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ABSTRACT

We study a holomorphic Poisson structure defined on the linear space $S(n,d) := \mathrm{Mat}_{n\times d}(\mathbb{C}) \times \mathrm{Mat}_{d\times n}(\mathbb{C})$ that is covariant under the natural left actions of the standard $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{GL}(d,\mathbb{C})$ Poisson–Lie groups. The Poisson brackets of the matrix elements contain quadratic and constant terms, and the Poisson tensor is non-degenerate on a dense subset. Taking the $d = 1$ special case gives a Poisson structure on $S(n, 1)$, and we construct a local Poisson map from the Cartesian product of d independent copies of $S(n, 1)$ into $S(n, d)$, which is a holomorphic diffeomorphism in a neighborhood of 0. The Poisson structure on $S(n, d)$ is the complexification of a real Poisson structure on $\mathrm{Mat}_{n\times d}(\mathbb{C})$ constructed by the authors and Marshall, where a similar decoupling into d independent copies was observed. We also relate our construction to a Poisson structure on $S(n, d)$ defined by Arutyunov and Olivucci in the treatment of the complex trigonometric spin Ruijsenaars–Schneider system by Hamiltonian reduction.

I. INTRODUCTION

In this paper, we prove a remarkable "decoupling property" of a holomorphic Poisson structure defined on the space

$$S(n,d) := \mathrm{Mat}_{n\times d}(\mathbb{C}) \times \mathrm{Mat}_{d\times n}(\mathbb{C}), \tag{1.1}$$

which appeared in recent derivations of trigonometric spin Ruijsenaars–Schneider models[1] by Hamiltonian reduction.[2,3] The decoupling means that the Poisson algebra of $S(n, d)$ will be realized using d independent (pairwise Poisson commuting) copies of the Poisson algebra of $S(n, 1)$. The spaces $S(n, d)$ are defined for arbitrary pairs of natural numbers, but the decoupling requires that both n and d are greater than 1. Our result is expected to be useful, for example, for the further studies of the holomorphic spin Ruijsenaars–Schneider systems.

To set the stage, for any natural number ℓ, we introduce the Drinfeld–Jimbo classical r-matrix r^ℓ by

$$r^\ell := \frac{1}{2} \sum_{1\le j<k\le \ell} E_{jk}(\ell) \wedge E_{kj}(\ell), \tag{1.2}$$

where $E_{jk}(\ell)$ is the usual elementary matrix of size $\ell \times \ell$. We also need

$$r^\ell_\pm := r^\ell \pm \frac{1}{2} I^\ell, \quad \text{with} \quad I^\ell := \sum_{j,k=1}^{\ell} E_{jk}(\ell) \otimes E_{kj}(\ell). \tag{1.3}$$

Note that for $\ell = 1$, $r^\ell = 0$ and I^ℓ can be viewed as $1 \otimes 1$. Denoting the elements of $S(n, d)$ as pairs (A, B) and employing the standard tensorial notation,[4,5] the pertinent Poisson bracket can be written as follows:

$$\begin{aligned} \{A_1, A_2\}_\kappa &= -\kappa\Big(r^n A_1 A_2 + A_1 A_2 r^d\Big), \\ \{B_1, B_2\}_\kappa &= -\kappa\Big(B_1 B_2 r^n + r^d B_1 B_2\Big), \\ \{A_1, B_2\}_\kappa &= \kappa\Big(B_2 r_+^n A_1 + A_1 r_+^d B_2 + C_{12}^{n\times d}\Big). \end{aligned} \tag{1.4}$$

Here, we use notations (1.2) and (1.3) together with

$$C_{12}^{n\times d} := \sum_{i=1}^{n}\sum_{\alpha=1}^{d} E_{i\alpha}^{n\times d} \otimes E_{\alpha i}^{d\times n}, \tag{1.5}$$

where $E_{i\alpha}^{n\times d} \in \mathrm{Mat}_{n\times d}(\mathbb{C})$ is the elementary matrix having a single non-zero entry, equal to 1, at the $i\alpha$ position. One could fix the arbitrary constant $\kappa \in \mathbb{C}^*$ without loss of generality, but it will be advantageous not to do so.

The Poisson structure (1.4) represents the complexification of a $\mathrm{U}(n) \times \mathrm{U}(d)$ covariant real Poisson structure on $\mathrm{Mat}_{n\times d}(\mathbb{C}) \simeq \mathbb{R}^{2nd}$ considered in Ref. 3. By simple changes of variables (see below), it also reproduces the holomorphic Poisson bracket defined on $S(n, d)$ by Arutyunov and Olivucci.[2] In the papers mentioned, it was natural to assume that $n > 1$, but here, we assume only that either n or d is greater than 1. The $d = 1$ (or $n = 1$) cases provide the building blocks from which the general $S(n, d)$ cases will be realized via the decoupling.

The above-mentioned Poisson brackets have remarkable Poisson–Lie covariance properties. (For background on the theory of Poisson–Lie groups, one may consult, for example, Refs. 4 and 6–8.) To describe these, we equip the group $\mathrm{GL}(\ell, \mathbb{C})$ with the standard multiplicative Poisson bracket given in the tensorial notation by

$$\{g_1, g_2\}_G^\kappa := \kappa\Big[g_1 g_2, r^\ell\Big]. \tag{1.6}$$

The subscript G expresses that this Poisson bracket lives on the group $G = \mathrm{GL}(\ell, \mathbb{C})$.

Then, the linear left-action of $\mathrm{GL}(n, \mathbb{C})$ on $S(n, d)$, defined by

$$\mathrm{GL}(n, \mathbb{C}) \times S(n, d) \ni (g, A, B) \mapsto \Big(gA, Bg^{-1}\Big) \in S(n, d), \tag{1.7}$$

enjoys the Poisson–Lie property, which means that map (1.7) is Poisson if $\mathrm{GL}(n, \mathbb{C})$ is equipped with bracket (1.6) for $\ell = n$ and $S(n, d)$ is equipped with bracket (1.4). Similarly, the linear left-action of $\mathrm{GL}(d, \mathbb{C})$, given by

$$\mathrm{GL}(d, \mathbb{C}) \times S(n, d) \ni (g, A, B) \mapsto \Big(Ag^{-1}, gB\Big) \in S(n, d), \tag{1.8}$$

also has the Poisson–Lie property, where $\mathrm{GL}(d, \mathbb{C})$ is equipped with bracket (1.6), for $\ell = d$.

Now, we describe our main result, which was motivated by an analogous result of Ref. 3. Let us introduce the group

$$D(\ell) := \mathrm{GL}(\ell, \mathbb{C}) \times \mathrm{GL}(\ell, \mathbb{C}). \tag{1.9}$$

This is the Drinfeld double of the Poisson–Lie group $\mathrm{GL}(\ell, \mathbb{C})$. The dual Poisson–Lie group, $\mathrm{GL}(\ell, \mathbb{C})^*$, is the subgroup of $D(\ell)$ consisting of pairs (h_+, h_-), where h_+ and h_- are invertible upper triangular and lower triangular matrices, respectively, whose respective diagonal entries are inverses of each other, i.e., $(h_-)_{jj} = 1/(h_+)_{jj}$ for $j = 1, \ldots, \ell$. Consider the space $S(n, 1)$, with elements (a^1, b^1), endowed with the Poisson bracket (1.4). Introduce the (locally defined) map

$$(g_+, g_-) : S(n, 1) \to \mathrm{GL}(n, \mathbb{C})^* \tag{1.10}$$

by the following definition:

$$(g_+)_{jj} = \sqrt{G_j / G_{j+1}}, \quad (g_+)_{jk} = \frac{a_j^1 b_k^1}{\sqrt{G_k G_{k+1}}} \text{ for } j < k \tag{1.11}$$

and

$$(g_-^{-1})_{jj} = \sqrt{G_j / G_{j+1}}, \quad (g_-^{-1})_{jk} = \frac{a_j^1 b_k^1}{\sqrt{G_j G_{j+1}}} \text{ for } j > k, \tag{1.12}$$

using the functions

$$G_j = 1 + \sum_{k=j}^{n} a_k^1 b_k^1, \quad G_0 = G_{n+1} = 1, \tag{1.13}$$

which are well-defined only locally, including a neighborhood of 0.

Theorem 1.1. *For any n and d greater than 1, take d copies of $S(n,1)$, each equipped with Poisson bracket (1.4), and denote their elements by (a^α, b^α), $\alpha = 1, \ldots, d$. Let (a, b) stand for the collection of the (a^α, b^α), and let A^α and B^α stand for the columns and the rows of the matrices $(A, B) \in S(n, d)$, respectively. Define the (local) map*

$$m : S(n,1) \times \cdots \times S(n,1) \to S(n,d) \tag{1.14}$$

by formulas $A^1(a,b) = a^1$ and $B^1(a,b) = b^1$, and for $\alpha \geq 2$,

$$A^\alpha(a,b) = g_+\left(a^1, b^1\right) \cdots g_+\left(a^{\alpha-1}, b^{\alpha-1}\right) a^\alpha, \tag{1.15a}$$

$$B^\alpha(a,b) = b^\alpha g_-^{-1}\left(a^{\alpha-1}, b^{\alpha-1}\right) \cdots g_-^{-1}\left(a^1, b^1\right). \tag{1.15b}$$

Then, the map m is a local, holomorphic Poisson diffeomorphism from the d-fold product Poisson space $S(n,1) \times \cdots \times S(n,1)$ to $S(n,d)$, where $S(n,1)$ and $S(n,d)$ are equipped with the relevant Poisson brackets of the form (1.4).

