A Unified Approach to Four Important Classes of Unary Operators

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Abstract

In this paper, we study operator dependent modifiers and we interpret the dual pair of modal operators based on an algebraic definition. It is a known fact that the substantiating and weakening modifier operators can be induced by repeating the arguments of conjunctive and disjunctive operators. We provide the conditions for which these modifier operators satisfy the requirements for a dual pair of necessity and possibility operators. Next, the necessary and sufficient condition for the distributivity of unary operators over conjunctive and disjunctive operators is presented. This also means that setting the distributivity as a requirement results in a unary operator that is identical to the modal operators mentioned above. Using this property, we establish an important connection between modal operators and linguistic hedges. Previously, we demonstrated that the unary operators induced by compositions of two strong negations satisfy the requirements for a dual pair of modal operators. Here, we view the negation operator as a modifier operator. Then, it is shown that (1) the strong negations, (2) the substantiating and weakening modifier operators, modal operators and linguistic hedges mentioned above, and (3) the unary operators, which are distributive over conjunctive and disjunctive operators, may be viewed as special cases of a unified unary operator class.

Keywords: Pliant negation, substantiating and weakening modifier operators, modal operators, distributive unary operators, hedges

1. Introduction

The unary operators that modify the continuous logical value of a statement or transform the membership function of a fuzzy set to a new membership function play a key role in many areas including fuzzy control, approximate reasoning, natural language query, etc. Not surprisingly, these operators have a variety of applications in continuous-valued logic and in the theory of fuzzy sets.

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Here, we will analyze the negation operators, the substantiating and weakening modifier operators, the modal operators and the fuzzy hedges.

The negation operators and the modal-like necessity and possibility operators have been intensely studied over the past few decades. Hájek presented a system called basic logic (BL) [1], and Gottwald and Hájek [2] published a survey paper on the state-of-art development of BL. Hájek [3] studied the fuzzy variant of the well-known modal logic S5, introduced three kinds of Kripke models and identified the corresponding deductive systems. Many authors contributed to the theory of negations and modal operators in continuous valued logic (see, e.g. Esteva *et al.* [4], Cintula *et al.* [5], Banerjee and Dubois [6], Cattaneo *et al.* [7], Vidal [8], Jain *et al.* [9]). Following the ideas of Łukasiewicz [10–12], Mattila [13, 14] presented the concepts of the substantiating and weakening modifier operators and the modifier logics based on graded modalities.

In the early 1970s, Zadeh [15] introduced a class of powering modifiers, which defined the concept of linguistic variables and hedges. He proposed the computing with words as an extension of fuzzy sets and logic theory [16–19]. Many researchers have contributed to the concept of computing with words (see, e.g. De Cock and Kerre [20], Huynh, Ho, and Nakamori [21], Yan *et al.* [22], Rubin [23]). Esteva *et al.* [24] proposed logics that accommodate most of the truth hedge functions used in the literature.

In this study, the concept of a dual pair of modal operators is interpreted by following the criteria for an algebraic version of necessity and possibility operators on De Morgan lattices given by Cattaneo, Ciucci and Dubois [7] (also, see [25]). Previously, we demonstrated that the unary operators induced by compositions of two strong negations satisfy the requirements for a dual pair of modal operators [26]. It is an acknowledged fact that the so-called substantiating and weakening modifier operators (see [13]) can be deduced by repeating the arguments of conjunctive and disjunctive operators (i.e. strict t-norms and t-conorms). In this paper, it is proved that if a conjunctive operator, a disjunctive operator and a strong negation build a De Morgan system, then the substantiating and weakening modifier operators induced by repeating the arguments of the conjunctive and disjunctive operators satisfy the requirements for a dual pair of modal operators. Next, the necessary and sufficient condition for the distributivity of a unary operator over conjunctive and disjunctive operators is presented. Here, an important connection between modal operators and linguistic hedges is demonstrated as well. Finally, it is shown that (1) the strong negations, (2) the substantiating and weakening modifier operators, modal operators and linguistic hedges mentioned above, and (3) the unary operators, which are distributive over conjunctive and disjunctive operators, may be viewed as special cases of a unified unary operator class.

This paper is structured as follows. In Section 2, the basic considerations of continuous-valued logic, which will be used later on, are described. In Section 3, we briefly review our previous results on how modal operators can be induced via compositions of strong negations. In Section 4, we describe how substantiating and weakening modifier operators can be induced by connectives. In Section 5, we show how the substantiating and weakening modifier operators can satisfy the algebraic criteria for modal operators. Also, some key consequences of our results are presented here. The necessary and sufficient condition for the distributivity of a unary operator over conjunctive and disjunctive operators is presented in Section 6. The connection between modal operators and linguistic hedges is described in Section 7. A unified form of these unary operators is presented in Section 8. Lastly, in Section 9, we shall summarize our conclusions.

2. Preliminaries

Here, we will briefly review the basic considerations of continuous-valued logic, which will be used later on. We will make use of the concepts of strict triangular norm (strict t-norm) and strict triangular conorm (strict t-conorm). Since we require that these operators are associative, we will use the following representation theorem of Aczél [27] (also see [28]).

Theorem 1. *A continuous and strictly increasing function* $F\colon [a,b]^2 \to [a,b]$ *is associative if and only if*

$$F(x,y) = f^{-1}\left(f(x) + f(y)\right),$$

where $f\colon [a,b] \to [0,\infty]$ *is a strictly decreasing continuous function. Here,* f *is called a generator function of* F*, and* F *is uniquely determined up to constant multiplier of* f*.*

The following definition of strict t-norms and strict t-conorms is based on Theorem 1.

Definition 1. *We say that the function* $o\colon [0,1]^2 \to [0,1]$ *is a strict t-norm (strict t-conorm, respectively) if* o *is continuous, and there exists a continuous and strictly decreasing (increasing, respectively) function* $f\colon [0,1] \to [0,\infty]$*, called a generator function of* o*, such that*

$$o(x,y) = f^{-1}\left(f(x) + f(y)\right),$$

for any $x, y \in [0,1]$*, and*

(a) for a strict t-norm c*,* $f = f_c$ *is strictly decreasing with* $f_c(1) = 0$ *and* $\lim_{x\to 0} f_c(x) = \infty$ *and*
(b) for a strict t-conorm d*,* $f = f_d$ *is strictly increasing with* $f_d(0) = 0$ *and* $\lim_{x\to 1} f_d(x) = \infty$*.*

Remark 1. *Hereafter, we will use the convention* $f_c(0) = \infty$ *and* $f_d(1) = \infty$*, and extend the arithmetic operations such that* $\frac{1}{0} = \infty$ *and* $\frac{1}{\infty} = 0$*.*

Note that the strict t-norm and strict t-conorm are special cases of the general t-norm and t-conorm classes, respectively. In our study:

(a) We do not use the pseudo inverse and ordinal sum to construct a general t-norm and t-conorm

(b) We do not use the commutativity axiom of the t-norm and t-conorm because it is always valid for the strict t-norm

(c) We do not use the boundary condition of the t-norm and t-conorm, just the compatibility condition with binary logic. (The boundary condition can be proved by applying the associativity property.)

In this article, we will refer to strict t-norms and t-conorms as conjuctive and disjunctive operators denoted by c and d, respectively. And from now on, the mapping $f\colon [0,1] \to [0,\infty]$ will always be a continuous, strictly increasing (or decreasing) generator function of a conjunctive or disjunctive operator.

Definition 2. *We say that $\eta\colon [0,1] \to [0,1]$ is a strong negation if η satisfies the following conditions:*

C1: η is bijective and continuous	*(Bijectivity and continuity)*
C2: $\eta(0) = 1$, $\eta(1) = 0$	*(Boundary conditions)*
C3: $\eta(x) < \eta(y)$ for $x > y$	*(Monotonicity)*
C4: $\eta(\eta(x)) = x$ for any $x \in [0,1]$	*(Involution).*

Remark 2. *Note that the boundary conditions in C2 can be inferred by using C1 and C3. Also note that from C1, C2 and C3, it follows that there exists a fixed point (or neutral value) $\nu \in (0,1)$ such that $\eta(\nu) = \nu$.*

Here, will use the following representation theorem for the strong negation given by Dombi (see [29, Theorem 5]).

