Multiplicity of zeros of polynomials*

Vilmos Totik

May 6, 2021

Abstract

Sharp bounds are given for the highest multiplicity of zeros of polynomials in terms of their norm on Jordan curves and arcs. The results extend a theorem of Erdős and Turán and solve a problem of them from 1940.

1 Introduction

According to Chebyshev's classical theorem, if $P_n(x) = x^n + \cdots$ is a polynomial of degree n with leading coefficient 1, then

$$\|P_n\|_{[-1,1]} \geq 2^{1-n}, \tag{1.1}$$

where $\|P_n\|_{[-1,1]}$ denotes the supremum norm of P_n on $[-1, 1]$. The equality is attained for the Chebyshev polynomials $2^{1-n}\cos(n \arccos x)$. It was Paul Erdős and Paul Turán who observed that if such a P_n has zeros in $[-1, 1]$ and its norm is not too much larger than the theoretical minimum, then the zeros are distributed like the zeros of the Chebyshev polynomials. More precisely, in [7] they verified that if $P_n(x) = x^n + \cdot$ have all their zeros x_j on $[-1, 1]$, then

$$\left| \frac{\#\{x_j \in [a, b]\}}{n} - \frac{\arcsin b - \arcsin a}{\pi} \right| \leq \frac{8}{\log 3} \sqrt{\frac{\log(2^n \|P_n\|_{[-1,1]})}{n}}. \tag{1.2}$$

As an immediate consequence they obtained that if $\|P_n\|_{[-1,1]} = O(1/2^n)$, then the largest multiplicity of any zero of P_n is at most $O(\sqrt{n})$. Indeed, if a is the zero in question, then the claim follows by applying (1.2) to the degenerated interval $a = b$. In connection with this observation Erdős and Turán wrote (see the paragraph before [7, (17)]): "We are of the opinion that ... there exists a polynomial $f(z) = z^n + \cdots$ of degree n, which has somewhere in $[-1, 1]$ a root of the multiplicity $[\sqrt{n}]$ and yet the inequality $|2^n f(x)| \leq B$ in $[-1, 1]$ holds."

This paper grew out of this problem of Erdős and Turán. In general, we shall relate the largest possible multiplicity of a zero of a polynomial on a set

*AMS Classification: 30C15, 31A15; Keywords: multiplicity of zeros, polynomials, potential theory

K to its supremum norm on K. We shall need to use some basic facts from potential theory, for all these see the books [3], [5], [12].

Recall ([12, Theorem 5.5.4]) that if $P_n(z) = z^n + \cdots$ is a monic polynomial and K is a compact subset of the plane then

$$\|P_n\|_K \geq \mathrm{cap}(K)^n \tag{1.3}$$

where $\mathrm{cap}(K)$ denotes the logarithmic capacity of K. Since $\mathrm{cap}([-1,1]) = 1/2$, we can see that in (1.2) the expression $2^n \|P_n\|_{[-1,1]}$ is the quantity

$$\|P_n\|_{[-1,1]}/\mathrm{cap}([-1,1])^n,$$

thus (1.2) is an estimate of the discrepancy of the distribution of the zeros from the arcsine distribution in terms of how much larger the norm of P_n is than the n-th power of capacity. Hence, in general, we shall compare the supremum norm of P_n on a compact set K with that of $\mathrm{cap}(K)^n$, and show that the multiplicity of any zero is governed by the ratio $\|P_n\|_K/\mathrm{cap}(K)^n$. Our first result is

Theorem 1.1 *Let K be a compact set consisting of pairwise disjoint $C^{1+\alpha}$-smooth Jordan curves or arcs lying exterior to each other. Then there is a constant C such that if $P_n(z) = z^n + \cdots$ is any monic polynomial of degree at most n, then the multiplicity m of any zero $a \in K$ of P_n satisfies*

$$m \leq C\sqrt{n \log \frac{\|P_n\|_K}{\mathrm{cap}(K)^n}}. \tag{1.4}$$

In the smoothness assumption $0 < \alpha < 1$ can be any small number. Recall also that a Jordan curve is a set homeomorphic to a circle while a Jordan arc is a set homeomorphic to a segment.

It is convenient to rewrite (1.4) in the form

$$\|P_n\|_K \geq e^{cm^2/n}\mathrm{cap}(K)^n, \tag{1.5}$$

which gives a lower bound for the norm of a monic polynomial on K in turn of the multiplicity of one of its zeros on K.

Our next theorem shows that this is sharp at least when K consists of one analytic component.

Theorem 1.2 *Let K be an analytic Jordan curve or arc, let $z_n \in K$ be prescribed points and $1 \leq m_n \leq n$ prescribed multiplicities for all n. There are constants A, c such that for every n there are polynomials $P_n = z^n + \cdots$ such that z_n is a zero of P_n of multiplicity m_n, and*

$$\|P_n\|_K \leq Ae^{cm_n^2/n}\mathrm{cap}(K)^n. \tag{1.6}$$

Furthermore, when K is a Jordan curve then we can set $A = 1$, and for a Jordan arc $A = 2$.

The Erdős-Turán conjecture mentioned above is the[1] $m_n = [\sqrt{n}]$, $K = [-1,1]$ special case of Theorem 1.2.

If K is the unit circle, then $\mathrm{cap}(K) = 1$ and $P_n(z) = z^n$ has supremum norm 1 on K, so the right-hand side of (1.4) is 0 even though $z = 0$ is a zero of P_n of multiplicity n. This indicates that the zero in Theorem 1.1 must lie on K to have the estimate (1.4), and this is why in Theorems 1.1 and 1.2 we concentrated on zeros on K. Note however, that in this example $a = 0$ lies in the inner domain of K, and, as we show in the next theorem, one does not need to assume $a \in K$ so long as a does not belong to the interior domains determined by K.

Theorem 1.3 *Let K and P_n be as in Theorem 1.1, and assume that P_n has a zero of multiplicity m which does not belong to any of the inner domains determined by the Jordan curve components of K. Then (1.4) holds true with a constant C depending only on K.*

Note that if K consists only of Jordan arcs, then there is no restriction whatsoever on the location of the zero a.

For small $m_n (\ll \sqrt{n})$ the factor m_n^2/n in the exponent in (1.6) is small, and then $\exp(cm_n^2/n) \approx 1$. In this case for analytic Jordan curves, for which $A = 1$, the polynomials in (1.6) are asymptotically minimal: $\|P_n\|_K = (1 + o(1))\mathrm{cap}(K)^n$. This is no longer true for arcs: when K is an arc then there are no polynomials $P_n(z) = z^n + \cdots$ whatsoever with $\|P_n\|_K = (1 + o(1))\mathrm{cap}(K)^n$ (see [17, Theorem 1]), in particular the constant A in (1.6) cannot be 1 when K is an arc. Therefore, Theorems 1.1 and 1.2 give finer estimates for the highest multiplicity of a zero on Jordan curves than on Jordan arcs. For example, if K is an analytic Jordan curve then, in view of Theorem 1.1, a single zero on K means that $\|P_n\| \geq (1 + c/n)\mathrm{cap}(K)^n$, and, conversely, $\|P_n\| \leq (1 + O(1/n))\mathrm{cap}(K)^n$ implies that the highest multiplicity of zeros on K is bounded by a constant. There are no such results for Jordan arcs: if K is a Jordan arc, then $\|P_n\|_K/\mathrm{cap}(K)^n$ in Theorem 1.1 is at least some constant $1 + \beta > 1$ ([17, Theorem 1]), so it cannot be $1 + O(1/n)$. In this case A in Theorem 1.2 must necessarily be bigger than 1, and for $K = [-1,1]$ the precise value is $A = 2$ (see below), so in this respect Theorem 1.2 is exact.

The Erdős–Turán theorem has the shortcoming that it cannot give better discrepancy estimate than $C/\sqrt{n}$, and, as a consequence, it cannot give a better upper bound for the multiplicity of a zero than $C\sqrt{n}$. This is due to the fact that Erdős and Turán compared $\|P_n\|_{[-1,1]}$ to $\mathrm{cap}^n([-1,1])$, and not to the theoretical minimum $2^{1-n} = 2\mathrm{cap}^n([-1,1])$. In fact, in view of (1.1), the right hand side in the estimate (1.2) is always $\geq c/\sqrt{n}$, i.e., the discrepancy given in the theorem is never better than $c/\sqrt{n}$. As a consequence, no matter how close $\|P_n\|_{[-1,1]}$ is to the theoretical minimum 2^{1-n}, we do not get from (1.2) a better estimate for the multiplicity of a zero than $\leq C\sqrt{n}$. Probably if one compares $\|P_n\|_{[-1,1]}$ not to $\mathrm{cap}^n([-1,1])$ but to the theoretical minimum $2\mathrm{cap}^n([-1,1])$,

[1] In what follows $[\cdot]$ denotes integral part.

then one can get better than $1/\sqrt{n}$ discrepancy rate and better multiplicity estimate than $C\sqrt{n}$. While we are not investigating such finer discrepancy results, we do verify the corresponding finer result in connection with multiplicity of the zeros.