The fundamental property of the map (g_+, g_-) (1.10) is the factorization identity,

$$\mathbf{1}_n + a^1 b^1 = g_+\left(a^1, b^1\right) g_-\left(a^1, b^1\right)^{-1}. \tag{1.16}$$

Introducing

$$\mathcal{G}_\pm(a,b) := g_\pm\left(a^1, b^1\right) \cdots g_\pm\left(a^d, b^d\right), \tag{1.17}$$

identity (1.16), and formulas of Theorem 1.1 imply the further identity,

$$\mathbf{1}_n + A(a,b)B(a,b) = \mathcal{G}_+(a,b)\mathcal{G}_-(a,b)^{-1}. \tag{1.18}$$

These properties, which are easily verified, actually motivated our construction. Their meaning will be enlightened in Sec. III (see Remark 3.5) utilizing the theory of Poisson–Lie moment maps.

We can also give an analogous realization of the Poisson bracket (1.4) on $S(n,d)$ in terms of n copies of the Poisson bracket on $S(1,d)$. Such a map can be obtained by combining Theorem 1.1 with the swap map ν from $S(n,d)$ to $S(d,n)$ that operates according to

$$\nu : (A,B) \mapsto \left(\eta^d B \eta^n, \eta^n A \eta^d\right), \tag{1.19}$$

where for any $\ell \in \mathbb{N}$, we let $\eta^\ell := \sum_{i=1}^{\ell} E_{i,\ell+1-i}(\ell)$. It is easily seen that

$$\nu : \left(S(n,d), \{\ ,\ \}_\kappa\right) \to \left(S(d,n), \{\ ,\ \}_{-\kappa}\right) \tag{1.20}$$

is a Poisson diffeomorphism.

In addition, we shall present decoupling results for the "oscillator Poisson brackets" of Arutyunov and Olivucci,[2] who introduced two Poisson structures on $S(n,d)$. Denoting the elements of $S(n,d)$ now as pairs $(\mathcal{A}, \mathcal{B})$, one of their Poisson structures, called $\{\ ,\ \}^+_\kappa$, is given by

$$\begin{aligned}
\{\mathcal{A}_1, \mathcal{A}_2\}^+_\kappa &= \kappa\left(r^n \mathcal{A}_1 \mathcal{A}_2 - \mathcal{A}_1 \mathcal{A}_2 r^d\right),\\
\{\mathcal{B}_1, \mathcal{B}_2\}^+_\kappa &= \kappa\left(\mathcal{B}_1 \mathcal{B}_2 r^n - r^d \mathcal{B}_1 \mathcal{B}_2\right),\\
\{\mathcal{A}_1, \mathcal{B}_2\}^+_\kappa &= \kappa\left(-\mathcal{B}_2 r^n_+ \mathcal{A}_1 + \mathcal{A}_1 r^d_- \mathcal{B}_2\right) - C^{n\times d}_{12}.
\end{aligned} \tag{1.21}$$

Their other Poisson bracket, called $\{\ ,\ \}^-_\kappa$, is obtained from this one by replacing $(\mathcal{A}, \mathcal{B})$ by $(\mathcal{A}\eta^d, \eta^d \mathcal{B})$ in the above formula. In fact, we have two different decoupling results for the Poisson bracket $\{\ ,\ \}^+_\kappa$. The first one is obtained by combining Theorem 1.1 with the following simple lemma.

Lemma 1.2. Let ξ_A and ξ_B be arbitrary constants for which $\xi_A \xi_B = -\frac{1}{\kappa}$. Then, the map

$$\xi : (A,B) \mapsto (\mathcal{A}, \mathcal{B}) := \left(\xi_A A \eta^d, \xi_B \eta^d B\right) \tag{1.22}$$

is a Poisson diffeomorphism from $\left(S(n,d), \{\ ,\ \}_\kappa\right)$ to $\left(S(n,d), \{\ ,\ \}^+_{-\kappa}\right)$.

An alternative decoupling map from $\left(S(n,1), \{\ ,\ \}_\kappa\right)^{\times d}$ to $\left(S(n,d), \{\ ,\ \}^+_\kappa\right)$ will be presented in Sec. V.

Remark 1.3. It is known that the brackets $\{\ ,\ \}_\kappa$ and $\{\ ,\ \}^+_\kappa$ satisfy the Jacobi identity, but the interested reader can also check this by routine calculation.

II. BASIC FACTS ABOUT POISSON–LIE GROUPS

We will recall the embedding of the Poisson–Lie group $\mathrm{GL}(\ell,\mathbb{C})$ and its dual into their Drinfeld double $D(\ell)$. Then, we will present the notion of the Poisson–Lie moment map. We do not give proofs here since the relevant statements can be found in many reviews.[4,6–8]

Let us consider the complex Lie group $D(\ell)$ (1.9) and equip its Lie algebra,

$$\mathcal{D}(\ell) := \mathfrak{gl}(\ell,\mathbb{C}) \oplus \mathfrak{gl}(\ell,\mathbb{C}), \tag{2.1}$$

with the non-degenerate, invariant bilinear form

$$\langle (U,V),(X,Y)\rangle_\kappa := \frac{1}{\kappa}(\mathrm{tr}(UX) - \mathrm{tr}(VY)), \tag{2.2}$$

using a constant $\kappa \in \mathbb{C}^*$. Let us also introduce the triangular decomposition

$$\mathfrak{gl}(\ell,\mathbb{C}) = \mathfrak{gl}(\ell,\mathbb{C})_> + \mathfrak{gl}(\ell,\mathbb{C})_0 + \mathfrak{gl}(\ell,\mathbb{C})_<, \tag{2.3}$$

where $\mathfrak{gl}(\ell,\mathbb{C})_0$ is the set of diagonal matrices, while $\mathfrak{gl}(\ell,\mathbb{C})_>$ [respectively, $\mathfrak{gl}(\ell,\mathbb{C})_<$] contains the upper (respectively, lower) triangular matrices with zero diagonal. Then, $\mathcal{D}(\ell)$ can be represented as the vector space direct sum of the isotropic subalgebras,

$$\mathfrak{gl}(\ell,\mathbb{C})_\delta := \{(X,X) \mid X \in \mathfrak{gl}(\ell,\mathbb{C})\} \tag{2.4}$$

and

$$\mathfrak{gl}(\ell,\mathbb{C})^*_\delta := \{(Y_> + Y_0, Y_< - Y_0) \mid Y_> \in \mathfrak{gl}(\ell,\mathbb{C})_>,\ Y_< \in \mathfrak{gl}(\ell,\mathbb{C})_<,\ Y_0 \in \mathfrak{gl}(\ell,\mathbb{C})_0\}. \tag{2.5}$$

In (2.4), the subscript δ indicates that $\mathfrak{gl}(\ell,\mathbb{C})_\delta$ is the diagonal embedding of $\mathfrak{gl}(\ell,\mathbb{C})$ into $\mathcal{D}(\ell)$ (2.1). We may identify $\mathfrak{gl}(\ell,\mathbb{C})$ with the diagonal subalgebra $\mathfrak{gl}(\ell,\mathbb{C})_\delta$ and identify its linear dual space with the subalgebra $\mathfrak{gl}(\ell,\mathbb{C})^*_\delta$. We also let $\mathrm{GL}(\ell,\mathbb{C})_\delta$ and $\mathrm{GL}(\ell,\mathbb{C})^*_\delta$ denote the subgroups of $D(\ell)$ corresponding to the subalgebras in the decomposition,

$$\mathcal{D}(\ell) = \mathfrak{gl}(\ell,\mathbb{C})_\delta + \mathfrak{gl}(\ell,\mathbb{C})^*_\delta. \tag{2.6}$$

The group $D(\ell)$ carries a natural multiplicative Poisson structure. To describe it, let us take arbitrary bases T^a of $\mathfrak{gl}(\ell,\mathbb{C})_\delta$ and T_a of $\mathfrak{gl}(\ell,\mathbb{C})^*_\delta$ that are in duality with respect to the pairing (2.2). The Poisson bracket of two holomorphic functions $\mathcal{F}$ and $\mathcal{H}$ on $D(\ell)$ is given by

$$\{\mathcal{F},\mathcal{H}\}^\kappa_D := \sum_{a=1}^{n^2}\big((\nabla_{T^a}\mathcal{F})(\nabla_{T_a}\mathcal{H}) - (\nabla'_{T^a}\mathcal{F})(\nabla'_{T_a}\mathcal{H})\big), \tag{2.7}$$

where for any $T \in \mathcal{D}(\ell)$, we have

$$(\nabla_T\mathcal{F})(p) = \frac{d}{dz}\Big|_{z=0}\mathcal{F}\Big(e^{zT}p\Big), \quad (\nabla'_T\mathcal{F})(p) = \frac{d}{dz}\Big|_{z=0}\mathcal{F}\Big(pe^{zT}\Big), \quad \forall p \in D(\ell). \tag{2.8}$$

It is well-known that $\mathrm{GL}(\ell,\mathbb{C})_\delta$ and $\mathrm{GL}(\ell,\mathbb{C})^*_\delta$ are Poisson submanifolds of $D(\ell)$, and we equip them with the inherited Poisson structures.

