On a Strong Negation-based Representation of Modalities

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Abstract

In this study, the concept of a dual pair of modal operators is interpreted following the criteria for an algebraic version of necessity and possibility operators on De Morgan lattices given by Cattaneo, Ciucci and Dubois, 2011. Here, a representation theorem is introduced which demonstrates that, in this algebraic model, a dual pair of modal operators can be represented by compositions of two strong negations, where one of them is stricter than the other. Then, the Pliant negation operator is utilized to derive dual modal operators. It is demonstrated that using the generator function of Dombi operators, the composition of two Pliant negations results in modal operators that have simple forms and easy-to-use characteristics. Next, we examine how the proposed modal operators are connected with the drastic necessity and possibility operators. Also, the necessary and sufficient condition for the distributivity of modal operators induced by compositions of strong negations over strict t-norms and strict t-conorms is presented. Lastly, a connection between the modal operators and hedges is highlighted.

Keywords: strong negations, Pliant negation, modal operators, drastic modal operators, hedges

1. Introduction

Since the modal-like necessity and possibility operators play an important role in reasoning, these modal operators have a variety of applications in classical and in continuous-valued logic.

A system called basic logic (BL) was defined in Hájek's book [1]. Later, a survey paper was published by Gottwald and Hájek [2] in which they discussed the state-of-art development of BL. In this logic, the implication is given by the residual of the t-norm, and the negation operator $\sim$ is defined by $\sim x := x \to 0$. It should be added that, in the strict operator case, this negation

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operator is not involutive. Hájek [3] studied the fuzzy variant of the well-known modal logic S5, introduced three kinds of Kripke models and identified the corresponding deductive systems. Esteva *et al.* [4] introduced logics with an involutive negation. In their approach negation is different from the implication-based negation "$\sim$", but it is not related to the residual implication in the strict monotonous operator case. Cintula *et al.* [5] presented a survey paper on propositional fuzzy logics extending SBL (BL plus the axiom schema $\varphi \wedge \neg\varphi \rightarrow \overline{0}$) with an additional involutive negation. With this approach, they improved the expressive potential of mathematical fuzzy logic. Banerjee and Dubois [6] used the syntax and axioms from the modal logic KD to establish an epistemic logic for reasoning about incomplete knowledge. Cattaneo *et al.* [7] introduced algebraic models of deviant modal operators based on De Morgan and Kleene lattices. Modal logic has also been used in rough set theory, where the sets are approximated by elements of a partition induced by an equivalence relation [8]. A natural choice for rough set logic is S5 (see [9]). Here, the possibility and necessity modalities can be viewed as upper and lower approximation operators. Esteva *et al.* [10] proposed logics that accommodate most of the truth hedge functions used in the literature. Zadeh [11] introduced a modal logic system called the finite-state model to highlight the fact that the concept of possibility has different roles in possibility theory and in modal logic. In a recent paper, Vidal [12] studied modal logics defined from valued Kripke frames, focusing on the computability and expressivity of modal logics of transitive Kripke frames evaluated over certain residuated lattices. Jain *et al.* [13] presented a new fuzzy modal logic to model and reason about transition systems involving uncertainty in behaviours.

It should be added that continuous-valued logic can be studied both from a logical point of view (axiomatization, completeness, possible extensions, predicat calculi, etc.) and from an algebraic point of view [14]. In the latter case, we need to solve functional equations to find conjunction, disjunction and negation operators, which a logical system is founded on. This also means that for a particular continuous-valued logical system, modal operators are provided.

In this study, we seek to find proper algebraic expressions for modal operators using some basic considerations. Namely, we interpret a dual pair of modal operators following the criteria for an algebraic version of necessity and possibility operators on De Morgan lattices given by Cattaneo, Ciucci and Dubois [7] (also, see [15]). Here, we provide a representation theorem, which demonstrates that, in our algebraic model, a dual pair of modal operators can be represented by compositions of two strong negations, where one of them is stricter than the other. Next, we use the Pliant negation operator to derive dual modal operators in a very simple way. Also, we show that by using the generator function of Dombi operators, the composition of two Pliant negations results in modal operators that have simple forms and easy-to-use characteristics. Next, we describe how the proposed modal operators are connected with the drastic necessity and possibility operators. Then, we present the necessary and sufficient condition for the distributivity of modal operators induced by compositions of strong negations over strict t-norms and strict t-conorms. Finally, we highlight a connection between the modal operators and hedges.

This study is structured as follows. In Section 2, the basic considerations of negations in continuous-valued logic are described. In Section 3, the basics of modal logic are briefly reviewed. A representation theorem for modal operators induced by compositions of strong negations is presented in Section 4. In section 5, it is demonstrated how the composition of two Pliant negations can be used to derive modal operators. The connection between the modal operators and the drastic modal operators is studied in Section 6. In Section 7, the necessary and sufficient condition for the distributivity of modal operators induced by compositions of strong negations over strict t-norms and strict t-conorms is presented. A connection between modalities and hedges is presented in Section 8. Lastly, in Section 9, we shall summarize our conclusions.

2. Basic considerations of negations in continuous-valued logic

Definition 1. *We say that* $\eta\colon [0,1] \to [0,1]$ *is a strong negation if and only if* η *satisfies the following conditions:*

C1: η *is bijective and continuous*	*(Bijectivity and continuity)*
C2: $\eta(0) = 1$, $\eta(1) = 0$	*(Boundary conditions)*
C3: $\eta(x) < \eta(y)$ *for* $x > y$	*(Monotonicity)*
C4: $\eta(\eta(x)) = x$ *for any* $x \in [0,1]$	*(Involution).*

Remark 1. *Note that the boundary condition C2 can be inferred by using C1 and C3.*

There are two representation theorems known for the strong negation given in Definition 1. Trillas [16] showed that every involutive negation operator $\eta\colon [0,1] \to [0,1]$ has the following form:

$$\eta(x) = g^{-1}(1 - g(x)),$$

where $g\colon [0,1] \to [0,1]$ is a continuous strictly increasing (or decreasing) function. Note that this generator function corresponds to the nilpotent operators (see [17] [18] [19]). Here, we will utilize another, parametric form of negation, which is known as the Dombi negation operator; and it is an element of the Pliant system [20, 21].

Definition 2 (Pliant negation)**.** *The negation operator* $\eta_\nu\colon [0,1] \to [0,1]$*, which is given by*

$$\eta_\nu(x) = f^{-1}\left(\frac{f^2(\nu)}{f(x)}\right) \tag{1}$$

is called the Pliant negation operator, where $\nu \in (0,1)$*,* $f\colon [0,1] \to [0,\infty]$ *is a continuous, strictly increasing (or decreasing) function and* f *is the generator function of a strict monotone t-norm, or t-conorm.*

Remark 2. *Note that we interpret $f(0)$ and $f(1)$ by the following limits*

$$f(0) = \lim_{x\to 0} f(x), \quad f(1) = \lim_{x\to 1} f(x).$$

Hereafter, η, η_1, η_2, etc. will denote strong negations and $\eta_\nu, \eta_{\nu_1}, \eta_{\nu_2}$, etc. will be used to denote Pliant negations.

Proposition 1. *The Pliant negation given in Definition 2 is a strong negation.*

Proof. For the proof see [20]. □

Remark 3. *It should be added that for any $\nu \in (0,1)$, $\eta_\nu(\nu) = \nu$. That is, the fix point of the Pliant negation η_ν is its parameter value ν.*

Definition 3. *We will say that the negation $\eta_1 \colon [0,1] \to [0,1]$ is stricter than the negation $\eta_2 \colon [0,1] \to [0,1]$ if and only if for any $x \in [0,1]$, $\eta_1(x) < \eta_2(x)$.*

Example 1. *Let f_c and f_d be the the generator functions of the probabilistic conjunctive and disjunctive operators, respectively. That is, functions $f_c, f_d \colon (0,1) \to (0,\infty)$ are given by*

$$f_c(x) = -\ln(x) \quad \text{and} \quad f_d(x) = -\ln(1-x).$$

Let $\nu \in (0,1)$. After direct calculation, we get that the Pliant negation operators $\eta_{\nu,c}, \eta_{\nu,d} \colon [0,1] \to [0,1]$ induced by function f_c and f_d are

$$\eta_{\nu,c}(x) = f_c^{-1}\left(\frac{f_c^2(\nu)}{f_c(x)}\right) = \nu^{\frac{\ln(\nu)}{\ln(x)}}, \quad \eta_{\nu,d}(x) = f_d^{-1}\left(\frac{f_d^2(\nu)}{f_d(x)}\right) = 1 - (1-\nu)^{\frac{\ln(1-\nu)}{\ln(1-x)}}.$$