Theorem 1.4 *Suppose that a polynomial $P_n(x) = x^n + \cdots$ has a zero in $[-1, 1]$ of multiplicity $m \geq 2$. Then*

$$\|P_n\|_{[-1,1]} \geq 2^{1-n} e^{cm^2/n} \tag{1.7}$$

with some absolute constant c.

Conversely, there is a constant $C > 0$ such that if $x_n \in [-1, 1]$ for all n and $2 \leq m_n \leq n$ are prescribed multiplicities, then there are polynomials $P_n(x) = x^n + \cdots$, $n = 1, 2, \ldots$, such that x_n is a zero of P_n of multiplicity m_n and

$$\|P_n\|_{[-1,1]} \leq 2^{1-n} e^{Cm_n^2/n}. \tag{1.8}$$

Note that in stating (1.7) we must assume $m \geq 2$ (as opposed to the Jordan curve case in Theorem 1.1 where a single zero raises the norm away from the theoretical minimum), just consider the classical Chebyshev polynomials for which the norm on $[-1, 1]$ is precisely 2^{1-n}.

There is no similar result on a set consisting of more than one intervals. Indeed, if $E \subset \mathbf{R}$ is such a set, then, by [14], for every polynomial P_n with leading coefficient 1 we have

$$\|P_n\|_E \geq 2\text{cap}(E)^n.$$

Therefore, the analogue of (1.8) would be to have polynomials $P_n(x) = x^n + \cdots$ with a zero of multiplicity m_n on E and with

$$\|P_n\|_E \leq 2\text{cap}(E)^n e^{Cm_n^2/n}. \tag{1.9}$$

But for $m_n = o(\sqrt{n})$ this is not possible, since there are no polynomials $P_n(z) = z^n + \cdots$, $n = 1, 2, \ldots$, for which

$$\|P_n\|_E = (1 + o(1))2\text{cap}(E)^n$$

is true, because, by [17, Theorem 3], the largest limit point of the sequence

$$\min_{P_n(x)=x^n+\cdots} \frac{\|P_n\|_E}{\text{cap}(E)^n}, \quad n = 1, 2, \ldots$$

as $n \to \infty$ is bigger than 2.

All the results above assumed smoothness of the underlying curves. Some kind of smoothness assumption is necessary as is shown by

Proposition 1.5 *Let $0 < \theta < 1$. There is a Jordan curve γ such that for infinitely many n, say for $n = n_1, n_2, \ldots$, there are polynomials $P_n(z) = z^n + \cdots$ such that P_n has a zero on γ of multiplicity at least n^θ, and yet*

$$\|P_n\|_\gamma = (1 + o(1))\text{cap}(\gamma)^n, \qquad n \to \infty,\ n = n_1, n_2, \ldots,$$

where $o(1)$ tends to 0 as $n \to \infty$.

Note that this is in sharp contrast to (1.5) because for smooth curves a zero of multiplicity $> n^\theta$ implies

$$\|P_n\|_K \geq e^{cn^{2\theta-1}}\text{cap}(K)^n,$$

and here the factor $\exp(cn^{2\theta-1})$ is large for $\theta > 1/2$.

Finally, we mention that for a single component Theorem 1.1 easily follows from results of V. V. Andrievskii and H-P. Blatt in [1, Ch. 4]. As for the converse, i.e., Theorem 1.2, the key will be a construction of G. Halász [8], see Proposition 4.1 below. For the unit circle Theorem 1.1 is a direct consequence of [15, Theorem 1], and Theorem 1.2 is a consequence of the just mentioned theorem of Halász. For related results when not the leading coefficient, but a value of P_n is fixed inside K see [2], [6], [16].

2 Proof of Theorem 1.1

Let K be as in the theorem, ds the arc measure on K, μ_K the equilibrium measure of K, Ω the unbounded component of $\overline{\mathbf{C}} \setminus K$ and $g_\Omega(z, \infty)$ the Green's function of Ω with pole at infinity. In the proof of the theorem we shall need the following lemma. Choose $\varepsilon > 0$ so that the closed ε-neighborhoods of the different connected components of K are disjoint, and let Γ be one of the connected components of K.

Lemma 2.1 I. If Γ is a Jordan curve, *then in the ε-neighborhood of Γ we have in the exterior of Γ the estimates*

$$c_0\text{dist}(z, \Gamma) \leq g_\Omega(z, \infty) \leq C_0\text{dist}(z, \Gamma) \tag{2.1}$$

with some positive constants c_0, C_0.

Furthermore, $d\mu_K/ds$ is continuous and positive on Γ.

II. If Γ is a Jordan arc, *then in the ε-neighborhood of Γ the Green's function behaves as described below. Let P, Q be the endpoints of Γ, let $Z \in \Gamma$ be (one of) the closest point to z in Γ, and assume that P is closer to Z than Q. Then*

$$c_0 H(z) \leq g_\Omega(z, \infty) \leq C_0 H(z) \tag{2.2}$$

with some positive constants c_0, C_0, where

$$H(z) = \begin{cases} \sqrt{|z-P|} & \text{if } |Z-P| \leq |z-Z|, \\ \text{dist}(z,\Gamma)/\sqrt{|Z-P|} & \text{if } |Z-P| > |z-Z|. \end{cases} \tag{2.3}$$

Furthermore,

$$\frac{d\mu(z)}{ds} \sim \frac{1}{\sqrt{|z-P|}} \tag{2.4}$$

on the "half" of Γ *that lies closer to* P *than to* Q.

In particular, if J is a subarc of Γ, then $\mu_K(J) \sim \sqrt{|J|}$ if J lies closer to P than its length $|J|$, while $\mu_K(J) \sim |J|/\sqrt{\mathrm{dist}(J,P)}$ in the opposite case (all this on the "half" of Γ that lies closer to P than to Q).

Here and in what follows, $A \sim B$ means that the ratio A/B is bounded away from 0 and infinity.

Lemma 2.1 is folklore, for completeness we shall give a short proof for it in the Appendix at the end of this paper.

Now let us proceed with the proof of Theorem 1.1.

First we mention that

$$n \log \mathrm{cap}(K) \le \int_K \log |P_n(z)| d\mu_K(t). \tag{2.5}$$

Indeed, from well-known properties of equilibrium measures (see e.g. [13, (I.4.8)] or [12, Sec. 4.4])

$$\int \log|z-t| d\mu_K(z) = \begin{cases} \log \mathrm{cap}(K) & \text{if } z \text{ lies in Pc(K)} \\ \log \mathrm{cap}(K) + g_\Omega(z,\infty) & \text{otherwise,} \end{cases} \tag{2.6}$$

where $\mathrm{Pc}(K) = \overline{\mathbf{C}} \setminus \Omega$ denotes the polynomial convex hull of K, which is the union of K with all the bounded connected components of $\mathbf{C} \setminus K$. Hence the left-hand side is always at least $\log \mathrm{cap}(K)$, which proves the inequality in (2.5) if we write $\log|P_n(z)|$ in the form $\sum_j \log|z - z_j|$ with the zeros z_j of P_n.

Let a be a zero of P_n on K of multiplicity m. Then a belongs to a component Γ of K, and first we consider the case when Γ is a Jordan curve.

Case I: Γ is a Jordan curve. Then in the ε-neighborhood of Γ as in Lemma 2.1

$$g_{\overline{\mathbf{C}}\setminus\Gamma}(\zeta,\infty) \le C_0 \mathrm{dist}(\zeta,\Gamma),$$

and for other ζ this is automatically true (if we increase C_0 somewhat if necessary). Hence, by the Bernstein-Walsh lemma [18, p. 77], for $|\zeta - a| \le \rho$ we have

$$|P_n(\zeta)| \le e^{ng_\Omega(\zeta,\infty)} \|P_n\|_K \le e^{C_0 n\rho} \|P_n\|_K. \tag{2.7}$$

Recall also that, by Cauchy's formula,

$$P_n^{(m)}(w) = \frac{m!}{2\pi i} \int_{|\zeta - w| = \rho/2} \frac{P_n(\zeta)}{(\zeta - w)^{m+1}} d\zeta \tag{2.8}$$

with integration on the circle with center at w and of radius $\rho/2$. As a consequence, for $|w-a| \le \rho/2$ we obtain

$$|P_n^{(m)}(w)| \le e^{C_0 n\rho} m! \frac{1}{(\rho/2)^m} \|P_n\|_K, \tag{2.9}$$

and here $\rho > 0$ is arbitrary.

