fields are plain from (3.64) and (3.65). Comparison with equation (1.8) in [18] shows that the bi-Hamiltonian vector fields (3.68) reproduce the spin Sutherland hierarchy studied earlier.  $\square$

**Remark 3.9.** It is worth noting that the vector fields  $\mathcal{Z}$  (3.61) representing infinitesimal gauge transformations are tangent to the submanifolds  $\mathfrak{M}_{0,\pm}^{\text{reg}}$ . Therefore the possibility to restrict a Hamiltonian vector field from  $\mathfrak{M}_0^{\text{reg}}$  to  $\mathfrak{M}_{0,\pm}^{\text{reg}}$  is independent of the ambiguity of adding an infinitesimal gauge transformation.

#### 4. Interpretation as spin Sutherland models

We below develop the physical interpretation of the reduced bi-Hamiltonian system. This interpretation will be reached via a suitable parameterization of the variable  $J$  for  $(e^q, J) \in \mathfrak{M}_0^{\text{reg}}$ . In this parameterization the first reduced Poisson structure and one of the reduced Hamiltonians take the form characteristic of spin Sutherland models. We first present the particular case corresponding to the Hamiltonian (1.7) and deal with the general case (1.15) subsequently. We will not spell out the expression of the second Poisson bracket in terms of the new variables, since those formulae are complicated and do not enhance our understanding.

##### 4.1. Spin Sutherland model on the subspace $\mathfrak{M}_{0,-}^{\text{reg}} \subset \mathfrak{M}_0^{\text{reg}}$

We have seen in proposition 3.8 that the evolutionary vector fields of the spin Sutherland hierarchy (given by (3.68)) result from our construction by restriction to the submanifold  $\mathfrak{M}_{0,-}^{\text{reg}}$ . Now we show that the bi-Hamiltonian structure found in [18] also results from this restriction. We start by demonstrating that the derivative of  $J^+$  vanishes along all Hamiltonian vector fields  $\mathcal{E}_h^i$

at those point where  $J^+ = 0$ , i.e. on the submanifold  $\mathfrak{M}_{0,-}^{\text{reg}}$  (3.66). To see this, we rearrange the Poisson brackets in the form

$$\{f, h\}_i^{\text{red}} = \langle \nabla_1 f, \mathcal{E}_h^i[q] \rangle + \langle (d_2 f)^-, \mathcal{E}_h^i[J^-] \rangle + \langle (d_2 f)^+, \mathcal{E}_h^i[J^+] \rangle, \tag{4.1}$$

and show that the functions

$$\mathcal{E}_h^i[J^+] : \mathfrak{M}_0^{\text{reg}} \rightarrow \mathcal{G}^+ \tag{4.2}$$

vanish for both Poisson brackets upon imposing the constraint  $J^+ = 0$ .

The following lemmas are obtained by straightforward, somewhat tedious, calculations. We sketch only the proof of the second lemma, which is the more complicated one.

**Lemma 4.1.** *The Hamiltonian vector field  $\mathcal{E}_h^1$  associated with the Poisson bracket (3.43) has the components*

$$\mathcal{E}_h^1[q] = (d_2 h)_0^-, \tag{4.3}$$

$$\mathcal{E}_h^1[J^-] = -\nabla_1 h + [(d_2 h)^-, J^+] + R(q)[J, d_2 h]^+ + [R(q)(d_2 h)^-, J^-], \tag{4.4}$$

$$\mathcal{E}_h^1[J^+] = [R(q)(d_2 h)^- - (d_2 h)^+, J^+]. \tag{4.5}$$

Consequently,  $\mathcal{E}_h^1$  is tangent to the submanifold  $\mathfrak{M}_{0,-}^{\text{reg}}$  (3.66) for every  $h \in C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ .

**Lemma 4.2.** *The Hamiltonian vector field  $\mathcal{E}_h^2$  associated with the Poisson bracket (3.49) has the components*

$$\mathcal{E}_h^2[q] = (J^-(d_2 h)^- + J^+(d_2 h)^+)_0^-, \tag{4.6}$$

$$\mathcal{E}_h^2[J^-] = (2J^-(d_2 h)^- J^+ - 2J^- R(q)((d_2 h)^- J^- + (d_2 h)^+ J^+) - \nabla_1 h J^-)^-, \tag{4.7}$$

$$\mathcal{E}_h^2[J^+] = (2J^+(d_2 h)^+ J^- - 2J^+ R(q)((d_2 h)^- J^- + (d_2 h)^+ J^+) - \nabla_1 h J^+)^+. \tag{4.8}$$

It follows that  $\mathcal{E}_h^2$  is tangent to both submanifolds  $\mathfrak{M}_{0,-}^{\text{reg}}$  (3.66) and  $\mathfrak{M}_{0,+}^{\text{reg}}$  (3.67).

**Proof.** We shall use that, for any  $X \in \mathcal{G}$ ,  $X^- = \frac{1}{2}(X + X^*)$  and  $X^+ = \frac{1}{2}(X - X^*)$ , together with obvious properties of the trace form (2.1), like  $\langle X^*, Y^* \rangle = \langle X, Y \rangle$ . By directly spelling it out, we find

$$(\nabla_2 h + \nabla'_2 h)^- = 2(J^-(d_2 h)^- + J^+(d_2 h)^+)^-. \tag{4.9}$$

Thus, the first term of  $\{f, h\}_2^{\text{red}}$  (3.49) gives (4.6), and the second term of (3.49) gives the last terms in both lines (4.7) and (4.8). In order to confirm the latter statement, note that

$$\begin{aligned}
 \frac{1}{2} \langle \nabla_1 h, (\nabla_2 f + \nabla_2' f)_0^- \rangle &= \langle \nabla_1 h, J^-(d_2 f)^- + J^+(d_2 f)^+ \rangle \\
 &= \langle \nabla_1 h J^-, (d_2 f)^- \rangle + \langle \nabla_1 h J^+, (d_2 f)^+ \rangle \\
 &= \langle (\nabla_1 h J^-)^-, (d_2 f)^- \rangle + \langle (\nabla_1 h J^+)^+, (d_2 f)^+ \rangle.
 \end{aligned}
 \tag{4.10}$$

Next, we inspect the terms of (3.49) that contain  $R(q)$  (3.34). We use the anti-symmetry of  $R(q)$  (3.35), the property  $(R(q)X)^* = -R(q)X^*$ , and that it maps  $\mathcal{G}^\pm$  into  $\mathcal{G}^\mp$ , respectively. Then we can write

$$\begin{aligned}
 \langle R(q)[d_2 f, J]^+, (\nabla_2 h + \nabla_2' h)^- \rangle &= 2 \langle R(q) \left( (d_2 f)^+ J^+ - J^+(d_2 f)^+ + (d_2 f)^- J^- - J^-(d_2 f)^- \right), J^-(d_2 h)^- + J^+(d_2 h)^+ \rangle \\
 &= 2 \langle R(q) \left( (d_2 f)^+ J^+ + (d_2 f)^- J^- \right), J^-(d_2 h)^- + J^+(d_2 h)^+ + (d_2 h)^- J^- + (d_2 h)^+ J^+ \rangle \\
 &= -2 \langle (d_2 f)^+, J^+ R(q) \left( J^-(d_2 h)^- + J^+(d_2 h)^+ + (d_2 h)^- J^- + (d_2 h)^+ J^+ \right) \rangle \\
 &\quad - 2 \langle (d_2 f)^-, J^- R(q) \left( J^-(d_2 h)^- + J^+(d_2 h)^+ + (d_2 h)^- J^- + (d_2 h)^+ J^+ \right) \rangle.
 \end{aligned}
 \tag{4.11}$$

Similarly, we obtain

$$\begin{aligned}
 - \langle R(q)[d_2 h, J]^+, (\nabla_2 f + \nabla_2' f)^- \rangle &= -2 \langle R(q)[d_2 h, J]^+, (d_2 f)^- J^- + (d_2 f)^+ J^+ \rangle \\
 &= -2 \langle (d_2 f)^+, J^+ R(q)[d_2 h, J]^+ \rangle - 2 \langle (d_2 f)^-, J^- R(q)[d_2 h, J]^+ \rangle.
 \end{aligned}
 \tag{4.12}$$