Example 2. *The generator function of the Dombi conjunction and disjunction operators is the function $g_\alpha \colon (0,1) \to (0,\infty)$ that is given by*

$$g_\alpha(x) = \left(\frac{1-x}{x}\right)^\alpha, \tag{2}$$

where $\alpha \neq 0$. If $\alpha > 0$, then g_α is the generator function of a conjunctive operator; and if $\alpha < 0$, then g_α is the generator function of a disjunctive operator (see, e.g. [22]). Now, let $\nu \in (0,1)$, $\alpha \neq 0$ and let $f(x) = g_\alpha(x)$ for any $x \in (0,1)$. Then, the Pliant negation operator $\eta_\nu \colon [0,1] \to [0,1]$ induced by function f is

$$\eta_\nu(x) = f^{-1}\left(\frac{f^2(\nu)}{f(x)}\right) = \frac{1}{1 + \left(\frac{1-\nu}{\nu}\right)^2 \frac{x}{1-x}}.$$

Here, we can see that η_ν is independent of the parameter α. That is, the Pliant negations induced by the generator functions of conjunctive and disjunctive Dombi operators coincide. Figure 1 shows typical plots of Pliant negations induced by the generator function of Dombi operators.

From now on, the mapping $f \colon [0,1] \to [0,\infty]$ will always be a continuous, strictly increasing (or decreasing) generator function of a strict monotone t-norm, or t-conorm. Later, we will make of use of the following property of the Pliant negation.

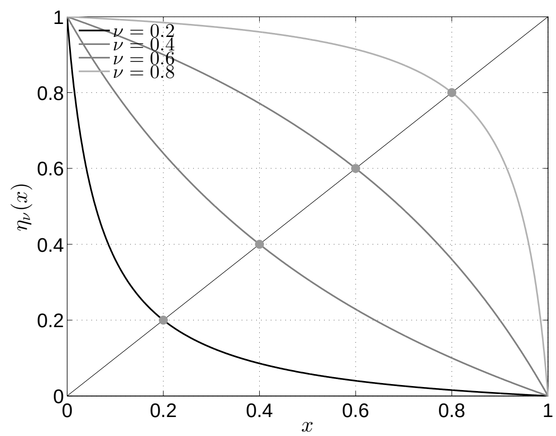

Figure 1: Pliant negations induced by the generator function of Dombi operators with various ν parameter values.

Proposition 2. *The Pliant negation η_{ν_1} is stricter than the Pliant negation η_{ν_2} if and only if $\nu_1 < \nu_2$.*

Proof. The proposition immediately follows from the definition of η_ν. $\square$

Later, we will also utilize the concept of drastic negation.

Definition 4 (Drastic negations)**.** *We say that the functions $\eta_{d,0}, \eta_{d,1} \colon [0,1] \to [0,1]$ are drastic negations if and only if $\eta_{d,0}$ and $\eta_{d,1}$ are given by*

$$\eta_{d,0}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases} \quad \text{and} \quad \eta_{d,1}(x) = \begin{cases} 1, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1. \end{cases}$$

Note that the drastic negations given in Definition 4 are not strong negations, but the drastic negations may be viewed as limit cases of Pliant negations. Namely, the following proposition is valid.

Proposition 3. *For any $x \in [0,1]$,*

$$\lim_{\nu \to 0} \eta_\nu(x) = \eta_{d,0}(x) \quad \text{and} \quad \lim_{\nu \to 1} \eta_\nu(x) = \eta_{d,1}(x).$$

Proof. Using the definition of η_ν, the proof is straightforward. $\square$

3. Basic considerations of modal logic

3.1. Basics of classical modal logic

Here, we will use the traditional notations $\Diamond$ and $\Box$ for the possibility and necessity operators of classical modal logic, respectively. Now, let P be a statement. For example, let P be the statement "It will rain today.". Then, $\Diamond P$ and $\Box P$ are the statements

$\Diamond P$: "It is possible that it will rain today."
$\Box P$: "It is necessary that it will rain today.".
There are two well-known identities of classical modal logic, namely,

$$\neg(\Diamond P) \equiv \Box(\neg P) \tag{3}$$

and

$$\neg(\Box P) \equiv \Diamond(\neg P), \tag{4}$$

where $\neg$ is the negation operator of classical logic. In our example, the identity in Eq. (3) means that the statements
$\neg(\Diamond P)$: "It is not possible that it will rain today."
$\Box(\neg P)$: "It is necessary that it not will rain today."
are equivalent. Also, Eq. (4) means that the statements
$\neg(\Box P)$: "It is not necessary that it will rain today."
$\Diamond(\neg P)$: "It is possible that it not will rain today."
are equivalent as well.

3.2. Algebraic criteria for modal operators in a continuous-valued logic

Following the criteria for an algebraic version of necessity and possibility operators on De Morgan lattices given in [7], we define the dual pair of necessity and possibility operators in continuous-valued logic as follows (also, see [15]). Note that here we will use the classical notations $\Box$ and $\Diamond$.

Definition 5. *The functions $\Box, \Diamond\colon [0,1] \to [0,1]$ are a dual pair of necessity and possibility operators, respectively, if and only if $\Box$ and $\Diamond$ satisfy the following requirements:*

$N1.$	$\Box(1) = 1$	$P1.$	$\Diamond(0) = 0$
$N2.$	$\Box(x) \leq x$	$P2.$	$x \leq \Diamond(x)$
$N3.$	$x \leq y$ *implies* $\Box(x) \leq \Box(y)$	$P3.$	$x \leq y$ *implies* $\Diamond(x) \leq \Diamond(y)$
$N4.$	$\eta(\Diamond(x)) = \Box(\eta(x))$	$P4.$	$\eta(\Box(x)) = \Diamond(\eta(x))$
$[N5.$	$\Diamond(x) = \Box(\Diamond(x))$	$P5.$	$\Box(x) = \Diamond(\Box(x))]$
$N5'.$	$\Box(\Diamond(x)) = x$	$P5'.$	$\Diamond(\Box(x)) = x$

for any $x \in [0,1]$, where $\eta\colon [0,1] \to [0,1]$ is a strong negation operator.

The requirements from $N1$ to $N5$ are called the N principle, T principle, K principle, DF$\Diamond$ principle and N^* principle, respectively. Also, the requirements from $P1$ to $P5$ are known as the P principle, T principle, K principle, DF$\Box$ principle and P^* principle, respectively.

Remark 4. *Note that in our approach, $N5$ and $P5$ will not be used. Instead of $N5$ and $P5$, our demand is the neutrality principle given by $N5'$ and $P5'$. Later, we will show that special cases of $\Box$ and $\Diamond$ meet the criteria $N4$ and $P4$ (see Remark 10). Also note that, according to $N5'$ and $P5'$, the functions $\Box$ and $\Diamond$ are inverse functions of each other.*

Remark 5. *Notice that $N4$ and $P4$ may be viewed as continuous-valued generalizations of the identities of classical modal logic in Eq. (3) and Eq. (4).*

4. Representing modal operators by compositions of strong negations

Now, following the ideas outlined in [14], we will present a representation theorem which demonstrates that there is an important connection between a dual pair of modal operators and the composition of two strong negations.

Theorem 1 (Representation)**.** *Let* $\Box\colon [0,1] \to [0,1]$ *and* $\Diamond\colon [0,1] \to [0,1]$ *be two continuous functions. Then,* $\Box$ *and* $\Diamond$ *are a dual pair of necessity and possibility operators, respectively, if and only if* $\Diamond$ *and* $\Box$ *have the form*

$$\Diamond = \eta_2 \circ \eta_1 \tag{5}$$

$$\Box = \eta_1 \circ \eta_2, \tag{6}$$

where $\eta_1 \colon [0,1] \to [0,1]$ *and* $\eta_2 \colon [0,1] \to [0,1]$ *are two strong negations such that* η_1 *is stricter than* η_2*.*

Here, we will prove the necessity and sufficiency conditions stated in Theorem 1 by proving Proposition 4 and Proposition 5.