The above Poisson structures can be conveniently presented in terms of the functions given by the matrix elements on the respective groups. Denoting the elements of $D(\ell)$ as pairs (u,v) and employing the tensorial notation of the Faddeev school, one has

$$\{u_1,u_2\}^\kappa_D = \kappa\Big[u_1u_2, r^\ell\Big],\ \{v_1,v_2\}^\kappa_D = \kappa\Big[v_1v_2, r^\ell\Big],\ \{u_1,v_2\}^\kappa_D = \kappa\Big[u_1v_2, r^\ell_+\Big], \tag{2.9}$$

using r-matrices (1.2) and (1.3). On the subgroup $\mathrm{GL}(\ell,\mathbb{C})_\delta$ with elements denoted as (g,g), this reduces to bracket (1.6). The group $\mathrm{GL}(\ell,\mathbb{C})^*_\delta$ consists of the pairs $(h_+,h_-) \in D(\ell)$ for which h_+ (respectively, h_-) is upper triangular (respectively, lower triangular) and the diagonal part of h_+ is the inverse of the diagonal part of h_-. Restriction from $D(\ell)$ gives the following Poisson bracket on this dual group:

$$\{h_{\pm,1},h_{\pm,2}\}^\kappa_* = \kappa\Big[h_{\pm,1}h_{\pm,2}, r^\ell\Big],\ \{h_{+,1},h_{-,2}\}^\kappa_* = \kappa\Big[h_{+,1}h_{-,2}, r^\ell_+\Big]. \tag{2.10}$$

We stress that $D(\ell)$, $\mathrm{GL}(\ell,\mathbb{C}) \equiv \mathrm{GL}(\ell,\mathbb{C})_\delta$, and $\mathrm{GL}(\ell,\mathbb{C})^* := \mathrm{GL}(\ell,\mathbb{C})^*_\delta$ with the above Poisson brackets are Poisson–Lie groups. This means, for example, that the group product $D(\ell)\times D(\ell) \to D(\ell)$ is a Poisson map.

Let us briefly explain how (2.9) follows from (2.8). For $T = (X,Y) \in \mathcal{D}(\ell)$, the derivatives of the matrix elements are

$$\nabla_T u_{ij} = (Xu)_{ij}, \quad \nabla'_T u_{ij} = (uX)_{ij}, \quad \nabla_T v_{ij} = (Yv)_{ij}, \quad \nabla'_T v_{ij} = (vY)_{ij}. \tag{2.11}$$

For any dual bases $T^a = (X^a, X^a)$ and $T_a = (Z_a, W_a)$, one can calculate that

$$X^a \otimes Z_a = -\kappa r^\ell_- \quad \text{and} \quad X^a \otimes W_a = -\kappa r^\ell_+. \tag{2.12}$$

By using these relations, one readily obtains (2.9) from (2.8).

There is an important mapping of $\mathrm{GL}(\ell,\mathbb{C})^*$ onto $\mathrm{GL}(\ell,\mathbb{C})$, which is given by

$$\chi : (h_+, h_-) \mapsto h := h_+ h_-^{-1}. \tag{2.13}$$

This mapping is 2^n to 1 since the image does not change if we replace (h_+, h_-) by $(h_+\tau, h_-\tau)$ for any diagonal matrix τ whose entries are taken from the set $\{+1, -1\}$. The map χ yields a holomorphic diffeomorphism[9] between respective neighborhoods of the identity elements. Moreover, it is a Poisson map with respect to the so-called Semenov–Tian–Shansky Poisson structure[4,7] on $\mathrm{GL}(\ell,\mathbb{C})$,

$$\{h_1, h_2\}^{\kappa}_{\mathrm{STS}} = \kappa\Big(h_1 r_-^{\ell} h_2 + h_2 r_+^{\ell} h_1 - h_1 h_2 r^{\ell} - r^{\ell} h_1 h_2\Big). \tag{2.14}$$

With this Poisson bracket, $\mathrm{GL}(\ell,\mathbb{C})$ can serve, at least locally, as a model of the dual Poisson–Lie group $\mathrm{GL}(\ell,\mathbb{C})^*$. We note in passing that this quadratic Poisson bracket naturally extends to a Poisson structure on $\mathfrak{gl}(\ell,\mathbb{C})$, which is compatible with its linear Lie–Poisson bracket.

We now recall[10] what is meant by a moment map for a Poisson action of $\mathrm{GL}(\ell,\mathbb{C})$. Suppose that $\mathrm{GL}(\ell,\mathbb{C})$ acts on a holomorphic Poisson manifold $(\mathcal{P}, \{\ ,\ \}_{\mathcal{P}})$ in such a way that the action map, $\mathrm{GL}(\ell,\mathbb{C}) \times \mathcal{P} \to \mathcal{P}$, is Poisson, where the product Poisson structure on $\mathrm{GL}(\ell,\mathbb{C}) \times \mathcal{P}$ is built from the bracket $\{\ ,\ \}^{\kappa}_G$ (1.6) on $\mathrm{GL}(\ell,\mathbb{C})$ and $\{\ ,\ \}_{\mathcal{P}}$ on $\mathcal{P}$. For any $X \in \mathfrak{gl}(\ell,\mathbb{C})$, let $X_{\mathcal{P}}$ be the vector field on $\mathcal{P}$ given by the flow of $\exp(tX)$. We can take the derivative $\mathcal{L}_{X_{\mathcal{P}}}\mathcal{F}$ of any holomorphic function on $\mathcal{P}$. We then say that a holomorphic map $(\phi_+, \phi_-) : \mathcal{P} \to \mathrm{GL}(\ell,\mathbb{C})^*_\delta$ is the Poisson–Lie moment map for the action if it satisfies the following two conditions. First, we must have the equality

$$\mathcal{L}_{X_{\mathcal{P}}}\mathcal{F} = \langle (X,X), \{\mathcal{F}, (\phi_+,\phi_-)\}_{\mathcal{P}} (\phi_+,\phi_-)^{-1}\rangle_{\kappa} \tag{2.15}$$

for all X and $\mathcal{F}$. Second, we also require that (ϕ_+, ϕ_-) is a Poisson map with respect to bracket (2.10) on the dual group. This second condition is equivalent to the requirement that the map

$$\phi := \phi_+\phi_-^{-1} : \mathcal{P} \to \mathrm{GL}(\ell,\mathbb{C}) \tag{2.16}$$

is Poisson with respect to the Semenov–Tian–Shansky bracket (2.14) on $\mathrm{GL}(\ell,\mathbb{C})$. Indeed, the Semenov–Tian–Shansky bracket is just the push-forward of the Poisson bracket (2.10) on the dual group $\mathrm{GL}(\ell,\mathbb{C})^*$. The first condition can also be recast in terms of the map ϕ, as we shall see in our concrete example in Sec. III.

III. COVARIANCE PROPERTIES OF THE POISSON STRUCTURE (1.4)

We now characterize the behavior of the Poisson bracket (1.4) on $S(n, d)$ under the natural left-action of $\mathrm{GL}(n,\mathbb{C})$. Throughout this section, $d \geq 1$ and $n \geq 2$; otherwise, they are arbitrary. The statements presented below can also be obtained as consequences of known[2,3] analogous properties of the Arutyunov–Olivucci Poisson bracket (1.21). We sketch the proofs in order to make this paper basically self-contained.

Proposition 3.1. Consider the natural action of $\mathrm{GL}(n,\mathbb{C})$ *on* $S(n, d)$*, defined by*

$$g \cdot (A, B) := \big(gA, Bg^{-1}\big), \quad \forall g \in \mathrm{GL}(n,\mathbb{C}),\ (A, B) \in S(n,d). \tag{3.1}$$

With respect to the Poisson structures (1.4) and (1.6), this is a Poisson action.

Proof. This is very easy and goes as follows. We can calculate $\{g_1A_1, g_2A_2\}$ using the product Poisson structure defined by combining (1.4) and (1.6). This gives

$$\begin{aligned}\{g_1A_1, g_2A_2\} &= \{g_1, g_2\}^{\kappa}_G A_1A_2 + g_1g_2\{A_1, A_2\}_{\kappa}\\ &= -\kappa(r^n g_1A_1g_2A_2 + g_1A_1g_2A_2 r^d),\end{aligned} \tag{3.2}$$

which agrees with the Poisson bracket on $S(n, d)$. The next line of (1.4) is handled in the same way. Finally, one needs to show that

$$\Big\{g_1A_1, B_2g_2^{-1}\Big\} = \kappa\Big(B_2g_2^{-1}r_+^n g_1A_1 + g_1A_1r_+^d B_2g_2^{-1} + C_{12}^{n\times d}\Big). \tag{3.3}$$

We refrain from spelling this out, but note that the direct verification of this equality relies on the identity $g_1C_{12}^{n\times d} = C_{12}^{n\times d}g_2$. □

Proposition 3.2. Suppose that $(\phi_+,\phi_-) : S(n,d) \to \mathrm{GL}(n,\mathbb{C})^*$ *is a (possibly only locally defined) moment map for the Poisson action (3.1). Then, condition (2.15) is equivalent to the following equalities:*

$$\{A_1, \phi_{\pm,2}\}_{\kappa} = -\kappa r_{\mp}^n A_1\phi_{\pm,2} \quad \textit{and} \quad \{B_1, \phi_{\pm,2}\}_{\kappa} = \kappa B_1 r_{\mp}^n \phi_{\pm,2}, \tag{3.4}$$

where the usual tensorial notation is employed. For $\phi := \phi_+\phi_-^{-1}$*, these relations imply*

$$\{A_1, \phi_2\}_{\kappa} = \kappa(\phi_2 r_+^n - r_-^n\phi_2)A_1 \quad \textit{and} \quad \{B_1, \phi_2\}_{\kappa} = \kappa B_1(r_-^n\phi_2 - \phi_2 r_+^n). \tag{3.5}$$

Proof. Let $T^a = (X^a, X^a)$ and $T_a = (Z_a, W_a)$ be dual bases of $\mathfrak{gl}(n,\mathbb{C})_\delta$ and $\mathfrak{gl}(n,\mathbb{C})^*_\delta$. Consider an arbitrary matrix element $A_{i\alpha}$ as a function on $S(n, d)$. Its derivative along the vector field induced by $X^a \in \mathfrak{gl}(n,\mathbb{C})$ equals $(X^a A)_{i\alpha}$, and (2.15) gives the following identity:

$$(X^a A)_{i\alpha} = \langle T^a, \{A_{i\alpha}, (\phi_+, \phi_-)\}_\kappa (\phi_+^{-1}, \phi_-^{-1})\rangle_\kappa. \tag{3.6}$$