By adding these two expressions, and taking into account the factor 2 on the left-hand side of (3.49), we get the terms containing  $R(q)$  in (4.7) and (4.8). The first terms on the right-hand sides of (4.7) and (4.8) result by expanding and collecting all terms coming from the last line of (3.49), which is laborious but fully straightforward. Some further details are given in appendix B.  $\square$

Let us remember [42] that a Poisson submanifold  $N$  of a Poisson manifold  $(M, \{, \}_M)$  is characterized by the property that if one considers any Hamiltonian vector field on  $M$  and restricts it to  $N$ , then the restricted vector field is tangent to  $N$ . Under this condition, one obtains a Poisson structure  $\{, \}_N$  on  $N$  as follows. Take any smooth functions  $\mathcal{F}, \mathcal{H}$  on  $N$  and extend them arbitrarily to smooth functions  $f, h$  on  $M$ . (It is sufficient to consider such extensions only locally, and  $N$  can be an immersed submanifold). Then the formula

$$\{ \mathcal{F}, \mathcal{H} \}_N(x) := \{ f, h \}_M(x), \quad \forall x \in N,
 \tag{4.13}$$

gives a well-defined Poisson bracket on  $N$ . It follows from lemma 4.1 and lemma 4.2 that this procedure can be applied in our situation, too, and thus we obtain well-defined Poisson brackets on  $C^\infty(\mathfrak{M}_{0,-}^{\text{reg}})^{\mathbb{T}^n}$  by restriction of the Poisson brackets on  $C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ . The variables of a function  $\mathcal{F} \in C^\infty(\mathfrak{M}_{0,-}^{\text{reg}})$  are given by the pair  $(e^q, J^-)$ . Mimicking the previous practice, we introduce the corresponding  $\mathcal{G}_0^-$ -valued and  $\mathcal{G}^-$ -valued derivatives  $\nabla_1 \mathcal{F}$  and  $d_2 \mathcal{F}$ , and also set  $\nabla_2 \mathcal{F} := J^- d_2 \mathcal{F}$  and  $\nabla_2' \mathcal{F} := d_2 \mathcal{F} J^-$ .

**Theorem 4.3.** *The compatible Poisson brackets given by (3.43) and (3.49) on  $C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$  can be restricted to  $C^\infty(\mathfrak{M}_{0,-}^{\text{reg}})^{\mathbb{T}^n}$ . For  $\mathcal{F}, \mathcal{H} \in C^\infty(\mathfrak{M}_{0,-}^{\text{reg}})^{\mathbb{T}^n}$  the restricted brackets, denoted  $\{ \mathcal{F}, \mathcal{H} \}_{i,-}^{\text{red}}$ , can be written as*

$$\begin{aligned} \{\mathcal{F}, \mathcal{H}\}_{1,-}^{\text{red}}(e^q, J^-) &= \langle \nabla_1 \mathcal{F}, d_2 \mathcal{H} \rangle - \langle \nabla_1 \mathcal{H}, d_2 \mathcal{F} \rangle \\ &+ \langle J^-, [R(q)d_2 \mathcal{F}, d_2 \mathcal{H}] + [d_2 \mathcal{F}, R(q)d_2 \mathcal{H}] \rangle \end{aligned} \tag{4.14}$$

and

$$\{\mathcal{F}, \mathcal{H}\}_{2,-}^{\text{red}}(e^q, J^-) = \langle \nabla_1 \mathcal{F}, \nabla_2 \mathcal{H} \rangle - \langle \nabla_1 \mathcal{H}, \nabla_2 \mathcal{F} \rangle + 2\langle \nabla_2 \mathcal{F}, R(q)\nabla_2 \mathcal{H} \rangle, \tag{4.15}$$

where the derivatives are evaluated at  $(e^q, J^-)$ . These formulae reproduce the compatible Poisson brackets of the hyperbolic spin Sutherland hierarchy described earlier in [18].

**Proof.** Suppose that  $\mathcal{F}, \mathcal{H}$  are the restrictions of  $f, h \in C^\infty(\mathfrak{M}_0^{\text{reg}})^{\mathbb{T}^n}$ . Then, according to the definition of the restricted brackets,

$$\{\mathcal{F}, \mathcal{H}\}_{i,-}^{\text{red}}(e^q, J^-) = (\langle \nabla_1 f, \mathcal{E}_h^i[q] \rangle + \langle (d_2 f)^-, \mathcal{E}_h^i[J^-] \rangle)(e^q, J^-), \tag{4.16}$$

where the formulae of the preceding two lemmas are applied, at  $J^+ = 0$ . Here, we substitute  $(d_2 f)^-(e^q, J^-) = d_2 \mathcal{F}(e^q, J^-)$  and analogous relations for the other derivatives. This readily leads to the above forms of the restricted Poisson brackets. For example, to obtain (4.15), we also use that

$$-\langle d_2 \mathcal{F}, 2J^- R(q)\nabla_2' \mathcal{H} \rangle = -2\langle \nabla_2' \mathcal{F}, R(q)\nabla_2' \mathcal{H} \rangle = 2\langle \nabla_2 \mathcal{F}, R(q)\nabla_2 \mathcal{H} \rangle. \tag{4.17}$$

This holds by virtue of the identities

$$\nabla_2 \mathcal{H} = (\nabla_2' \mathcal{H})^*, \quad R(q)X^* = -(R(q)X)^*, \quad \langle X^*, Y^* \rangle = \langle X, Y \rangle, \quad \forall X, Y \in \mathcal{G}. \tag{4.18}$$

Taking into account some obvious differences of notation, one sees by direct comparison that the Poisson brackets in (4.14) and (4.15) coincide with those in theorem 1 of [18].  $\square$

**Remark 4.4.** The Poisson bracket (4.15) was obtained in [18] by suitably rewriting the Poisson structure of an example of models derived by Li [33] applying a rather complicated method based on dynamical Poisson–Lie groupoids. Then it was directly shown to be compatible with the first Poisson bracket (4.14) extracted from [23]. The present derivation is simpler and it highlights that both Poisson brackets originate from a single reduction in a unified manner. If we parameterize the Hermitian matrix  $J = J^-$  in the form

$$(J^-)_{ij} = p_i \delta_{ij} - (1 - \delta_{ij}) \frac{\xi_{ij}}{\sinh(q_i - q_j)}, \tag{4.19}$$

where the  $p_i$  are arbitrary real numbers and  $\xi$  in an off-diagonal anti-Hermitian matrix, then the reduced Hamiltonian  $\frac{1}{2} \text{tr}(J^2)$  reproduces (1.7). The spin  $\xi$  matters up to the gauge transformations  $\xi \mapsto \tau \xi \tau^{-1}$ ,  $\forall \tau \in \mathbb{T}^n$ . Under the reduced first Poisson bracket (4.14), the  $\mathbb{T}^n$  invariant functions of  $\xi$  are those arising from the  $u(n)$  Lie–Poisson bracket reduced by the first class constraints  $\xi_{kk} = 0$  for all  $k$ . The  $q_i, p_i$  ( $i = 1, \dots, n$ ) form canonical pairs with respect to the reduced first Poisson bracket, and they Poisson commute with the  $\mathbb{T}^n$  invariant functions of  $\xi$ . These statements are proved in [18, 23], and will be generalized in the next subsection.

The combined message of theorem 4.3 and proposition 3.8 is summarized in the next corollary.