Theorem 2. *For any strong negation η_ν, there exists a function f and a constant $\nu \in (0,1)$ such that*

$$\eta_\nu(x) = f^{-1}\left(\frac{f^2(\nu)}{f(x)}\right) \tag{1}$$

holds for any $x \in [0,1]$, where f is a generator function of a conjunctive or disjunctive operator and ν is the fixed point of η_ν (i.e., $\eta_\nu(\nu) = \nu$).

Exploiting the result of Theorem 2, we will use the parametric form of negation given by Eq. (1), which is known as the Dombi form of negation (or Pliant negation); and it is an element of the Pliant system [29, 30].

Remark 3. *Since for any $\nu \in (0,1)$, $\eta_\nu(\nu) = \nu$, the Pliant negation η_ν is characterized by its fixed point, which is its parameter value ν.*

Definition 3. *We will say that the negation $\eta_1\colon [0,1] \to [0,1]$ is stricter than the negation $\eta_2\colon [0,1] \to [0,1]$ if for any $x \in (0,1)$, $\eta_1(x) < \eta_2(x)$.*

Proposition 1. *The Pliant negation η_{ν_1} is stricter than the Pliant negation η_{ν_2} if and only if $\nu_1 < \nu_2$.*

Proof. The proposition immediately follows from the definition of η_ν. □

It should be added that another representation theorem for the strong negation given in Definition 2 was first presented by Trillas [31].

Later, we will also utilize the concept of drastic negation.

Definition 4. *We say that the functions $\eta_0, \eta_1 \colon [0,1] \to [0,1]$ are drastic negations if η_0 and η_1 are given by*

$$\eta_0(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases} \quad \text{and} \quad \eta_1(x) = \begin{cases} 1, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1. \end{cases}$$

Note that the drastic negations given in Definition 4 are not strong negations, but the drastic negations may be viewed as limit cases of Pliant negations. Namely, the following proposition is valid.

Proposition 2. *For any $x \in [0,1]$,*

$$\lim_{\nu \to 0} \eta_\nu(x) = \eta_0(x) \quad \text{and} \quad \lim_{\nu \to 1} \eta_\nu(x) = \eta_1(x).$$

Proof. Using the definition of η_ν, the proof is straightforward. □

Later, we will use the following theorem, which characterizes those strict operator systems that have infinitely many negations and build a De Morgan system (see [29]).

Theorem 3. *Let $c \colon [0,1]^2 \to [0,1]$ be a conjunctive operator with the generator function $f_c \colon [0,1] \to [0,\infty]$, let $d \colon [0,1]^2 \to [0,1]$ be a disjunctive operator with the generator function $f_d \colon [0,1] \to [0,\infty]$ and let $\eta_{\nu_*} \colon [0,1] \to [0,1]$ be a Pliant negation operator, where $\nu_* \in (0,1)$. Then, c and d build a De Morgan class with η_{ν_*} if and only if*

$$f_c(x) f_d(x) = 1$$

for any $x \in (0,1)$.

2.1. Algebraic criteria for modal operators in a continuous-valued logic

Here, we will use the traditional notation $\Diamond$ and $\Box$ for the possibility and necessity operators of classical modal logic, respectively. There are two well-known identities of classical modal logic, namely,

$$\neg(\Diamond P) \equiv \Box(\neg P) \tag{2}$$

and

$$\neg(\Box P) \equiv \Diamond(\neg P), \tag{3}$$

where $\neg$ is the negation operator of classical logic.

Following the criteria for an algebraic version of necessity and possibility operators on De Morgan lattices given in [7], we define the dual pair of necessity and possibility operators in continuous-valued logic as follows (also, see [25]). Note that here we will use the classical notations $\Box$ and $\Diamond$ for necessity and possibility operators, respectively.

Definition 5. *The functions* $\Box, \Diamond \colon [0,1] \to [0,1]$ *are a dual pair of necessity and possibility operators, respectively, if* $\Box$ *and* $\Diamond$ *satisfy the following requirements:*

$N1.$	$\Box(1) = 1$	$P1.$	$\Diamond(0) = 0$
$N2.$	$\Box(x) \leq x$	$P2.$	$x \leq \Diamond(x)$
$N3.$	$x \leq y$ *implies* $\Box(x) \leq \Box(y)$	$P3.$	$x \leq y$ *implies* $\Diamond(x) \leq \Diamond(y)$
$N4.$	$\eta(\Diamond(x)) = \Box(\eta(x))$	$P4.$	$\eta(\Box(x)) = \Diamond(\eta(x))$
$[N5.$	$\Diamond(x) = \Box(\Diamond(x))$	$P5.$	$\Box(x) = \Diamond(\Box(x))]$
$N5'.$	$\Box(\Diamond(x)) = x$	$P5'.$	$\Diamond(\Box(x)) = x$

for any $x \in [0,1]$*, where* $\eta \colon [0,1] \to [0,1]$ *is a strong negation operator.*

The requirements from $N1$ to $N5$ are called the N principle, T principle, K principle, DF$\Diamond$ principle and N^* principle, respectively. Also, the requirements from $P1$ to $P5$ are known as the P principle, T principle, K principle, DF$\Box$ principle and P^* principle, respectively.

Remark 4. *Note that in our approach,* $N5$ *and* $P5$ *will not be used. Instead of* $N5$ *and* $P5$*, our demand is the neutrality principle given by* $N5'$ *and* $P5'$*. In the special case, where in the composition* $\Box \circ \Diamond$ *(*$\Diamond \circ \Box$*, respectively)* $\Diamond$ *(*$\Box$*, respectively) is a drastic operator, the composition meets the criteria* $N4$ *and* $P4$ *(see [26]). Also note that, according to* $N5'$ *and* $P5'$*, the functions* $\Box$ *and* $\Diamond$ *are inverse functions of each other.*

Remark 5. *Notice that* $N4$ *and* $P4$ *may be viewed as continuous-valued generalizations of the identities of classical modal logic (see Eq.* (2) *and Eq.* (3)*).*

3. Modalities induced by compositions of two Pliant negations

Here, we will summarize our previous results on how modal operators can be induced by compositions of strong negations. In [26], we proved that the following proposition is valid.

Proposition 3. *Let* $\nu_1, \nu_2 \in (0,1)$*, let* $f \colon [0,1] \to [0,\infty]$ *be a generator function of a conjunctive operator or disjunctive operator and let* $\eta_{\nu_*}, \eta_{\nu_1}, \eta_{\nu_2} \colon [0,1] \to [0,1]$ *be Pliant negation operators induced by the generator function* f*. Let the functions* $\tau_{\nu_1,\nu_2} \colon [0,1] \to [0,1]$ *and* $\tau_{\nu_2,\nu_1} \colon [0,1] \to [0,1]$ *be given by*

$$\tau_{\nu_1,\nu_2} = \eta_{\nu_1} \circ \eta_{\nu_2} \quad \textit{and} \quad \tau_{\nu_2,\nu_1} = \eta_{\nu_2} \circ \eta_{\nu_1}. \tag{4}$$

Then, τ_{ν_1,ν_2} *and* τ_{ν_2,ν_1} *satisfy the requirements for a dual pair of modal operators given in Definition 5 with the negation* η_{ν_*}*. If* η_{ν_1} *is stricter than* η_{ν_2}*, then* τ_{ν_1,ν_2} *is a necessity operator and* τ_{ν_2,ν_1} *is a possibility operator, and vice versa.*

Remark 6. *After direct calculation, we get that*

$$\tau_{\nu_1,\nu_2}(x) = f^{-1}\left(\frac{f^2(\nu_1)}{f^2(\nu_2)} f(x)\right) \quad \textit{and} \quad \tau_{\nu_2,\nu_1}(x) = f^{-1}\left(\frac{f^2(\nu_2)}{f^2(\nu_1)} f(x)\right)$$

for any $x \in [0,1]$. *Now, let* $\nu_0 \in (0,1)$ *be arbitrarily chosen. If we define* ν *as*

$$\nu = f^{-1}\left(f(\nu_0) \frac{f^2(\nu_2)}{f^2(\nu_1)} \right),$$

then $\nu \in (0,1)$, *and* τ_{ν_1,ν_2} *and* τ_{ν_2,ν_1} *can be written as*

$$\underline{\tau}_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0) \frac{f(x)}{f(\nu)} \right) \quad \text{and} \quad \overline{\tau}_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu) \frac{f(x)}{f(\nu_0)} \right) \tag{5}$$

for any $x \in [0,1]$, *respectively.*

4. Modifier operators induced by connectives

Following the approach presented by Mattila [13], we define the substantiating modifier operator and the weakening modifier operator as follows.