Since $P_n(z)$ has a zero at a of order m, we can write

$$P_n(z) = \int_a^z \int_a^{w_1} \cdots \int_a^{w_{m-1}} P_n^{(m)}(w) dw dw_{m-1} \cdots dw_1 \tag{2.10}$$

with integration along the segment connecting a and z. Hence, for $z \in \Gamma$, $|z - a| \le \rho/2$ we have (note that during m-fold integration the factor $1/m!$ emerges)

$$|P_n(z)| \le e^{C_0 n\rho} m! \frac{1}{(\rho/2)^m} \frac{|a-z|^m}{m!} \|P_n\|_K \le e^{C_0 n \rho} \left(\frac{|a-z|}{\rho/2} \right)^m \|P_n\|_K. \tag{2.11}$$

Now this gives[2] for $\rho = m/n$ and $|z - a| \le (m/n)/2e \cdot e^{C_0}$

$$|P_n(z)| \le \left(\frac{1}{e} \right)^m \|P_n\|_K,$$

i.e., on the arc J of Γ on which $|a - z| \le (m/n)/2e \cdot e^{C_0}$, the estimate

$$\log |P_n(z)| \le \log \|P_n\|_K - m \tag{2.12}$$

holds. Elsewhere we use $|P_n(z)| \le \|P_n\|_K$. The μ_K-measure of J is $\ge c_1(m/n)/e \cdot e^{C_0}$ with some c_1 depending only on K (see Lemma 2.1), hence we obtain from (2.5) and (2.12)

$$\begin{aligned} n \log \operatorname{cap}(K) &\le \int \log |P_n| d\mu_K \le \log \|P_n\|_K - \big(c_1(m/n)/e \cdot e^{C_0}\big) m \\ &\le \log \|P_n\|_K - c_2 m^2/n, \end{aligned} \tag{2.13}$$

which proves (1.5).

Case II: Γ is a Jordan arc. The proof is along the previous lines, though the computations are somewhat more complicated. Suppose that P is the endpoint of Γ that lies closer to a than the other endpoint, and let d be the distance from a to P. First consider the case when $d \le (m/n)^2$. In that case we set $\rho = (m/n)^2$. In this situation (i.e., a lies closer to P than ρ) if $|\zeta - a| \le \rho$, then, by Lemma 2.1, $g_{\mathbf{C}\setminus\Gamma}(\zeta, \infty) \le C_0\sqrt{2\rho}$, so instead of (2.7) and (2.9) we have for $|w - a| \le \rho/2$

$$|P_n^{(m)}(w)| \le e^{2C_0 n\sqrt{\rho}} m! \frac{1}{(\rho/2)^m} \|P_n\|_K, \tag{2.14}$$

and, as a consequence, instead of (2.11) we derive for $|z - a| \le \rho/2$ the estimate

$$|P_n(z)| \le e^{2C_0 n\sqrt{\rho}} m! \frac{1}{(\rho/2)^m} \frac{|a-z|^m}{m!} \|P_n\|_K \le e^{2C_0 n\sqrt{\rho}} \left(\frac{|a-z|}{\rho/2} \right)^m \|P_n\|_K. \tag{2.15}$$

[2]We may assume that $m/n \le \varepsilon$, for the $m/n > \varepsilon$ case of Theorem 1.1 follows from its $m = [\varepsilon n]$ case. The same remark applies in similar situations to be discussed below.

Since $\rho = (m/n)^2$, on the arc J of Γ on which $|a-z| \leq (m/n)^2/2e \cdot e^{2C_0}$ we have (2.12). The μ_K-measure of J in this case is

$$\mu_K(J) \geq c_1\sqrt{|J|} \geq c_1(m/n)/\sqrt{2e \cdot e^{2C_0}},$$

hence (2.13) is true again, and that proves the claim in the theorem.

The just given proof works also when $a = P$, i.e when $d = 0$.

Finally, let us assume that $d > (m/n)^2$, in which case we set $\rho = (m/n)\sqrt{d}$. Now for $|\zeta - a| = \rho$ we have $g_{\mathbf{C}\setminus\Gamma}(\zeta,\infty) \leq C_0\rho/\sqrt{d}$ (see Lemma 2.1), so instead of (2.9) and (2.14) we get for $|w-a| \leq \rho/2$ the inequality

$$|P_n^{(m)}(w)| \leq e^{C_0 n\rho/\sqrt{d}} m! \frac{1}{(\rho/2)^m} \|P_n\|_K, \tag{2.16}$$

and instead of (2.11) and (2.15) we have for $|z-a| \leq \rho/2$

$$|P_n(z)| \leq e^{C_0 n\rho/\sqrt{d}} m! \frac{1}{(\rho/2)^m} \frac{|a-z|^m}{m!} \|P_n\|_K \leq e^{C_0 n\rho/\sqrt{d}} \left(\frac{|a-z|}{\rho/2}\right)^m \|P_n\|_K. \tag{2.17}$$

Since $\rho = (m/n)\sqrt{d}$, we obtain that on the arc J of Γ on which

$$|a-z| \leq (m/n)\sqrt{d}/2e \cdot e^{C_0}$$

we have (2.12). The μ_K-measure of J in this case is

$$\mu_K(J) \geq c_1|J|/\sqrt{d} \geq c_1(m/n)/2e \cdot e^{C_0},$$

hence (2.13) is true again, which proves the theorem.

■

3 Proof of Theorem 1.3

As before, let Ω be the unbounded component of $\overline{\mathbf{C}} \setminus K$. The assumption in the theorem on the location of the zero a is equivalent to $a \in \overline{\Omega} = K \cup \Omega$. Let $\varepsilon > 0$ be again a small number such that the closed ε-neighborhoods of the different connected components of K do not intersect. The Green's function $g_\Omega(z,\infty)$ has a positive lower bound in Ω away from K, so there is a $\beta > 0$ such that if $a \in K \cup \Omega$ does not belong to the ε-neighborhood of K, then $g_\Omega(a,\infty) > \beta$. Hence we obtain from (2.6)

$$\int \log|P_n| d\mu_K \geq n \log \operatorname{cap}(K) + m\beta,$$

which implies

$$\|P_n\| \geq e^{m\beta} \operatorname{cap}(K)^n,$$

and that is stronger than (1.5).

Thus, in what follows we may assume that a lies closer than ε to K, say lies closer than ε to the component Γ of K.

Case I: Γ is a Jordan curve. Let $A \in \Gamma$ be (one of) the closest point to a in Γ. We fix a small $\theta < 1/2$ to be determined below, and we distinguish two cases.

Case 1: $|a - A| \le \theta(m/n)$. In this case we case we follow the proof of Theorem 1.1. As there, we set $\rho = (m/n)$. We have the analogue of (2.7):

$$|P_n(\zeta)| \le e^{ng_\Omega(\zeta,\infty)}\|P_n\|_K \le e^{C_0 n\rho}\|P_n\|_K, \qquad |\zeta - A| \le \rho,$$

and from here we get as in (2.9)

$$|P_n^{(m)}(w)| \le e^{C_0 n\rho} m! \frac{1}{(\rho/2)^m}\|P_n\|_K, \qquad |w - A| \le \rho/2. \tag{3.1}$$

Now if $|z - A| \le \rho/2$ and z belongs to Γ, then integrating along the segment connecting a and z we obtain as in (2.10)–(2.11) from (3.1) and from the fact that a is a zero of P_n of multiplicity m the estimate

$$|P_n(z)| \le e^{C_0 n\rho}\left(\frac{|a - z|}{\rho/2}\right)^m \|P_n\|_K. \tag{3.2}$$

This gives for $\rho = m/n$ and $|a - z| \le (m/n)/2e \cdot e^{C_0}$

$$|P_n(z)| \le \left(\frac{1}{e}\right)^m \|P_n\|_K,$$

i.e., on the arc J of Γ for which $|a - z| \le (m/n)/2e \cdot e^{C_0}$, we have

$$\log|P_n(z)| \le \log\|P_n\|_K - m. \tag{3.3}$$

However, if $|a - A| \le \theta(m/n)$ and here $\theta = 1/4e \cdot e^{C_0}$, then every $z \in \Gamma$ with $|z - A| \le \theta(m/n)$ belongs to J, so we have (3.3) at those points. Since the μ_K-measure of these points is $\ge c_1\theta(m/n)$ with some $c_1 > 0$, we obtain (2.13) in the form

$$n \log \operatorname{cap}(K) \le \log\|P_n\|_K - c_2 m^2/n, \tag{3.4}$$

and that proves (1.5).

This argument used $\theta = 1/4e \cdot e^{C_0}$, and that is how we choose θ.