Since T^a is a basis of $\mathfrak{gl}(n,\mathbb{C})_\delta$, this implies that

$$\{A_{i\alpha}, (\phi_+, \phi_-)\}_\kappa (\phi_+^{-1}, \phi_-^{-1}) = (X^a A)_{i\alpha} T_a. \tag{3.7}$$

This is equivalent to the following relations:

$$\{A_{i\alpha}, \phi_+\}_\kappa = \left(X^a A\right)_{i\alpha} Z_a \phi_+ \quad \text{and} \quad \{A_{i\alpha}, \phi_-\}_\kappa = \left(X^a A\right)_{i\alpha} W_a \phi_-. \tag{3.8}$$

By using identities (2.12), these two equations are just the componentwise form of the first tensorial formulas in (3.4). The relations involving B are verified in the same way. The equalities in (3.4) are converted into those in (3.5) by a short calculation. Since the matrix elements of A and B form a coordinate system on $S(n, d)$, the proof is complete. ◻

Proposition 3.3. Define the map $\Gamma : S(n,d) \to \mathfrak{gl}(n,\mathbb{C})$ by the following formula:

$$\Gamma(A,B) = \mathbf{1}_n + AB. \tag{3.9}$$

As a consequence of the Poisson brackets (1.4), this map satisfies the relation

$$\{\Gamma_1, \Gamma_2\}_\kappa = \kappa\left(\Gamma_1 r_-^n \Gamma_2 + \Gamma_2 r_+^n \Gamma_1 - \Gamma_1 \Gamma_2 r^n - r^n \Gamma_1 \Gamma_2\right) \tag{3.10}$$

together with

$$\{A_1, \Gamma_2\}_\kappa = \kappa(\Gamma_2 r_+^n - r_-^n \Gamma_2) A_1 \quad \text{and} \quad \{B_1, \Gamma_2\}_\kappa = \kappa B_1 (r_-^n \Gamma_2 - \Gamma_2 r_+^n). \tag{3.11}$$

In a neighborhood of 0, Γ can be represented in the form $\Gamma = \Gamma_+ \Gamma_-^{-1}$ so that $(\Gamma_+, \Gamma_-) : S(n,d) \to \mathrm{GL}(n,\mathbb{C})^*$ serves (locally) as the Poisson–Lie moment map for the action (3.1).

Proof. Equalities (3.10) and (3.11) can be verified by an easy calculation. For this, one needs to use the identities $A_1 A_2 I^d = I^n A_1 A_2$ and $I^n A_1 = A_2 C_{12}^{n\times d}$. Since $\Gamma(0) = \mathbf{1}_n$, it is clear that Γ admits a unique factorization of the form $\Gamma = \Gamma_+ \Gamma_-^{-1}$ if we restrict (A, B) to be near enough to 0, require the continuity of $\Gamma_\pm$, and impose condition $\Gamma_\pm(0) = \mathbf{1}_n$. The so-obtained (Γ_+, Γ_-) can be written as holomorphic functions of the matrix entries of Γ. Equation (3.10) entails that (Γ_+, Γ_-) gives a Poisson map into the dual group $\mathrm{GL}(\ell,\mathbb{C})^*$ carrying brackets (2.10), and relations (3.11) are equivalent to the moment map conditions given in (3.4). Here, we used the coincidence of (3.5) with (3.11) and that Γ and (Γ_+, Γ_-) are related by a local Poisson diffeomorphism. ◻

Remark 3.4. A natural generalization of Proposition 3.3 holds around an arbitrary point $(A_0, B_0) \in S(n, d)$ for which $(\mathbf{1}_n + A_0 B_0)$ is an invertible matrix. See Refs. 9 and 11 for the construction of (Γ_+, Γ_-).

Remark 3.5. We observe from Proposition 3.3 and the first factorization identity (1.16) that (g_+, g_-) given by (1.10) is nothing but the (local) $\mathrm{GL}(n,\mathbb{C})^*$-valued Poisson–Lie moment map on $S(n, 1)$. For $d > 1$, the meaning of the second factorization identity (1.18) is that the (local) moment map (Γ_+, Γ_-) mentioned in Proposition 3.3 satisfies the following equality:

$$(\Gamma_\pm \circ m)(a,b) = \mathcal{G}_\pm(a,b), \tag{3.12}$$

where m is the map of Theorem 1.1 and $\mathcal{G}_\pm$ are defined in (1.17).

IV. DERIVATION OF THEOREM 1.1

The Poisson structure is derived in Sec. IV A, and we prove that m is a diffeomorphism in Sec. IV B. The strategy of the derivation is analogous to Ref. 3, Lemma 5.1.

A. The Poisson structure

1. Preparation and notations

The product Poisson structure on $S(n,1)^{\times d}$ can be written in the tensor notation using the pairs (a^α, b^α) as follows:

$$\left\{a_1^\alpha, a_2^\beta\right\}_\kappa = -\kappa\delta_{\alpha\beta} r^n a_1^\alpha a_2^\beta, \qquad \left\{b_1^\alpha, b_2^\beta\right\}_\kappa = -\kappa\delta_{\alpha\beta} b_1^\alpha b_2^\beta r^n, \tag{4.1a}$$

$$\left\{a_1^\alpha, b_2^\beta\right\}_\kappa = \kappa\delta_{\alpha\beta}\left(b_2^\beta r_+^n a_1^\alpha + \frac{1}{2} a_1^\alpha b_2^\beta + C_{12}^{n\times 1}\right). \tag{4.1b}$$

For fixed $\alpha \in \{1,\ldots,d\}$, we use the pair (a^α, b^α) to define G_j^α locally by (1.13) and then the upper and lower triangular matrices $g_{+,\alpha}, g_{-,\alpha}^{-1}$ by (1.11) and (1.12). Using (1.16) in the form

$$g_{+,\alpha} g_{-,\alpha}^{-1} = \mathbf{1}_n + a^\alpha b^\alpha, \tag{4.2}$$

we can combine Propositions 3.2 and 3.3 for each copy $S(n,1)$, and we obtain

$$\{a_1^\alpha, (g_{+,\beta})_2\}_\kappa = -\kappa\delta_{\alpha\beta}\, r_-^n\, a_1^\alpha (g_{+,\beta})_2, \quad \{a_1^\alpha, (g_{-,\beta}^{-1})_2\}_\kappa = \kappa\delta_{\alpha\beta} (g_{-,\beta}^{-1})_2\, r_+^n\, a_1^\alpha, \tag{4.3a}$$

$$\{b_1^\alpha, (g_{+,\beta})_2\}_\kappa = \kappa\delta_{\alpha\beta} b_1^\alpha\, r_-^n\, (g_{+,\beta})_2, \quad \{b_1^\alpha, (g_{-,\beta}^{-1})_2\}_\kappa = -\kappa\delta_{\alpha\beta} b_1^\alpha (g_{-,\beta}^{-1})_2\, r_+^n. \tag{4.3b}$$

We also use (2.10) to write

$$\{(g_{+,\alpha})_1, (g_{+,\beta})_2\}_\kappa = \kappa\delta_{\alpha\beta}[(g_{+,\alpha})_1 (g_{+,\beta})_2, r^n], \tag{4.4a}$$

$$\{(g_{-,\alpha}^{-1})_1, (g_{-,\beta}^{-1})_2\}_\kappa = -\kappa\delta_{\alpha\beta}[(g_{-,\alpha}^{-1})_1 (g_{-,\beta}^{-1})_2, r^n], \tag{4.4b}$$

$$\{(g_{+,\alpha})_1, (g_{-,\beta}^{-1})_2\}_\kappa = -\kappa\delta_{\alpha\beta}\big((g_{+,\alpha})_1 r_+^n (g_{-,\beta}^{-1})_2 - (g_{-,\beta}^{-1})_2 r_+^n (g_{+,\alpha})_1\big). \tag{4.4c}$$

Remark 4.1. The involution $\iota : S(n,1)^{\times d} \to S(n,1)^{\times d}$ defined by

$$\iota\big(a^1, b^1, \ldots, a^d, b^d\big) = \Big((b^1)^T, (a^1)^T, \ldots, (b^d)^T, (a^d)^T\Big) \tag{4.5}$$

is an anti-Poisson automorphism by Remark A.3 in the Appendix. Using this map, we observe the following identities of matrix-valued functions:

$$g_{+,\alpha} \circ \iota = (g_{-,\alpha}^{-1})^T, \quad g_{-,\alpha}^{-1} \circ \iota = g_{+,\alpha}^T. \tag{4.6}$$

The consistency of the anti-Poisson property with the Poisson brackets collected above is straightforward to check.