Definition 6. *We say that* $\tau\colon [0,1] \to [0,1]$ *is a substantiating modifier operator if*

$$\tau(x) < x \tag{6}$$

and τ *is a weakening modifier operator if*

$$\tau(x) > x \tag{7}$$

for any $x \in (0,1)$, *and* $\tau(0) = 0$, $\tau(1) = 1$.

Using a negation operator, we can associate a given modifier operator with its dual modifier operator according to the following definition.

Definition 7. *Let* τ *and* τ_* *be two modifier operators. We say that* τ_* *is the dual modifier operator associated with* τ *if*

$$\tau_*(x) = \eta(\tau(\eta(x))), \tag{8}$$

for any $x \in [0,1]$, *where* η *is a strong negation given by Definition 2.*

It is easy to see that if τ is a substantiating modifier then its dual τ_* is a weakening modifier, and vice versa.

We can generate substantiating or weakening modifier operators using conjunctive or disjunctive operators that build a De Morgan class (see [32]) with a strong negation. Let c be a conjunctive operator, let d be a disjunctive operator and let η be a strong negation such that (c, d, η) is a De Morgan class. This means that

$$d(x,y) = \eta(c(\eta(x), \eta(y)))$$

holds for any $x, y \in [0,1]$. Utilizing the fact that (c, d, η) is a De Morgan class, c generates the disjunctive operator d as its dual t-norm. As the conjunctive operator c and the disjunctive operator d are Archimedian, i.e., $c(x,x) < x$ and $d(x,x) > x$ for any $x \in (0,1)$), by repeating the arguments of c and d, we get the substantiating and weakening modifier operators $\tau_{c,m}\colon [0,1] \to [0,1]$ and $\tau_{d,m}\colon [0,1] \to [0,1]$, respectively:

$$\tau_{c,m}(x) = c(\underbrace{x,\ldots,x}_{m}) \quad \text{and} \quad \tau_{d,m}(x) = d(\underbrace{x,\ldots,x}_{m}). \tag{9}$$

Depending on the number of arguments of the operators, we get different substantiating and weakening modifier operators. The more arguments there are, the more substantiating and more weakening the resulting modifier operators are. Note that the original idea of obtaining modifiers by repeating the arguments comes from Pavelka's seminal papers [33–35]. It should be added that Łukasiewicz introduced the modal operators $\Diamond$ and $\Box$ like so:

$$\Diamond \stackrel{\text{def}}{=} \neg\phi \to \phi \quad \text{and} \quad \Box \stackrel{\text{def}}{=} \neg\Diamond\neg\phi,$$

from which, by using the definition of implication, we have

$$\Diamond \equiv \phi \vee \phi \quad \text{and} \quad \Box \equiv \neg(\phi \to \neg\phi) \equiv \phi \wedge \phi.$$

These considerations support the idea of deriving modifier operators by repeating the arguments of conjuctive and disjunctive operators (also see [1, 10–12]).

Now, let $f_c\colon [0,1] \to [0,\infty]$ and $f_d\colon [0,1] \to [0,\infty]$ be the generator functions of the conjunctive operator c and of the disjunctive operator d, respectively. Then, using the representation of these connectives (see Definition 1), from Eq. (9), we get

$$\tau^{(c)}(x) = f_c^{-1}\left(m f_c(x)\right) \quad \text{and} \quad \tau^{(d)}(x) = f_d^{-1}\left(m f_d(x)\right)$$

for any $x \in [0,1]$. Since (c,d,η) is a De Morgan triple, we have

$$\tau^{(d)}(x) = f_d^{-1}\left(m f_d(x)\right) = \eta\left(f_c^{-1}\left(m f_c(\eta(x))\right)\right) = \eta\left(\tau^{(c)}\left(\eta(x)\right)\right),$$

which, by noting Definition 7, means that $\tau^{(c)}$ and $\tau^{(d)}$ are a dual pair of substantiating and weakening modifier operators, respectively.

Here, we generalize the modifier operators $\tau^{(c)}$ and $\tau^{(d)}$ such that m is replaced by the positive real valued numbers

$$\frac{f_c(\nu_0)}{f_c(\nu_c)} \quad \text{and} \quad \frac{f_d(\nu_0)}{f_d(\nu_d)}$$

in $\tau^{(c)}$ and $\tau^{(d)}$, respectively, where $\nu_c, \nu_d, \nu_0 \in (0,1)$. Therefore, we get the operators $\tau_{\nu_c,\nu_0}\colon [0,1] \to [0,1]$ and $\tau_{\nu_d,\nu_0}\colon [0,1] \to [0,1]$:

$$\tau_{\nu_c,\nu_0}(x) = f_c^{-1}\left(f_c(\nu_0)\frac{f_c(x)}{f_c(\nu_c)}\right) \tag{10}$$

$$\tau_{\nu_d,\nu_0}(x) = f_d^{-1}\left(f_d(\nu_0)\frac{f_d(x)}{f_d(\nu_d)}\right). \tag{11}$$

which are characterized by the parameters $\nu_0, \nu_c, \nu_d \in (0,1)$. Notice that $\tau_{\nu_c,\nu_0}(\nu_c) = \nu_0$ and $\tau_{\nu_d,\nu_0}(\nu_d) = \nu_0$.

Proposition 4. *Let $\nu_0, \nu_c, \nu_d \in (0,1)$, let $f_c\colon [0,1] \to [0,\infty]$ be a generator function of a conjunctive operator c and let $f_d\colon [0,1] \to [0,\infty]$ be a generator function of a disjunctive operator d. Let the operators $\tau_{\nu_c,\nu_0}\colon [0,1] \to [0,1]$ and $\tau_{\nu_d,\nu_0}\colon [0,1] \to [0,1]$ be given by Eq. (10) and Eq. (11), respectively. Then, the following are valid:*

(a) If $\nu_c > \nu_0$ ($\nu_c < \nu_0$, respectively), then τ_{ν_c,ν_0} is a substantiating (weakening, respectively) modifier operator

(b) If $\nu_d > \nu_0$ ($\nu_d < \nu_0$, respectively), then τ_{ν_d,ν_0} is a substantiating (weakening, respectively) modifier operator.

Proof. By taking into account the properties of f_c and f_d, the proof of both is straightforward. □

Note that if $\nu_c = \nu$, then τ_{ν_c,ν_0} is the identity operator and similarly, if $\nu_d = \nu$, then τ_{ν_d,ν_0} is the identity operator.

5. Modal operators induced by connectives

In the previous section, we showed that the operators τ_{ν_c,ν_0} and τ_{ν_d,ν_0} given by Eq. (10) and Eq. (11), respectively, can be used as substantiating and weakening modifier operators. Now, we will give the sufficient condition for having the operators τ_{ν_c,ν_0} and τ_{ν_d,ν_0} satisfy the requirements for a dual pair of modal operators given by Definition 5.

Remark 7. *Let $f_c\colon [0,1] \to [0,\infty]$ be a generator function of a conjunctive operator c and let $f_d\colon [0,1] \to [0,\infty]$ be a generator function of a disjunctive operator d. If*

$$f_c(x) f_d(x) = 1 \tag{12}$$

for any $x \in (0,1)$, then

$$f_c^{-1}(z) = f_d^{-1}\left(\frac{1}{z}\right) \tag{13}$$

for any $z \in (0,\infty]$.