Case 2: $|a - A| \ge \theta(m/n)$. In this case, in view of Lemma 2.1, we have $g_\Omega(a, \infty) \ge c_0\theta(m/n)$, so (2.6) yields

$$\int \log|P_n| d\mu_K \ge n \log \operatorname{cap}(K) + mc_0\theta(m/n),$$

which gives again (1.5).

Case II: Γ is a Jordan arc, with endpoints, say, P and Q. In this case the behavior of the Green's function g_Ω and of the equilibrium measure is described in the second part of Lemma 2.1.

Let again A be a closest point in Γ to a, and let the endpoint P be closer to A than the other endpoint of Γ.

If $d = |A - P|$ is the distance from A to P, then we distinguish three cases.

Case 1: $d \le (m/n)^2$. Set $\rho = (m/n)^2$ and choose again a small $\theta > 0$ as below.

If $|a - A| \le \theta(m/n)^2$, then follow the proof for Theorem 1.1 for the Jordan arc case. As there, for $|w - A| \le \rho/2$ we obtain

$$|P_n^{(m)}(w)| \le e^{2C_0 n\sqrt{\rho}} m! \frac{1}{(\rho/2)^m} \|P_n\|_K$$

(see (2.14)) and for $|A - z| \le \rho/2$

$$|P_n(z)| \le e^{2C_0 n\sqrt{\rho}} \left(\frac{|a - z|}{\rho/2}\right)^m \|P_n\|_K$$

(see (2.15)). Since $\rho = (m/n)^2$, on the arc J of Γ on which

$$|a - z| \le (m/n)^2/2e \cdot e^{2C_0} \tag{3.5}$$

we have (2.12). But if $\theta = 1/4e \cdot e^{2C_0}$, then every point $z \in \Gamma$ with $|z - A| \le \theta(m/n)^2$ satisfies (3.5) and the μ_K-measure of these points is $\ge c_1\sqrt{\theta}(m/n)$, hence (2.13) is true again, proving (1.5).

If, on the other hand $|a - A| \ge \theta(m/n)^2$, then in view of (2.2)–(2.3) and (2.6) we obtain

$$\int \log|P_n| d\mu_K \ge n \log \operatorname{cap}(K) + m\tilde{c}_0\sqrt{\theta}(m/n)$$

with some constant $\tilde{c}_0 > 0$ (consider separately when $d \le |a - A|$ and when $|a - A| < d$) implying again (1.5).

Case 2: $d > (m/n)^2$ and $|a - A| \le d$. In this case we set $\rho = (m/n)\sqrt{d}$ and select again a small $\theta > 0$ as below.

If $|a - A| \le \theta\rho$, then, as before, follow the proof of Theorem 1.1 leading to (2.16) and (2.17). We get as in (2.17)

$$|P_n(z)| \le e^{C_0 n\rho/\sqrt{d}} \left(\frac{|a - z|}{\rho/2}\right)^m \|P_n\|_K \tag{3.6}$$

for $|z - A| \le \rho/2$. Therefore, for $\theta = 1/4e \cdot e^{C_0}$ and for $|z - a| \le \theta\rho$ the inequality

$$|P_n(z)| \le \left(\frac{1}{e}\right)^n \|P_n\|_K$$

holds for all

$$z \in J := \{z \in \Gamma \,|\, |A - z| \le \theta\rho\}.$$

So in this case (2.12) is true on J, and since

$$\mu_K(J) \ge c_1|J|/\sqrt{d} \ge c_1\theta(m/n),$$

we conclude (2.13), and that proves (1.5).

If, however, $d \geq |a - A| > \theta\rho$, then, in view of (2.2)–(2.3)

$$g_\Omega(z, \infty) \geq c_0 |a - A| / \sqrt{d},$$

and we obtain from (2.6)

$$\begin{aligned} \int \log |P_n| d\mu_K &\geq n \log \operatorname{cap}(K) + m c_0 |a - A| / \sqrt{d} \\ &\geq n \log \operatorname{cap}(K) + m c_0 \theta (m/n) \sqrt{d} / \sqrt{d} \end{aligned}$$

and (1.5) follows.

Case 3: $|a - A| > d > (m/n)^2$. In view of Lemma 2.1 we have then

$$g_\Omega(a, \infty) \geq c_0 \sqrt{|a - P|} \geq c_0 \sqrt{|a - A|} \geq c_0 (m/n),$$

so we get from (2.6)

$$\int \log |P_n| d\mu_K \geq n \log \operatorname{cap}(K) + m c_0 (m/n)$$

giving again (1.5).

■

4 Proof of Theorem 1.2

We need to extend the following theorem of Gábor Halász.

Proposition 4.1 *For every n there is a polynomial $Q_n(z) = z^n + \cdots$ such that Q_n has a zero at 1, and*

$$\|Q_n\|_{C_1} \leq e^{2/n}, \tag{4.1}$$

where C_1 denotes the unit circle.

We are going to show the following variant.

Proposition 4.2 *If γ is an analytic Jordan curve, then there is a C such that if $z_0 \in \gamma$ is given, then for every n there are polynomials $S_n(z) = z^n + \cdots$ which have a zero at z_0 and for which*

$$\|S_n\|_\gamma \leq e^{C/n} \operatorname{cap}(\gamma)^n.$$

Proof. The claim can be reduced to Halász' result by the Faber-type argument given below. For large n the construction gives $C = 5$ independently of the curve γ.

First of all, for the Q_n in Halász' result we may assume that they decrease geometrically in n on compact subsets of the open unit disk on the price that in (4.1) the exponent $2/n$ is replaced by $4/n$. In fact, it is enough to consider $Q_n^*(z) = z^{[n/2]} Q_{[(n+1)/2]}(z)$. For these we have $Q_n^*(1) = 0$,

$$\|Q_n^*\|_{C_1} \leq e^{4/n} \tag{4.2}$$

and

$$|Q_n^*(z)| \leq C(\sqrt{r})^n, \qquad \text{if } |z| \leq r < 1. \tag{4.3}$$

By simple rotation, i.e., considering $Q_{n,\zeta}^*(z) = \zeta^n Q_n(\zeta^{-1} z)$, the zero can be moved from 1 to any point ζ of the unit circle.

Now let γ be an analytic Jordan curve, and let Φ the conformal map from the exterior Ω of γ onto the exterior $\overline{\mathbf{C}} \setminus \overline{\Delta}$ of the unit disk that leaves the point infinity invariant. Without loss of generality we may assume γ to have logarithmic capacity 1, in which case the Laurent expansion of Φ around the point ∞ is of the form $\Phi(z) = z + c_0 + c_{-1}/z + \cdots$. Since γ is analytic, Φ can be extended to some domain that contains γ (see [11, Proposition 3.1]), hence for $r < 1$ sufficiently close to 1 the level set $\gamma_r := \{z \mid |\Phi(z)| = r\}$ is defined, and it is an analytic curve inside γ. Fix such an r. Let the image of z_0 under Φ be $\zeta \in C_1$, and consider the polynomial S_n^* which is the polynomial part of $Q_{n,\zeta}^*(\Phi(z))$. Set $R_n^*(z) = Q_{n,\zeta}^*(\Phi(z)) - S_n^*(z)$, which is the Laurent-part of $Q_{n,\zeta}^*(\Phi(z))$. By Cauchy's formula we have for $z \in \gamma$

$$R_n^*(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{Q_{n,\zeta}^*(\Phi(\xi))}{\xi - z} d\xi \tag{4.4}$$

with clockwise orientation on γ_r (note that the corresponding integral with $Q_{n,\zeta}^*(\Phi(\xi))$ replaced by $S_n^*(\xi)$ vanishes since then the integrand is analytic inside γ_r), and since γ_r is mapped by Φ into the circle $|z| = r < 1$, (4.3) shows that $R_n^*(z)$ is exponentially small on γ: $|R_n^*(z)| \leq C\sqrt{r}^n$. Now

$$S_n(z) := S_n^*(z) + R_n^*(z_0) = Q_{n,\zeta}^*(\Phi(z)) - R_n^*(z) + R_n^*(z_0)$$

is a monic polynomial of degree n, on γ it has norm

$$\leq e^{4/n} + 2C\sqrt{r}^n \leq e^{C/n},$$

and $S_n(z_0) = Q_{n,\zeta}^*(\Phi(z_0)) = Q_{n,\zeta}^*(\zeta) = 0$.

∎

Based on the polynomials S_n from Proposition 4.2, the proof of Theorem 1.2 for an analytic curve K is now easy. Set $\gamma = K$ and with the just constructed S_n for γ and z_n define $P_n(z) = S_{[n/m_n]}(z)^{m_n}$. $P_n(z)$ is a monic polynomial, but its degree may not be n, it is $[n/m_n]m_n =: n - k$ with some $0 \leq k < m_n$. To have exact degree n suitably modify one of the factors in P_n, i.e., use $S_{[n/m_n]+k}(z)$ instead of $S_{[n/m_n]}(z)$. Since

$$\|P_n\|_\gamma \leq \left(e^{C/[n/m_n]}\right)^{m_n} \leq e^{2Cm_n^2/n},$$

it is clear that P_n satisfies (1.6) with $A = 1$, and it has at z_n a zero of multiplicity m_n.

