Saddle-Node Bifurcation of Periodic Orbits for a Delay Differential Equation

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Abstract

We consider the scalar delay differential equation

$$\dot{x}(t) = -x(t) + f_K(x(t-1))$$

with a nondecreasing feedback function f_K depending on a parameter K, and we verify that a saddle-node bifurcation of periodic orbits takes place as K varies.

The nonlinearity f_K is chosen so that it has two unstable fixed points (hence the dynamical system has two unstable equilibria), and these fixed points remain bounded away from each other as K changes. The generated periodic orbits are of large amplitude in the sense that they oscillate about both unstable fixed points of f_K.

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1. Introduction

Numerous scientific works have studied the existence and the bifurcation of periodic orbits for delay differential equations, see the books [1, 2, 3] and the survey paper [4] of Walther. In paper [5], Krisztin gives a detailed summary on known results for equations of the special form

$$\dot{x}(t) = -x(t) + f_K(x(t-1)), \tag{1.1}$$

where f_K is monotone nonlinearity. Hopf-bifurcation is a widely studied phenomenon [4]. A well-known example is due to Krisztin, Walther and Wu: periodic orbits of (1.1) arise via a series of Hopf-bifurcations for strictly monotone increasing nonlinearities, e.g., for $f_K(x) = K\tanh(x)$ or for $f_K(x) = K\tan^{-1}(x)$

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as K increases, see [6, 7, 8]. Other types of bifurcations involving periodic orbits are rarely studied. An interesting example is given by Walther in [9]: he studies a delay equation coming from a prize model, and he shows the bifurcation of periodic orbits with small amplitudes and with periods descending from infinity. To the best of the authors' knowledge, no one has verified saddle-node bifurcation of periodic orbits for equation (1.1). López Nieto has an analogous result for another class of delay differential equations [10].

In this paper we consider (1.1) in the so-called positive feedback case; f_K is supposed to be a nondecreasing continuous function such that

$$f_K|_{(-\infty,-1-\varepsilon]} = -K, \quad f_K|_{[-1,1]} = 0 \quad \text{and} \quad f_K|_{[1+\varepsilon,\infty)} = K,$$

where ε is a fixed positive number and K is the bifurcation parameter. For technical simplicity, we define f_K to be a piecewise linear continuous function:

$$f_K(x) = \frac{K}{\varepsilon}(x+1) \quad \text{for } x \in (-1-\varepsilon,-1),$$

and

$$f_K(x) = \frac{K}{\varepsilon}(x-1) \quad \text{for } x \in (1,1+\varepsilon),$$

see Fig. 1.1.

The results of the paper are expected to hold if the nondecreasing continuous function f_K is defined differently on $(-1-\varepsilon,-1)\cup(1,1+\varepsilon)$, or if the coefficient of the linear term on the right hand side of (1.1) is $-\mu$ with $\mu > 0$.

The phase space for (1.1) is the Banach space $C = C([-1,0],\mathbb{R})$ with the maximum norm. If for some $t \in \mathbb{R}$, the interval $[t-1,t]$ is in the domain of a continuous function x, then the segment $x_t \in C$ is defined by $x_t(s) = x(t+s)$ for $-1 \leq s \leq 0$.

If $K > 1+\varepsilon$, then f_K has two fixed points $\chi_- \in (-1-\varepsilon,-1)$ and $\chi_+ \in (1,1+\varepsilon)$ with $f_K'(\chi_-) > 1$ and $f_K'(\chi_+) > 1$. Thus the constant elements

$$[-1,0] \ni s \mapsto \chi_- \in \mathbb{R} \text{ and } [-1,0] \ni s \mapsto \chi_+ \in \mathbb{R}$$