Theorem 4. *Let $\nu_*, \nu_0, \nu_c, \nu_d \in (0,1)$, let $f_c\colon [0,1] \to [0,\infty]$ be a generator function of a conjunctive operator c, let $f_d\colon [0,1] \to [0,\infty]$ be a generator function of a disjunctive operator d and let $\eta_{\nu_*}\colon [0,1] \to [0,1]$ be a Pliant negation operator induced by the generator function f_c or f_d such that c, d and η_{ν_*} form a De Morgan class. Let the modifier operators $\tau_{\nu_c,\nu_0}\colon [0,1] \to [0,1]$ and $\tau_{\nu_d,\nu_0}\colon [0,1] \to [0,1]$ be given by Eq. (10) and Eq. (11), respectively. If*

$$\frac{f_c(\nu_0)}{f_c(\nu_c)} = \frac{f_d(\nu_0)}{f_d(\nu_d)}$$

then τ_{ν_c,ν_0} and τ_{ν_d,ν_0} satisfy the requirements for a dual pair of modal operators given in Definition 5 with the negation η_{ν_*}.

Proof. In order to demonstrate that τ_{ν_c,ν_0} and τ_{ν_d,ν_0} are a dual pair of modal operators with the negation η_{ν_*}, we need to show that one of them meets the criteria $N1$–$N4$ and $N5'$ given in Definition 5, and the other one satisfies the requirements $P1$–$P4$ and $P5'$ given in Definition 5. Suppose that

$$\frac{f_c(\nu_0)}{f_c(\nu_c)} = \frac{f_d(\nu_0)}{f_d(\nu_d)} = a.$$

Taking into account the properties of f_c and f_d, and the condition that $\nu_0, \nu_c, \nu_d \in (0,1)$, we have that $a > 0$. Here, we can distinguish three cases: $a > 1$, $a = 1$, and $0 < a < 1$.

If $a = 1$, then τ_{ν_c,ν_0} and τ_{ν_d,ν_0} are both the identity operator, which trivially satisfies the requirements of Definition 5. In this case, τ_{ν_c,ν_0} and τ_{ν_d,ν_0} are a dual pair of necessity (possibility) and possibility (necessity) operators, respectively, with the negation η_{ν_*}.

Now, we will show that if $a > 1$, then τ_{ν_c,ν_0} and τ_{ν_d,ν_0} are a dual pair of necessity and possibility operators, respectively, with the negation η_{ν_*}. By noting the properties of f_c and f_d (see Definition 1), the proof that τ_{ν_c,ν_0} satisfies $N1$, $N2$ and $N3$, and the proof that τ_{ν_d,ν_0} satisfies $P1$, $P2$ and $P3$ are both straightforward.

Proof of $N4$ and $P4$. Since (c, d, η_{ν_*}) is a De Morgan triple, based on Theorem 3, we have that Eq. (12) holds; i.e., $f_c(x)f_d(x) = 1$ for any $x \in (0,1)$. Making use of Remark 7 with $z = f_c^2(\nu_*)f_d(x)$, we get

$$f_c^{-1}\left(f_c^2(\nu_*)f_d(x)\right) = f_d^{-1}\left(\frac{1}{f_c^2(\nu_*)f_d(x)}\right) \tag{14}$$

for any $x \in (0,1)$. By noting the fact that $f_d(x) = \frac{1}{f_c(x)}$ for any $x \in (0,1)$ and $\frac{1}{f_c^2(\nu_*)} = f_d^2(\nu_*)$, from Eq. (14), we have

$$f_c^{-1}\left(\frac{f_c^2(\nu_*)}{f_c(x)}\right) = f_d^{-1}\left(\frac{f_d^2(\nu_*)}{f_d(x)}\right)$$

for any $x \in [0,1]$. This means that the Pliant negations induced by f_c and f_d are identical, if $f_c(x)f_d(x) = 1$ holds for any $x \in (0,1)$. Noting this result, we have

$$\eta_{\nu_*}(x) = f_c^{-1}\left(\frac{f_c^2(\nu_*)}{f_c(x)}\right) = f_d^{-1}\left(\frac{f_d^2(\nu_*)}{f_d(x)}\right)$$

for any $x \in (0,1)$. Hence,

$$\eta_{\nu_*}\left(\tau_{\nu_c,\nu_0}(x)\right) = \eta_{\nu_*}\left(f_c^{-1}\left(af_c(x)\right)\right) = f_c^{-1}\left(\frac{1}{a}\frac{f_c^2(\nu_*)}{f_c(x)}\right) \tag{15}$$

and

$$\tau_{\nu_d,\nu_0}\left(\eta_{\nu_*}(x)\right) = f_d^{-1}\left(af_d\left(\eta_{\nu_*}(x)\right)\right) = f_d^{-1}\left(a\frac{f_d^2(\nu_*)}{f_d(x)}\right) \tag{16}$$

for any $x \in (0,1)$. Now, using Remark 7 with

$$z = \frac{f_c^2(\nu_*) f_d(x)}{a}$$

we have

$$f_c^{-1}\left(\frac{f_c^2(\nu_*) f_d(x)}{a}\right) = f_d^{-1}\left(\frac{a}{f_c^2(\nu_*) f_d(x)}\right).$$

By noting the fact that $f_d(x) = \frac{1}{f_c(x)}$ for any $x \in (0,1)$ and $\frac{1}{f_c^2(\nu_*)} = f_d^2(\nu_*)$, from the previous equation, we have

$$f_c^{-1}\left(\frac{1}{a}\frac{f_c^2(\nu_*)}{f_c(x)}\right) = f_d^{-1}\left(a\frac{f_d^2(\nu_*)}{f_d(x)}\right) \tag{17}$$

for any $x \in (0,1)$. Now, by taking into account Eq. (15), Eq. (16) and Eq. (17), we get that

$$\eta_{\nu_*}\left(\tau_{\nu_c,\nu_0}(x)\right) = \tau_{\nu_d,\nu_0}\left(\eta_{\nu_*}(x)\right)$$

holds for any $x \in [0,1]$, which proves $P4$. The proof of $N4$ is similar to that of $P4$.

Proof of $N5'$ and $P5'$. Here, we will prove $N5'$, the proof of $P5'$ is similar to that of $N5'$. Using Eq. (10), Eq. (11) and Remark 7, we can write

$$\begin{aligned} \tau_{\nu_c,\nu_0}\left(\tau_{\nu_d,\nu_0}(x)\right) &= f_c^{-1}\left(a f_c\left(f_d^{-1}\left(a f_d(x)\right)\right)\right) = \\ &= f_c^{-1}\left(\frac{1}{f_d(x)}\right) = f_c^{-1}\left(f_c(x)\right) = x \end{aligned} \tag{18}$$

for any $x \in [0,1]$, which proves that $N5'$ holds.

The proof of the statement that if $a < 1$, then τ_{ν_c,ν_0} and τ_{ν_d,ν_0} are a dual pair of possibility and necessity operators, respectively, with the negation η_{ν_*} is similar to that of the case where $a > 1$. □

Remark 8. *The fact that τ_{ν_c,ν_0} and τ_{ν_d,ν_0} satisfy the requirements $N4$ and $P4$ for a modal operator given in Definition 5, i.e., $\eta_{\nu_*}\left(\tau_{\nu_c,\nu_0}(x)\right) = \tau_{\nu_d,\nu_0}\left(\eta_{\nu_*}(x)\right)$ and $\eta_{\nu_*}\left(\tau_{\nu_d,\nu_0}(x)\right) = \tau_{\nu_c,\nu_0}\left(\eta_{\nu_*}(x)\right)$ for any $x \in [0,1]$ means that, according to Definition 7, τ_{ν_c,ν_0} is the dual modifier operator associated with τ_{ν_d,ν_0}, and vice versa, with the negation η_{ν_*}.*

The following theorem is a key consequence of Theorem 4.