**Corollary 4.5.** *The restrictions of the Hamiltonians  $h_k$  (3.56) on  $\mathfrak{M}_{0,-}^{\text{reg}}$ , given by  $\mathcal{H}_k(e^q, J^-) = \frac{1}{k} \text{tr}((J^-)^k)$ , together with the compatible Poisson brackets of theorem 4.3 reproduce the bi-Hamiltonian vector fields (3.68) of the spin Sutherland hierarchy. By using the parameterization (4.19),  $\mathcal{H}_2(e^q, J^-)$  turns into<sup>2</sup>  $\mathcal{H}_{\text{spin-1}}$  (1.7). The Hamiltonians  $\tilde{h}_k$  (3.56) vanish on  $\mathfrak{M}_{0,-}^{\text{reg}}$ .*

Finally, let us observe that  $J^+$  and  $J^-$  appear rather symmetrically in the formula of lemma 4.2. In particular,  $\mathcal{E}_h^2[J^-] = 0$  holds after restriction to the submanifold  $\mathfrak{M}_{0,+}^{\text{reg}}$  (3.67). Therefore we can restrict the second reduced Poisson bracket on this submanifold. Moreover,  $n$  out of the  $2n$  commuting Hamiltonians (3.56) survives this restriction, in correspondence to the vector fields in (3.70). It can be verified that the substitution  $J^+ = iJ^-$ , with the new variable  $J^-$ , converts the restricted Poisson bracket on  $C^\infty(\mathfrak{M}_{0,+}^{\text{reg}})^{\mathbb{T}^n}$  into the restricted second Poisson bracket on  $C^\infty(\mathfrak{M}_{0,-}^{\text{reg}})^{\mathbb{T}^n}$ . This means that we do not obtain anything new from this restriction, and hence we omit its more detailed description.

4.2. The general case: Sutherland model with two spins

We now explain how the generalized spin Sutherland Hamiltonian (1.15) arises from our reduced system. For this purpose, we take  $(e^q, J) \in \mathfrak{M}_0^{\text{reg}}$  and (applying (2.7)) define the new variables

$$\xi^l := -J^+, \quad \xi^r := (e^{-q} J e^q)^+, \quad p_k := J_{kk}^-, \quad k = 1, \dots, n. \tag{4.20}$$

We observe that the pair  $(\xi^l, \xi^r)$  obeys the constraints

$$\xi_{kk}^l + \xi_{kk}^r = 0, \quad k = 1, \dots, n. \tag{4.21}$$

**Lemma 4.6.** *The matrix  $J$  can be reconstructed from  $(q, p, \xi^l, \xi^r)$  defined by (4.20) according to the formula*

$$J_{ij} = p_i \delta_{ij} - (1 - \delta_{ij}) \left( \coth(q_i - q_j) \xi_{ij}^l + \xi_{ij}^r / \sinh(q_i - q_j) \right) - \xi_{ij}^l, \quad \forall 1 \leq i, j \leq n. \tag{4.22}$$

*This expression provides a parameterization of  $\mathfrak{M}_0^{\text{reg}}$  (3.23) by the variables  $(q, p, \xi^l, \xi^r)$ , where  $q_1 > q_2 > \dots > q_n$ , the  $p_k \in \mathbb{R}$  are arbitrary and  $(\xi^l, \xi^r) \in \mathfrak{u}(n) \oplus \mathfrak{u}(n)$  is subject to the constraints (4.21). The residual gauge transformations act on  $(q, p, \xi^l, \xi^r)$  according to*

$$(q, p, \xi^l, \xi^r) \mapsto (q, p, \tau \xi^l \tau^{-1}, \tau \xi^r \tau^{-1}), \quad \forall \tau \in \mathbb{T}^n. \tag{4.23}$$

*In terms of these variables, the reduced Hamiltonian coming from  $H_1$  in (2.27) becomes  $\mathfrak{R}tr(J) = \sum_{k=1}^n p_k$ , while  $H_2 = \frac{1}{2} \mathfrak{R}tr(J^2)$  yields the generalized spin Sutherland Hamiltonian  $\mathcal{H}_{\text{spin-2}}$  (1.15) displayed in the introduction.*

**Proof.** By decomposing  $J$  as  $J = J^+ + J^-$  using (2.7), we can write

<sup>2</sup>This justifies calling the system ‘spin Sutherland hierarchy’.

$$(e^{-q}J e^q)_{ij} = (\cosh(q_i - q_j)J_{ij}^+ - \sinh(q_i - q_j)J_{ij}^-) + (\cosh(q_i - q_j)J_{ij}^- - \sinh(q_i - q_j)J_{ij}^+). \tag{4.24}$$

Since the first two terms give the anti-Hermitian part  $(e^{-q}J e^q)^+$ , we obtain from (4.20)

$$\xi_{ij}^r = -\cosh(q_i - q_j)\xi_{ij}^l - \sinh(q_i - q_j)J_{ij}^-. \tag{4.25}$$

This relation can be solved for  $J_{ij}^-$  as

$$J_{ij}^- = -\coth(q_i - q_j)\xi_{ij}^l - \xi_{ij}^r / \sinh(q_i - q_j), \quad \text{if } i \neq j, \tag{4.26}$$

and  $J_{ii}^- = p_i$  by definition. Thus we have derived (4.22) and it is easy to see that this yields a smooth bijection between  $(e^q, J) \in \mathfrak{M}_0^{\text{reg}}$  and the set of quadruplets  $(q, p, \xi^l, \xi^r)$  satisfying the conditions stated by the lemma. The transformation rule (4.23) is equivalent to (3.25). The forms of  $\mathfrak{R}tr(J)$  and  $\frac{1}{2}\mathfrak{R}tr(J^2)$  then follow by straightforward evaluation. To obtain the formula (1.15), one applies the identity  $2 \cosh x / \sinh^2 x = 1 / \sinh^2(x/2) - 2 / \sinh^2 x$ .  $\square$

It turns out that under the reduced first Poisson bracket (3.43) the  $q_i, p_i$  form canonically conjugate pairs, the Poisson brackets of the  $\mathbb{T}^n$  invariant functions of  $\xi^l, \xi^r$  are governed by the Lie–Poisson bracket of  $\mathfrak{u}(n) \oplus \mathfrak{u}(n)$  reduced by the constraints (4.21), and these two sets of variables decouple under  $\{\cdot, \cdot\}_1^{\text{red}}$ . This result can be obtained as a consequence of the symplectic reduction approach adopted in [24]. Alternatively, we can directly perform the required change of variables in the formula (3.43). In order to make this paper self-contained, we present the second method, but relegate all computational details to appendix C. Incidentally, we have verified that the two methods give the same result, which provides an excellent check on our considerations.

Thus, by using (4.22), we parameterize  $\mathfrak{M}_0^{\text{reg}}$  by the quintets of variables

$$(q, p, \xi_{\perp}^l, \xi_{\perp}^r, \xi_0) \quad \text{where} \quad \xi_0 \in \mathcal{G}_0^+ \quad \text{and} \quad \xi^l = \xi_{\perp}^l + \xi_0, \quad \xi^r = \xi_{\perp}^r - \xi_0. \tag{4.27}$$

For any smooth, real function  $F$  of these variables, we have the ‘partial gradients’

$$d_{\xi_{\perp}^l} F \in \mathcal{G}_{\perp}^+, \quad d_{\xi_{\perp}^r} F \in \mathcal{G}_{\perp}^+, \quad d_{\xi_0} F \in \mathcal{G}_0^+, \tag{4.28}$$

which are defined in the natural manner using the restriction of the pairing (2.1) to  $\mathcal{G}_{\perp}^+$  and to  $\mathcal{G}_0^+$  (remember that  $\mathcal{G}^+ = \mathfrak{u}(n)$ ). For arbitrary  $\mathbb{T}^n$  invariant functions  $f, h$  of the old variables  $(e^q, J)$  we write

$$f(e^q, J) = F(q, p, \xi_{\perp}^l, \xi_{\perp}^r, \xi_0) \quad \text{and} \quad h(e^q, J) = H(q, p, \xi_{\perp}^l, \xi_{\perp}^r, \xi_0), \tag{4.29}$$

and calculate  $\{F, H\}_1^{\text{red}}$  from the identity

$$\{F, H\}_1^{\text{red}}(q, p, \xi_{\perp}^l, \xi_{\perp}^r, \xi_0) := \{f, h\}_1^{\text{red}}(e^q, J). \tag{4.30}$$

**Proposition 4.7.** *Let  $F, H$  be  $\mathbb{T}^n$  invariant smooth functions on  $\mathfrak{M}_0^{\text{reg}}$ , parameterized by the variables (4.27) that transform according to*

$$(q, p, \xi_{\perp}^l, \xi_{\perp}^r, \xi_0) \mapsto (q, p, \tau \xi_{\perp}^l \tau^{-1}, \tau \xi_{\perp}^r \tau^{-1}, \xi_0), \quad \forall \tau \in \mathbb{T}^n. \tag{4.31}$$