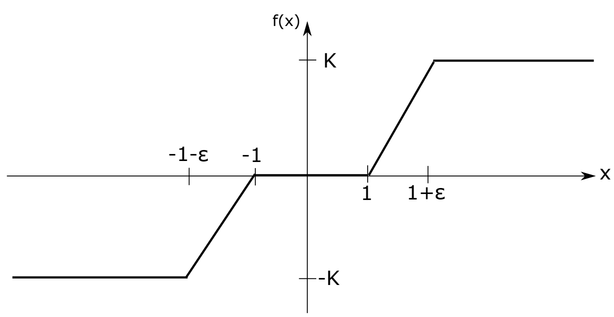

Figure 1.1: The plot of f_K.

of C are unstable equilibria. We know that there exist periodic solutions oscillating about either χ_- or χ_+ if K is sufficiently large, and these periodic orbits appear via Hopf-bifurcations [11].

We say that a periodic solution has large amplitude if it oscillates about both χ_- and χ_+. The corresponding orbit is a large-amplitude periodic orbit. The existence of a pair of large-amplitude periodic orbits has been first shown in [12] for a similar nonlinearity f_K with K large enough. More complicated configurations of such periodic orbits has appeared in [13]. A third work in this topic, the paper [14] has described the complicated geometric structure of the unstable set of a large-amplitude periodic orbit in detail. These works have not explained how these periodic orbits bifurcate as the parameter K changes. Apparently they cannot appear via Hopf bifurcation in a neighborhood of an unstable equilibrium. In this paper we verify that for the nonlinearity f_K defined above, large-amplitude periodic orbits arise via a saddle-node bifurcation. The following theorem has already appeared in [12] as a conjecture.

Theorem 1.1. *(Saddle-node bifurcation of periodic orbits) For all sufficiently small positive ε, one can give a threshold parameter $K^* = K^*(\varepsilon) \in (6.5, 7)$, a large-amplitude periodic solution $p = p(\varepsilon) : \mathbb{R} \to \mathbb{R}$ of (1.1) for parameter $K = K^*$, an open neighborhood $B = B(\varepsilon)$ of its initial segment p_0 in C, and a constant $\delta = \delta(\varepsilon) > 0$ such that*

(i) if $K \in (K^* - \delta, K^*)$, then no periodic orbit for (1.1) has segments in B;

(ii) if $K = K^*$, then $\mathcal{O} = \{p_t : t \in \mathbb{R}\}$ is the only periodic orbit with segments in B;

(iii) if $K \in (K^*, K^* + \delta)$, then there are exactly two periodic orbits with segments in B, and both of them are of large-amplitude.

Let K_0 be that solution of equation

$$(K-1)(K+1)^3 = e\left(K^2 - 2K - 1\right)^2 \tag{1.2}$$

that belongs to the interval $(6.5, 7)$. It is easy to show that K_0 is unique, see Section 3 of [12]. Numerical computation shows that $K_0 \approx 6.87$. We will see that the limit of the bifurcation parameter $K^*(\varepsilon)$ is K_0 as $\varepsilon \to 0^+$.

The proof is organized as follows. Let $\varepsilon \in (0, 1)$. In Section 2 we introduce a one-dimensional map F depending also on parameters K and ε. In Section 3 we show that the fixed points of $F(\cdot, K, \varepsilon)$ determine large-amplitude periodic solutions for equation (1.1). Then we show in Section 4 that F undergoes a saddle-node bifurcation as K varies if ε is a fixed and sufficiently small positive number. We also need to show that – locally – all periodic solutions can be obtained as fixed points of $F(\cdot, K, \varepsilon)$. This is done Section 5. Theorem 1.1 immediately follows from these results, see Section 6. The Appendix contains certain lengthy but straightforward calculations used in Section 4.

In the saddle-node bifurcation of F, a neutral fixed point splits into two fixed points, one attracting and one repelling. This does not imply that we have one stable and one unstable periodic orbit for $K > K^*$. We know that if f_K is a C^1-function with nonnegative derivative, then all periodic orbits are unstable, see e.g., Proposition 7.1 in [13]. Hence we presume that the periodic orbits given by the above theorem are also unstable.