We still need to consider the case when K is an analytic arc γ. First assume that z_n is not one of the endpoints of γ. We may assume that the endpoints of γ are ± 2, and consider the standard mapping $Z = \frac{1}{2}(z + \sqrt{z^2 - 4})$, where we take that branch (analytic on $\mathbf{C} \setminus \gamma$) of $\sqrt{z^2 - 4}$ for which $Z \sim z$ for $|z| \sim \infty$. This "opens up" γ, and it maps γ into a Jordan curve Γ (cf. [19, p. 206 and Lemma 11.1]) with the same logarithmic capacity as γ (and maps $\mathbf{C} \setminus \gamma$ into the unbounded component of $\mathbf{C} \setminus \Gamma$). Furthermore, it is not difficult to show that if γ is analytic then so is Γ, see e.g. the discussion in [9, Proposition 5]. The point z_n is considered to belong to both sides of γ, and then it is mapped into two points $Z_n^{\pm}$ on Γ, for which $Z_n^- = 1/Z_n^+$. Now for each of these points and for the analytic Jordan curve Γ construct the polynomials P_n above but for degree $[n/2]$ (more precisely, for one of them of degree $[n/2]$ and for the other one of degree $[(n+1)/2]$ to have precise degree n in their product), let these be $P_n^{\pm}$. Thus, P_n^+ has a zero at Z_n^+ of multiplicity m_n, P_n^- has a zero at Z_n^- of multiplicity m_n, and their norm on Γ is at most

$$\exp(Cm_n^2/[n/2])\mathrm{cap}(\Gamma)^{[n/2]} \le \exp(3Cm_n^2/n)\mathrm{cap}(\Gamma)^{[n/2]}$$

respectively

$$\exp(Cm_n^2/[(n+1)/2])\mathrm{cap}(\Gamma)^{[(n+1)/2]} \le \exp(2Cm_n^2/n)\mathrm{cap}(\Gamma)^{[(n+1)/2]}.$$

Consider now the product

$$P_n^*(Z) = P_n^+(Z)P_n^-(Z) = Z^n + \cdots,$$

which has a zero of multiplicity m_n at both $Z^{\pm}$, and it has norm

$$\|P_n^*\|_K \le \exp(5Cm_n^2/n)\mathrm{cap}(\Gamma)^n.$$

Note that $z \to \frac{1}{2}(z - \sqrt{z^2 - 4}) = 1/Z$ also maps γ into Γ (mapping $\mathbf{C} \setminus \gamma$ into the bounded component of $\mathbf{C} \setminus \Gamma$) and z_n is mapped by this mapping again into $Z_n^{\pm}$ (but the images of the two sides of γ are interchanged, i.e., if z_n on one side of γ was mapped into Z_n^+ by $z \to \frac{1}{2}(z + \sqrt{z^2 - 4})$, then under this second mapping it is mapped into $Z_n^- = 1/Z_n^+$). Now

$$P_n(z) = P_n^*\left(\frac{1}{2}(z + \sqrt{z^2 - 4})\right) + P_n^*\left(\frac{1}{2}(z - \sqrt{z^2 - 4})\right)$$

is a polynomial of degree n with leading coefficient 1 (just consider its behavior at ∞), and for its norm on γ we have

$$\|P_n\|_\gamma \le 2\|P_n^*\|_\Gamma \le 2\exp(5Cm_n^2/n)\mathrm{cap}(\Gamma)^n = 2\exp(5Cm_n^2/n)\mathrm{cap}(\gamma)^n.$$

Finally, since $Z_n^{\pm} = 1/Z_n^{\mp}$ and since $(Z - Z_n^{\pm})^{m_n}$ are factors in P_n^*, and as $z \to z_n$ we have

$$z + \sqrt{z^2 - 4} = Z \to Z_n^+, \qquad z - \sqrt{z^2 - 4} = 1/Z \to Z_n^-$$

or $Z \to Z_n^-$, $1/Z \to Z_n^+$ (depending on which side of γ the point z is approaching z_n), and then, since z_n is not an endpoint,

$$|z - z_n| \sim |Z - Z_n^+| \sim \left| \frac{1}{Z} - Z_n^- \right|$$

resp.

$$|z - z_n| \sim |Z - Z_n^-| \sim \left| \frac{1}{Z} - Z_n^+ \right|,$$

it follows that $P_n(z)$ divided $(z - z_n)^{m_n}$ is bounded around z_n, hence z_n is a zero of P_n of multiplicity m_n.

If z_n coincides with one of the endpoints, say $z_n = 2$, then the preceding $\sim$ relations are not true and we have instead e.g.

$$|z - z_n| \sim |Z - Z_n^+|^2 \sim \left| \frac{1}{Z} - Z_n^- \right|^2.$$

But since then $Z_+ = Z_- = 2$ is also satisfied, we get again a zero of multiplicity m_n at $z_n = 2$.

■

5 Proof Theorem 1.4

Since $\mathrm{cap}([-1, 1]) = 1/2$, the second part follows from (1.6) with $A = 2$. Therefore, we shall deal only with the first part (which is *not covered* by Theorem 1.1).

Suppose that a is a zero of P_n of multiplicity $m \geq 2$. We set $\nu = [m/2]$, so P_n has a zero at a of multiplicity $\geq 2\nu$. The idea of the proof is to transform $[(\nu + 1)/2]$ of the zeros at a to the point 1 without raising the norm, and then to get a lower estimate for the norm on $[-1, 1]$ from the information that 1 is a zero of multiplicity $\geq [(\nu + 1)/2]$. This will be carried out in several steps.

Step 1. The point a lies in an interval $[\cos(\pi(k + 1)/n), \cos(k\pi/n)]$, $0 \leq k < n$. If a coincides with one of the endpoints, then go to Step 2 setting there $S_n = P_n$, otherwise let

$$\varepsilon = \min(a - \cos(\pi(k + 1)/n), \cos(k\pi/n) - a)$$

and

$$S_n(x) = \frac{P_n(x)}{(x - a)^{2\nu}}(x - a - \varepsilon)^\nu (x - a + \varepsilon)^\nu.$$

This is a polynomial of degree n with leasing coefficient 1 which has a zero either at $\cos(\pi(k + 1)/n)$ or at $\cos(\pi k/n)$ of multiplicity at least ν. We claim that $\|S_n\|_{[-1,1]} \le \|P_n\|_{[-1,1]}$. Indeed, it is clear that $|S_n(x)| \le |P_n(x)|$ for all $x \notin (a-\varepsilon, a+\varepsilon)$, so it is sufficient to show that $|S_n|$ takes its maximum in $[-1,1]$ on the set $[-1, a-\varepsilon] \cup [a+\varepsilon, 1]$. For that purpose it is sufficient to prove that if

$$S_{n,\varepsilon'}(x) = \frac{P_n(x)}{(x-a)^{2\nu}}(x-a-\varepsilon')^\nu (x-a+\varepsilon')^\nu, \qquad 0 < \varepsilon' < \varepsilon,$$

then $|S_{n,\varepsilon'}|$ takes its maximum in $[-1,1]$ only on the set $[-1, a-\varepsilon'] \cup [a+\varepsilon', 1]$, for then the claim for S_n follows by letting ε' tend to ε.

Now suppose to the contrary that $|S_{n,\varepsilon'}|$ takes its maximum in $[-1,1]$ somewhere in $(a-\varepsilon', a+\varepsilon')$, say at the point b. Then the trigonometric polynomial $S_{n,\varepsilon'}(\cos t)$ takes its maximum modulus on $\mathbf{R}$ at the point $\arccos b \in \big(\arccos(a+\varepsilon'), \arccos(a-\varepsilon')\big)$, so, by Riesz' lemma ([4, 5.1.E13]) it cannot have a zero in the interval $(\arccos b - \pi 2/n, \arccos b + \pi/2n)$. However,

$$\frac{k\pi}{n} < \arccos(a+\varepsilon') < \arccos b < \arccos(a-\varepsilon') < \frac{(k+1)\pi}{n},$$

so either $\big(\arccos(a-\varepsilon') - \arccos b\big)$ or $\big(\arccos b - \arccos(a+\varepsilon')\big)$ is smaller than $\pi/2n$. Thus, we obtain a contradiction to Riesz' lemma because $S_{n,\varepsilon'}(\cos t)$ is zero at $\arccos(a \pm \varepsilon')$, and this contradiction proves the claim.