In terms of these variables, the first reduced Poisson bracket (3.43) can be written as

$$\begin{aligned} \{F, H\}_1^{\text{red}}(q, p, \xi_\perp^l, \xi_\perp^r, \xi_0) &= \sum_{i=1}^n \left( \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} \right) \\ &+ \langle \xi_\perp^l + \xi_0, [d_{\xi_\perp^l} F + d_{\xi_0} F, d_{\xi_\perp^l} H + d_{\xi_0} H] \rangle + \langle \xi_\perp^r - \xi_0, [d_{\xi_\perp^r} F, d_{\xi_\perp^r} H] \rangle. \end{aligned} \tag{4.32}$$

**Remark 4.8.** The  $\mathbb{T}^n$  invariance of  $H$  implies that

$$\langle \xi_\perp^l, [d_{\xi_0} F, d_{\xi_\perp^l} H] \rangle + \langle \xi_\perp^r, [d_{\xi_0} F, d_{\xi_\perp^r} H] \rangle = 0. \tag{4.33}$$

By using this identity, and its counterpart with  $F$  and  $H$  exchanged, one may write the terms containing  $d_{\xi_0}$  in the second line of (4.32) in many alternative ways. Upon imposing the constraint  $\xi^l = 0$ , and putting  $\xi := \xi_\perp^r$ ,  $J$  in (4.22) reproduces  $J^-$  in (4.19). Setting also all derivatives with respect to  $\xi_0$  and  $\xi_\perp^l$  to zero, the Poisson bracket (4.32) reproduces the first reduced Poisson bracket of the spin Sutherland model (1.7) described in remark 4.4. Of course, by setting  $\xi^r = 0$  instead of  $\xi^l = 0$  one reaches the same model.

**Remark 4.9.** Let us recall from remark 2.4 that the master system behind the spin Sutherland model (1.15) is a degenerate integrable system. It is known [27, 53] that degenerate integrability is generically preserved under Hamiltonian reduction. The degenerate integrability of the model (1.15) on generic symplectic leaves of the first Poisson structure follows as a special case of results of [45]. It would be nice to enhance these results by explicitly exhibiting the required number of independent constants of motion. We here restrict ourselves to displaying a large number of  $U(n) \times U(n)$  invariant elements of the ring  $\mathfrak{A}$  of unreduced constants of motion, discussed in remark 2.4, which descend to constants of motion of the reduced system. Namely, let  $P$  be an arbitrary product of non-negative powers of  $J^+$  and  $J^-$ . That is,  $P$  has the form

$$P = (J^+)^{k_1} (J^-)^{k_2} (J^+)^{k_3} (J^-)^{k_4} \dots, \tag{4.34}$$

with non-negative integers  $k_1, k_2, k_3, k_4$  etc. On account of the transformation rules (3.1) and (3.2), the real and imaginary parts of  $\text{tr}(P)$  are  $U(n) \times U(n)$  invariant elements of  $\mathfrak{A}$ . Similarly, using (2.34), one obtains  $U(n) \times U(n)$  invariant unreduced constants of motion by taking the trace of an arbitrary product of powers of  $\tilde{J}^+$  and  $\tilde{J}^-$ . Degenerate integrable systems are also known to be Liouville integrable under very general conditions [27]. Considering the restriction of the model to generic symplectic leaves, a construction of sufficient number of constant of motion in involution can be found in [29].

### 5. Conclusion

We here introduced a bi-Hamiltonian hierarchy on the cotangent bundle of the real Lie group  $GL(n, \mathbb{C})$  and analyzed its quotient with respect to the symmetry group  $U(n) \times U(n)$ . We described the form of the compatible reduced Poisson brackets (theorems 3.4 and 3.5) as well as the bi-Hamiltonian vector fields generated by the commuting reduced Hamiltonians (propositions 3.6 and 3.8). We found that the restriction of the reduced bi-Hamiltonian hierarchy to a joint Poisson subspace of its two Poisson brackets reproduces the hyperbolic spin Sutherland hierarchy associated with the Hamiltonian (1.7). In the general case, the reduced system was

identified as a Sutherland model coupled to two spin variables according to the Hamiltonian (1.15). Thus the commuting flows of the model that are generated by the spectral invariants of the Lax matrix (4.22) all admit bi-Hamiltonian description, which may be considered as our main result.

The spin Sutherland interpretation arose from using the variables  $(q, p, \xi^l, \xi^r)$  instead of  $(e^q, J)$ , in which the first reduced Poisson bracket takes the form displayed in proposition 4.7. It is in principle possible to present also the second Poisson bracket in these variables, but the resulting formulae are not expected to have a transparent structure. At least on the subspace where  $J$  is Hermitian and positive definite, it should be possible to construct an alternative parameterization that would allow  $\text{tr}(J)$  to be interpreted as a spin Ruijsenaars–Schneider Hamiltonian. In the corresponding trigonometric case, such a change of variables is known [19], and it permits one to recover the spinless trigonometric Ruijsenaars–Schneider model on a special symplectic leaf of the reduced second Poisson bracket. Thus we suspect that the spinless hyperbolic Ruijsenaars–Schneider model should be found on a symplectic leaf of the second Poisson structure described in theorem 4.3. However, we do not know how to characterize the symplectic leaves of this Poisson structure. We encountered difficulties when trying to find them by ‘analytic continuation’ from the trigonometric to the hyperbolic case. This poses a very interesting open problem for future work. It is worth mentioning that, at least to our knowledge, no derivation of the real, hyperbolic Ruijsenaars–Schneider model by Hamiltonian reduction is known at present, as opposed to the real trigonometric model and its complex holomorphic counterpart, for which several reduction treatments are available [3, 8, 22, 25, 40]. It would be important to construct such a derivation, and finding the symplectic leaves of the Poisson bracket (4.15) could help to resolve this long-standing conundrum.

## Acknowledgments

I wish to thank Maxime Fairon for several useful remarks on the manuscript. This work was supported in part by the NKFIH research Grant K134946. Open access funding was provided by the University of Szeged.

## Appendix A. An explanation of the second Poisson bracket

We below outline how the second Poisson bracket (2.16) arises by a change of variables from Semenov–Tian–Shansky’s Poisson bracket [49] on the Heisenberg double  $G \times G$ . We will be brief since this explanation closely follows the appendix in [20]. However, note that in [20] we considered holomorphic complex functions, while now we deal with real smooth functions.

We begin by introducing a non-degenerate, invariant bilinear form on the real Lie algebra  $\mathcal{G} \oplus \mathcal{G}$  by

$$\langle (X_1, X_2), (Y_1, Y_2) \rangle_2 := \langle X_1, Y_1 \rangle - \langle X_2, Y_2 \rangle, \quad (\text{A.1})$$

where  $(X_1, X_2)$  and  $(Y_1, Y_2)$  are from  $\mathcal{G} \oplus \mathcal{G}$ , and (2.1) is applied. It is not difficult to see that  $\mathcal{G} \oplus \mathcal{G}$  is the vector space direct sum of the subalgebras

$$\mathcal{G}^\delta := \{(X, X) | X \in \mathcal{G}\} \quad (\text{A.2})$$

and

$$\mathcal{G}^* := \{(r_+(X), r_-(X)) | X \in \mathcal{G}\}. \quad (\text{A.3})$$

Recall that  $r_{\pm}$  are defined in (2.5), i.e.  $r_+(X) = X_{>} + \frac{1}{2}X_0$  and  $r_-(X) = -X_{<} - \frac{1}{2}X_0$  for  $X$  written as in (2.2). These are isotropic subalgebras, meaning that the bilinear form (A.1) vanishes on them separately. After identifying  $\mathcal{G}$  with  $\mathcal{G}^\delta$ , we can use the bilinear form to take  $\mathcal{G}^*$  (A.3) as a model of the dual space of  $\mathcal{G}$ , which explains the notation.