To ease computations, we introduce

$$h_+^\alpha = g_{+,1} \cdots g_{+,\alpha}, \quad h_-^\alpha = g_{-,\alpha}^{-1} \cdots g_{-,1}^{-1}, \quad 1 \le \alpha \le d, \tag{4.7}$$

with $h_+^0 = \mathbf{1}_n = h_-^0$ so that (1.15a) and (1.15b) become

$$A^\alpha = h_+^{\alpha-1} a^\alpha, \quad B^\alpha = b^\alpha h_-^{\alpha-1}, \quad 1 \le \alpha \le d. \tag{4.8}$$

It will also be convenient to introduce for $1 \le \alpha \le \gamma \le d$,

$$h_+^{\alpha;\gamma} = g_{+,\alpha} \cdots g_{+,\gamma}, \quad h_+^{\alpha;\alpha} = g_{+,\alpha}, \quad h_-^{\alpha;\gamma} = g_{-,\gamma}^{-1} \cdots g_{-,\alpha}^{-1}, \quad h_-^{\alpha;\alpha} = g_{-,\alpha}^{-1}, \tag{4.9}$$

and we set $h_+^{\alpha;\alpha-1} = \mathbf{1}_n = h_-^{\alpha;\alpha-1}$. We note, in particular, that under the involution ι (4.5), which satisfies (4.6), we can write

$$h_+^{\alpha;\gamma} \circ \iota = (h_-^{\alpha;\gamma})^T, \quad h_-^{\alpha;\gamma} \circ \iota = (h_+^{\alpha;\gamma})^T. \tag{4.10}$$

2. Preliminary lemmas

Lemma 4.2. The following identities hold:

$$\{a_1^\alpha, (h_+^\beta)_2\}_\kappa = -\kappa\delta_{(\alpha\le\beta)} (h_+^{\alpha-1})_2\, r_-^n\, a_1^\alpha (h_+^{\alpha;\beta})_2, \tag{4.11a}$$

$$\{b_1^\alpha, (h_+^\beta)_2\}_\kappa = \kappa\delta_{(\alpha\le\beta)} (h_+^{\alpha-1})_2 b_1^\alpha\, r_-^n\, (h_+^{\alpha;\beta})_2, \tag{4.11b}$$

$$\{a_1^\alpha, (h_-^\beta)_2\}_\kappa = \kappa\delta_{(\alpha\le\beta)} (h_-^{\alpha;\beta})_2\, r_+^n\, a_1^\alpha (h_-^{\alpha-1})_2, \tag{4.11c}$$

$$\{b_1^\alpha, (h_-^\beta)_2\}_\kappa = -\kappa\delta_{(\alpha\le\beta)} b_1^\alpha (h_-^{\alpha;\beta})_2\, r_+^n\, (h_-^{\alpha-1})_2. \tag{4.11d}$$

Here, the value of $\delta_{(\alpha\le\beta)}$ is 1 if the condition $\alpha \le \beta$ holds and is zero otherwise.

Proof. We have from (4.3a) that $\{a_1^\alpha,(g_{+,\gamma})_2\}_\kappa$ vanishes identically if $\gamma\neq\alpha$. By definition of h_+^β, we thus get

$$\left\{a_1^\alpha,\left(h_+^\beta\right)_2\right\}_\kappa=\left(h_+^{\alpha-1}\right)_2\{a_1^\alpha,(g_{+,\alpha})_2\}_\kappa\left(h_+^{\alpha+1;\beta}\right)_2 \tag{4.12}$$

if $\alpha\leq\beta$, while it vanishes for $\beta<\alpha$. We then get the first equality from (4.3a). The second equality is found in the same way, and the following two are obtained by applying the anti-Poisson automorphism ι. □

Lemma 4.3. The following identities hold:

$$\left\{\left(h_+^\alpha\right)_1,\left(h_+^\beta\right)_2\right\}_\kappa\overset{\alpha\leq\beta}{=}\kappa\left(\left(h_+^\alpha\right)_1\left(h_+^\alpha\right)_2r^n\left(h_+^{\alpha+1;\beta}\right)_2-r^n\left(h_+^\alpha\right)_1\left(h_+^\beta\right)_2\right), \tag{4.13a}$$

$$\left\{\left(h_+^\alpha\right)_1,\left(h_-^\beta\right)_2\right\}_\kappa\overset{\alpha\leq\beta}{=}\kappa\left(\left(h_-^\beta\right)_2r_+^n\left(h_+^\alpha\right)_1-\left(h_+^\alpha\right)_1\left(h_-^{\alpha+1;\beta}\right)_2r_+^n\left(h_-^\alpha\right)_2\right), \tag{4.13b}$$

$$\left\{\left(h_+^\alpha\right)_1,\left(h_-^\beta\right)_2\right\}_\kappa\overset{\alpha\geq\beta}{=}\kappa\left(\left(h_-^\beta\right)_2r_+^n\left(h_+^\alpha\right)_1-\left(h_+^\beta\right)_1r_+^n\left(h_+^{\beta+1;\alpha}\right)_1\left(h_-^\beta\right)_2\right). \tag{4.13c}$$

Proof. For the first identity, since $\alpha\leq\beta$, we use the decomposition

$$\left\{\left(h_+^\alpha\right)_1,\left(h_+^\beta\right)_2\right\}_\kappa=\sum_{\gamma=1}^{\alpha}\left(h_+^{\gamma-1}\right)_1\left(h_+^{\gamma-1}\right)_2\{(g_{+,\gamma})_1,(g_{+,\gamma})_2\}_\kappa\left(h_+^{\gamma+1;\alpha}\right)_1\left(h_+^{\gamma+1;\beta}\right)_2,$$

and note that the Poisson bracket appearing in the sum is given by (4.4a). This directly leads to the claimed result. For the second identity, we write for $\alpha\leq\beta$,

$$\left\{\left(h_+^\alpha\right)_1,\left(h_-^\beta\right)_2\right\}_\kappa=\sum_{\gamma=1}^{\alpha}\left(h_+^{\gamma-1}\right)_1\left(h_-^{\gamma+1;\beta}\right)_2\{(g_{+,\gamma})_1,(g_{-,\gamma}^{-1})_2\}_\kappa\left(h_+^{\gamma+1;\alpha}\right)_1\left(h_-^{\gamma-1}\right)_2,$$

and then, we use (4.4c) to get the desired result. The case $\alpha\geq\beta$ is obtained in a similar way. □

Note that the identities from Lemmas 4.2 and 4.3 can be used with $h_\pm^0=\mathbf{1}_n$ as well.

3. The Poisson brackets $\{A_1,A_2\}_\kappa$ and $\{B_1,B_2\}_\kappa$

We note that obtaining $\{A_1,A_2\}_\kappa$ in (1.4) is equivalent to deriving

$$\left\{A_1^\alpha,A_2^\beta\right\}_\kappa=-\kappa\left(r^nA_1^\alpha A_2^\beta+\frac{1}{2}\mathrm{sgn}(\alpha-\beta)A_1^\beta A_2^\alpha\right). \tag{4.14}$$

This follows by spelling out the action of r^d using (1.2). In order to get (4.14), we note that (4.8) yields

$$\begin{aligned}\left\{A_1^\alpha,A_2^\beta\right\}_\kappa=&\left\{\left(h_+^{\alpha-1}\right)_1,\left(h_+^{\beta-1}\right)_2\right\}_\kappa a_1^\alpha a_2^\beta+\left(h_+^{\beta-1}\right)_2\left\{\left(h_+^{\alpha-1}\right)_1,a_2^\beta\right\}_\kappa a_1^\alpha\\&+\left(h_+^{\alpha-1}\right)_1\left\{a_1^\alpha,\left(h_+^{\beta-1}\right)_2\right\}_\kappa a_2^\beta+\left(h_+^{\alpha-1}\right)_1\left(h_+^{\beta-1}\right)_2\left\{a_1^\alpha,a_2^\beta\right\}_\kappa.\end{aligned} \tag{4.15}$$

We can then use (4.1a) and Lemmas 4.2 and 4.3 to reduce this expression. If $\alpha=\beta$, we directly get

$$\left\{A_1^\alpha,A_2^\alpha\right\}_\kappa=-\kappa r^nA_1^\alpha A_2^\alpha. \tag{4.16}$$

If $\alpha<\beta$, we get

$$\left\{A_1^\alpha,A_2^\beta\right\}_\kappa=-\kappa r^nA_1^\alpha A_2^\alpha+\frac{\kappa}{2}\left(h_+^{\alpha-1}\right)_1\left(h_+^{\alpha-1}\right)_2I^n\,a_1^\alpha\left(h_+^{\alpha;\beta-1}\right)_2a_2^\beta. \tag{4.17}$$

Upon using the identity

$$I^n\,a_1^\alpha\left(h_+^{\alpha;\beta-1}a^\beta\right)_2=a_2^\alpha\left(h_+^{\alpha;\beta-1}a^\beta\right)_1, \tag{4.18}$$

we find

$$\left\{A_1^\alpha,A_2^\beta\right\}_\kappa=-\kappa\,r^nA_1^\alpha A_2^\beta+\frac{\kappa}{2}A_1^\beta A_2^\alpha. \tag{4.19}$$

Thus, we have derived (4.14) for all $\alpha\leq\beta$, and hence, it holds for all α,β by antisymmetry. We can check that we obtain the claimed Poisson bracket for $\{B_1,B_2\}_\kappa$ either by a direct computation or using the anti-Poisson automorphism ι (4.5) under which $A\circ\iota=B^T$.