Thus, S_n has a zero either at $\cos(\pi(k+1)/n)$ or at $\cos(\pi k/n)$ of multiplicity at least ν, and its supremum norm on $[-1,1]$ is at most as large as the norm of P_n. For definiteness assume e.g. that S_n has a zero at $\cos(\pi k/n)$ of multiplicity at least ν.

Step 2. Define

$$T_n(t) = S_n(\cos t) = (\cos t)^n + \cdots = 2^{1-n}\cos nt + \cdots.$$

This is an even trigonometric polynomial of degree n which has a zero at $k\pi/n$ of multiplicity at least ν. Then

$$\tilde{T}_n(t) = T_n(t + k\pi/n) = 2^{1-n}\cos(n(t+k\pi/n)) + \cdots = (-1)^k 2^{1-n}\cos nt + \cdots$$

is a trigonometric polynomial (not necessarily even) of degree n which has a zero at 0 of multiplicity at least ν. Then the same is true of $\tilde{T}_n(-t)$, and hence also of

$$T_n^*(t) = \frac{1}{2}(T_n(t) + T_n(-t)) = (-1)^k 2^{1-n}\cos nt + \cdots,$$

which is already an even trigonometric polynomial of degree at most n. However, the multiplicity of a zero at 0 of an even trigonometric polynomials is necessarily even, so T_n^* has a zero at 0 of multiplicity at least $2[(\nu+1)/2] \ge 2$, which means that $T_n^*(t)/(\cos t - 1)^{[(\nu+1)/2]}$ is bounded around 0.

Therefore, by setting

$$R_n(x) = (-1)^k T_n^*(\arccos x) = x^n + \cdots$$

we get a monic polynomial of degree n which has a zero at $x = 1$ of multiplicity at least $\kappa := [(\nu + 1)/2]$.

Note that this R_n has norm

$$\|R_n\|_{[-1,1]} \leq \|S_n\|_{[-1,1]} \leq \|P_n\|_{[-1,1]}.$$

Step 3. From now on we work with the monic polynomial R_n which has a zero at 1 of multiplicity $\geq \kappa = [(\nu + 1)/2]$. By the Bernstein-Walsh lemma ([18, p. 77]) we have for all z

$$|R_n(z)| \leq \|R_n\|_{[-1,1]} |z + \sqrt{z^2 - 1}|^n,$$

hence if $0 < \rho < 1$ is given, then

$$|R_n(z)| \leq \|R_n\|_{[-1,1]} (1 + 3\sqrt{\rho})^n \leq \|R_n\|_{[-1,1]} e^{3\sqrt{\rho} n}$$

for $|z - 1| \leq \rho$. So, by Cauchy's integral formula for he κ-th derivative using integration over the circle with center at t and of radius $\rho/2$ (cf. (2.8)), we get for $1 \leq t \leq 1 + \rho/2$ the bound

$$|R^{(\kappa)}(t)| \leq \|R_n\|_{[-1,1]} \kappa! \frac{e^{3\sqrt{\rho} n}}{(\rho/2)^\kappa},$$

and hence for $x \in [1, 1 + \rho/8]$

$$\begin{aligned} |R_n(x)| &= \left| \int_1^x \int_1^{x_1} \cdots \int_1^{x_{\kappa-1}} R_n(t)^{(\kappa)} dt dx_{k-1} \cdots dx_1 \right| \\ &\leq \|R_n\|_{[-1,1]} \kappa! \frac{e^{3\sqrt{\rho} n}}{(\rho/2)^\kappa} \frac{(x-1)^\kappa}{\kappa!} \leq \|R_n\|_{[-1,1]} \left(\frac{1}{4}\right)^\kappa e^{3\sqrt{\rho} n}. \end{aligned}$$

By selecting here $\rho = (\kappa/3n)^2$ we obtain that

$$|R_n(x)| \leq \|R_n\|_{[-1,1]} \qquad \text{for} \quad x \in [1, 1 + (\kappa/3n)^2/8],$$

i.e., if $I = [-1, 1 + (\kappa/3n)^2/8]$, then

$$\|R_n\|_I \leq \|R_n\|_{[-1,1]} \leq \|P_n\|_{[-1,1]}.$$

Now

$$\|P_n\|_{[-1,1]} \geq 2^{1-n} \exp\left(\frac{\kappa^2}{n288}\right)$$

follows because, by Chebyshev's theorem,

$$\|R_n\|_I \geq 2 \left(\frac{|I|}{4}\right)^n = 2 \left(\frac{1}{2} + \left(\frac{\kappa}{n}\right)^2 \frac{1}{288}\right)^n = 2^{1-n} \left(1 + \frac{\kappa^2}{n^2 144}\right)^n$$

and because $1 + \tau \geq e^{\tau/2}$ for $0 \leq \tau \leq 1$. Since here $\kappa = [(\nu+1)/2] \geq \nu/2 \geq m/4$, the inequality (1.7) has been proven with $c = 1/4 \cdot 288$.

■

6 Proof of Proposition 1.5

We sketch the construction. We shall consider Jordan curves σ with 2π-periodic parametrizations $\sigma : \mathbf{R} \to \mathbf{C}$, where σ is a continuous 2π-periodic function which maps $[0, 2\pi)$ in a one-to-one manner into the complex plane. We shall often use σ also for the range $\{\sigma(t) \mid t \in \mathbf{R}\}$. The curve σ is analytic if $\sigma(t)$, $t \in \mathbf{R}$, is analytic and $\sigma' \neq 0$. First we show the following.

Lemma 6.1 *Let σ be an analytic Jordan curve and $\varepsilon > 0$, $0 < \theta < 1$. There are an analytic Jordan curve σ^*, a point $Z^* \in \sigma^*$, a natural number n and a polynomial $P_n^*(z) = z^n + \cdots$ such that*

(i) *Z^* is a zero of P_n^* of multiplicity at least n^θ,*

(ii) *$|\sigma(t) - \sigma^*(t)| < \varepsilon$ for all $t \in \mathbf{R}$, and*

(iii) $\|P_n^*\|_{\sigma^*} < (1+\varepsilon)\mathrm{cap}(\sigma^*)^n$.

Furthermore, there is an $\eta^* > 0$ such that if γ is a Jordan curve with $|\gamma - \sigma^*| < \eta^*$, then there are a point $Z \in \gamma$, $|Z - Z^*| < \eta^*$, and a polynomial $P_n(z) = z^n + \cdots$ such that Z is a zero of P_n of multiplicity at least n^θ, and

$$\|P_n\|_\gamma < (1+\varepsilon)\mathrm{cap}(\gamma)^n.$$

Proof. Without loss of generality we may assume $\mathrm{cap}(\sigma) = 1$ and $\theta > 1/2$. Consider a conformal map Φ from the exterior of σ onto the exterior of the unit circle that leaves the point ∞ invariant. As in the proof of Theorem 1.2 this Φ can be extended to a conformal map of a domain G that contains σ, and let γ_r be the inverse image under Φ of the circle $\{z \mid |z| = r\}$ for some $r < 1$ lying close to 1. For a positive integer m let S_m be the polynomial part of $\Phi(z)^m$ — it is a monic polynomial. As in (4.4) we have the representation

$$\Phi(z)^m - S_m(z) = \frac{1}{2\pi i}\int_{\gamma_r} \frac{\Phi(\xi)^m}{\xi - z} d\xi \tag{6.1}$$

for all z lying outside γ_r, so at every such point the left-hand side is $O(r^m)$ in absolute value. This gives

$$|S_m(z)| \leq 1 + C_1 r^m, \qquad z \in \sigma,$$

with some C_1 independent of m.

Let $\tau < \varepsilon/6$ be a small positive number, and $\tilde{Z} \in G$ a point inside σ and outside γ_r the distance of which to σ is smaller than τ. Then (6.1) gives with some C_2 the bound $|\Phi(\tilde{Z})^m - S_m(\tilde{Z})| \leq C_2 r^m$, and since $|\Phi(\tilde{Z})| < 1$, we obtain $|S_m(\tilde{Z})| \leq C_3 r_1^m$ with some $C_3 > C_1$ and $r < r_1 < 1$. Hence, for the monic polynomial $Q_m(z) = S_m(z) - S_m(\tilde{Z})$ we obtain

$$\|S_m(z)\|_\sigma \leq 1 + 2C_3 r_1^m, \tag{6.2}$$

and $\tilde{Z}$ is a zero of S_m.