Proposition 4. *If* $\Box$ *and* $\Diamond$ *are a dual pair of necessity and possibility operators, respectively, then* $\Diamond$ *and* $\Box$ *have the forms given by Eq.* (5) *and Eq.* (6)*, where* $\eta_1 \colon [0,1] \to [0,1]$ *and* $\eta_2 \colon [0,1] \to [0,1]$ *are two strong negations such that* η_1 *is stricter than* η_2*.*

Proof. Since $\Box$ and $\Diamond$ are a dual pair of necessity and possibility operators, respectively, $\Box$ and $\Diamond$ satisfy the requirements given in Definition 5. Therefore, based on $N4$ and $P4$, we have

$$\eta\left(\Diamond(x)\right) = \Box\left(\eta(x)\right) \tag{7}$$

$$\eta\left(\Box(x)\right) = \Diamond\left(\eta(x)\right) \tag{8}$$

for any $x \in [0,1]$, where $\eta\colon [0,1] \to [0,1]$ is a strong negation operator. Now, let $\eta_1 \colon [0,1] \to [0,1]$ and $\eta_2 \colon [0,1] \to [0,1]$ be given by

$$\eta_1(x) = \Box\left(\eta(x)\right) \tag{9}$$

$$\eta_2(x) = \eta(x) \tag{10}$$

for any $x \in [0,1]$. By using $N1$–$N4$ and $N5'$, it can be readily verified that η_1 is a strong negation. Also, by using $N2$, we have $\Box\left(\eta(x)\right) \leq \eta(x)$ for any $x \in [0,1]$, which means that η_1 is stricter than η. Therefore η_1 and η_2 are two strong negations and η_1 is stricter than η_2. Note that here, $\eta_2(x)$ represents *not*(x), while $\eta_1(x)$ can be interpreted as *necessarily*(*not*(x)). It also means that *necessarily not* is a stricter negation than *not*. Using Eq. (9) and Eq. (10), Eq. (7) and Eq. (8) can be written as

$$\eta_2\left(\Diamond(x)\right) = \eta_1(x) \tag{11}$$

$$\eta_2\left(\Box(x)\right) = \Diamond\left(\eta_2(x)\right). \tag{12}$$

Now, by applying η_2 to both sides of Eq. (11) and Eq. (12), we get

$$\eta_2\left(\eta_2\left(\lozenge(x)\right)\right) = \eta_2\left(\eta_1(x)\right) \tag{13}$$

and

$$\eta_2\left(\eta_2\left(\square(x)\right)\right) = \eta_2\left(\lozenge\left(\eta_2(x)\right)\right) \tag{14}$$

for any $x \in [0,1]$. Next, by noting that η_2 is an involution, from Eq. (13) we have

$$\lozenge(x) = \eta_2\left(\eta_1(x)\right) \tag{15}$$

for any $x \in [0,1]$, which means that $\lozenge$ has a form given by Eq. (5). And since $\eta_1(x)$ represents $necessarily(not(x))$ and $\eta_2(x)$ represents $not(x)$, Eq. (11) can be interpreted as $not(possibly(x)) = necessarily(not(x))$. Now, by taking into account the fact that η_2 is an involution, from Eq. (14) we have

$$\square(x) = \eta_2\left(\lozenge\left(\eta_2(x)\right)\right) \tag{16}$$

for any $x \in [0,1]$. By using Eq. (11) and the substitution $y = \eta_2(x)$, Eq. (16) can be written as

$$\square(x) = \eta_2\left(\lozenge\left(\eta_2(x)\right)\right) = \eta_2\left(\lozenge\left(y\right)\right) = \eta_1(y) = \eta_1\left(\eta_2(x)\right), \tag{17}$$

which means that $\square$ has a form given by Eq. (6). $\square$

Proposition 5. *If the functions* $\square\colon [0,1] \to [0,1]$ *and* $\lozenge\colon [0,1] \to [0,1]$ *have the forms given by Eq. (5) and Eq. (6), where* $\eta_1\colon [0,1] \to [0,1]$ *and* $\eta_2\colon [0,1] \to [0,1]$ *are two strong negations such that* η_1 *is stricter than* η_2*, then* $\square$ *and* $\lozenge$ *are a dual pair of necessity and possibility operators, respectively.*

Proof. Since $\lozenge$ and $\square$ have the forms given by Eq. (5) and Eq. (6), respectively, we have

$$\lozenge(x) = \eta_2\left(\eta_1(x)\right) \tag{18}$$

$$\square(x) = \eta_1\left(\eta_2(x)\right) \tag{19}$$

for any $x \in [0,1]$, where $\eta_1\colon [0,1] \to [0,1]$ and $\eta_2\colon [0,1] \to [0,1]$ are two strong negations such that η_1 is stricter than η_2.

In order to demonstrate that $\square$ and $\lozenge$ are a dual pair of necessity and possibility operators, respectively, we need to show that $\square$ meets the criteria $N1$–$N4$ and $N5'$ given in Definition 5, and that $\lozenge$ satisfies the requirements $P1$–$P4$ and $P5'$ given in Definition 5.

Proof of $N1$ *and* $P1$*.* Exploiting the fact that η_1 and η_2 are two strong negations, we have

$$\square(1) = \eta_1(\eta_2(1)) = \eta_1(0) = 1$$

and

$$\lozenge(0) = \eta_2(\eta_1(0)) = \eta_2(1) = 0.$$

Proof of $N2$ *and* $P2$. Noting that η_1 is a stricter negation than η_2, we have

$$\eta_1(x) \le \eta_2(x) \tag{20}$$

for any $x \in [0,1]$. By applying η_1 to both sides of Eq. (20) and noting that η_1 is strictly decreasing and involutive, we get

$$x \ge \eta_1(\eta_2(x)) = \Box(x)$$

for any $x \in [0,1]$. Similarly, by applying η_2 to both sides of Eq. (20), we get

$$\eta_2(\eta_1(x)) \ge x,$$

which means that $\Diamond(x) \ge x$ for any $x \in [0,1]$.

Proof of $N3$ *and* $P3$. Let $x \le y$, $x, y \in [0,1]$. By taking into account the fact that η_1 and η_2 are two strong negations, if $x \le y$, then

$$\eta_1(\eta_2(x)) \le \eta_1(\eta_2(y)) \quad \text{and} \quad \eta_2(\eta_1(x)) \le \eta_2(\eta_1(y)),$$

which means that

$$\Box(x) \le \Box(y) \quad \text{and} \quad \Diamond(x) \le \Diamond(y),$$

respectively. Moreover, based on a similar consideration, since η_1 and η_2 are two strong negations, $x < y$ implies that

$$\Box(x) < \Box(y) \quad \text{and} \quad \Diamond(x) < \Diamond(y).$$

Proof of $N4$ *and* $P4$. Let $\Box^{-1}$ denote the inverse of the strictly increasing function $\Box$ and let $\eta\colon [0,1] \to [0,1]$ be given by

$$\eta(x) = \Box^{-1}(\eta_1(x)) \tag{21}$$

for any $x \in [0,1]$. Since $\Box^{-1}(x) = \eta_2(\eta_1(x))$ for any $x \in [0,1]$, and by noting that η_1 is an involution, from Eq. (21) we also have

$$\eta(x) = \eta_2(\eta_1(\eta_1(x))) = \eta_2(x) \tag{22}$$

for any $x \in [0,1]$. Hence, η is a strong negation. By applying $\Box$ to both sides of Eq. (21) we get

$$\Box(\eta(x)) = \eta_1(x), \tag{23}$$

for any $x \in [0,1]$. Next, by using Eq. (22) and Eq. (23), Eq. (18) can be written as

$$\Diamond(x) = \eta(\Box(\eta(x))). \tag{24}$$

By applying η to both sides of Eq. (24) and noting that η is an involution, we get

$$\eta(\Diamond(x)) = \eta(\eta(\Box(\eta(x)))) = \Box(\eta(x)), \tag{25}$$

for any $x \in [0,1]$, which is identical to $N4$. Also, by noting again that η is an involution, using the substitution $y = \eta(x)$ and Eq. (25), we can write

$$\Box(x) = \Box\left(\eta\left(\eta(x)\right)\right) = \Box\left(\eta(y)\right) = \eta\left(\Diamond(y)\right) = \eta\left(\Diamond\left(\eta(x)\right)\right).$$

Now, applying η to both sides of the last equation and noting that η is an involution, we get

$$\eta\left(\Box(x)\right) = \Diamond\left(\eta(x)\right)$$

for any $x \in [0,1]$, which is identical to $P4$.