4. The Poisson bracket $\{A_1, B_2\}_\kappa$

We now use that obtaining $\{A_1, B_2\}_\kappa$ in (1.4) is equivalent to deriving

$$\left\{A_1^\alpha, B_2^\beta\right\}_\kappa = \kappa\left(B_2^\beta r_+^n A_1^\alpha + \frac{1}{2}\delta_{\alpha\beta}A_1^\alpha B_2^\beta + \delta_{\alpha\beta}\sum_{\mu<\alpha} A_1^\mu B_2^\mu + \delta_{\alpha\beta}C_{12}^{n\times 1}\right). \tag{4.20}$$

We have by (4.8) that

$$\begin{aligned}\left\{A_1^\alpha, B_2^\beta\right\}_\kappa = {} & b_2^\beta\left\{\left(h_+^{\alpha-1}\right)_1, \left(h_-^{\beta-1}\right)_2\right\}_\kappa a_1^\alpha + \left\{\left(h_+^{\alpha-1}\right)_1, b_2^\beta\right\}_\kappa a_1^\alpha\left(h_-^{\beta-1}\right)_2 \\ & + \left(h_+^{\alpha-1}\right)_1 b_2^\beta\left\{a_1^\alpha, \left(h_-^{\beta-1}\right)_2\right\}_\kappa + \left(h_+^{\alpha-1}\right)_1\left\{a_1^\alpha, b_2^\beta\right\}_\kappa\left(h_-^{\beta-1}\right)_2,\end{aligned} \tag{4.21}$$

which can be computed using (4.1b) and Lemmas 4.2 and 4.3. If $\alpha < \beta$, only the first and third sums in (4.21) do not trivially vanish, and we find that

$$\left\{A_1^\alpha, B_2^\beta\right\}_\kappa = \kappa B_2^\beta r_+^n A_1^\alpha. \tag{4.22}$$

If $\alpha > \beta$, we also obtain (4.22) by a similar computation. If $\alpha = \beta$, only the first and fourth sums in (4.21) are nonzero, and we obtain

$$\left\{A_1^\alpha, B_2^\alpha\right\}_\kappa = \kappa\left(B_2^\alpha r_+^n A_1^\alpha + \frac{1}{2}A_1^\alpha B_2^\alpha + \left(h_+^{\alpha-1}\right)_1 C_{12}^{n\times 1}\left(h_-^{\alpha-1}\right)_2\right). \tag{4.23}$$

We deduce from (4.2) that $h_+^{\alpha-1}h_-^{\alpha-1} = \mathbf{1}_n + \sum_{\mu<\alpha} A^\mu B^\mu$, which implies

$$\left(h_+^{\alpha-1}\right)_1 C_{12}^{n\times 1}\left(h_-^{\alpha-1}\right)_2 = C_{12}^{n\times 1} + \sum_{\mu<\alpha} A_1^\mu B_2^\mu. \tag{4.24}$$

Thus, we can write $\left\{A_1^\alpha, B_2^\beta\right\}_\kappa$ for all α, β in the desired form (4.20).

B. Diffeomorphism property

Let us consider a point where the map m (1.14) is well-defined, i.e., we can construct $g_{\pm,\alpha} = g_\pm(a^\alpha, b^\alpha)$ for $\alpha = 1, \ldots, d$ using (1.11) and (1.12) with (1.13). In a sufficiently small neighborhood of this point, the entries of the matrices $g_{\pm,\alpha}$ are analytic functions; hence, m is holomorphic. From the image of this neighborhood, we can define inductively

$$\begin{aligned} & a^1 = A^1, \quad b^1 = B^1, \qquad a^2 = g_{+,1}^{-1}A^2, \quad b^2 = B^2 g_{-,1}, \quad \ldots, \\ & a^d = g_{+,d-1}^{-1}\cdots g_{+,1}^{-1}A^d, \quad b^d = B^d g_{-,1}\cdots g_{-,d-1}, \end{aligned} \tag{4.25}$$

which is the inverse of the map m. The inverse map is holomorphic since the elements $g_{\pm,\alpha}$ are analytic functions in (A_i^β, B_i^β) for $\beta \le \alpha$.

V. A DECOUPLING PROPERTY OF THE ARUTYUNOV–OLIVUCCI BRACKET

We now derive an alternative realization of the Poisson algebra (1.21), as was promised after Lemma 1.2. For this purpose, we take d copies of $(S(n,1), \{\ ,\ \}_\kappa)$ with variables (a^α, b^α) for $\alpha = 1, \ldots, d$ and define the new variables $(\hat{A}^\alpha, \hat{B}^\alpha)$ as follows:

$$\hat{A}^\alpha(a,b) = g_+^{-1}\left(a^d, b^d\right)\cdots g_+^{-1}\left(a^\alpha, b^\alpha\right)a^\alpha, \tag{5.1a}$$

$$\hat{B}^\alpha(a,b) = b^\alpha g_-\left(a^\alpha, b^\alpha\right)\cdots g_-\left(a^d, b^d\right), \tag{5.1b}$$

using the functions introduced previously in (1.11) and (1.12).

Theorem 5.1. *The map $F : (a,b) \mapsto (\hat{A}, \hat{B})$ given by (5.1a) and (5.1b), where $(\hat{A}^\alpha, \hat{B}^\alpha)$ denote the columns and the rows, respectively, of the matrices $(\hat{A}, \hat{B}) \in S(n,d)$, is a local Poisson diffeomorphism,*

$$F : \left(S(n,1), \{\ ,\ \}_\kappa\right)^{\times d} \to \left(S(n,d), \{\ ,\ \}_\kappa'\right), \tag{5.2}$$

where $\{\ ,\ \}_\kappa'$ denotes the Poisson structure on $S(n, d)$ defined by

$$\begin{aligned} \left\{\hat{A}_1, \hat{A}_2\right\}_\kappa' &= \kappa\left(r^n \hat{A}_1\hat{A}_2 - \hat{A}_1\hat{A}_2 r^d\right), \\ \left\{\hat{B}_1, \hat{B}_2\right\}_\kappa' &= \kappa\left(\hat{B}_1\hat{B}_2 r^n - r^d\hat{B}_1\hat{B}_2\right), \\ \left\{\hat{A}_1, \hat{B}_2\right\}_\kappa' &= \kappa\left(-\hat{B}_2 r_+^n\hat{A}_1 + \hat{A}_1 r_-^d\hat{B}_2 + C_{12}^{n\times d}\right). \end{aligned} \tag{5.3}$$

Proof. The calculation of the Poisson brackets of the functions in (5.1a) and (5.1b) is, in principle, straightforward and follows the derivation of Theorem 1.1 made in Sec. IV A.

The fact that the map F is a local diffeomorphism is similar to the argument used in Sec. IV B. We begin by observing the following identities:

$$\mathbf{1}_n - g_{+,\alpha+1}\cdots g_{+,d}\hat{A}^\alpha\hat{B}^\alpha g_{-,d}^{-1}\cdots g_{-,\alpha+1}^{-1} = g_{+,\alpha}^{-1}g_{-,\alpha} \quad \text{for} \quad \alpha = d,\ldots,1, \tag{5.4}$$

which follow from (5.1a) and (5.1b) using $g_{\pm,\alpha} = g_\pm(a^\alpha, b^\alpha)$, with $g_{\pm,d+1} := \mathbf{1}_n$, and applying the analog of (1.16) for all α. Then, picking $\hat{A}$ and $\hat{B}$ near 0, we define the functions $(\hat{g}_{+,\alpha}, \hat{g}_{-,\alpha}) \in \mathrm{GL}(n,\mathbb{C})^*$ for $1 \le \alpha \le d$ by considering the factorization problems,

$$\mathbf{1}_n - \hat{A}^d\hat{B}^d = \hat{g}_{+,d}^{-1}\hat{g}_{-,d}, \tag{5.5}$$

and iteratively,

$$\mathbf{1}_n - \hat{g}_{+,\alpha+1}\cdots\hat{g}_{+,d}\hat{A}^\alpha\hat{B}^\alpha\hat{g}_{-,d}^{-1}\cdots\hat{g}_{-,\alpha+1}^{-1} = \hat{g}_{+,\alpha}^{-1}\hat{g}_{-,\alpha} \quad \text{for} \quad \alpha = d-1,\ldots,1. \tag{5.6}$$

This procedure uniquely specifies $\hat{g}_{\pm,\alpha}$ for all α if we set $\hat{g}_{\pm,\alpha} = \mathbf{1}_n$ for vanishing $\hat{A}$ and $\hat{B}$ and further require that these matrices depend continuously on $\hat{A}, \hat{B}$ in an open neighborhood of 0. As the final step, we define

$$a^\alpha = \hat{g}_{+,\alpha}\hat{g}_{+,\alpha+1}\cdots\hat{g}_{+,d}\hat{A}^\alpha, \quad b^\alpha = \hat{B}^\alpha\hat{g}_{-,d}^{-1}\cdots\hat{g}_{-,\alpha+1}^{-1}\hat{g}_{-,\alpha}^{-1}. \tag{5.7}$$

The definitions guarantee that if on the left-hand sides of (5.5) and (5.6) we use (5.1a) and (5.1b), then we obtain

$$\hat{g}_{+,\alpha} = g_+(a^\alpha, b^\alpha), \quad \hat{g}_{-,\alpha} = g_-(a^\alpha, b^\alpha), \tag{5.8}$$

and hence, the map that we constructed by (5.7) is indeed the local inverse of F. □

It should be noted that although the map F from Theorem 5.1 is only a local diffeomorphism, formulas (5.3) yield a holomorphic Poisson structure on the full space $S(n, d)$.