Let us define the linear operator  $\rho$  on  $\mathcal{G} \oplus \mathcal{G}$  by

$$\rho := \frac{1}{2} (P_{\mathcal{G}^\delta} - P_{\mathcal{G}^*}) \tag{A.4}$$

using the projections  $P_{\mathcal{G}^\delta}$  onto  $\mathcal{G}^\delta$  and  $P_{\mathcal{G}^*}$  onto  $\mathcal{G}^*$  associated with the vector space direct sum  $\mathcal{G} \oplus \mathcal{G} = \mathcal{G}^\delta + \mathcal{G}^*$ . It features in two well-known [49] Poisson brackets on  $C^\infty(G \times G, \mathbb{R})$ . For  $\mathcal{F} \in C^\infty(G \times G, \mathbb{R})$ , the  $\mathcal{G} \oplus \mathcal{G}$ -valued left- and right-derivatives are determined by

$$\begin{aligned} \langle \mathcal{D}\mathcal{F}(g_1, g_2), (X_1, X_2) \rangle_2 &:= \left. \frac{d}{dt} \right|_{t=0} \mathcal{F}(e^{tX_1}g_1, e^{tX_2}g_2), \\ \langle \mathcal{D}'\mathcal{F}(g_1, g_2), (X_1, X_2) \rangle_2 &:= \left. \frac{d}{dt} \right|_{t=0} \mathcal{F}(g_1e^{tX_1}, g_2e^{tX_2}), \end{aligned} \tag{A.5}$$

where  $t \in \mathbb{R}$ ,  $(X_1, X_2)$  runs over  $\mathcal{G} \oplus \mathcal{G}$  and  $(g_1, g_2) \in G \times G$ . With these notations, the two Poisson brackets read

$$\{\mathcal{F}, \mathcal{H}\}_\pm := \langle \mathcal{D}\mathcal{F}, \rho\mathcal{D}\mathcal{H} \rangle_2 \pm \langle \mathcal{D}'\mathcal{F}, \rho\mathcal{D}'\mathcal{H} \rangle_2. \tag{A.6}$$

The minus bracket is called the Drinfeld double bracket, and the plus one the Heisenberg double bracket. The former makes  $G \times G$  into a Poisson–Lie group, and the latter is symplectic in a neighbourhood of the identity [49].

Now we introduce new variables in a neighbourhood of  $(1_n, 1_n) \in G \times G$ . We need the connected subgroups of  $G$  corresponding to the subalgebras in (2.3). These are denoted  $G_{>}$ ,  $G_{<}$  and  $G_0$ , respectively, where  $G_{>}$  contains the upper triangular complex matrices in  $G$  having 1 in their diagonal entries, and  $G_0$  contains the diagonal matrices in  $G$ . Then the connected Lie subgroups of  $G \times G$  associated with the subalgebras  $\mathcal{G}^\delta$  (A.2) and  $\mathcal{G}^*$  (A.3) are

$$G^\delta := \{g_\delta | g_\delta := (g, g), g \in G\} \tag{A.7}$$

and

$$G^* = \{\eta_* | \eta_* := (\eta_{>}\eta_0, (\eta_0\eta_{<})^{-1}), \eta_{>} \in G_{>}, \eta_0 \in G_0, \eta_{<} \in G_{<}\}. \tag{A.8}$$

Since  $\mathcal{G} \oplus \mathcal{G} = \mathcal{G}^\delta + \mathcal{G}^*$ , there exist open neighbourhoods of the identity in  $G \times G$  whose elements can be factorized uniquely as

$$(g_1, g_2) = g_{\delta L}\eta_{*R}^{-1} = \eta_{*L}g_{\delta R}^{-1}, \tag{A.9}$$

where  $(g_{\delta L}, \eta_{*R})$  and  $(g_{\delta R}, \eta_{*L})$  vary in corresponding open sets around the identity in  $G^\delta \times G^*$ . Let us write

$$g_{\delta R} = (g, g) \quad \text{and} \quad \eta_{*R} = (\eta_{>}\eta_0, (\eta_0\eta_{<})^{-1}), \tag{A.10}$$

and use the factorizations (A.9) to define the map

$$\psi : (g_1, g_2) \mapsto (g, J) \quad \text{with} \quad J := \eta_{>}\eta_0^2\eta_{<}. \tag{A.11}$$

By suitably choosing its domain,  $\psi$  gives a diffeomorphism between open subsets of  $G \times G$  containing  $(1_n, 1_n)$ , but it is advantageous for us to regard its image as a subset of  $G \times \mathcal{G}$ .

The following statement can be verified by essentially the same calculations that were presented in [20].

**Proposition A.1.** *Let  $\mathcal{F}$  and  $\mathcal{H}$  be smooth real functions on the domain of the local diffeomorphism  $\psi$  (A.11). Referring to (A.6), for  $F := \mathcal{F} \circ \psi^{-1}$  and  $H := \mathcal{H} \circ \psi^{-1}$  define*

$$\{F, H\}_2 := \{\mathcal{F}, \mathcal{H}\}_+ \circ \psi^{-1}. \tag{A.12}$$

Then  $\{F, H\}_2$  (A.12) yields a Poisson bracket for smooth functions defined locally around  $(1_n, 1_n) \in G \times \mathcal{G}$ , and it has the explicit form displayed in (2.16).

The proposition guarantees that the Jacobi identity holds for the restriction of the second Poisson bracket (2.16) to an open set around  $(1_n, 1_n)$ . This implies that (2.16) gives a Poisson bracket on  $C^\infty(\mathfrak{M}, \mathbb{R})$ , too. Indeed, as determined by (2.16), the Poisson brackets of the coordinate functions on  $\mathfrak{M}$  provided by the matrix elements (2.19) and the components of  $J$  are real-analytic functions on  $\mathfrak{M}$ , and thus the Jacobi identity holds for them globally since we know from the proposition that it holds on an open set.

**Remark A.2.** Incidentally, by writing  $\nabla_1 F(g, J) = g d_1 F(g, J)$ , where  $d_1 F$  is defined similarly to (2.13) using that  $G$  is an open subset of  $\mathcal{G}$ , one may extend both Poisson brackets (2.15) and (2.16) to  $C^\infty(\mathcal{G} \times \mathcal{G}, \mathbb{R})$  as well.

**Remark A.3.** For the aficionados of Poisson–Lie groups, we remark that  $u(n)^\delta < \mathcal{G}^\delta$  has the property that its annihilator inside  $\mathcal{G}^*$  (A.3) with respect to the pairing (A.1) is a Lie subalgebra of  $\mathcal{G}^*$ . By applying the general theory [49], one can trace back the closure statements of lemmas 3.1 and 3.2 to this property. We gave direct proofs of these lemmas, and thus there is no need to elaborate this point.

### Appendix B. A remark on the proof of lemma 4.2

The goal of this appendix is to help those readers who wish to go through the details of the last step of the proof of lemma 4.2, which requires the demonstration of the following identity:

$$\begin{aligned} X &:= \langle (\nabla_2 f)^-, (\nabla_2' h)^- \rangle + \langle (\nabla_2' f)^+, (\nabla_2 h)^+ \rangle - \mathcal{E}(f, h) \\ &= 4 \langle (d_2 f)^+, (J^+ (d_2 h)^+ J^-)^+ \rangle + 4 \langle (d_2 f)^-, (J^- (d_2 h)^- J^+)^- \rangle, \end{aligned} \tag{B.1}$$

where  $\mathcal{E}(f, h)$  stands for the two terms obtained by exchanging  $f$  and  $h$ . We note that  $X$  as defined above can be expressed in the alternative form

$$X = \langle [d_2 f, J]^+, (J d_2 h + d_2 h J)^+ \rangle + \langle J d_2 f, d_2 h J \rangle - \mathcal{E}(f, h). \tag{B.2}$$

This was also used in the proof of theorem 3.5, and one can easily check it. When spelling out the expression (B.2), it is convenient to write

$$f^\pm := (d_2 f)^\pm \quad \text{and} \quad h^\pm := (d_2 h)^\pm. \tag{B.3}$$

With this notation, we have

$$\begin{aligned} [d_2 f, J]^+ &= [f^+, J^+] + [f^-, J^-], \\ (J d_2 h + d_2 h J)^- &= (J^+ h^+ + h^+ J^+) + (J^- h^- + h^- J^-), \end{aligned} \tag{B.4}$$

$$(Jd_2h + d_2hJ)^+ = (J^-h^+ + h^+J^-) + (J^+h^- + h^-J^+). \tag{B.5}$$

To verify the subsequent statements, one only needs to use the cyclic property of the trace form (2.1) and that  $\mathcal{G}^+$  is perpendicular to  $\mathcal{G}^-$ .