Theorem 5. *Let $\nu_*, \nu_0, \nu \in (0,1)$, let $f\colon [0,1] \to [0,\infty]$ be a generator function of a conjunctive operator or disjunctive operator and let $\eta_{\nu_*}\colon [0,1] \to [0,1]$ be the Pliant negation operator induced by the generator function f. Let the functions $\underline{\tau}_{\nu,\nu_0}\colon [0,1] \to [0,1]$ and $\overline{\tau}_{\nu,\nu_0}\colon [0,1] \to [0,1]$ be given by*

$$\underline{\tau}_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0)\frac{f(x)}{f(\nu)}\right) \quad \text{and} \quad \overline{\tau}_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu)\frac{f(x)}{f(\nu_0)}\right) \tag{19}$$

for any $x \in [0,1]$. Then, $\underline{\tau}_{\nu,\nu_0}$ and $\overline{\tau}_{\nu,\nu_0}$ satisfy the requirements for a dual pair of modal operators given in Definition 5 with the negation η_{ν_*}. Also,

(a) *If $\nu \geq \nu_0$, then $\underline{\tau}_\nu$ and $\overline{\tau}_\nu$ are a dual pair of necessity and possibility operators, respectively*

(b) *If $\nu \leq \nu_0$, then $\underline{\tau}_\nu$ and $\overline{\tau}_\nu$ are a dual pair of possibility and necessity operators, respectively.*

Proof. Here, we distinguish two cases: (1) f is a strictly decreasing continuous function; i.e., f is the generator function of a conjunctive operator and (2) f is a strictly increasing continuous function, i.e., f is the generator function of a disjunctive operator. We will prove the theorem for case (1), the proof for case (2) is similar to that of case (1).

Let $f\colon [0,1] \to [0,\infty]$ be a strictly decreasing continuous function, and let the functions $f_c\colon [0,1] \to [0,\infty]$ and $f_d\colon [0,1] \to [0,\infty]$ be given by

$$f_c(x) = f(x) \quad \text{and} \quad f_d(x) = \frac{1}{f(x)} \tag{20}$$

for any $x \in (0,1)$. That is, f_c is the generator function of a conjunctive operator c and f_d is the generator function of a disjunctive operator d. Now, let

$$\nu_c = \nu \quad \text{and} \quad \nu_d = f^{-1}\left(\frac{f^2(\nu_0)}{f(\nu)}\right). \tag{21}$$

Since $\nu \in (0,1)$, by noting the properties of f, from Eq. (21), we readily have that $\nu_c, \nu_d \in (0,1)$. Next, by taking into account Eq. (20) and Eq. (21), we get

$$\frac{f(\nu_0)}{f(\nu)} = \frac{f_c(\nu_0)}{f_c(\nu_c)} = \frac{f_d(\nu_0)}{f_d(\nu_d)}. \tag{22}$$

By noting Eq. (20), we also have

$$f_c^{-1}(x) = f^{-1}(x) \quad \text{and} \quad f_d^{-1}(x) = f^{-1}\left(\frac{1}{x}\right), \tag{23}$$

for any $x \in (0,\infty]$. Now, by using Eq. (20), Eq. (22) and Eq. (23), $\underline{\tau}_{\nu,\nu_0}$ and $\overline{\tau}_{\nu,\nu_0}$ can be written as

$$\underline{\tau}_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0)\frac{f(x)}{f(\nu)}\right) = f_c^{-1}\left(f_c(\nu_0)\frac{f_c(x)}{f_c(\nu_c)}\right) \tag{24}$$

$$\overline{\tau}_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu)\frac{f(x)}{f(\nu_0)}\right) = f_d^{-1}\left(f_d(\nu_0)\frac{f_d(x)}{f_d(\nu_d)}\right), \tag{25}$$

for any $x \in [0,1]$. Since $f_c(x)f_d(x) = 1$ (see Eq. (20)), based on Theorem 3, c and d builds a De Morgan system with the Pliant negation η_{ν_*}. Therefore, noting Eq. (22), Eq. (24) and Eq. (25), based on Theorem 4, we get that $\underline{\tau}_{\nu,\nu_0}$ and $\overline{\tau}_{\nu,\nu_0}$ are a dual pair of modal operators with the Pliant negation η_{ν_*}. Also, noting Eq. (24) and Eq. (25), we immediately have (a) and (b). □

Example 1. *The generator function of the Dombi conjunction and disjunction operators is the function $g_\alpha\colon (0,1) \to (0,\infty)$ that is given by*

$$g_\alpha(x) = \left(\frac{1-x}{x}\right)^\alpha, \tag{26}$$

where $\alpha \neq 0$. If $\alpha > 0$, then g_α is the generator function of a strict conjunctive operator; and if $\alpha < 0$, then g_α is the generator function of a strict disjunctive operator (see, e.g. [36]). Let $\alpha > 0$ and let $f_c(x) = g_\alpha(x)$, $f_d(x) = g_{-\alpha}(x)$ for any $x \in [0,1]$. Let $\nu_0, \nu_c, \nu_d \in (0,1)$. The modifier operators τ_{ν_c,ν_0} and τ_{ν_d,ν_0}, induced by f_c and f_d, respectively, are

$$\tau_{\nu_c,\nu_0}(x) = \frac{1}{1+\frac{1-\nu_0}{\nu_0}\left(\frac{1-x}{x}\frac{\nu_c}{1-\nu_c}\right)} \quad \text{and} \quad \tau_{\nu_d,\nu_0}(x) = \frac{1}{1+\frac{1-\nu_0}{\nu_0}\left(\frac{1-x}{x}\frac{\nu_d}{1-\nu_d}\right)}.$$

Here, $f_c(x)f_d(x) = 1$ holds for any $x \in (0,1)$. Therefore, if we require

$$\frac{f_c(\nu_0)}{f_c(\nu_c)} = \frac{f_d(\nu_0)}{f_d(\nu_d)}, \tag{27}$$

then, based on Theorem 4, τ_{ν_c,ν_0} and τ_{ν_d,ν_0} are a dual pair of modal operators with the Pliant negation induced by f_c or f_d. Here, the requirement in Eq. (27) *is equivalent to*

$$\frac{1-\nu_c}{\nu_c}\frac{1-\nu_d}{\nu_d} = \left(\frac{1-\nu_0}{\nu_0}\right)^2. \tag{28}$$

Now, let $\nu_c = \nu$, where $\nu \in (0,1)$. Then, after noting Eq. (28)*, we have*

$$\begin{aligned}\tau_{\nu_c,\nu_0}(x) = \underline{\tau}_{\nu,\nu_0}(x) &= \frac{1}{1+\frac{1-\nu_0}{\nu_0}\left(\frac{1-x}{x}\frac{\nu}{1-\nu}\right)}\\ \tau_{\nu_d,\nu_0}(x) = \overline{\tau}_{\nu,\nu_0}(x) &= \frac{1}{1+\frac{\nu_0}{1-\nu_0}\left(\frac{1-x}{x}\frac{1-\nu}{\nu}\right)}.\end{aligned}$$

The following proposition is a consequence of Theorem 5.

Proposition 5. *Let $\nu_*, \nu \in (0,1)$, let $f\colon [0,1] \to [0,\infty]$ be a generator function of a conjunctive operator or disjunctive operator and let $\eta_{\nu_*}\colon [0,1] \to [0,1]$ be the Pliant negation operator induced by the generator function f. Let the functions $\underline{\tau}_\nu\colon [0,1] \to [0,1]$ and $\overline{\tau}_\nu\colon [0,1] \to [0,1]$ be given by*

$$\underline{\tau}_\nu(x) = f^{-1}\left(\frac{f(x)}{f(\nu)}\right) \quad \text{and} \quad \overline{\tau}_\nu(x) = f^{-1}\left(f(\nu)f(x)\right) \tag{29}$$

for any $x \in [0,1]$. Then, $\underline{\tau}_\nu$ and $\overline{\tau}_\nu$ satisfy the requirements for a dual pair of modal operators given in Definition 5 with the negation η_{ν_*}. Also,

(a) If $\nu \geq f^{-1}(1)$, then $\underline{\tau}_\nu$ and $\overline{\tau}_\nu$ are a dual pair of necessity and possibility operators, respectively

(b) If $\nu \le f^{-1}(1)$, then $\underline{\tau}_\nu$ and $\overline{\tau}_\nu$ are a dual pair of possibility and necessity operators, respectively.