Now for a large n set

$$\tilde{P}_n(z) = S_{n^{1-\theta}}(z)^{n^\theta}, \tag{6.3}$$

more precisely let $\tilde{P}_n$ be the product of $[n^\theta]+1$ copies of $Q_{[n^{1-\theta}]-1}$, $Q_{[n^{1-\theta}]}$ or $Q_{[n^{1-\theta}]+1}$ in such a way that P_n has degree precisely n, but for simplicity we shall just use the form (6.3). This has at $\tilde{Z}$ a zero of multiplicity at least n^θ, and its norm on σ is at most

$$\|\tilde{P}_n(z)\|_\sigma \le (1 + 2C_3 r_1^{n^{1-\theta}})^{n^\theta} < 1 + C_4 r_1^{n^{1-\theta}/2}. \tag{6.4}$$

We choose and fix n so large that

$$\|\tilde{P}_n(z)\|_\sigma < 1 + \tau, \tag{6.5}$$

which is possible in view of (6.4).

The point $\tilde{Z}$ is inside σ and now we make a Jordan curve $\tilde{\sigma}$ lying inside but close to σ with capacity close to 1 that contains $\tilde{Z}$. Indeed, let J be a small arc on σ lying in the τ-neighborhood of $\tilde{Z}$, remove J from σ and connect the two endpoints of J to $\tilde{Z}$ via two segments. This way we get a Jordan curve $\tilde{\sigma}$ that lies in the τ-neighborhood of σ, $\tilde{\sigma}$ already contains $\tilde{Z}$, and it is clear from the construction that we can choose a parametrization of $\tilde{\sigma}$ so that for all $t \in \mathbf{R}$ we have

$$|\tilde{\sigma}(t) - \sigma(t)| < \tau. \tag{6.6}$$

Furthermore, if J is sufficiently small, then the capacity of $\tilde{\sigma}$ will be so close to $\mathrm{cap}(\sigma) = 1$, that along with (6.5) we also have

$$\|\tilde{P}_n(z)\|_{\tilde{\sigma}} < (1+\tau)\mathrm{cap}(\tilde{\sigma})^n. \tag{6.7}$$

Choose now for a $\rho > 0$ an analytic Jordan curve[3] σ^* such that for all $t \in \mathbf{R}$ we have

$$|\sigma^*(t) - \tilde{\sigma}(t)| < \rho, \tag{6.8}$$

which implies (ii) if $\tau + \rho < \varepsilon$ (see (6.6)). Then $\tilde{Z}$ lies closer to σ^* than ρ, so we can translate $\tilde{Z}$ by at most of distance ρ to get a point Z^* on σ^*. Now if we set

$$P_n^*(z) = \tilde{P}_n(z + \tilde{Z} - Z^*),$$

then for sufficiently small ρ we will have

$$\|P_n^*(z)\|_{\sigma^*} < (1+\tau)\mathrm{cap}(\sigma^*)^n \tag{6.9}$$

(see (6.7)), hence (iii) (as well as (i)) is also true.

The last statement concerning η^* is clear if we make a translation of Z^* to a point $Z \in \gamma$ such that $|Z - Z^*| < \eta^*$ and consider

$$P_n(z) = \tilde{P}_n^*(z + Z^* - Z))$$

[3]Say a level line of a conformal mapping from the outer domain of $\tilde{\sigma}$ to the unit disk or first approximate $\tilde{\sigma}$ by a C^2 smooth Jordan curve σ_1 with $\sigma_1' \neq 0$, then approximate σ_1' by trigonometric polynomials and then integrate them.

(apply the just used translation argument).

∎

After this let us return to the proof of Proposition 1.5. The γ in that proposition will be the uniform limit of analytic Jordan curves γ_j, $j = 1, 2, \ldots$. To each γ_j there is also associated a positive number ε_j. Suppose that γ_j and ε_j are given, and set $\sigma = \gamma_j$, $\varepsilon = \varepsilon_j$ in Lemma 6.1. The lemma provides a σ^*, a Z^*, an n, a P_n^* and an η^* that have the properties listed in the lemma. We set $\gamma_{j+1} = \sigma^*$, $\eta_{j+1}^* = \eta^*$,

$$\varepsilon_{j+1} = \min(\varepsilon_j/3, \eta_{j+1}^*/3), \tag{6.10}$$

$z_{j+1}^* = Z^*$, $n_{j+1} := n$ and $P_{n_{j+1}}^* = P_n^*$. So z_{j+1}^* is a zero of $P_{n_{j+1}}^*$ of multiplicity at least n_{j+1}^θ. Furthermore,

$$\gamma(t) = \lim_{j\to\infty} \gamma_j(t)$$

satisfies, in view of (6.10), the estimate

$$|\gamma(t) - \gamma_{j+1}(t)| < \sum_{k=j+1}^{\infty} \varepsilon_k < \eta_{j+1}^*. \tag{6.11}$$

Therefore, by the choice of $\eta^* = \eta_{j+1}^*$, there is a $z_{j+1} \in \gamma$ of distance smaller than η_{j+1}^* from z_{j+1}^* and a polynomial $P_{n_{j+1}} = z^{n_{j+1}} + \cdots$ such that z_{j+1} is a zero of $P_{n_{j+1}}$ of multiplicity at least n_{j+1}^θ and

$$\|P_{n_{j+1}}\|_\gamma < (1 + \varepsilon_j)\mathrm{cap}(\gamma)^{n_{j+1}}.$$

This seemingly completes the proof of Proposition 1.5, but there is a problem, namely the uniform limit of Jordan curves is not necessarily a Jordan curve. We ensure that $\gamma = \lim \gamma_j$ is a Jordan curve as follows. Let

$$\delta_{j+1} = \frac{1}{2} \min \left\{ |\gamma_{j+1}(u) - \gamma_{j+1}(t)| \,\middle|\, |u - t| \geq 1/(j+1) \pmod{2\pi} \right\}.$$

This is a positive number because γ_{j+1} is a Jordan curve. Now if η_{j+1}^* is sufficiently small, then for all Jordan curves γ for which $|\gamma - \gamma_{j+1}| < \eta_{j+1}^*$ we will have by the definition of δ_{j+1} the inequality

$$\min \left\{ |\gamma(u) - \gamma(t)| \,\middle|\, |u - t| \geq 1/(j+1) \pmod{2\pi} \right\} > \delta_{j+1}, \tag{6.12}$$

and we make sure that the η_{j+1}^* above is so small that this additional property is also satisfied. Then, by (6.11), the limit curve γ satisfies (6.12) for all $j \geq 2$, which shows that $\gamma : \mathbf{R} \to \mathbf{C}$ is, indeed, one-to-one on $[0, 2\pi)$, i.e., γ is a Jordan curve.

∎

7 Appendix

We briefly give the proof of Lemma 2.1. Let Ω_Γ be the outer domain to Γ, and $\gamma \subset \mathbf{C} \setminus K$ a Jordan curve that contains Γ in its interior, but all other components of Γ are exterior to γ. The Green's functions $g_\Omega(z,\infty)$ and $g_{\Omega_\Gamma}(z,\infty)$ are bounded away from zero and infinity on γ, hence

$$\alpha g_{\Omega_\Gamma}(z,\infty) \le g_\Omega(z,\infty) \le g_{\Omega_\Gamma}(z,\infty), \qquad z \in \gamma, \tag{7.1}$$

with an $\alpha > 0$. Since both functions are 0 on Γ, the maximum principle yields that (7.1) remains valid also in the domain G enclosed by Γ and γ. This shows that when we deal with g_Ω, we may assume $K = \Gamma$.

As for the equilibrium measure, the situation is similar. In fact, μ_K is the harmonic measure with respect to the point ∞ in Ω, and hence (see e.g. [10, II.(4.1)]) on Γ

$$\frac{d\mu_K(z)}{ds} = \frac{1}{2\pi}\frac{\partial g_\Omega(z,\infty)}{\partial \mathbf{n}},$$

where $\mathbf{n}$ denotes the normal at $z \in \Gamma$ pointing towards the interior of Ω (when Γ is an arc we must consider both of its sides, so actually then we have

$$\frac{d\mu_K(z)}{ds} = \frac{1}{2\pi}\left(\frac{\partial g_\Omega(z,\infty)}{\partial \mathbf{n}_+} + \frac{\partial g_\Omega(z,\infty)}{\partial \mathbf{n}_-}\right)$$

with $\mathbf{n}_\pm$ being the two normals) and a similar formula holds for μ_Γ. Since both $g_\Omega(z,\infty)$ and $g_{\Omega_\Gamma}(z,\infty)$ are zero on Γ, the inequality (7.1) extends to their normal derivatives on Γ, i.e., we have

$$\alpha\frac{d\mu_\Gamma(z)}{ds} \le \frac{d\mu_K(z)}{ds} \le \frac{d\mu_\Gamma(z)}{ds}, \qquad z \in \Gamma.$$

Thus, it is sufficient to prove the lemma for $K = \Gamma$, in which case Ω is simply connected. Let Φ be a conformal map from Ω onto the exterior of the unit disk that leaves the point infinity invariant. Then $g_\Omega(z) = \log|\Phi(z)|$ (just check the defining properties of Green's functions for $\log|\Phi(z)|$). Now we distinguish the curve and arc cases.