**Lemma B.1.** *The following identity holds:*

$$\begin{aligned} \langle Jd_2f, d_2hJ \rangle - \mathcal{E}(f, h) &= \langle [f^+, h^+] + [f^-, h^-], J^+J^- + J^-J^+ \rangle \\ &\quad + \langle [f^+, h^-] + [f^-, h^+], (J^-)^2 + (J^+)^2 \rangle. \end{aligned} \tag{B.6}$$

**Lemma B.2.** *The following identity holds:*

$$\begin{aligned} \langle [d_2f, J]^+, (Jd_2h + d_2hJ)^+ \rangle - \mathcal{E}(f, h) &= 2\langle f^+, J^+h^+J^- - J^-h^+J^+ \rangle \\ &\quad + 2\langle f^-, J^-h^-J^+ - J^+h^-J^- \rangle \\ &\quad + \langle [h^+, f^+] + [h^-, f^-], J^+J^- + J^-J^+ \rangle \\ &\quad + \langle [h^-, f^+] + [h^+, f^-], (J^-)^2 + (J^+)^2 \rangle. \end{aligned} \tag{B.7}$$

Finally, the claimed formula (B.1) follows by combining the statements displayed above.

### Appendix C. Proof of proposition 4.7

We introduce the notations

$$S(x) := \sinh x, \quad C(x) := \cosh x, \quad \hat{q} := \text{ad}_q, \tag{C.1}$$

and parameterize  $J$  according to (4.22), i.e.

$$J = p - R(q)\xi_{\perp}^1 - S(\hat{q})^{-1}\xi_{\perp}^r - \xi^1, \quad \xi^1 = \xi_{\perp}^1 + \xi_0, \tag{C.2}$$

where  $p = \text{diag}(p_1, \dots, p_n) \in \mathcal{G}_0^-$  and  $R(q) = \text{coth } \hat{q}$  on  $\mathcal{G}_{\perp}$ . Note that  $S(\hat{q})^{-1}$  denotes the inverse of the restriction of  $S(\hat{q})$  on  $\mathcal{G}_{\perp}$ . Referring to the functional calculus of linear operators, we have the identity  $S(\hat{q})^{-1} = (1/S)(\hat{q})$  on  $\mathcal{G}_{\perp}$ . We shall use that  $R(q)$  and  $S(\hat{q})$  both map  $\mathcal{G}_{\perp}^{\pm}$  to  $\mathcal{G}_{\perp}^{\mp}$  (3.29), respectively, and they vanish on  $\mathcal{G}_0$ , while  $\mathcal{G}^{\pm}$  are invariant subspaces of  $C(\hat{q})$ .

We put

$$f(e^q, J) = F(q, p, \xi_{\perp}^1, \xi_{\perp}^r, \xi_0), \quad h(e^q, J) = H(q, p, \xi_{\perp}^1, \xi_{\perp}^r, \xi_0). \tag{C.3}$$

This leads to

$$(d_2f)^- = d_pF + S(\hat{q})d_{\xi_{\perp}^r}F, \tag{C.4}$$

$$(d_2f)^+ = -d_{\xi_0}F - d_{\xi_{\perp}^1}F + C(\hat{q})d_{\xi_{\perp}^r}F, \tag{C.5}$$

and

$$\nabla_1 f = d_qF - \left[ R(q)S(\hat{q})^{-1}\xi_{\perp}^r + S(\hat{q})^{-2}\xi_{\perp}^1, S(\hat{q})d_{\xi_{\perp}^r}F \right]_0^-. \tag{C.6}$$

Here,  $d_q F, d_p F$  belong to  $\mathcal{G}_0^-$ , and for the other derivatives see (4.28). We recall the formula

$$\begin{aligned} \{f, h\}_1^{\text{red}}(e^q, J) &= \langle \nabla_1 f, (d_2 h)_0^- \rangle - \langle \nabla_1 h, (d_2 f)_0^- \rangle \\ &\quad + \langle R(q)[d_2 f, J]^+, (d_2 h)^- \rangle - \langle R(q)[d_2 h, J]^+, (d_2 f)^- \rangle \\ &\quad + \langle J^+, [(d_2 f)^-, (d_2 h)^-] - [(d_2 f)^+, (d_2 h)^+] \rangle, \end{aligned} \tag{C.7}$$

which we have to rewrite in terms of  $F$  and  $H$  (C.3). To make the subsequent equations shorter, we denote

$$\begin{aligned} F_q &:= d_q F, & F_p &:= d_p F, & F_0 &:= d_{\xi_0} F, \\ F_\perp^r &:= d_{\xi_\perp^r} F, & F_\perp^l &:= d_{\xi_\perp^l} F, & F^1 &:= F_\perp^1 + F_0, \end{aligned} \tag{C.8}$$

and will often omit the argument  $q, \hat{q}$  in  $R(q), S(\hat{q})$  etc. We shall write  $\mathcal{E}(F, H)$  for any expression obtained by exchanging the roles of  $F$  and  $H$ .

We start by inspecting

$$\langle R[d_2 f, J]^+, (d_2 h)^- \rangle = \langle R[(d_2 f)^-, J^-], (d_2 h)^- \rangle + \langle R[(d_2 f)^+, J^+], (d_2 h)^- \rangle. \tag{C.9}$$

**Lemma C.1.** *We have the identity*

$$\begin{aligned} \langle R[(d_2 f)^-, J^-], (d_2 h)^- \rangle - \mathcal{E}(f, h) &= \langle \xi_\perp^r, [F_\perp^r, H_\perp^r] \rangle + \langle C\xi_\perp^l, [F_\perp^r, H_\perp^r] \rangle \\ &\quad + (\langle F_p, [S^{-1}\xi_\perp^r, CH_\perp^r] \rangle + [R\xi_\perp^l, CH_\perp^r] - \mathcal{E}(F, H)). \end{aligned} \tag{C.10}$$

**Proof.** We first note that

$$\langle R[(d_2 f)^-, J^-], (d_2 h)^- \rangle = -\langle [(d_2 f)^-, J^-], R(d_2 h)^- \rangle, \tag{C.11}$$

and

$$R(d_2 h)^- = R(d_p H + SH_\perp^r) = CH_\perp^r. \tag{C.12}$$

Then we substitute  $(d_2 f)^-$  and  $J^-$ , which gives

$$\begin{aligned} \langle R[(d_2 f)^-, J^-], (d_2 h)^- \rangle - \mathcal{E}(f, h) &= (\langle [F_p + SF_\perp^r, S^{-1}\xi_\perp^r - p], CH_\perp^r \rangle - \mathcal{E}(F, H)) \\ &\quad + (\langle F_p, [R\xi_\perp^l, CH_\perp^r] \rangle + \langle CH_\perp^r, [SF_\perp^r, R\xi_\perp^l] \rangle - \mathcal{E}(F, H)). \end{aligned} \tag{C.13}$$

By expanding  $S$  and  $C$  in terms of exponentials, one can check that

$$[SF_\perp^r, CH_\perp^r] - \mathcal{E}(F, H) = S[F_\perp^r, H_\perp^r]. \tag{C.14}$$

By combining this with the equality  $\langle V, S(\hat{q})W \rangle = -\langle S(\hat{q})V, W \rangle$  ( $\forall V, W \in \mathcal{G}$ ), we find that

$$\begin{aligned} \langle [F_p + SF_\perp^r, S^{-1}\xi_\perp^r - p], CH_\perp^r \rangle - \mathcal{E}(F, H) &= \langle \xi_\perp^r, [F_\perp^r, H_\perp^r] \rangle + (\langle F_p, [S^{-1}\xi_\perp^r, CH_\perp^r] \rangle - \mathcal{E}(F, H)), \end{aligned} \tag{C.15}$$

and

$$\langle [CH_\perp^r, SF_\perp^r], R\xi_\perp^l \rangle - \mathcal{E}(F, H) = \langle [F_\perp^r, H_\perp^r], C\xi_\perp^l \rangle. \tag{C.16}$$

To get the last equality, we also used that  $S(\hat{q})R(q)\xi_{\perp}^1 = C(\hat{q})\xi_{\perp}^1$ . Putting together these identities, the claim is proved.  $\square$

The other terms in the second and third lines of (C.7) can be spelled out straightforwardly, and the result is summarized as follows.