Remark 5.2. By using $\mathcal{G}_\pm$ (1.17), the formula

$$(a,b) \mapsto (\mathcal{G}_+(a,b), \mathcal{G}_-(a,b)) \tag{5.9}$$

defines a local Poisson map from $(S(n,1), \{\ ,\ \}_\kappa)^{\times d}$ to $(\mathrm{GL}(n,\mathbb{C})^*, \{\ ,\ \}_*^\kappa)$. This map satisfies the following identity:

$$\mathcal{G}_+(a,b)^{-1}\mathcal{G}_-(a,b) = \mathbf{1}_n - \hat{A}(a,b)\hat{B}(a,b), \tag{5.10}$$

which is a counterpart of identity (1.18). The left-action of $\mathrm{GL}(n,\mathbb{C})$ on $S(n, d)$ has the Poisson–Lie property with respect to the bracket $\{\ ,\ \}'_\kappa$ (5.3) on $S(n, d)$ and the bracket $\{\ ,\ \}_G^{-\kappa}$ (1.6) on $\mathrm{GL}(n,\mathbb{C})$. The map $(\hat{A},\hat{B}) \mapsto \hat{\Gamma}(\hat{A},\hat{B}) := \mathbf{1}_n - \hat{A}\hat{B}$ represents the (densely defined) moment map associated with this action, using the standard mapping (2.13) of $\mathrm{GL}(n,\mathbb{C})^*$ into $\mathrm{GL}(n,\mathbb{C})$. To put it more explicitly, $\hat{A}, \hat{B}$ and $\hat{\Gamma}$ satisfy

$$\{\hat{\Gamma}_1, \hat{\Gamma}_2\}'_\kappa = -\kappa\big(\hat{\Gamma}_1 r_-^n\hat{\Gamma}_2 + \hat{\Gamma}_2 r_+^n\hat{\Gamma}_1 - \hat{\Gamma}_1\hat{\Gamma}_2 r^n - r^n\hat{\Gamma}_1\hat{\Gamma}_2\big) \tag{5.11}$$

together with

$$\{\hat{A}_1, \hat{\Gamma}_2\}'_\kappa = -\kappa\big(\hat{\Gamma}_2 r_+^n - r_-^n\hat{\Gamma}_2\big)\hat{A}_1 \quad \text{and} \quad \{\hat{B}_1, \hat{\Gamma}_2\}'_\kappa = -\kappa\hat{B}_1\big(r_-^n\hat{\Gamma}_2 - \hat{\Gamma}_2 r_+^n\big). \tag{5.12}$$

These relations have the same form as those in (3.10) and (3.11), taking into account that now we are referring to Poisson–Lie symmetry with respect to the bracket $\{\ ,\ \}_G^{-\kappa}$ on $\mathrm{GL}(n,\mathbb{C})$. The interested reader can verify all these equalities by direct calculation.

Finally, we state the sought after decoupling property of the Arutyunov–Olivucci Poisson bracket (1.21).

Corollary 5.3. Let θ_A and θ_B be arbitrary constants satisfying $\theta_A\theta_B = -\frac{1}{\kappa}$. Then, the rescaling

$$\theta : (\hat{A},\hat{B}) \mapsto (\mathcal{A},\mathcal{B}) := (\theta_A\hat{A}, \theta_B\hat{B}) \tag{5.13}$$

gives a Poisson diffeomorphism from $(S(n,d), \{\ ,\ \}'_\kappa)$ (5.3) to $(S(n,d), \{\ ,\ \}_\kappa^+)$ (1.21). Composing this with the map F from Theorem 5.1, we get a local Poisson diffeomorphism,

$$\theta\circ F : \big(S(n,1), \{\ ,\ \}_\kappa\big)^{\times d} \to \big(S(n,d), \{\ ,\ \}_\kappa^+\big). \tag{5.14}$$

This map enjoys the following identity:

$$(\mathbf{1}_n + \kappa\mathcal{A}\mathcal{B})\circ\theta\circ F = \mathcal{G}_+^{-1}\mathcal{G}_-. \tag{5.15}$$

The observation that $(\mathbf{1}_n + \kappa\mathcal{A}\mathcal{B})$ can be realized by applying mapping (2.13) on the inverse $(\mathcal{G}_+, \mathcal{G}_-)^{-1}$ of a Poisson map $(\mathcal{G}_+, \mathcal{G}_-)$ into $(\mathrm{GL}(n,\mathbb{C})^*, \{\ ,\ \}_*^\kappa)$ played an important role in the derivation of the trigonometric complex spin Ruijsenaars–Schneider model by Arutyunov and Olivucci.[2] [To be precise, they locally realized $(\mathcal{G}_+, \mathcal{G}_-)$ as a moment map generating a Poisson–Lie action of $(\mathrm{GL}(n,\mathbb{C}), \{\ ,\ \}_G^\kappa)$ on $(S(n,d), \{\ ,\ \}_\kappa^+)$.] Our result (5.15) provides decoupled variables $((a^\alpha, b^\alpha)$ for $\alpha = 1, \ldots, d)$ that give such a realization explicitly. These new variables (a, b) are expected to be useful for further studies of the reduction treatment of the complex spin Ruijsenaars–Schneider model, similarly as proved to be the case for the real form of this important integrable Hamiltonian system.[3]

VI. CONCLUSION

In this paper, we presented a detailed analysis of the $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(d,\mathbb{C})$ covariant Poisson structures (1.4) and (1.21) on the linear space $S(n, d)$ (1.1) for arbitrary natural numbers n and d. Our main results are encapsulated by Theorem 1.1 and Theorem 5.1 with Corollary 5.3 that provide new realizations of the corresponding Poisson algebras in terms of d independent copies of "elementary spin variables" living in $S(n, 1)$. The Appendix highlights further relevant properties of these Poisson structures, especially by giving the underlying symplectic form on a dense open subset of $S(n, 1)$. These results may contribute, for example, to deepening the understanding of Ruijsenaars–Schneider-type integrable many-body models with spin having hidden $\mathrm{GL}(n,\mathbb{C})$ Poisson–Lie symmetry. It is also an interesting open question to search for their quantum mechanical analogs in the future.

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APPENDIX: ADDITIONAL PROPERTIES OF THE POISSON BRACKET ON $S(n, d)$

In this appendix, we show that the Poisson bracket (1.4) on $S(n, 1)$ can be seen as a particular example of the complexification of Zakrzewski's $\mathrm{U}(n)$ covariant Poisson brackets on $\mathbb{C}^n$.[12] We also point out that the Poisson bracket (1.4) on $S(n, d)$ is never globally symplectic and present the symplectic form that corresponds to this Poisson bracket in a neighborhood of 0.

1. The Zakrzewski Poisson brackets

Let us introduce a real anti-symmetric biderivation on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, which is written on the components of $u \in \mathbb{C}^n$ as

$$\begin{aligned}\{u_i, u_j\} &= -\epsilon \mathrm{i}\, \mathrm{sgn}(i-j) u_i u_j,\\ \{u_i, \bar{u}_l\} &= -\epsilon \mathrm{i}\delta_{il} F + \epsilon \mathrm{i} G u_i \bar{u}_l - \epsilon \mathrm{i}\delta_{il} \sum_{r=1}^n \mathrm{sgn}(r-i)|u_r|^2,\end{aligned} \tag{A1}$$

where $\epsilon \in \mathbb{R}^*$ and $F = F(|u|^2)$, $G = G(|u|^2)$ are two arbitrary functions. Denote by F', G' the derivatives $F'(t) = \frac{d}{dt}F(t)$, $G'(t) = \frac{d}{dt}G(t)$. We recall the following result due to Zakrzewski.

Lemma A.1 (Ref. 12). The anti-symmetric biderivation (A1) is always Poisson for $n = 1$, while for $n \geq 2$, it is Poisson if and only if

$$FF' + G(F - F'|u|^2) = |u|^2. \tag{A2}$$

Furthermore, the action $\mathrm{U}(n) \times \mathbb{C}^n \to \mathbb{C}^n$ defined by left multiplication yields a Poisson map if $\mathrm{U}(n)$ is equipped with the multiplicative Poisson bracket, satisfying $\{g_1, g_2\}_{\mathrm{U}(n)} = -2\mathrm{i}\epsilon[g_1 g_2, r^n]$.

In complete analogy, we define an anti-symmetric, holomorphic biderivation on $\mathbb{C}^{2n}$ endowed with coordinates (a_i, b_i) (where we see a and b, respectively, as a vector and a covector) by

$$\{a_i, a_j\} = \frac{\kappa}{2}\mathrm{sgn}(i-j) a_i a_j, \quad \{b_i, b_j\} = -\frac{\kappa}{2}\mathrm{sgn}(i-j) b_i b_j, \tag{A3a}$$

$$\{a_i, b_l\} = \frac{\kappa}{2}\delta_{il} F - \frac{\kappa}{2} G a_i b_l + \frac{\kappa}{2}\delta_{il} \sum_{r=1}^n \mathrm{sgn}(r-i) a_r b_r, \tag{A3b}$$

where $\kappa \in \mathbb{C}^*$ and $F = F(t)$, $G = G(t)$ are two arbitrary holomorphic functions of $t := \sum_{r=1}^n a_r b_r$. We denote by F', G' the derivatives of F, G with respect to t. The following result can then be proved as Lemma A.1.

Lemma A.2. The anti-symmetric biderivations (A3a) and *(A3b) are always Poisson for $n = 1$, while for $n \geq 2$, it is Poisson if and only if*

$$FF' + G(F - F't) = t, \qquad t := \sum_{r=1}^{n} a_r b_r. \tag{A4}$$

Furthermore, the action

$$\tau : \mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^{2n} \to \mathbb{C}^{2n}, \quad \tau_g(a, b) = (ga, bg^{-1}) \tag{A5}$$

is a Poisson map when $\mathrm{GL}(n, \mathbb{C})$ is equipped with the Poisson bracket (1.6).

The Poisson bracket (1.4) on $S(n, 1)$ is an example of the complex Zakrzewski Poisson brackets of Lemma A.2 since it corresponds to the cases $F(t) = 2 + t$ and $G(t) = -1$.

Remark A.3. The involution

$$\iota : \mathbb{C}^{2n} \to \mathbb{C}^{2n}, \quad \iota(a, b) = (b^T, a^T) \tag{A6}$$

is an anti-Poisson automorphism. This follows from a direct verification of this property on the evaluation functions a_i, b_i; see (A3a) and (A3b).