In the previous paper [12], we obtained large-amplitude periodic solutions also as fixed points of finite dimensional maps. We emphasize that here we construct F in a different way. The current approach is simpler because it yields shorter calculations. The advantage of the construction used in [12] is the following: the eigenvalues of the derivatives of the finite dimensional maps in [12] at the fixed points coincide with the Floquet multipliers of the corresponding periodic orbits. Hence those finite dimensional maps give precise information on the stability properties of the periodic orbits. This is not true here.

The reader may find other examples, in which the existence of a periodic orbit is shown by handling a finite dimensional fixed point problem, in the papers [15, 16, 17].

2. The map F

Let $\varepsilon \in (0,1)$ and $K \in (6.5, 7)$. In this section we define a periodic function p as the concatenation of certain auxiliary functions $y_1, y_2, ..., y_{10}$ such that if p is a solution of the delay equation (1.1), then $y_1, y_2, ..., y_{10}$ satisfy a system of ordinary differential equations with boundary conditions. Then we reduce this ODE system to a single fixed point equation of the form $F(L_2, K, \varepsilon) = L_2$, where L_2 is a parameter corresponding to p.

Assume that

(H1) $L_i > 0$ for $i \in \{1, 2, ..., 5\}$,

(H2) $2L_1 + 5L_2 + 5L_3 + 3L_4 + 3L_5 = 1$,

(H3) $\theta_i > 1 + \varepsilon$ for $i \in \{1, 2, 3, 4\}$, and $\theta_i \in (1, 1 + \varepsilon)$ for $i \in \{5, 6\}$.

Consider the subsequent continuous functions:

(H4) $y_1 \in C([0, L_1], \mathbb{R})$ with $y_1(0) = 1 + \varepsilon$ and $y_1(L_1) = \theta_1$,
$y_2 \in C([0, L_2], \mathbb{R})$ with $y_2(0) = \theta_1$ and $y_2(L_2) = \theta_2$,
$y_3 \in C([0, L_3], \mathbb{R})$ with $y_3(0) = \theta_2$ and $y_3(L_3) = \theta_3$,
$y_4 \in C([0, L_4], \mathbb{R})$ with $y_4(0) = \theta_3$ and $y_4(L_4) = \theta_4$,
$y_5 \in C([0, L_5], \mathbb{R})$ with $y_5(0) = \theta_4$ and $y_5(L_5) = 1 + \varepsilon$,
$y_6 \in C([0, L_2], \mathbb{R})$ with $y_6(0) = 1 + \varepsilon$ and $y_6(L_2) = \theta_5$,
$y_7 \in C([0, L_3], \mathbb{R})$ with $y_7(0) = \theta_5$ and $y_7(L_3) = \theta_6$,
$y_8 \in C([0, L_4], \mathbb{R})$ with $y_8(0) = \theta_6$ and $y_8(L_4) = 1$,
$y_9 \in C([0, L_2 + L_5], \mathbb{R})$ with $y_9(0) = 1$ and $y_9(L_2 + L_5) = -1$,
$y_{10} \in C([0, L_3], \mathbb{R})$ with $y_{10}(0) = -1$ and $y_{10}(L_3) = -1 - \varepsilon$,

(H5) if $i \in \{1, 2, ..., 5\}$, then $y_i(s) > 1 + \varepsilon$ for all s in the interior of the domain of y_i,
if $i \in \{6, 7, 8\}$, then $y_i(s) \in (1, 1 + \varepsilon)$ for all s in the interior of the domain of y_i,
$y_9(s) \in (-1, 1)$ for all $s \in (0, L_2 + L_5)$,
$y_{10}(s) \in (-1 - \varepsilon, -1)$ for all $s \in (0, L_3)$.

Fig. 2.1 plots certain horizontal translations of $y_1, ..., y_{10}$.