Proof of $N5'$ and $P5'$. Noting that η_1 and η_2 are involutive functions, from Eq. (18) and Eq. (19) we have

$$\Box(\Diamond(x)) = \eta_1(\eta_2(\eta_2(\eta_1(x)))) = x$$

and

$$\Diamond(\Box(x)) = \eta_2(\eta_1(\eta_1(\eta_2(x)))) = x$$

for any $x \in [0,1]$, which means that $N5'$ and $P5'$ hold. $\Box$

5. Modal operators represented by compositions of two Pliant negations

Here, we will show how the compositions of two Pliant negations can be used to derive modal operators and describe the properties of these modal operators.

Definition 6. *The function $M_{\nu_1,\nu_2}\colon [0,1] \to [0,1]$ is given by*

$$M_{\nu_1,\nu_2} = \eta_{\nu_1} \circ \eta_{\nu_2},$$

where $\nu_1, \nu_2 \in (0,1)$ and η_{ν_1} and η_{ν_2} are two Pliant negation operators given by Definition 2.

Remark 6. *It immediately follows from the definition of η_ν in Eq. (1) that if $\nu_1 = \nu_2$, then M_{ν_1,ν_2} is the identity function.*

Proposition 6. *Let $\nu_1, \nu_2 \in (0,1)$. The functions M_{ν_1,ν_2} and M_{ν_2,ν_1}, both given by Definition 6, can be written as*

$$M_{\nu_1,\nu_2}(x) = f^{-1}\left(\frac{f^2(\nu_1)}{f^2(\nu_2)} f(x)\right) \tag{26}$$

and

$$M_{\nu_2,\nu_1}(x) = f^{-1}\left(\frac{f^2(\nu_2)}{f^2(\nu_1)} f(x)\right) \tag{27}$$

for any $x \in [0,1]$, and

(a) If $\nu_1 < \nu_2$, then M_{ν_1,ν_2} and M_{ν_2,ν_1} are a dual pair of necessity and possibility operators, respectively

(b) If $\nu_1 > \nu_2$, then M_{ν_1,ν_2} and M_{ν_2,ν_1} are a dual pair of possibility and necessity operators, respectively

(c) If $\nu_1 = \nu_2$, then M_{ν_1,ν_2} and M_{ν_2,ν_1} are both the identity function, and at the same time, they are a dual pair of necessity (possibility) and possibility (necessity) operators, respectively.

Proof. Using the definitions of M_{ν_1,ν_2} and M_{ν_2,ν_1} (see Definition 6), and the definition of the Pliant negation operator η_ν in Definition 2, after direct calculation we get Eq. (26) and Eq. (27). By noting Proposition 1, Proposition 2 and Theorem 1, we immediately have (a) and (b). Next, if $\nu_1 = \nu_2$, then $M_{\nu_1,\nu_2}(x) = M_{\nu_2,\nu_1}(x) = x$ for any $x \in [0,1]$. In this case M_{ν_1,ν_2} and M_{ν_2,ν_1} trivially satisfy the requirements $N1$–$N4$, $N5'$, $P1$–$P4$ and $P5'$ for a dual pair of necessity (possibility) and possibility (necessity) operators in Definition 5. □

Remark 7. *Based on Proposition 6, we can call M_{ν_1,ν_2} and M_{ν_2,ν_1} a dual pair of modal operators.*

Proposition 7. *Let $\nu_1, \nu_2 \in (0,1)$ and let M_{ν_1,ν_2} and M_{ν_2,ν_1} be a a dual pair of modal operators. Then, there exists a unique $\nu \in (0,1)$ such that for any $x \in [0,1]$, either*

$$M_{\nu_1,\nu_2}(x) = \underline{M}_\nu(x), \quad M_{\nu_2,\nu_1}(x) = \overline{M}_\nu(x), \tag{28}$$

or

$$M_{\nu_1,\nu_2}(x) = \overline{M}_\nu(x), \quad M_{\nu_2,\nu_1}(x) = \underline{M}_\nu(x) \tag{29}$$

holds, where $\underline{M}_\nu, \overline{M}_\nu \colon [0,1] \to [0,1]$ and $\underline{M}_\nu$, $\overline{M}_\nu$ are given by

$$\underline{M}_\nu(x) = f^{-1}\left(\frac{f(x)}{f^2(\nu)}\right), \quad \overline{M}_\nu(x) = f^{-1}\left(f^2(\nu) f(x)\right). \tag{30}$$

Proof. Based on Proposition 6, for any $x \in [0,1]$, we have

$$M_{\nu_1,\nu_2}(x) = f^{-1}\left(\frac{f^2(\nu_1)}{f^2(\nu_2)} f(x)\right) \tag{31}$$

and

$$M_{\nu_2,\nu_1}(x) = f^{-1}\left(\frac{f^2(\nu_2)}{f^2(\nu_1)} f(x)\right). \tag{32}$$

Now, by noting that $f \colon [0,1] \to [0,\infty]$ is a continuous and strictly monotone function, we see that there exists a unique $\nu \in (0,1)$ such that either

$$\frac{f(\nu_1)}{f(\nu_2)} = \frac{1}{f(\nu)} \quad \text{and} \quad \frac{f(\nu_2)}{f(\nu_1)} = f(\nu),$$

or

$$\frac{f(\nu_1)}{f(\nu_2)} = f(\nu) \quad \text{and} \quad \frac{f(\nu_2)}{f(\nu_1)} = \frac{1}{f(\nu)}.$$

Therefore, by using the equation pair in Eq. (30), we have either the equation pair in Eq. (28), or the equation pair in Eq. (29). □

Proposition 8. *Let $\nu \in (0,1)$ and let $\underline{M}_\nu \colon [0,1] \to [0,1]$ and $\overline{M}_\nu \colon [0,1] \to [0,1]$ be given by*

$$\underline{M}_\nu(x) = f^{-1}\left(\frac{f(x)}{f^2(\nu)}\right), \quad \overline{M}_\nu(x) = f^{-1}\left(f^2(\nu)f(x)\right).$$

Then,

(a) if f is strictly decreasing and $f(\nu) \leq 1$, then $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of necessity and possibility operators, respectively
(b) if f is strictly decreasing and $f(\nu) \geq 1$, then $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of possibility and necessity operators, respectively
(c) if f is strictly increasing and $f(\nu) \leq 1$, then $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of possibility and necessity operators, respectively
(d) if f is strictly increasing and $f(\nu) \geq 1$, then $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of necessity and possibility operators, respectively.

Proof. Here, we will prove case (a), the proofs for the other three cases being similar to that of case (a). Let $\nu_1 \in (0,1)$ have an arbitrary fixed value and let ν_2 be given by

$$\nu_2 = f^{-1}\left(f(\nu_1)f(\nu)\right). \tag{33}$$

Then, by noting the properties of the generator function f, from Eq. (33), we have that $\nu_2 \in (0,1)$. Also, applying f to both sides of Eq. (33), after direct calculation, we get

$$\frac{1}{f^2(\nu)} = \frac{f^2(\nu_1)}{f^2(\nu_2)}, \tag{34}$$

and so

$$\underline{M}_\nu(x) = f^{-1}\left(\frac{1}{f^2(\nu)}f(x)\right) = f^{-1}\left(\frac{f^2(\nu_1)}{f^2(\nu_2)}f(x)\right)$$

for any $x \in [0,1]$. Next, by noting the fact that $f(\nu) \leq 1$ and the fact that f is strictly decreasing, from Eq. (34), we have $\nu_1 \leq \nu_2$. Therefore, based on Proposition 6, $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of necessity and possibility operators, respectively. $\square$

Remark 8. *When $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of necessity and possibility operators, respectively, we will use the notations*

$$\Box_\nu = \underline{M}_\nu, \quad \text{and} \quad \Diamond_\nu = \overline{M}_\nu.$$

Also, when $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of possibility and necessity operators, respectively, we will use the notations

$$\Diamond_\nu = \underline{M}_\nu, \quad \text{and} \quad \Box_\nu = \overline{M}_\nu.$$