Γ is a Jordan curve. If Γ is a $C^{1+\alpha}$ Jordan curve, then Φ' can be extended to Γ to a nonvanishing continuous function (see [11, Theorem 3.6]) so (2.2) follows. Since μ_K is the harmonic measure with respect to the point ∞ in Ω, we obtain from the conformal invariance of harmonic measures that μ_K is the pull-back of the normalized arc measure on the unit circle under the mapping Φ (i.e., $\mu(E) = |\Phi(E)|/2\pi$ where $|\cdot|$ denotes arc-length), which proves the statement in the lemma concerning μ_K.

Γ is a Jordan arc. In this case we may assume that its endpoints are -2 and 2. The Joukowski mapping $\psi(z) = \frac{1}{2}(z + \sqrt{z^2-4})$ maps Γ into a $C^{1+\alpha}$-smooth Jordan curve (see [19, Lemma 11.1]) Γ^* with outer domain Ω_{Γ^*}. By the conformal invariance of Green's functions we have

$$g_\Omega(z,\infty) = g_{\Omega_{\Gamma^*}}(\psi(z),\infty),$$

and here, by the just proven first part,

$$g_{\Omega_{\Gamma^*}}(\psi(z),\infty) \sim \mathrm{dist}(\psi(z),\Gamma^*),$$

from which the relation (2.3) can be easily deduced. As before, μ_E is the pull-back of the arc measure on the unit circle under the mapping $\Phi^* \circ \psi$ where Φ^* is the conformal map from $\Gamma^* = \psi(\Gamma)$ onto the exterior of the unit disk. We have already seen that Φ^* is continuously differentiable with nonvanishing derivative up to Γ^*, hence (2.4) follows from the form of ψ.

An alternative proof can be given via some known distortion theorems of conformal maps. Indeed, assume we want to prove the claim in the lemma around a point $P = 0$. The most complicated situation is when Γ is a Jordan arc and P is one of its endpoint, so let us just consider that case. Let δ be so small that the disk $D_{2\delta} = \{z \mid |z| \le 2\delta\}$ intersects only the component Γ of K and the other endpoint of Γ lies outside $D_{2\delta}$. Let $E_1 = \Gamma$, Ω_1 its complement, and consider a conformal map Φ_1 from Ω_1 onto the exterior of the unit disk that leaves the point ∞ invariant, and let, say, $\Phi_1(0) = 1$. By [11, Corollary 2.2] this Φ_1 can be continuously extended to (the two sides of) E_1, and if φ_1 is its inverse, then [11, Theorem 3.9] with $\alpha = 2$ gives that $\varphi(w)/(w-1)^2$ and $\varphi'(w)/(w-1)$, $|w| \ge 1$, are continuous and non-vanishing functions in a neighborhood of 1. This translates to the continuity of $(\Phi_1(z)-1)^2/z$ and $\Phi_1'(z)(\Phi_1(z)-1)$ in a neighborhood of 0. Therefore, $|\Phi_1(z)-1| \sim \sqrt{|z|}$, and then $|\Phi_1'(z)| \sim 1/\sqrt{|z|}$. Since $\log|\Phi_1(z)|$ is the Green's function g_{Ω_1} of Ω_1 with pole at infinity, the behavior (2.2)–(2.3) follow (at this moment only) for g_{Ω_1}.

The equilibrium measure μ_{E_1} is the pull-back of the normalized arc measure on the unit circle under the mapping $w = \Phi_1(z)$, hence it follows that

$$\frac{d\mu_{E_1}(z)}{ds} \sim 1/\sqrt{|z|}. \tag{7.2}$$

in $\Gamma \cap D_\delta$.

The just given relations will be the suitable upper bounds for g_Ω and μ_K. The matching lower bounds follow in a similar manner. In fact, connect the different components of K by smooth arcs so that we obtain a connected set E_2 containing K for which $E_2 \cap D_{2\delta} = E_1 \cap D_{2\delta} = \Gamma \cap D_{2\delta}$, and let Ω_2 be the unbounded component of the complement of E_2. This Ω_2 is again simply connected, and let Φ_2 be the conformal map from Ω_2 onto the exterior of the unit disk that leaves ∞ invariant and for which $\Phi_2(0) = 1$. Everything we have just said about E_1 holds also for E_2 because [11, Theorem 3.9] is a local theorem and in the neighborhood $D_{2\delta}$ of 0 the two sets are the same. Therefore, we obtain again the behavior (2.2)–(2.3) for g_{Ω_2}, and on $\Gamma \cap D_\delta$

$$\frac{d\mu_{E_2}(z)}{ds} \sim 1/\sqrt{|z|}. \tag{7.3}$$

Finally, since $\Omega_2 \subset \Omega \subset \Omega_1$ we have $g_{\Omega_2}(z,\infty) \le g_\Omega(z,\infty) \le g_{\Omega_1}(z,\infty)$, so the (2.2)–(2.3) behavior for g_Ω follows from the similar behavior for g_{Ω_1} and g_{Ω_2}.

As for μ_K, it is the harmonic measure of the point ∞ in the outer domain Ω, and since we have $\Omega_2 \subset \Omega \subset \Omega_1$ and $\Gamma \cap D_\delta$ is a common arc on the boundaries of Ω, Ω_1 and Ω_2, we have the relation (see [11, Corollary 4.16])

$$\mu_{E_2}\Big|_{\Gamma \cap D_\delta} \le \mu_K\Big|_{\Gamma \cap D_\delta} \le \mu_{E_1}\Big|_{\Gamma \cap D_\delta},$$

so the claim in the lemma regarding the equilibrium measure follows from (7.2) and (7.3).

■

References

[1] V. V. Andrievskii and H-P. Blatt, *Discrepancy of Signed Measures and Polynomial Approximation,* Springer Monographs in Mathematics. Springer-Verlag, New York, 2002.

[2] V. V. Andrievskii and H-P. Blatt, Polynomials with prescribed zeros on an analytic curve, *Acta Math. Hungar.,* **128**(2010), 221–238.

[3] D. H. Armitage and S. J. Gardiner, *Classical Potential Theory*, Springer Verlag, Berlin, Heidelberg, New York, 2002.

[4] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Graduate Texts in Mathematics, **161**, Springer Verlag, New York, 1995.

[5] J. B. Garnett and D. E. Marshall, *Harmonic Measure*, Cambridge University Press, New mathematical monographs, Cambridge, New York, 2005.

[6] T. Erdélyi, An improvement of the Erdős–Turán theorem on the distribution of zeros of polynomials, *C. R. Math. Acad. Sci. Paris*, **346**(2008), 267–270.

[7] P. Erdős and P. Turán, On the uniform-dense distribution of certain sequences of points, *Ann. Math.*, **41**(1940), 162–173.

[8] G. Halász, On the first and second main theorem in Turán's theory of power sums, *Studies in pure mathematics*, 259–269, Birkhäuser, Basel, 1983.

[9] S. I. Kalmykov and B. Nagy, Polynomial and rational inequalities on analytic Jordan arcs and domains, *J. Math. Anal. Appl.*, **430**(2015), 874–894.

[10] R. Nevanlinna, *Analytic Functions*, Grundlehren der mathematischen Wissenschaften, **162**, Springer Verlag, Berlin, 1970.

[11] Ch. Pommerenke, *Boundary Behavior of Conformal Mappings*, Grundlehren der mathematischen Wissenschaften, **299**, Springer Verlag, Berlin, Heidelberg New York, 1992.

[12] T. Ransford, *Potential Theory in the Complex plane*, Cambridge University Press, Cambridge, 1995

[13] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Grundlehren der mathematischen Wissenschaften, **316**, Springer-Verlag, New York/Berlin, 1997.

[14] K. Schiefermayr, A lower bound for the minimum deviation of the Chebyshev polynomial on a compact real set, *East J. Approx.*, **14**(2008), 65–75.

[15] V. Totik and P. Varjú, Polynomials with prescribed zeros and small norm, *Acta Sci. Math.*, (Szeged) **73**(2007), 593–612.

[16] V. Totik, Polynomials with zeros and small norm on curves, *Proc. Amer Math. Soc.*, **140**(2012), 3531-3539.

[17] V. Totik, Chebyshev polynomials on compact sets, *Potential Analysis*, **40**(2014), 511–524.

[18] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, third edition, Amer. Math. Soc. Colloquium Publications, **XX**, Amer. Math. Soc., Providence, 1960.

[19] H. Widom, Extremal polynomials associated with a system of curves in the complex plane, *Adv. Math.*, **3**(1969), 127–232.

MTA-SZTE Analysis and Stochastics Research Group
Bolyai Institute
University of Szeged
Szeged
Aradi v. tere 1, 6720, Hungary

totik@math.u-szeged.hu