Set $0 < \tau_1 < \tau_2 < \tau_3 < \omega < 1$ as

$$\begin{aligned} \tau_1 &= \sum_{i=1}^{5} L_i, \\ \tau_2 &= \tau_1 + L_2 + L_3 + L_4, \\ \tau_3 &= \tau_2 + L_2 + L_5, \\ \omega &= \tau_3 + L_3. \end{aligned} \tag{2.1}$$

Introduce a 2ω-periodic function $p : \mathbb{R} \to \mathbb{R}$ as follows. Set p on $[-1, -1+\omega]$ such that

$$\begin{aligned} p(t-1) &= y_1(t) \quad \text{for } t \in [0, L_1], \\ p(t-1+L_1) &= y_2(t) \quad \text{for } t \in [0, L_2], \\ p(t-1+L_1+L_2) &= y_3(t) \quad \text{for } t \in [0, L_3], \\ p(t-1+L_1+L_2+L_3) &= y_4(t) \quad \text{for } t \in [0, L_4], \\ p(t-1+L_1+L_2+L_3+L_4) &= y_5(t) \quad \text{for } t \in [0, L_5], \\ p(t-1+\tau_1) &= y_6(t) \quad \text{for } t \in [0, L_2], \\ p(t-1+\tau_1+L_2) &= y_7(t) \quad \text{for } t \in [0, L_3], \\ p(t-1+\tau_1+L_2+L_3) &= y_8(t) \quad \text{for } t \in [0, L_4], \\ p(t-1+\tau_2) &= y_9(t) \quad \text{for } t \in [0, L_2+L_5], \\ p(t-1+\tau_3) &= y_{10}(t) \quad \text{for } t \in [0, L_3]. \end{aligned} \tag{P.1}$$

Let

$$p(t) = -p(t-\omega) \quad \text{for all } t \in [-1+\omega, -1+2\omega]. \tag{P.2}$$

Then extend p to the real line 2ω-periodically. See Fig. 2.1 for the plot of p on $[-1, 1]$. It is clear that p is of large amplitude.

Our first goal is to find what conditions hold for $L_1, ..., L_5, \theta_1, ..., \theta_6$ and $y_1, ..., y_{10}$ if p satisfies equation (1.1) for all $t \in \mathbb{R}$. As $p(t) = -p(t-\omega)$ for all real t and f_K is odd, we do not lose information if we restrict our examinations to the interval $[0, \omega]$. So consider

$$\dot{p}(t) = -p(t) + f_K(p(t-1)) \quad \text{for } t \in [0, \omega]. \tag{2.2}$$

We study (2.2) first on the interval $[0, \tau_1]$, then on $[\tau_1, \tau_2]$, $[\tau_2, \tau_3]$ and $[\tau_3, \omega]$.

1. The interval $[0, \tau_1]$. The way we extended p from $[-1, -1+\omega]$ to $\mathbb{R}$ and condition (H2) together imply that

$$\begin{aligned} p(t) &= -y_8(t) \quad \text{for } t \in [0, L_4], \\ p(t+L_4) &= -y_9(t) \quad \text{for } t \in [0, L_2+L_5], \\ p(t+L_4+L_2+L_5) &= -y_{10}(t) \quad \text{for } t \in [0, L_3], \\ p(t+L_4+L_2+L_5+L_3) &= y_1(t) \quad \text{for } t \in [0, L_1], \end{aligned}$$

see Fig. 2.1. Also observe – using (P.1) and (H3)-(H5) – that

$$p(t) \geq 1+\varepsilon \quad \text{for } t \in [-1, -1+\tau_1],$$

and thus (2.2) is in the form $\dot{p}\,(t) = -p\,(t) + K$ on $[0, \tau_1]$. We conclude that (2.2) holds for $t \in [0, \tau_1]$ if and only if the subsequent four equations are satisfied:

$$\dot{y}_8(t) = -\,y_8(t) - K, \quad t \in [0, L_4], \tag{2.3}$$

$$\dot{y}_9(t) = -\,y_9(t) - K, \quad t \in [0, L_2 + L_5], \tag{2.4}$$

$$\dot{y}_{10}(t) = -\,y_{10}(t) - K, \quad t \in [0, L_3], \tag{2.5}$$

$$\dot{y}_1(t) = -\,y_1(t) + K, \quad t \in [0, L_1]. \tag{2.6}$$

2. The interval $[\tau_1, \tau_2]$. By the definition of p and hypothesis (H2),

$$p(t+\tau_1) = y_2(t) \quad \text{for } t \in [0, L_2],$$
$$p(t+\tau_1+L_2) = y_3(t) \quad \text{for } t \in [0, L_3]$$

and

$$p(t+\tau_1+L_2+L_3) = y_4(t) \quad \text{for } t \in [0, L_4].$$

We also know from (P.1) that

$$p(t-1+\tau_1) = y_6(t) \quad \text{for } t \in [0, L_2],$$
$$p(t-1+\tau_1+L_2) = y_7(t) \quad \text{for } t \in [0, L_3],$$
$$p(t-1+\tau_1+L_2+L_3) = y_8(t) \quad \text{for } t \in [0, L_4].$$

Hypotheses (H3)-(H5) then guarantee that

$$p(t) \in [1, 1+\varepsilon] \quad \text{for } t \in [-1+\tau_1, -1+\tau_2].$$

Using the definition of f_K, we obtain that (2.2) holds on $[\tau_1, \tau_2]$ if and only if

$$\dot{y}_2(t) = -\,y_2(t) + \frac{K}{\varepsilon}(y_6(t) - 1) \quad \text{for } t \in [0, L_2], \tag{2.7}$$

$$\dot{y}_3(t) = -\,y_3(t) + \frac{K}{\varepsilon}(y_7(t) - 1) \quad \text{for } t \in [0, L_3] \tag{2.8}$$

and

$$\dot{y}_4(t) = -y_4(t) + \frac{K}{\varepsilon}(y_8(t) - 1) \quad \text{for } t \in [0, L_4]. \tag{2.9}$$

3. The interval $[\tau_2, \tau_3]$. Next observe that

$$p(t+\tau_2) = y_5(t) \quad \text{for } t \in [0, L_5],$$
$$p(t+\tau_2+L_5) = y_6(t) \quad \text{for } t \in [0, L_2]$$

and

$$p(t) \in [-1, 1] \quad \text{for } t \in [-1+\tau_2, -1+\tau_3].$$

Therefore

$$\dot{y}_5(t) = -y_5(t) \quad \text{for } t \in [0, L_5] \tag{2.10}$$

and

$$\dot{y}_6(t) = -y_6(t) \quad \text{for } t \in [0, L_2]. \tag{2.11}$$

4. The interval $[\tau_3, \omega]$. At least observe that

$$p(t + \tau_3) = y_7(t) \quad \text{for } t \in [0, L_3],$$
$$p(t - 1 + \tau_3) = y_{10}(t) \quad \text{for } t \in [0, L_3],$$

and

$$p(t) \in [-1 - \varepsilon, -1] \quad \text{for } t \in [-1 + \tau_3, -1 + \omega].$$

So on the interval $[\tau_3, \omega]$, equation (2.2) is equivalent to

$$\dot{y}_7(t) = -y_7(t) + \frac{K}{\varepsilon}(y_{10}(t) + 1), \quad t \in [0, L_3]. \tag{2.12}$$