Example 3. *Let f_c and f_d be the the generator functions of the probabilistic conjunctive and disjunctive operators, respectively. That is,*

$$f_c(x) = -\ln(x) \quad \text{and} \quad f_d(x) = -\ln(1-x)$$

for any $x \in (0,1)$*. Let* $\nu \in (0,1)$*. Then, by using Eq.* (30)*, after direct calculation, we get that the dual pair of modal operators* $\underline{M}_{\nu,c}\colon (0,1) \to (0,1)$ *and* $\overline{M}_{\nu,c}\colon (0,1) \to (0,1)$ *induced by the generator function* f_c *are*

$$\underline{M}_{\nu,c}(x) = f_c^{-1}\left(\frac{f_c(x)}{f_c^2(\nu)}\right) = x^{\frac{1}{\ln^2(\nu)}}$$
$$\overline{M}_{\nu,c}(x) = f_c^{-1}\left(f_c^2(\nu) f_c(x)\right) = x^{\ln^2(\nu)}.$$

Since f_c *is strictly decreasing, based on Proposition 8, we have that*

(a) If $f_c(\nu) \geq 1$*, which means that* $\nu \leq \mathrm{e}^{-1}$*, then* $\underline{M}_{\nu,c}$ *and* $\overline{M}_{\nu,c}$ *are a dual pair of possibility and necessity operators, respectively; i.e.* $\underline{M}_{\nu,c} = \Diamond_\nu$ *and* $\overline{M}_{\nu,c} = \Box_\nu$

(b) If $f_c(\nu) \leq 1$*, which means that* $\nu \geq \mathrm{e}^{-1}$*, then* $\underline{M}_{\nu,c}$ *and* $\overline{M}_{\nu,c}$ *are a dual pair of necessity and possibility operators, respectively; i.e.* $\underline{M}_{\nu,c} = \Box_\nu$ *and* $\overline{M}_{\nu,c} = \Diamond_\nu$*.*

Similarly, by using Eq. (30)*, the dual pair of modal operators* $\underline{M}_{\nu,d}\colon (0,1) \to (0,1)$ *and* $\overline{M}_{\nu,d}\colon (0,1) \to (0,1)$ *induced by the generator function* f_d *are*

$$\underline{M}_{\nu,d}(x) = f_d^{-1}\left(\frac{f_d(x)}{f_d^2(\nu)}\right) = 1-(1-x)^{\frac{1}{\ln^2(1-\nu)}}$$
$$\overline{M}_{\nu,d}(x) = f_d^{-1}\left(f_d^2(\nu) f_d(x)\right) = 1-(1-x)^{\ln^2(\nu)}.$$

As f_d *is strictly increasing, by noting Proposition 8, we have that*

(a) If $f_d(\nu) \leq 1$*, which means that* $\nu \leq 1-\mathrm{e}^{-1}$*, then* $\underline{M}_{\nu,d}$ *and* $\overline{M}_{\nu,d}$ *are a dual pair of possibility and necessity operators, respectively; i.e.* $\underline{M}_{\nu,d} = \Diamond_\nu$ *and* $\overline{M}_{\nu,d} = \Box_\nu$

(b) If $f_d(\nu) \geq 1$*, which means that* $\nu \geq 1-\mathrm{e}^{-1}$*, then* $\underline{M}_{\nu,d}$ *and* $\overline{M}_{\nu,d}$ *are a dual pair of necessity and possibility operators, respectively; i.e.* $\underline{M}_{\nu,d} = \Box_\nu$ *and* $\overline{M}_{\nu,d} = \Diamond_\nu$*.*

Example 4. *Now, let* $\nu \in (0,1)$*,* $\alpha \neq 0$ *and let* f *be the generator function of the Dombi conjunction and disjunction operators given in Eq.* (2)*. That is,*

$$f(x) = g_\alpha(x) = \left(\frac{1-x}{x}\right)^\alpha$$

for any $x \in (0,1)$*. Then, after direct calculation, we get that the dual pair of modal operators* $\underline{M}_\nu\colon (0,1) \to (0,1)$ *and* $\overline{M}_\nu\colon (0,1) \to (0,1)$ *induced by the generator function* f *are*

$$\underline{M}_\nu(x) = f^{-1}\left(\frac{f(x)}{f^2(\nu)}\right) = \frac{1}{1+\left(\frac{\nu}{1-\nu}\right)^2 \frac{1-x}{x}} \tag{35}$$

and

$$\overline{M}_\nu(x) = f^{-1}\left(f^2(\nu) f(x)\right) = \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2 \frac{1-x}{x}}. \tag{36}$$

Notice that $\underline{M}_\nu$ *and* $\overline{M}_\nu$ *are independent of* α*.*

(a) If $\alpha > 0$, then f is strictly decreasing and, based on Proposition 8, we have that:

(a1) If $f(\nu) \geq 1$, which means that $\nu \leq 0.5$, then $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of possibility and necessity operators, respectively; i.e. $\underline{M}_\nu = \Diamond_\nu$ and $\overline{M}_\nu = \Box_\nu$

(a2) If $f(\nu) \leq 1$, which means that $\nu \geq 0.5$, then $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of necessity and possibility operators, respectively; i.e. $\underline{M}_\nu = \Box_\nu$ and $\overline{M}_\nu = \Diamond_\nu$.

(b) If $\alpha < 0$, then f is strictly increasing and, by using Proposition 8, we have that:

(b1) If $f(\nu) \leq 1$, which means that $\nu \leq 0.5$, then $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of possibility and necessity operators, respectively; i.e. $\underline{M}_\nu = \Diamond_\nu$ and $\overline{M}_\nu = \Box_\nu$

(b2) If $f(\nu) \geq 1$, which means that $\nu \geq 0.5$, then $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of necessity and possibility operators, respectively; i.e. $\underline{M}_\nu = \Box_\nu$ and $\overline{M}_\nu = \Diamond_\nu$.

Therefore, the findings in (a) and (b) can be summarized as follows:

- *If $\nu \leq 0.5$, then $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of possibility and necessity operators, respectively, i.e. $\underline{M}_\nu = \Diamond_\nu$ and $\overline{M}_\nu = \Box_\nu$*
- *If $\nu \geq 0.5$, then $\underline{M}_\nu$ and $\overline{M}_\nu$ are a dual pair of necessity and possibility operators, respectively, i.e. $\underline{M}_\nu = \Box_\nu$ and $\overline{M}_\nu = \Diamond_\nu$.*

Figure 2 shows example plots of functions $\underline{M}_\nu$ and $\overline{M}_\nu$.

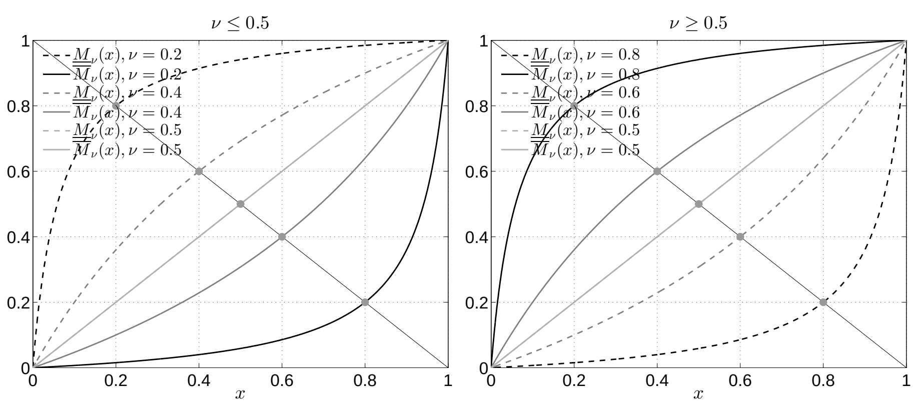

Figure 2: Dual pairs of modal operators induced by compositions of two Pliant negations with various ν parameter values when f is the generator function of Dombi operators.

Note that if $\nu = 0.5$, then $\underline{M}_\nu(x) = \overline{M}_\nu(x) = x$ for any $x \in (0, 1)$.

It should be added that

$$\underline{M}_\nu(\nu) = 1 - \nu \quad \text{and} \quad \overline{M}_\nu(1 - \nu) = \nu.$$

This means that when f is the generator function of Dombi operators, then one of the dual modal operators induced by compositions of two Pliant negations intersects the line $1-x$ at $x=\nu$ and the other one intersects the line $1-x$ at $x=1-\nu$.