Proof. Notice that this Proposition is a special case of Theorem 5. Namely, if we apply Theorem 5 with a $\nu_0 \in (0,1)$ for which $f(\nu_0) = 1$, then we get this proposition. □

Remark 9. *When $\underline{\tau}_\nu$ and $\overline{\tau}_\nu$ are a dual pair of necessity and possibility operators, respectively, we will use the notation*

$$\Box_\nu = \underline{\tau}_\nu, \quad \text{and} \quad \Diamond_\nu = \overline{\tau}_\nu.$$

Also, when $\underline{\tau}_\nu$ and $\overline{\tau}_\nu$ are a dual pair of possibility and necessity operators, respectively, we will use the notation

$$\Diamond_\nu = \underline{\tau}_\nu, \quad \text{and} \quad \Box_\nu = \overline{\tau}_\nu.$$

Example 2. *Let f_c and f_d be the generator functions of the probabilistic conjunctive and disjunctive operators, respectively. That is,*

$$f_c(x) = -\ln(x) \quad \text{and} \quad f_d(x) = -\ln(1-x)$$

for any $x \in (0,1)$. Let $\nu \in (0,1)$. Then, by using Eq. (29), after direct calculation, we get that the dual pair of modal operators $\underline{\tau}_{\nu,c}\colon (0,1) \to (0,1)$ and $\overline{\tau}_{\nu,c}\colon (0,1) \to (0,1)$ induced by the generator function f_c are

$$\underline{\tau}_{\nu,c}(x) = x^{-\frac{1}{\ln(\nu)}} \quad \text{and} \quad \overline{\tau}_{\nu,c}(x) = x^{-\ln(\nu)}.$$

Since f_c is strictly decreasing, based on Proposition 5, we have that

(a) If $\nu \le \mathrm{e}^{-1}$, then $\underline{\tau}_{\nu,c} = \Diamond_\nu$ and $\overline{\tau}_{\nu,c} = \Box_\nu$
(b) If $\nu \ge \mathrm{e}^{-1}$, then $\underline{\tau}_{\nu,c} = \Box_\nu$ and $\overline{\tau}_{\nu,c} = \Diamond_\nu$.

Similarly, by using Eq. (29), the dual pair of modal operators $\underline{\tau}_{\nu,d}\colon (0,1) \to (0,1)$ and $\overline{\tau}_{\nu,d}\colon (0,1) \to (0,1)$ induced by the generator function f_d are

$$\underline{\tau}_{\nu,d}(x) = 1 - (1-x)^{-\frac{1}{\ln(1-\nu)}} \quad \text{and} \quad \overline{\tau}_{\nu,d}(x) = 1 - (1-x)^{-\ln(1-\nu)}.$$

As f_d is strictly increasing, after noting Proposition 5, we have that

(a) If $\nu \le 1 - \mathrm{e}^{-1}$, then $\underline{\tau}_{\nu,d} = \Diamond_\nu$ and $\overline{\tau}_{\nu,d} = \Box_\nu$
(b) If $\nu \ge 1 - \mathrm{e}^{-1}$, then $\underline{\tau}_{\nu,d} = \Box_\nu$ and $\overline{\tau}_{\nu,d} = \Diamond_\nu$.

Example 3. *Let $\nu \in (0,1)$, $\alpha \ne 0$ and let $f(x) = g_\alpha(x)$ for any $x \in (0,1)$, where g_α is the generator function of the Dombi operators given in Eq. (26). Then, after direct calculation, we get that the dual pair of modal operators $\underline{\tau}_\nu\colon (0,1) \to (0,1)$ and $\overline{\tau}_\nu\colon (0,1) \to (0,1)$ induced by the generator function f are*

$$\underline{\tau}_\nu(x) = \frac{1}{1 + \frac{\nu}{1-\nu}\frac{1-x}{x}} \quad \text{and} \quad \overline{\tau}_\nu(x) = \frac{1}{1 + \frac{1-\nu}{\nu}\frac{1-x}{x}}. \tag{30}$$

Notice that $\underline{\tau}_\nu$ and $\overline{\tau}_\nu$ are independent of α. Next, based on Proposition 5, we have that

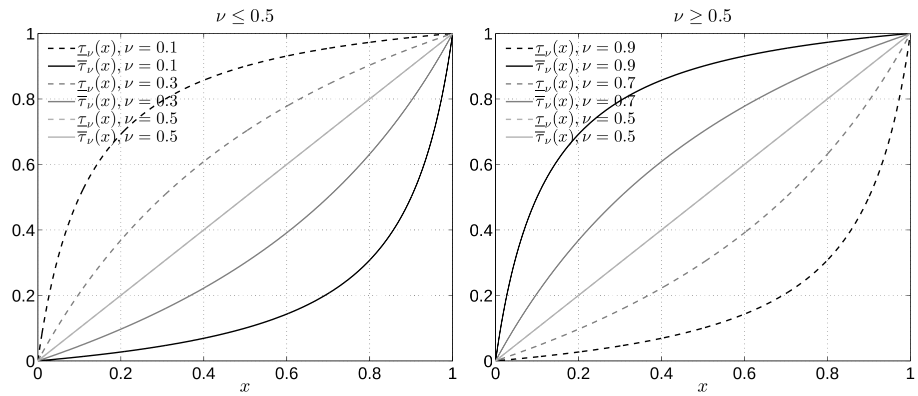

Figure 1: Dual pairs of modal operators induced by the generator function of Dombi operators with various ν values.

(a) If $\nu \leq 0.5$*, then* $\underline{\tau}_\nu = \Diamond_\nu$ *and* $\overline{\tau}_\nu = \Box_\nu$

(b) If $\nu \geq 0.5$*, then* $\underline{\tau}_\nu = \Box_\nu$ *and* $\overline{\tau}_\nu = \Diamond_\nu$*.*

Figure 1 shows example plots of functions $\underline{\tau}_\nu$ *and* $\overline{\tau}_\nu$*.*
Note that if $\nu = 0.5$*, then* $\underline{\tau}_\nu(x) = \overline{\tau}_\nu(x) = x$ *for any* $x \in (0,1)$*.*

Remark 10. *We should mention that a dual pair of modal operators* $\Box_\nu$ *and* $\Diamond_\nu$ *induced by Eq.* (29) *using the generator function of Dombi operators are very simple and easy-to-use. Noting Eq.* (29)*, based on Proposition 5, we get that the modal operators* $\underline{\tau}_{\nu_1}$ *(*$\overline{\tau}_{\nu_1}$*, respectively) and* $\underline{\tau}_{\nu_2}$ *(*$\overline{\tau}_{\nu_2}$*, respectively), both induced by the same generator function of Dombi operators, are a dual pair of modal operators with a Pliant negation induced by the same generator function, if and only if*

$$\nu_1 + \nu_2 = 1.$$

6. Unary operators with distributivity property

The following theorem characterizes the form of unary operators that are distributive over a conjunctive or disjunctive operator o, which is given by Definition 1.

Theorem 6. *Let* $o\colon [0,1]^2 \to [0,1]$ *be a conjunctive or disjunctive operator with the generator function* $f\colon [0,1] \to [0,\infty]$*. The unary operator* $\tau\colon [0,1] \to [0,1]$ *is distributive over the operator* o *if and only if* τ *has the form*

$$\tau(x) = f^{-1}\left(kf(x)\right), \tag{31}$$

where $k \in \mathbb{R}$ *is a constant and* $k > 0$*.*

Proof. The distributivity of τ over o means that

$$\tau\left(o(x,y)\right) = o\left(\tau(x), \tau(y)\right) \tag{32}$$

holds for any $(x, y) \in [0,1]^2$. Noting the fact that the operator o has the form $o(x, y) = f^{-1}(f(x) + f(y))$, Eq, (32) can be written as

$$\tau\left(f^{-1}(f(x) + f(y))\right) = f^{-1}\left(f(\tau(x)) + f(\tau(y))\right) \tag{33}$$

holds for any $(x, y) \in [0,1]^2$. Applying f to both sides of Eq. (33) and using the identity $x = f^{-1}(f(x))$ on the right hand side of Eq. (33), we get

$$f\left(\tau\left(f^{-1}(f(x) + f(y))\right)\right) = f\left(\tau\left(f^{-1}(f(x))\right)\right) + f\left(\tau\left(f^{-1}(f(y))\right)\right). \tag{34}$$