**Lemma C.2.** *The following equalities hold. First,*

$$\begin{aligned} \langle R[(d_2f)^+, J^+], (d_2h)^- \rangle - \mathcal{E}(f, h) &= -2\langle \xi^1, [CF_{\perp}^r, CH_{\perp}^r] \rangle \\ &\quad + (\langle \xi^1, [F^1, CH_{\perp}^r] \rangle - \mathcal{E}(F, H)). \end{aligned} \quad (C.17)$$

*Second,*

$$\langle J^+, [(d_2f)^-, (d_2h)^-] \rangle = -\langle \xi^1, [SF_{\perp}^r, SH_{\perp}^r] \rangle + (\langle F_p, [\xi_{\perp}^1, SH_{\perp}^r] \rangle - \mathcal{E}(F, H)). \quad (C.18)$$

*Third,*

$$\begin{aligned} -\langle J^+, [(d_2f)^+, (d_2h)^+] \rangle &= \langle \xi^1, [F^1, H^1] + [CF_{\perp}^r, CH_{\perp}^r] \rangle \\ &\quad + (\langle \xi^1, [CH_{\perp}^r, F^1] \rangle - \mathcal{E}(F, H)). \end{aligned} \quad (C.19)$$

The next statement results by collecting the terms from the preceding two lemmas.

**Lemma C.3.** *Denote by  $X$  the sum of the last two lines of (C.7). Then we have*

$$\begin{aligned} X &= \langle \xi^1, [F^1, H^1] \rangle + \langle \xi^r, [F_{\perp}^r, H_{\perp}^r] \rangle \\ &\quad + (\langle F_p, [R\xi_{\perp}^1 + S^{-1}\xi_{\perp}^r, CH_{\perp}^r] + [\xi_{\perp}^1, SH_{\perp}^r] \rangle - \mathcal{E}(F, H)) \\ &\quad + \langle C\xi^1, [F_{\perp}^r, H_{\perp}^r] \rangle - \langle \xi^1, [CF_{\perp}^r, CH_{\perp}^r] + [SF_{\perp}^r, SH_{\perp}^r] \rangle. \end{aligned} \quad (C.20)$$

To obtain (C.20), we used that  $C\xi_0 = \xi_0$ , and rewrote  $\langle \xi_{\perp}^r, [F_{\perp}^r, H_{\perp}^r] \rangle$  coming from lemma C.1 as

$$\langle \xi_{\perp}^r, [F_{\perp}^r, H_{\perp}^r] \rangle = \langle \xi^r, [F_{\perp}^r, H_{\perp}^r] \rangle + \langle C\xi_0, [F_{\perp}^r, H_{\perp}^r] \rangle \quad \text{with} \quad \xi^r = \xi_{\perp}^r - \xi_0. \quad (C.21)$$

By spelling out  $C$  and  $S$  in terms of  $e^{\pm\hat{q}}$  and using  $[e^{\hat{q}}V, e^{\hat{q}}W] = e^{\hat{q}}[V, W]$  ( $\forall V, W \in \mathcal{G}$ ) we get

$$[CF_{\perp}^r, CH_{\perp}^r] + [SF_{\perp}^r, SH_{\perp}^r] = C[F_{\perp}^r, H_{\perp}^r], \quad (C.22)$$

which implies that the third line in lemma C.3 vanishes.

**Lemma C.4.** *The following equalities hold:*

$$\langle \nabla_1 f, (d_2h)_0^- \rangle = \langle F_q, H_p \rangle + \langle [R\xi_{\perp}^r + S^{-1}\xi_{\perp}^1, F_{\perp}^r], H_p \rangle, \quad (C.23)$$

*and*

$$\langle H_p, [R\xi_{\perp}^1 + S^{-1}\xi_{\perp}^r, CF_{\perp}^r] + [\xi_{\perp}^1, SF_{\perp}^r] \rangle = \langle [R\xi_{\perp}^r + S^{-1}\xi_{\perp}^1, F_{\perp}^r], H_p \rangle. \quad (C.24)$$

**Proof.** We use the identity (C.6) to write

$$\begin{aligned} \langle \nabla_1 f, (d_2 h)_0^- \rangle &= \langle F_q - [RS^{-1}\xi_\perp^r + S^{-2}\xi_\perp^l, SF_\perp^r], H_p \rangle \\ &= \langle F_q, H_p \rangle - \langle RS^{-1}\xi_\perp^r + S^{-2}\xi_\perp^l, S[F_\perp^r, H_p] \rangle \\ &= \langle F_q, H_p \rangle + \langle [R\xi_\perp^r + S^{-1}\xi_\perp^l, F_\perp^r], H_p \rangle. \end{aligned} \quad (\text{C.25})$$

Regarding the second equality, we have

$$\begin{aligned} &\langle H_p, [R\xi_\perp^l + S^{-1}\xi_\perp^r, CF_\perp^r] + [\xi_\perp^l, SF_\perp^r] \rangle \\ &= \langle R[H_p, \xi_\perp^l] + S^{-1}[H_p, \xi_\perp^r], CF_\perp^r \rangle - \langle S[H_p, \xi_\perp^l], F_\perp^r \rangle \\ &= \langle C^2 S^{-1}[H_p, \xi_\perp^l] + R[H_p, \xi_\perp^r], F_\perp^r \rangle - \langle [H_p, S\xi_\perp^l], F_\perp^r \rangle \\ &= \langle H_p, [(S + S^{-1})\xi_\perp^l + R\xi_\perp^r, F_\perp^r] \rangle - \langle H_p, [S\xi_\perp^l, F_\perp^r] \rangle \\ &= \langle H_p, [R\xi_\perp^r + S^{-1}\xi_\perp^l, F_\perp^r] \rangle, \end{aligned} \quad (\text{C.26})$$

which finishes the proof.  $\square$

In terms of the new variables, we have  $\{F, H\}_1^{\text{red}}(q, p, \xi_\perp^l, \xi_\perp^r, \xi_0) := \{f, h\}_1^{\text{red}}(e^q, J)$ . The combination of lemma C.3, where the third line of the formula was shown to vanish, with lemma C.4 gives

$$\{F, H\}_1^{\text{red}} = \langle F_q, H_p \rangle - \langle H_q, F_p \rangle + \langle \xi^l, [F^l, H^l] \rangle + \langle \xi^r, [F_\perp^r, H_\perp^r] \rangle. \quad (\text{C.27})$$

This is the same as (4.32), and thus proposition 4.7 is proved.