Remark 9. *We should emphasize that a dual pair of modal operators $\square_\nu$ and $\lozenge_\nu$ induced by compositions of two Pliant negations using the generator function of Dombi operators are very simple and easy-to-use. Namely,*

(a) If $\nu \geq 0.5$, then

$$\square_\nu(x) = \frac{1}{1+\left(\frac{\nu}{1-\nu}\right)^2 \frac{1-x}{x}} \quad \text{and} \quad \lozenge_\nu(x) = \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2 \frac{1-x}{x}}$$

(b) If $\nu \leq 0.5$, then

$$\lozenge_\nu(x) = \frac{1}{1+\left(\frac{\nu}{1-\nu}\right)^2 \frac{1-x}{x}} \quad \text{and} \quad \square_\nu(x) = \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2 \frac{1-x}{x}},$$

where $x, \nu \in (0,1)$.

It follows from Proposition 8 that the modal operators M_{ν_1} and M_{ν_2}, both induced by a composition of two Pliant negations using the generator function of Dombi operators, are a dual pair of modal operators if and only if

$$\nu_1 + \nu_2 = 1.$$

6. Drastic modal operators

In this section, we will describe how the drastic modal operators are connected with the modal operators induced by compositions of two Pliant negations. The drastic modal operators are defined as follows.

Definition 7 (Drastic modal operators)**.** *We say that the functions $\square_1, \lozenge_0 \colon [0,1] \to [0,1]$ are drastic necessity and possibility operators, respectively, if and only if $\square_1$ and $\lozenge_0$ are given by*

$$\square_1(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \neq 1 \end{cases} \quad \text{and} \quad \lozenge_0(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Figure 3 shows the plots of drastic modal operators $\square_1$ and $\lozenge_0$.

Note that $\square_1$ and $\lozenge_0$ are known as the Baas-Monteiro Δ operator and its dual ∇, respectively. Δ and ∇ are definable by an involution and strict negation [2].

The following proposition provides a connection between the drastic negation operators and the drastic modal operators.

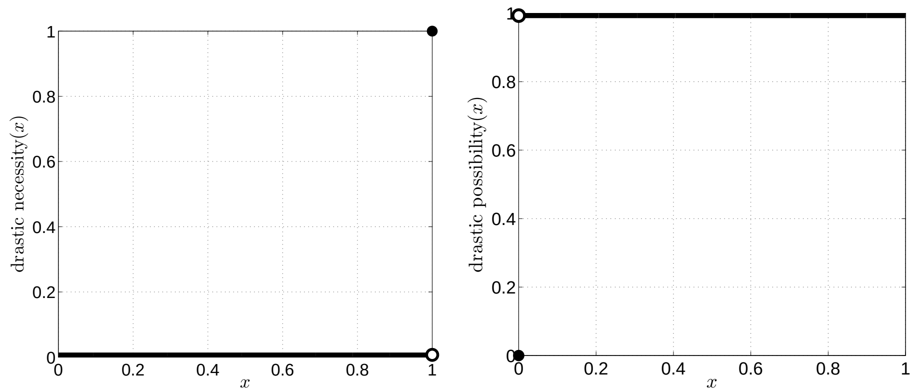

Figure 3: Plots of drastic modal operators.

Proposition 9. *The drastic modal operators* $\Box_1$ *and* $\Diamond_0$ *given in Definition 7 can be written as*

$$\Box_1 = \eta_{d,0} \circ \eta_{d,1} \quad \text{and} \quad \Diamond_0 = \eta_{d,1} \circ \eta_{d,0},$$

where $\eta_{d,0}$ *and* $\eta_{d,1}$ *are the drastic negation operators given by Definition 4.*

Proof. By direct calculation, we have

$$\Box_1(x) = \eta_{d,0}\left(\eta_{d,1}(x)\right) \quad \text{and} \quad \Diamond_0(x) = \eta_{d,1}\left(\eta_{d,0}(x)\right)$$

for any $x \in [0, 1]$. □

Proposition 9 tells us that an appropriate composition of two drastic negations results in a drastic modal operator. Since the drastic negations may be viewed as limits of Pliant negations, we can state the following connection between the Pliant negation operator and the drastic modal operators.

Proposition 10. *The drastic modal operators* $\Box_1$ *and* $\Diamond_0$ *given in Definition 7 can be written as*

$$\Box_1 = \underline{\eta}_\nu \circ \overline{\eta}_\nu \quad \text{and} \quad \Diamond_0 = \overline{\eta}_\nu \circ \underline{\eta}_\nu,$$

where

$$\underline{\eta}_\nu = \lim_{\nu \to 0} \eta_\nu \quad \text{and} \quad \overline{\eta}_\nu = \lim_{\nu \to 1} \eta_\nu,$$

and η_ν *is the Pliant negation operator given by Definition 2.*

Proof. This proposition immediately follows from Proposition 3 and Proposition 9. □

The following proposition tells us that the drastic modal operators may be viewed as limit cases of modal operators that are induced by a composition of two Pliant negations.

Proposition 11. *Let $\nu \in (0,1)$ and let $\underline{M}_\nu \colon [0,1] \to [0,1]$ and $\overline{M}_\nu \colon [0,1] \to [0,1]$ be given by*

$$\underline{M}_\nu(x) = f^{-1}\left(\frac{f(x)}{f^2(\nu)}\right), \quad \overline{M}_\nu(x) = f^{-1}\left(f^2(\nu) f(x)\right).$$

Let $\Box_\nu$ and $\Diamond_\nu$ be a dual pair of necessity and possibility operators, respectively, interpreted according to Remark 8. Then, for the drastic modal operators $\Box_1$ and $\Diamond_0$, which are given by Definition 7, the following are valid:

(a) If f is strictly decreasing, then

$$\lim_{\nu \to 1} \underline{M}_\nu(x) = \lim_{\nu \to 1} \Box_\nu(x) = \Box_1(x)$$

and

$$\lim_{\nu \to 0} \underline{M}_\nu(x) = \lim_{\nu \to 0} \Diamond_\nu(x) = \Diamond_0(x)$$

(b) If f is strictly increasing, then

$$\lim_{\nu \to 1} \overline{M}_\nu(x) = \lim_{\nu \to 1} \Box_\nu(x) = \Box_1(x)$$

and

$$\lim_{\nu \to 0} \overline{M}_\nu(x) = \lim_{\nu \to 0} \Diamond_\nu(x) = \Diamond_0(x)$$

for any $x \in (0,1)$.

Proof. By noting the definitions of $\underline{M}_\nu$ and $\overline{M}_\nu$, and by making use of Proposition 8, Remark 8 and the definitions of the drastic modal operators in Definition 7, the proofs are straightforward. □

Here, we will list some properties of the dual modal operators related to compositions of these operators.

Proposition 12. *Let $\Box_\nu, \Diamond_\nu$ be a dual pair of necessity and possibility operators, respectively, both induced by a composition of two Pliant negations, as described in Proposition 8 and Remark 8, where $\nu \in (0,1)$. Also, let $\Box_1$ and $\Diamond_0$ be a drastic necessity operator and a drastic possibility operator, respectively, given by Definition 7. Then the following are valid:*

(a) $\Box_1 \circ \Diamond_\nu = \Box_1$
(b) $\Box_\nu \circ \Diamond_0 = \Diamond_0$
(c) $\Diamond_0 \circ \Box_\nu = \Diamond_0$
(d) $\Diamond_\nu \circ \Box_1 = \Box_1$
(e) $\Box_1 \circ \Diamond_0 = \Diamond_0$
(f) $\Diamond_0 \circ \Box_1 = \Box_1$.

Proof. By using Proposition 8, Remark 8 and the definitions of the drastic modal operators in Definition 7, the proofs can be obtained via direct calculations. □

Remark 10. *It is worth adding that (b) and (e) in Proposition 12 correspond to $N5$ in Definition 5. Also, (d) and (f) in Proposition 12 correspond to $P5$ in Definition 5.*

7. Distributivity of modal operations over strict t-norms and strict t-conorms

It is a well-known fact that a strict t-norm $c\colon [0,1]^2 \to [0,1]$ has the form

$$c(x,y) = f_c^{-1}\left(f_c(x) + f_c(y)\right), \tag{37}$$

where $f_c\colon [0,1] \to [0,\infty]$ is a strictly decreasing continuous function with $f_c(1) = 0$ and $\lim_{x\to 0} f_c(x) = \infty$. Also, a strict t-conorm $d\colon [0,1]^2 \to [0,1]$ has the form

$$d(x,y) = f_d^{-1}\left(f_d(x) + f_d(y)\right), \tag{38}$$

where $f_d\colon [0,1] \to [0,\infty]$ is a strictly increasing continuous function with $f_d(0) = 0$ and $\lim_{x\to 1} f_d(x) = \infty$. Here, f_c and f_d are called the generator functions of the strict t-norm c and the strict t-conorm d, respectively [22]. Note that the functions f_c and f_d are determined up to a multiplicative constant.