2. Degeneracy of the Poisson bracket

We note that the Poisson bracket (1.4) on $S(n, d)$ can be written in the coordinates $A_i^\alpha := A_{i\alpha}$, $B_i^\alpha := B_{\alpha i}$ as

$$\left\{A_i^\alpha, A_k^\beta\right\} = \frac{\kappa}{2}\mathrm{sgn}(i-k)A_k^\alpha A_i^\beta - \frac{\kappa}{2}\mathrm{sgn}(\alpha-\beta)A_k^\alpha A_i^\beta, \tag{A7a}$$

$$\left\{B_i^\alpha, B_k^\beta\right\} = -\frac{\kappa}{2}\mathrm{sgn}(i-k)B_k^\alpha B_i^\beta + \frac{\kappa}{2}\mathrm{sgn}(\alpha-\beta)B_k^\alpha B_i^\beta, \tag{A7b}$$

$$\begin{aligned}\left\{A_i^\alpha, B_k^\beta\right\} =& \frac{\kappa}{2}\delta_{ik}A_i^\alpha B_k^\beta + \kappa\delta_{ik}\sum_{s>i} A_s^\alpha B_s^\beta \\ &+ \frac{\kappa}{2}\delta_{\alpha\beta}A_i^\alpha B_k^\beta + \kappa\delta_{\alpha\beta}\sum_{\mu<\alpha} A_i^\mu B_k^\mu + \kappa\delta_{\alpha\beta}\delta_{ik}.\end{aligned} \tag{A7c}$$

If we consider a point $p \in S(n, d)$ contained in the one-dimensional subset where

$$1 + A_n^1 B_n^1 = 0, \quad A_j^\alpha, B_j^\alpha = 0 \text{ for } (j, \alpha) \neq (n, 1), \tag{A8}$$

we directly see that the Poisson brackets (A7a) and (A7b) evaluated at p are 0. Furthermore, decomposing (A7c) as

$$\left\{A_i^\alpha, B_k^\beta\right\} = 0, \qquad i \neq k, \alpha \neq \beta, \tag{A9a}$$

$$\left\{A_i^\alpha, B_k^\alpha\right\} = \frac{\kappa}{2}A_i^\alpha B_k^\alpha + \kappa\sum_{\mu<\alpha} A_i^\mu B_k^\mu, \quad i \neq k,\ \alpha = \beta, \tag{A9b}$$

$$\left\{A_i^\alpha, B_i^\beta\right\} = \frac{\kappa}{2}A_i^\alpha B_k^\beta + \kappa\sum_{s>i} A_s^\alpha B_s^\beta, \quad i = k,\ \alpha \neq \beta, \tag{A9c}$$

$$\left\{A_i^\alpha, B_i^\alpha\right\} = \kappa A_i^\alpha B_i^\alpha + \kappa\sum_{s>i} A_s^\alpha B_s^\alpha + \kappa\sum_{\mu<\alpha} A_i^\mu B_i^\mu + \kappa, \quad i = k,\ \alpha = \beta, \tag{A9d}$$

we get that (A9a)–(A9c) vanish at p, while we can write (A9d) as

$$\left\{A_i^\alpha, B_i^\alpha\right\}(p) = \kappa[1 - \delta_{in}][1 - \delta_{\alpha 1}]. \tag{A10}$$

Hence, at p, the rank of the Poisson structure is $2(n-1)(d-1)$ and the Poisson bracket (1.4) is not globally non-degenerate.

3. The symplectic form on a dense open subset

The holomorphic Poisson bracket (1.4) is non-degenerate on a dense subset of $S(n, d)$ since it is non-degenerate at the origin. We now present the corresponding symplectic form for $d = 1$. For $d \geq 2$, the symplectic form can be obtained around the origin by combining this result with Theorem 1.1.

Proposition A.4. Consider $S(n, 1)$ with the Poisson bracket (1.4), and denote its elements by (a, b). On the open subset where

$$G_i := 1 + \sum_{r\geq i} a_r b_r \neq 0, \quad \forall\, 1 \leq i \leq n, \tag{A11}$$

the Poisson structure is non-degenerate, and the associated symplectic form can be written as

$$\omega = -\frac{1}{\kappa}\sum_{i=1}^{n}\frac{da_i \wedge db_i}{G_i} + \frac{1}{2\kappa}\sum_{i=1}^{n}\sum_{s>i}\frac{1}{G_i G_{i+1}}(b_i da_i - a_i db_i) \wedge (b_s da_s + a_s db_s), \tag{A12}$$

where we set $a_{n+1} = b_{n+1} = 0$ and $G_{n+1} = 1$.

Proof. Without loss of generality, we take $\kappa = 2$. We will prove using the two-form (A12) that for any $1 \leq j \leq n$,

$$\iota_{X_{a_j}}\omega = -da_j, \tag{A13}$$

where $X_{a_j} = \{-, a_j\}$ denotes the Hamiltonian vector field of a_j, which is given by

$$X_{a_j} = \sum_{l\neq j}\mathrm{sgn}(l-j)a_j a_l \frac{\partial}{\partial a_l} - \sum_{l\neq j} a_j b_l \frac{\partial}{\partial b_l} - 2G_j\frac{\partial}{\partial b_j}. \tag{A14}$$

By symmetry between a and b, we will also have that $\iota_{X_{b_j}}\omega = -db_j$. These conditions then imply that ω is non-degenerate and corresponds to the Poisson bracket on $S(n, 1)$; hence, it is also closed.

To prove (A13), let us denote the three terms appearing in the vector field (A14) as X_1, X_2, X_3. Contracting with the two-form, we compute

$$\begin{aligned}\iota_{X_1}\omega =& -\frac{1}{2}\sum_{l\neq j}\mathrm{sgn}(l-j)\frac{a_j a_l}{G_l}db_l \\ &+ \frac{1}{4}\sum_{l\neq j}\mathrm{sgn}(l-j)\sum_{s>l} a_j\frac{a_l b_l}{G_l G_{l+1}}(b_s da_s + a_s db_s) \\ &- \frac{1}{4}\sum_{l\neq j}\mathrm{sgn}(l-j)\sum_{i<l} a_j\frac{a_l b_l}{G_i G_{i+1}}(b_i da_i - a_i db_i),\end{aligned} \tag{A15}$$

then

$$\begin{aligned}\iota_{X_2}\omega =& -\frac{1}{2}\sum_{l\neq j}\frac{a_j b_l}{G_l}da_l + \frac{1}{4}\sum_{l\neq j}\sum_{s>l} a_j\frac{a_l b_l}{G_l G_{l+1}}(b_s da_s + a_s db_s) \\ &+ \frac{1}{4}\sum_{l\neq j}\sum_{i<l} a_j\frac{a_l b_l}{G_i G_{i+1}}(b_i da_i - a_i db_i),\end{aligned} \tag{A16}$$

and finally

$$\begin{aligned}\iota_{X_3}\omega =& -da_j + \frac{1}{2}\sum_{s>j} a_j\frac{1}{G_{j+1}}(b_s da_s + a_s db_s) \\ &+ \frac{1}{2}\sum_{i<j} a_j\frac{G_j}{G_i G_{i+1}}(b_i da_i - a_i db_i).\end{aligned} \tag{A17}$$

After summing together the last two terms from (A15) and (A16), we find

$$\begin{aligned}\iota_{X_{a_j}}\omega =& -da_j - \frac{1}{2}\sum_{s\neq j}\mathrm{sgn}(s-j)\frac{a_j a_s}{G_s}db_s - \frac{1}{2}\sum_{s\neq j}\frac{a_j b_s}{G_s}da_s \\ &+ \frac{1}{2}\sum_{s>l>j} a_j\frac{a_l b_l}{G_l G_{l+1}}(b_s da_s + a_s db_s) + \frac{1}{2}\sum_{s<l<j} a_j\frac{a_l b_l}{G_s G_{s+1}}(b_s da_s - a_s db_s) \\ &+ \frac{1}{2}\sum_{s>j} a_j\frac{1}{G_{j+1}}(b_s da_s + a_s db_s) + \frac{1}{2}\sum_{s<j} a_j\frac{G_j}{G_s G_{s+1}}(b_s da_s - a_s db_s).\end{aligned} \tag{A18}$$

For fixed s, j, we note the following identities:

$$\sum_{l=j+1}^{s-1} \frac{a_l b_l}{G_l G_{l+1}} = \frac{1}{G_s} - \frac{1}{G_{j+1}}, \qquad \sum_{l=s+1}^{j-1} a_l b_l = G_{s+1} - G_j, \tag{A19}$$

which allow us to write the line in the middle of (A18) as

$$+\frac{1}{2}\sum_{s>j} a_j \left(\frac{1}{G_s} - \frac{1}{G_{j+1}}\right)(b_s da_s + a_s db_s) + \frac{1}{2}\sum_{s<j} a_j \frac{G_{s+1} - G_j}{G_s G_{s+1}}(b_s da_s - a_s db_s). \tag{A20}$$

This can be simplified with the last line of (A18), and we get

$$\begin{aligned} \iota_{X_{a_j}}\omega = &- da_j - \frac{1}{2}\sum_{s\neq j} \operatorname{sgn}(s-j)\frac{a_j a_s}{G_s} db_s - \frac{1}{2}\sum_{s\neq j}\frac{a_j b_s}{G_s} da_s \\ &+ \frac{1}{2}\sum_{s>j} a_j \frac{a_s b_s}{G_s}\left(\frac{da_s}{a_s} + \frac{db_s}{b_s}\right) + \frac{1}{2}\sum_{s<j} a_j \frac{a_s b_s}{G_s}\left(\frac{da_s}{a_s} - \frac{db_s}{b_s}\right), \end{aligned} \tag{A21}$$

which is just $-da_j$ as the other four terms cancel out. □

Remark A.5. It can be shown that the Poisson tensor corresponding to the bracket (1.4) on $S(n, 1)$ is degenerate precisely on the zero set of the function $\prod_{i=1}^n G_i$, which is the complement of the set considered in Proposition A.4. If in (A12), we put $b_j = \bar{a}_j$ and $\kappa = 2\mathrm{i}$, then we recover the real symplectic form on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ given by Ref. 3, Proposition A.6, from which our formula was obtained by complexification.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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