## References

- [1] Aniceto I, Avan J and Jevicki A 2010 Poisson structures of Calogero–Moser and Ruijsenaars–Schneider models *J. Phys. A: Math. Theor.* **43** 185201
- [2] Arthamonov S and Reshetikhin N 2021 Superintegrable systems on moduli spaces of flat connections *Commun. Math. Phys.* **386** 1337–81
- [3] Arutyunov G 2019 *Elements of Classical and Quantum Integrable Systems* (Berlin: Springer)
- [4] Arutyunov G and Olivucci E 2020 Hyperbolic spin Ruijsenaars–Schneider model from Poisson reduction *Proc. Steklov Inst. Math.* **309** 31–45
- [5] Bartocci C, Falqui G, Mencattini I, Ortenzi G and Pedroni M 2010 On the geometric origin of the Bi-Hamiltonian structure of the Calogero–Moser system *Int. Math. Res. Not.* **2010** 279–96
- [6] De Sole A, Kac V G and Valeri D 2018 Classical affine  $W$ -algebras and the associated integrable Hamiltonian hierarchies for classical Lie algebras *Commun. Math. Phys.* **360** 851–918
- [7] Calogero F 1971 Solution of the one-dimensional  $N$ -body problem with quadratic and/or inversely quadratic pair potentials *J. Math. Phys.* **12** 419–36
- [8] Chalykh O and Fairon M 2017 Multiplicative quiver varieties and generalised Ruijsenaars–Schneider models *J. Geom. Phys.* **121** 413–37
- [9] Chalykh O and Fairon M 2020 On the Hamiltonian formulation of the trigonometric spin Ruijsenaars–Schneider system *Lett. Math. Phys.* **110** 2893–940
- [10] van Diejen J F and Görbe T F 2021 Elliptic Ruijsenaars difference operators on bounded partitions *Int. Math. Res. Notices* **rnab251**
- [11] van Diejen J F and Vinet L (ed) 2000 *Calogero–Moser–Sutherland Models* (Berlin: Springer)
- [12] Etingof P 2007 *Calogero–Moser Systems and Representation Theory* (Zurich: European Mathematical Society)
- [13] Fairon M 2021 Integrable systems on multiplicative quiver varieties from cyclic quivers (arXiv:2108.02496 [nlin.SI])



- [14] Fairon M, Fehér L and Marshall I 2021 Trigonometric real form of the spin RS model of Krichever and Zabrodin *Ann. Henri Poincaré* **22** 615–75
- [15] Falqui G and Mencattini I 2017 Bi-Hamiltonian geometry and canonical spectral coordinates for the rational Calogero–Moser system *J. Geom. Phys.* **118** 126–37
- [16] Fehér L 2013 An application of the reduction method to Sutherland type many-body systems *Geometric Methods in Physics* (Basel: Birkhäuser) pp 109–17
- [17] Fehér L 2019 Poisson–Lie analogues of spin Sutherland models *Nucl. Phys. B* **949** 114807
- [18] Fehér L 2019 Bi-Hamiltonian structure of a dynamical system introduced by Braden and Hone *Nonlinearity* **32** 4377–94
- [19] Fehér L 2020 Reduction of a bi-Hamiltonian hierarchy on  $T^*U(n)$  to spin Ruijsenaars–Sutherland models *Lett. Math. Phys.* **110** 1057–79
- [20] Fehér L 2021 Bi-Hamiltonian structure of spin Sutherland models: the holomorphic case *Ann. Henri Poincaré* **22** 4063–85
- [21] Fehér L and Klimčík C 2009 On the duality between the hyperbolic Sutherland and the rational Ruijsenaars–Schneider models *J. Phys. A: Math. Theor.* **42** 185202
- [22] Fehér L and Klimčík C 2011 Poisson–Lie interpretation of trigonometric Ruijsenaars duality *Commun. Math. Phys.* **301** 55–104
- [23] Fehér L and Puzsai B G 2006 Spin Calogero models associated with Riemannian symmetric spaces of negative curvature *Nucl. Phys. B* **751** 436–58
- [24] Fehér L and Puzsai B G 2007 A class of Calogero type reductions of free motion on a simple Lie group *Lett. Math. Phys.* **79** 263–77
- [25] Fock V V and Rosly A A 1999 Poisson structure on moduli of flat connections on Riemann surfaces and the  $R$ -matrix *Moscow Seminar in Mathematical Physics (AMS Transl. Ser. 2 vol 191)* (Providence, RI: American Mathematical Society) pp 67–86
- [26] Gibbons J and Hermsen T 1984 A generalisation of the Calogero–Moser system *Physica D* **11** 337–48
- [27] Jovanovic B 2008 Symmetries and integrability *Publ. Inst. Math.* **84** 1–36
- [28] Kazhdan D, Kostant B and Sternberg S 1978 Hamiltonian group actions and dynamical systems of Calogero type *Commun. Pure Appl. Math.* **31** 481–507
- [29] Kharchev S, Levin A, Olshanetsky M and Zotov A 2018 Quasi-compact Higgs bundles and Calogero–Sutherland systems with two types of spins *J. Math. Phys.* **59** 103509
- [30] Krichever I and Zabrodin A 1995 Spin generalization of the Ruijsenaars–Schneider model, the non-abelian 2D Toda chain, and representations of the Sklyanin algebra *Russ. Math. Surv.* **50** 1101–50
- [31] Langmann E, Noumi M and Shiraishi J 2020 Construction of eigenfunctions for the elliptic Ruijsenaars difference operators (arXiv:2012.05664 [math-ph])
- [32] Laurent-Gengoux C, Miranda E and Vanhaecke P 2011 Action-angle coordinates for integrable systems on Poisson manifolds *Int. Math. Res. Not.* **1839–69**
- [33] Li L-C 2006 Poisson involutions, spin Calogero–Moser systems associated with symmetric Lie subalgebras and the symmetric space spin Ruijsenaars–Schneider models *Commun. Math. Phys.* **265** 333–72
- [34] Li L-C and Xu P 2002 A class of integrable spin Calogero–Moser systems *Commun. Math. Phys.* **231** 257–86
- [35] Magri F 1978 A simple model of the integrable Hamiltonian equation *J. Math. Phys.* **19** 1156–62
- [36] Magri F, Casati P, Falqui G and Pedroni M 2004 Eight lectures on integrable systems *Integrability of Nonlinear Systems* ed Y Kosmann-Schwarzbach *et al* (Berlin: Springer) pp 209–50
- [37] Mischenko A S and Fomenko A T 1978 Generalized Liouville method for integrating Hamiltonian systems *Funct. Anal. Appl.* **12** 113–25
- [38] Moser J 1975 Three integrable Hamiltonian systems connected with isospectral deformations *Adv. Math.* **16** 197–220
- [39] Nekhoroshev N N 1972 Action-angle variables and their generalizations *Trans. Moscow Math. Soc.* **26** 180–97
- [40] Oblomkov A 2004 Double affine Hecke algebras and Calogero–Moser spaces *Represent. Theory* **8** 243–66
- [41] Olshanetsky M A and Perelomov A M 1976 Explicit solutions of some completely integrable systems *Lett. Nuovo Cimento* **17** 97–101
- [42] Ortega J-P and Ratiu T 2004 *Momentum Maps and Hamiltonian Reduction* (Basel: Birkhäuser)

- [43] Prokofev V and Zabrodin A 2021 Elliptic solutions to matrix KP hierarchy and spin generalization of elliptic Calogero–Moser model *J. Math. Phys.* **62** 061502
- [44] Reshetikhin N 2016 Degenerately integrable systems *J. Math. Sci.* **213** 769–85
- [45] Reshetikhin N 2021 Spin Calogero–Moser models on symmetric spaces *Integrability, Quantization, and Geometry: I. Integrable Systems* ed S Novikov *et al* (Providence, RI: American Mathematical Society) pp 377–402
- [46] Reshetikhin N and Stokman J 2020  $N$ -point spherical functions and asymptotic boundary KZB equations (arXiv:2002.02251 [math.RT])
- [47] Ruijsenaars S N M 1999 Systems of Calogero–Moser type *Proceedings of the 1994 CRM-Banff Summer School: Particles and Fields* (Berlin: Springer) pp 251–352
- [48] Ruijsenaars S N M and Schneider H 1986 A new class of integrable systems and its relation to solitons *Ann. Phys., NY* **170** 370–405
- [49] Semenov-Tian-Shansky M A 1985 Dressing transformations and Poisson group actions *Publ. Res. Inst. Math. Sci.* **21** 1237–60
- [50] Smirnov R G 1997 Bi-Hamiltonian formalism: a constructive approach *Lett. Math. Phys.* **41** 333–47
- [51] Suris Y B 2003 *The Problem of Integrable Discretization: Hamiltonian Approach* (Basel: Birkhäuser)
- [52] Sutherland B 1971 Exact results for a quantum many-body problem in one dimension *Phys. Rev. A* **4** 2019–21
- [53] Zung N T 2006 Torus actions and integrable systems *Topological Methods in the Theory of Integrable Systems* (Cambridge: Cambridge Science Publishing) pp 289–328