The following theorem concerns the distributivity property of modal operators given in Eq. (5) (or in Eq. (6)) over strict t-norms and strict t-conorms.

Theorem 2 (Distributivity)**.** *Let the modal operator $M\colon [0,1] \to [0,1]$ be given by $M = \eta_1 \circ \eta_2$, where $\eta_1\colon [0,1] \to [0,1]$ and $\eta_2\colon [0,1] \to [0,1]$ are two strong negations such that one of them is stricter than the other. Let $c\colon [0,1]^2 \to [0,1]$ be a strict t-norm with the generator function $f_c\colon [0,1] \to [0,\infty]$ and let $d\colon [0,1]^2 \to [0,1]$ be a strict t-conorm with the generator function $f_d\colon [0,1] \to [0,\infty]$. Then, the modal operator M is distributive over the strict t-norm c and over the strict t-conorm d if and only if*

$$f_c(x) f_d(x) = 1 \tag{39}$$

for any $x \in [0,1]$.

Proof. The distributivity of M over c and d means that

$$M\left(c(x,y)\right) = c\left(M(x), M(y)\right) \tag{40}$$

and

$$M\left(d(x,y)\right) = d\left(M(x), M(y)\right) \tag{41}$$

hold for any $(x,y) \in [0,1]^2$.

Let $M_1 = \eta_{1,1} \circ \eta_{1,2}$ and $M_2 = \eta_{2,1} \circ \eta_{2,2}$ such that $M_1(x) \neq M_2(x)$ for any $x \in (0,1)$, where $\eta_{1,1}\colon [0,1] \to [0,1]$ and $\eta_{1,2}\colon [0,1] \to [0,1]$ are two strong negations such that one of them is stricter than the other; and $\eta_{2,1}\colon [0,1] \to [0,1]$ and $\eta_{2,2}\colon [0,1] \to [0,1]$ are also two strong negations such that one of them is stricter than the other. Based on Theorem 1, M_1 and M_2 are two modal operators, and so, they are two strictly increasing bijective functions.

Let $c_1(x,y)$ and $c_2(x,y)$ be given by

$$c_1(x,y) = M_1^{-1}\left(c\left(M_1(x), M_1(y)\right)\right) \tag{42}$$

$$c_2(x,y) = M_2^{-1}\left(c\left(M_2(x), M_2(y)\right)\right), \tag{43}$$

where $(x,y) \in [0,1]^2$. Using the definitions of M_1 and M_2, we have that

$$M_1^{-1}(x) = \eta_{1,2}\left(\eta_{1,1}(x)\right) \quad \text{and} \quad M_2^{-1}(x) = \eta_{2,2}\left(\eta_{2,1}(x)\right)$$

for any $x \in [0,1]$, and noting the fact that c is a strict t-norm, $c_1(x,y)$ and $c_2(x,y)$ can be written as

$$c_1(x,y) = M_1^{-1}\left(f_c^{-1}\left(f_c\left(M_1(x)\right) + f_c\left(M_1(y)\right)\right)\right)$$

$$c_2(x,y) = M_2^{-1}\left(f_c^{-1}\left(f_c\left(M_2(x)\right) + f_c\left(M_2(y)\right)\right)\right).$$

Let x' and y' be given by

$$x' = f_c\left(M_1(x)\right) \quad \text{and} \quad y' = f_c\left(M_1(y)\right),$$

where $x, y \in [0,1]$. From these two equations we have

$$\begin{aligned} f_c\left(M_2(x)\right) &= f_c\left(M_2\left(M_1^{-1}\left(f_c^{-1}(x')\right)\right)\right) \\ f_c\left(M_2(y)\right) &= f_c\left(M_2\left(M_1^{-1}\left(f_c^{-1}(y')\right)\right)\right). \end{aligned} \tag{44}$$

Suppose that

$$c_1(x,y) = c_2(x,y),$$

from which we also have

$$f_c\left(M_2\left(c_1(x,y)\right)\right) = f_c\left(M_2\left(c_2(x,y)\right)\right). \tag{45}$$

Using Eq. (42), Eq. (43) and Eq. (44), we have

$$f_c\left(M_2\left(c_1(x,y)\right)\right) = f_c\left(M_2\left(M_1^{-1}\left(f_c^{-1}(x'+y')\right)\right)\right) \tag{46}$$

and

$$\begin{aligned} &f_c\left(M_2\left(c_2(x,y)\right)\right) = \\ &= f_c\left(M_2\left(M_1^{-1}\left(f_c^{-1}(x')\right)\right)\right) + f_c\left(M_2\left(M_1^{-1}\left(f_c^{-1}(y')\right)\right)\right). \end{aligned} \tag{47}$$

Next, noting Eq. (46) and Eq. (47), Eq. (45) can be written as

$$f_c\left(t\left(f_c^{-1}(x'+y')\right)\right) = f_c\left(t\left(f_c^{-1}(x')\right)\right) + f_c\left(t\left(f_c^{-1}(y')\right)\right), \tag{48}$$

where the function $t\colon [0,1] \to [0,1]$ is given by $t(x) = M_2\left(M_1^{-1}(x)\right)$. Based on the properties of M_1 and M_2, function t is strictly increasing, $t(0) = 0$ and $t(1) = 1$. Also, $t(x) \neq x$ for any $x \in (0,1)$. This is because if $t(x) = x$ held for at least one $x \in (0,1)$, then we would have $M_2\left(M_1^{-1}(x)\right) = x$, from which $M_1(x) = M_2(x)$, and it would contradict the assumption that $M_1(x) \neq M_2(x)$ for any $x \in (0,1)$.
Now, let the function $F_c\colon [0,\infty] \to [0,\infty]$ be given by

$$F_c(x) = f_c\left(t\left(f_c^{-1}(x)\right)\right). \tag{49}$$

Then, Eq. (48) has the form

$$F_c(x+y) = F_c(x) + F_c(y), \tag{50}$$

which is the well-known Cauchy functional equation. The solution of the functional equation in Eq. (50) is

$$F_c(x) = a_c x,$$

where a_c is a constant with an arbitrary value. Therefore, by noting Eq. (49), we have

$$f_c\left(t\left(f_c^{-1}(x)\right)\right) = a_c x, \tag{51}$$

where $x \in [0, \infty]$, and by applying the substitution $x = f_c(z)$ we get

$$f_c\left(t(z)\right) = a_c f_c(z), \tag{52}$$

where $z \in [0, 1]$.
Similar considerations lead to

$$f_d\left(t(z)\right) = a_d f_d(z), \tag{53}$$

where $z \in [0, 1]$ and a_d is a constant with an arbitrary value. Suppose that $a_c, a_d \neq 0$. Now, multiplying Eq. (52) by Eq. (53) and letting $f_c(x)f_d(x) = g(x)$, where $g\colon [0, 1] \to [0, 1]$, we have

$$g\left(t(x)\right) = a_c a_d g(x), \tag{54}$$

where $x \in [0, 1]$. Since t is a strictly increasing function, $t(x) \neq x$ for any $x \in (0, 1)$, and a_c and a_d are constants, the solution of Eq. (54) is

$$g(x) = k,$$

where k is a constant and $a_d = \frac{1}{a_c}$. Noting that $g(x) = f_c(x)f_d(x)$, we have

$$f_c(x)f_d(x) = k$$

for any $x \in [0, 1]$, and since the generator function is determined up to a multiplicative constant, we can get Eq. (39). □

8. Modal operators and hedges

A linguistic hedge or modifier is a unary operation that changes the meaning of a linguistic term (see [23–25]). Let X be the domain of discourse and let A be a continuous linguistic term for the input variable $x \in X$ with the membership function $\mu_A\colon X \to [0, 1]$. Then A^s, which is given by the membership function

$$\mu_{A^s}(x) = \left(\mu_A(x)\right)^p \tag{55}$$

for any $x \in X$, is interpreted as a modified version of A, where $p > 0$. Here, p denotes the linguistic hedge value.