Now, let $X = f(x)$, $Y = f(y)$ and let the function $F : [0, \infty] \to [0, \infty]$ be given by

$$F(X) = f\left(\tau\left(f^{-1}(X)\right)\right). \tag{35}$$

Then, Eq. (34) can be written in the form

$$F(X + Y) = F(X) + F(Y), \tag{36}$$

which is the well-known Cauchy functional equation. The solution of Eq. (36) is

$$F(X) = kX, \tag{37}$$

where $k \in \mathbb{R}$. Next, noting Eq. (35), Eq. (37) and the fact that $X = f(x)$, we have

$$f(\tau(x)) = kf(x) \tag{38}$$

for any $x \in [0,1]$. Applying f^{-1} to both members of Eq. (38), we get Eq. (31). □

Remark 11. *Since for any $k > 0$ there exist $\nu, \nu_0 \in (0,1)$ such that*

$$k = \frac{f(\nu_0)}{f(\nu)},$$

(where f is the generator function of a conjunctive or disjunctive operator) and by exploiting the result of Theorem 6, we may conclude that any unary operator $\tau\colon [0,1] \to [0,1]$ that is distributive over a conjunctive or disjunctive operator can be written in the form

$$\tau(x) = \tau_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0)\frac{f(x)}{f(\nu)}\right)$$

for any $x \in [0,1]$.

7. Linguistic hedges and modal operators

The concept of linguistic hedges and modifier operators appears at the very beginning of fuzzy set theory. They are related to an attempt to model meanings like ‘very’, ‘more or less’, ‘somewhat’, ‘rather’ and ‘quite’.

A linguistic hedge modifies the shape of the fuzzy set, causing a change in the membership function. Thus, a linguistic hedge transforms one fuzzy set into another fuzzy set. Modifiers (modalities) modify the truth values of a fuzzy logical statement. That is, hedges work on fuzzy sets and modifiers work on logical variables. In the fuzzy concept, the variables are membership functions. Therefore, there is a simple correspondence between modalities and linguistic hedges. Namely, if we apply a modal operator to a membership function, then we get a hedge. Here, we would like to stress the importance of distributivity of a modal operator over a logical connective. Suppose that $\mu_A\colon X \to [0,1]$ and $\mu_B\colon X \to [0,1]$ are the membership functions of the fuzzy sets A and B, respectively, $o\colon [0,1]^2 \to [0,1]$ is a strict monotone connective and $\tau\colon [0,1] \to [0,1]$ is a modal operator which is distributive over o. Then we can write

$$\tau\left(o\left(\mu_A(x), \mu_B(x)\right)\right) = o\left(\tau\left(\mu_A(x)\right), \tau\left(\mu_B(x)\right)\right) \tag{39}$$

for any $x \in X$. On the right hand side of Eq. (39), the modal operator τ is applied to fuzzy sets; that is, the modal operator acts as a hedge. In this case, for a given $x \in X$, the arguments of τ are values from the range of the membership functions μ_A and μ_B. On the left hand side of Eq. (39), the argument of the modal operator τ is a value from the domain $[0,1]$ of logical variables.

A linguistic hedge or modifier is a unary operation that changes the meaning of a linguistic term (see [37–39]). Let X be the domain of discourse and let A be a continuous linguistic term for the input variable $x \in X$ with the membership function $\mu_A\colon X \to [0,1]$. Then A^s, which is given by the membership function

$$\mu_{A^s}(x) = \left(\mu_A(x)\right)^p \tag{40}$$

for any $x \in X$, is interpreted as a modified version of A, where $p > 0$. Here, p denotes the linguistic hedge value.

In Example 2, we showed that the dual pair of modal operators $\underline{\tau}_{\nu,c}\colon (0,1) \to (0,1)$ and $\overline{\tau}_{\nu,c}\colon (0,1) \to (0,1)$, which are induced by the generator function $f_c(x) = -\ln(x)$ of the probabilistic conjunctive operator using Eq. (29), are

$$\underline{\tau}_{\nu,c}(x) = x^{-\frac{1}{\ln(\nu)}} \quad \text{and} \quad \overline{\tau}_{\nu,c}(x) = x^{-\ln(\nu)},$$

where $x, \nu \in (0,1)$. We can readily see that applying these modal operators to the membership function μ_A, we get

$$\underline{\tau}_{\nu,c}(\mu_A(x)) = (\mu_A(x))^{-\frac{1}{\ln(\nu)}} \quad \text{and} \quad \overline{\tau}_{\nu,c}(\mu_A(x)) = (\mu_A(x))^{-\ln(\nu)},$$

which, with $p = -\frac{1}{\ln(\nu)}$ and $p = -\ln(\nu)$, respectively, have the form of a hedge given by Eq. (40).

The following example demonstrates that the application of a modal operator, which is induced by the generator function of Dombi operators using Eq. (29), to the membership function of a fuzzy set may be viewed as a hedge as well.

Example 4. *Let the membership function of the fuzzy set 'young person' be given by the sigmoid-like function* $\sigma_{a,b}^{(\lambda)} \colon [0,\infty) \to [0,1]$*:*

$$\sigma_{a,b}^{(\lambda)}(x) = \begin{cases} 1, & \text{if } x = 0 \\ \frac{1}{1+\mathrm{e}^{-\lambda(x-a)}}, & \text{if } 0 < x < b \\ 0, & \text{if } x \geq b, \end{cases}$$

where $\lambda = -0.5$*,* $a = 30$ *and* $b = 70$*. (Note that* $0 < a < b$*.) Here,* x *is in years and* $\sigma_{a,b}^{(\lambda)}(x)$ *represents the truth value of the soft inequality* $x < 30$*. If someone is much younger than 30 years, then this person has a high membership value in the fuzzy set 'young person'; and conversely, if a person is much elder than 30 years, then this person has a low membership value in the fuzzy set 'young person'. With these parameters, we have* $\sigma_{a,b}^{(\lambda)}(0) = 1$*, which means that a newborn baby has the membership degree of 1 to the set of young persons. Also,* $\sigma_{a,b}^{(\lambda)}(70) = 0$*, which indicates that a 70 years old person has the membership degree of 0 to the set of young persons.*

Let $\nu \in [0.5, 1)$ *and let* $\Box_\nu$ *and* $\Diamond_\nu$ *be a dual pair of modal operators induced by Eq. (29) using the generator function of Dombi operators (see Example 3). That is,* $\Box_\nu$ *and* $\Diamond_\nu$ *given by Eq. (30) are as follows:*

$$\Box_\nu(x) = \frac{1}{1 + \frac{\nu}{1-\nu}\frac{1-x}{x}} \quad \text{and} \quad \Diamond_\nu(x) = \frac{1}{1 + \frac{1-\nu}{\nu}\frac{1-x}{x}}.$$

By applying $\Box_\nu$ *and* $\Diamond_\nu$ *to* $\sigma_{a,b}^{(\lambda)}$*, we get the membership functions* $\Box_\nu \circ \sigma_{a,b}^{(\lambda)}$ *and* $\Diamond_\nu \circ \sigma_{a,b}^{(\lambda)}$ *of the fuzzy sets 'necessarily young person' and 'possibly young person', respectively. After a direct calculation, we get*

$$\Box_\nu\left(\sigma_{a,b}^{(\lambda)}(x)\right) = \begin{cases} 1, & \text{if } x = 0 \\ \frac{1}{1+\mathrm{e}^{-\lambda\left(x-\left(a+\frac{1}{\lambda}\ln\left(\frac{\nu}{1-\nu}\right)\right)\right)}} & \text{if } 0 < x < b \\ 0, & \text{if } x \geq b \end{cases}$$

and

$$\Diamond_\nu\left(\sigma_{a,b}^{(\lambda)}(x)\right) = \begin{cases} 1, & \text{if } x = 0 \\ \frac{1}{1+\mathrm{e}^{-\lambda\left(x-\left(a+\frac{1}{\lambda}\ln\left(\frac{1-\nu}{\nu}\right)\right)\right)}} & \text{if } 0 < x < b \\ 0, & \text{if } x \geq b \end{cases}$$