In Example 3, we showed that the modal operators induced by compositions of two Pliant negations using the generator function $f_c(x) = -\ln(x)$ of the probabilistic strict t-norms are

$$\underline{M}_{\nu,c}(x) = x^{\frac{1}{\ln^2(\nu)}} \quad \text{and} \quad \overline{M}_{\nu,c}(x) = x^{\ln^2(\nu)},$$

where $x, \nu \in (0, 1)$. We can readily see that applying these modal operators to the membership function μ_A, we get

$$\underline{M}_{\nu,c}(\mu_A(x)) = (\mu_A(x))^{\frac{1}{\ln^2(\nu)}} \quad \text{and} \quad \overline{M}_{\nu,c}(\mu_A(x)) = (\mu_A(x))^{\ln^2(\nu)},$$

which, with $p = \frac{1}{\ln^2(\nu)}$ and $p = \ln^2(\nu)$, respectively, have the form of a hedge given by Eq. (55).

The following example demonstrates that the application of a modal operator, which is induced by a composition of two Pliant negations using the generator function of Dombi operators, to the membership function of a fuzzy set may be viewed as a hedge as well.

Example 5. *Let the membership function of the fuzzy set "tall person" be given by the sigmoid function $\sigma_a^{(\lambda)}\colon (-\infty, \infty) \to (0, 1)$:*

$$\sigma_a^{(\lambda)}(x) = \frac{1}{1 + \mathrm{e}^{-\lambda(x-a)}},$$

where $\lambda = 0.5$ and $a = 180$. Here, x is in centimeters and $\sigma_a^{(\lambda)}(x)$ represents the truth value of the soft inequality $x > 180$. If someone is much taller than 180cm, then he or she has a high membership value in the fuzzy set "tall person"; and conversely, if a person is much shorter than 180cm, then this person has a low membership value in the fuzzy set "tall person".

Let $\nu \in [0.5, 1)$ and let $\Box_\nu$ and $\Diamond_\nu$ be a dual pair of modal operators induced by compositions of two Pliant negations using the generator function of Dombi operators (see Example 4). That is, $\Box_\nu$ and $\Diamond_\nu$ given by Eq. (35) and Eq. (36) are as follows:

$$\Box_\nu(x) = \frac{1}{1 + \left(\frac{\nu}{1-\nu}\right)^2 \frac{1-x}{x}} \quad \text{and} \quad \Diamond_\nu(x) = \frac{1}{1 + \left(\frac{1-\nu}{\nu}\right)^2 \frac{1-x}{x}}.$$

By applying $\Box_\nu$ and $\Diamond_\nu$ to $\sigma_a^{(\lambda)}$, we get the membership functions $\Box_\nu \circ \sigma_a^{(\lambda)}$ and $\Diamond_\nu \circ \sigma_a^{(\lambda)}$ of the fuzzy sets "necessarily tall person" and "possibly tall person", respectively. After a direct calculation, we get

$$\Box_\nu\left(\sigma_a^{(\lambda)}(x)\right) = \frac{1}{1 + \mathrm{e}^{-\lambda\left(x - \left(a + \frac{2}{\lambda}\ln\left(\frac{\nu}{1-\nu}\right)\right)\right)}} = \sigma^{(\lambda)}_{a+\frac{2}{\lambda}\ln\left(\frac{\nu}{1-\nu}\right)}(x)$$

and

$$\Diamond_\nu\left(\sigma_a^{(\lambda)}(x)\right) = \frac{1}{1 + \mathrm{e}^{-\lambda\left(x - \left(a + \frac{2}{\lambda}\ln\left(\frac{1-\nu}{\nu}\right)\right)\right)}} = \sigma^{(\lambda)}_{a+\frac{2}{\lambda}\ln\left(\frac{1-\nu}{\nu}\right)}(x).$$

We can see that both $\Box_\nu \circ \sigma_a^{(\lambda)}$ and $\Diamond_\nu \circ \sigma_a^{(\lambda)}$ are sigmoid functions. It means that the modal operators $\Box_\nu$ and $\Diamond_\nu$ shift the membership function $\sigma_a^{(\lambda)}$ along the horizontal axis upwards and downwards, respectively.

Figure 4 shows typical plots of the membership functions of fuzzy sets "necessarily tall person" and "possibly tall person", which have been derived by applying the modal operators $\Box_\nu$ and $\Diamond_\nu$, respectively, to the membership functions of fuzzy set "tall person".

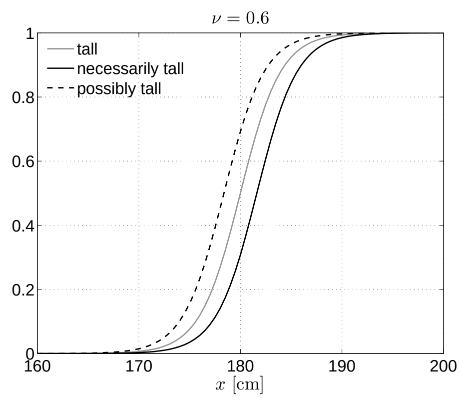

Figure 4: Effects of applications of modal operators to the sigmoid membership function of fuzzy set "tall person".

Remark 11. *We can see that in Example 5, the modal operators $\Box_\nu$ and $\Diamond_\nu$ play the role of modifier operators. Even though these operators are not power functions, and so they do not have the form of a traditional hedge given in Eq.* (55)*, their application to the membership function of a fuzzy set results in fuzzy hedges.*

8.1. A note on distributivity

Here, we would like to highlight the importance of distributivity. For example, let the truth values of the statements "x is young." and "x is tall." be given by the membership function values $\mu_y(x)$ and $\mu_t(x)$, respectively. Then, $c(\mu_y(x), \mu_t(x))$ can be interpreted as the truth value of the statement "x is young and tall.". Now, by applying the possibility operator $\Diamond$ to $c(\mu_y(x), \mu_t(x))$, we get $\Diamond(c(\mu_y(x), \mu_t(x)))$, which can be interpreted as the truth value of the statement "It is possible that x is young and tall.". Next, by assuming the distributivity of $\Diamond$ over c, we have $c(\Diamond(\mu_y(x)), \Diamond(\mu_t(x)))$, which can be interpreted as the truth value of the statement "x is possibly young and possibly tall.". Notice that in the first case, the modal operator $\Diamond$ is applied to a connective, while in the second case, $\Diamond$ is applied to two continuous valued logical statements. This means that in the second case, the modal operator modifies the truth values of two statements, and then these modified values are connected by a conjunction operation. Therefore, $\Diamond$ may be viewed as a linguistic hedge, and the distributivity of $\Diamond$ over c means that the linguistic terms "It is possible that x is young and tall." and "x is possibly young and possibly tall." have the same truth values. That is, owing to the distributivity, we can express the same meaning by using two different, but logically equivalent linguistic terms.

Remark 12. *Let $\alpha > 0$ and let g_α be the generator function of the Dombi operators given in Eq. (2). Then, $f_c = g_\alpha$ is the generator function of a strict t-norm and $f_d = g_{-\alpha}$ is the generator function of a strict t-conorm. Since*

$$f_c(x) f_d(x) = g_\alpha(x) g_{-\alpha}(x) = 1$$

for any $x \in (0,1)$, based on Theorem 2, the modal operators induced by compositions of two Pliant negations using the generator function of Dombi operators are distributive over the strict t-norm and strict t-conorm induced by the generator functions f_c and f_d, respectively.

9. Conclusions

The main findings of our study can be summarized as follows.

(a) In this study, we interpreted a dual pair of modal operators following the criteria for an algebraic version of necessity and possibility operators on De Morgan lattices given by Cattaneo, Ciucci and Dubois [7] (also, see [15]).
(b) Here, we provided a representation theorem, which demonstrates that, in our algebraic model, a dual pair of modal operators can be represented by compositions of two strong negations, where one of them is stricter than the other.
(c) Also, we used the Pliant negation operator to derive dual modal operators in a very simple way.
(d) Next, we showed that by using the generator function of Dombi operators, the composition of two Pliant negations results in modal operators that have simple forms and easy-to-use characteristics.
(e) Here, we described how the proposed modal operators are connected with the drastic necessity and possibility operators.
(f) Also, we presented the necessary and sufficient condition for the distributivity of modal operators induced by compositions of two strong negations over strict t-norms and strict t-conorms.
(g) Lastly, we highlighted a connection between the modal operators and hedges.

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