We can see that both $\Box_\nu \circ \sigma_{a,b}^{(\lambda)}$ *and* $\Diamond_\nu \circ \sigma_{a,b}^{(\lambda)}$ *have the same form as* $\sigma_{a,b}^{(\lambda)}$*. It means that the modal operators* $\Box_\nu$ *and* $\Diamond_\nu$ *shift the membership function* $\sigma_{a,b}^{(\lambda)}$ *along the horizontal axis upwards and downwards, respectively.*

Figure 2 shows typical plots of the membership functions of fuzzy sets 'necessarily young person' and 'possibly young person', which have been derived by applying the modal operators $\Box_\nu$ *and* $\Diamond_\nu$*, respectively, to the membership functions of fuzzy set 'young person'.*

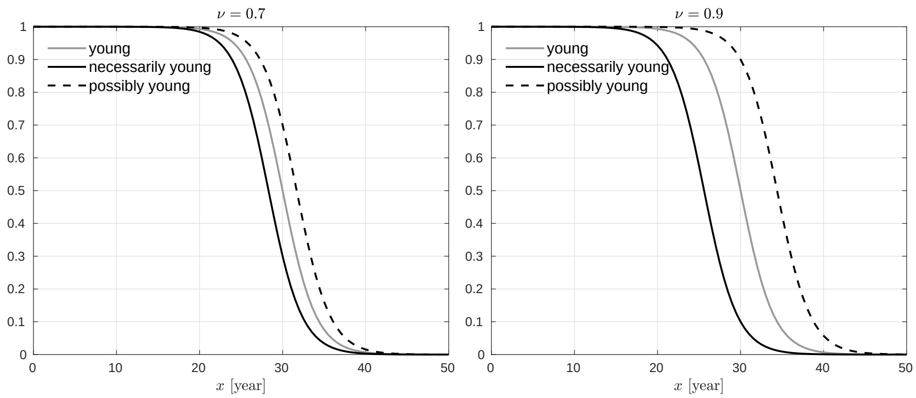

Figure 2: Effects of applications of modal operators on the membership function of fuzzy set "young person".

It is worth noting that we can get a very similar results, if the membership function of the fuzzy set 'young person' is given by the sigmoid-like function $\sigma^{(\lambda)}_{a,b,c}\colon [0,\infty) \to [0,1]$*:*

$$\sigma^{(\lambda)}_{a,b,c}(x) = \max\left(\min\left(\frac{1+b}{1+\mathrm{e}^{-\lambda(x-a)}} - c, 1\right), 0\right),$$

where $\lambda = -0.5$*,* $a = 30$*,* $b = 10^{-6}$ *and* $c = 3 \cdot 10^{-9}$*. In this case, the membership functions* $\Box_\nu \circ \sigma^{(\lambda)}_{a,b,c}$ *and* $\Diamond_\nu \circ \sigma^{(\lambda)}_{a,b,c}$ *of the fuzzy sets 'necessarily young person' and 'possibly young person' are:*

$$\Box_\nu\left(\sigma^{(\lambda)}_{a,b,c}(x)\right) = \max\left(\min\left(\frac{1}{1+\frac{\nu}{1-\nu}\frac{(1+c)\mathrm{e}^{-\lambda(x-a)}-b+c}{1-c\mathrm{e}^{-\lambda(x-a)}+b-c}}, 1\right), 0\right)$$

and

$$\Diamond_\nu\left(\sigma^{(\lambda)}_{a,b,c}(x)\right) = \max\left(\min\left(\frac{1}{1+\frac{1-\nu}{\nu}\frac{(1+c)\mathrm{e}^{-\lambda(x-a)}-b+c}{1-c\mathrm{e}^{-\lambda(x-a)}+b-c}}, 1\right), 0\right).$$

It can be verified that the plots of $\Box_\nu \circ \sigma^{(\lambda)}_{a,b,c}$ *and* $\Diamond_\nu \circ \sigma^{(\lambda)}_{a,b,c}$ *are very similar to those in Figure 2.*

Remark 12. *We can see that in Example 4, the modal operators* $\Box_\nu$ *and* $\Diamond_\nu$ *play the role of substantiating and weakening modifier operators, respectively, as well. The membership function of the fuzzy set 'necessarily young' can be interpreted as the membership function of a fuzzy set 'very young'. Similarly, the membership function of the fuzzy set 'possibly young' may be viewed as the membership function of a fuzzy set 'somewhat young'. That is, even though* $\Box_\nu$ *and* $\Diamond_\nu$ *are not power functions, and so they do not have the form of a traditional hedge given in Eq.* (40)*, their application to the membership function of a fuzzy set results in fuzzy hedges.*

8. A unified form of strong negations, substantiating and weakening modifier operators, modalities and linguistic hedges

In Section 2, we showed that the operator $\eta_\nu \colon [0,1] \to [0,1]$ given by

$$\eta_\nu(x) = f^{-1}\left(f(\nu)\frac{f(\nu)}{f(x)}\right), \tag{41}$$

where $\nu \in (0,1)$ and f is a generator function of a conjunctive or disjunctive operator, is a strong negation operator (i.e., the Pliant negation).

Based on the results presented in Section 3, Section 4, Section 5 and Section 7, the operator $\tau_{\nu,\nu_0} \colon [0,1] \to [0,1]$ given by

$$\tau_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0)\frac{f(x)}{f(\nu)}\right), \tag{42}$$

where $\nu, \nu_0 \in (0,1)$ and f is a generator function of a conjunctive or disjunctive operator, may be viewed as the common form of substantiating and weakening modifier operators, modal operators and linguistic hedges. In Section 6, we demonstrated that the unary operators that are distributive over a conjunctive or disjunctive operator, which is induced by the generator function f, also have the form given by Eq. (42).

Following these findings, we define a unified form of unary operators and demonstrate that the operators given by Eq. (41) and Eq. (42) are special cases of the unified form of unary operators.

Definition 8. *The unified unary operator* $u^{(\varepsilon)}_{\nu,\nu_0} \colon [0,1] \to [0,1]$ *is given by*

$$u^{(\varepsilon)}_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0)\left(\frac{f(x)}{f(\nu)}\right)^{\varepsilon}\right), \tag{43}$$

where $\nu, \nu_0 \in (0,1)$*,* $\varepsilon \in \{-1, 1\}$ *and* f *is a generator function of a conjunctive or disjunctive operator.*

Theorem 7. *The negation operator given by Eq.* (41)*, and the common form of substantiating and weakening modifier operators, modal operators and linguistic hedges given by Eq.* (42) *are special cases of the unified unary operator* $u^{(\varepsilon)}_{\nu_*,\nu_0}$ *given by Definition 8.*

Proof. After basic considerations, we readily get the following.

(a) If $\varepsilon = -1$ and $\nu_0 = \nu$, then $u^{(\varepsilon)}_{\nu,\nu_0}$ is identical to the Pliant negation operator given by Eq. (41).

(b) If $\varepsilon = 1$, then $u^{(\varepsilon)}_{\nu,\nu_0}$ is identical to the common form of substantiating and weakening modifier operators, modal operators and linguistic hedges given by Eq. (42).

□

9. Conclusions

The main findings of our study can be summarized as follows.

(a) The results of this study are related to operator dependent modifiers.

(b) Here, the concept of a dual pair of modal operators is interpreted following the criteria for an algebraic version of necessity and possibility operators on De Morgan lattices given by Cattaneo, Ciucci and Dubois, 2011.

(c) We proved that if a conjunctive operator c, a disjunctive operator d and a strong negation η_{ν_*}, which is induced by using the generator function of c or d, build a De Morgan system, then the substantiating and weakening modifier operators induced by repeating the arguments of c and d satisfy the requirements for a dual pair of modal operators with the negation η_{ν_*}.

(d) Next, we presented the necessary and sufficient condition for the distributivity of a unary operator over conjunctive and disjunctive operators.

(e) Also, we highlighted an important connection between modal operators and linguistic hedges.

(f) Then, we demonstrated that (1) the strong negations, (2) the substantiating and weakening modifier operators, modal operators and linguistic hedges mentioned above, and (3) the unary operators, which are distributive over conjunctive and disjunctive operators, may be viewed as special cases of a unified unary operator class.

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