We see that under hypotheses (H1)-(H5), equation (2.2) is equivalent to a system of linear ordinary differential equations. It worth solving equations (2.3)-(2.6) and (2.10)-(2.11) first because they are independent from the other ones. Then we can solve (2.7), (2.9) and (2.12) using the solutions of (2.11), (2.3) and (2.5), respectively. At last, using the solution of (2.12), we can find the solution of (2.8). Applying the boundary conditions given by (H4) for $t = 0$, we obtain that

$$y_1(t) = K - (K - 1 - \varepsilon)e^{-t}, \ t \in [0, L_1], \tag{Y.1}$$

$$y_2(t) = \theta_1 e^{-t} + \frac{K}{\varepsilon}\left((1+\varepsilon)te^{-t} + e^{-t} - 1\right), \ t \in [0, L_2], \tag{Y.2}$$

$$y_3(t) = \theta_2 e^{-t} + \frac{K}{\varepsilon}\left((\theta_5 t + 1)e^{-t} - 1\right) - \frac{K^2}{\varepsilon^2}(K-1)\left(1 - \left(1 + t + \frac{t^2}{2}\right)e^{-t}\right), \ t \in [0, L_3], \tag{Y.3}$$

$$y_4(t) = \theta_3 e^{-t} + \frac{K}{\varepsilon}\left((K + \theta_6)te^{-t} - (K+1)\left(1 - e^{-t}\right)\right), \ t \in [0, L_4], \tag{Y.4}$$

$$y_5(t) = \theta_4 e^{-t}, \ t \in [0, L_5], \tag{Y.5}$$

$$y_6(t) = (1+\varepsilon)e^{-t}, \ t \in [0, L_2], \tag{Y.6}$$

$$y_7(t) = \theta_5 e^{-t} - \frac{K}{\varepsilon}(K-1)\left(1 - (1+t)e^{-t}\right), \ t \in [0, L_3], \tag{Y.7}$$

$$y_8(t) = (K + \theta_6)e^{-t} - K, \ t \in [0, L_4], \tag{Y.8}$$

$$y_9(t) = (K+1)e^{-t} - K, \ t \in [0, L_2 + L_5], \tag{Y.9}$$

$$y_{10}(t) = (K-1)e^{-t} - K, \ t \in [0, L_3]. \tag{Y.10}$$

If we apply the boundary conditions given for the right end points of the domains of y_i, $i \in \{1, ..., 10\}$, then we get the following relations:

$$\theta_1 = K - (K - 1 - \varepsilon)e^{-L_1}, \tag{B.1}$$

$$\theta_2 = \theta_1 e^{-L_2} + \frac{K}{\varepsilon}\left((1+\varepsilon)L_2 e^{-L_2} + e^{-L_2} - 1\right), \tag{B.2}$$

$$\theta_3 = \theta_2 e^{-L_3} + \frac{K}{\varepsilon}\left((\theta_5 L_3 + 1)e^{-L_3} - 1\right) - \frac{K^2}{\varepsilon^2}(K-1)\left(1 - \left(1 + L_3 + \frac{L_3^2}{2}\right)e^{-L_3}\right), \tag{B.3}$$

$$\theta_4 = \theta_3 e^{-L_4} + \frac{K}{\varepsilon}\left((K+\theta_6)L_4 e^{-L_4} - (K+1)\left(1 - e^{-L_4}\right)\right), \tag{B.4}$$

$$1 + \varepsilon = \theta_4 e^{-L_5}, \tag{B.5}$$

$$\theta_5 = (1+\varepsilon)e^{-L_2}, \tag{B.6}$$

$$\theta_6 = \theta_5 e^{-L_3} - \frac{K}{\varepsilon}(K-1)\left(1 - (1+L_3)e^{-L_3}\right), \tag{B.7}$$

$$1 = (K+\theta_6)e^{-L_4} - K, \tag{B.8}$$

$$-1 = (K+1)e^{-L_2-L_5} - K, \tag{B.9}$$

$$-1 - \varepsilon = (K-1)e^{-L_3} - K. \tag{B.10}$$

Next we reduce the algebraic system of equations (H2), (B.1)-(B.10) to a single equation for L_2, K and ε. Meanwhile, we express L_1, L_3, L_4, L_5 and $\theta_1, \theta_2, ..., \theta_6$ as functions of L_2, K and ε.

By (B.10),

$$L_3 = \ln \frac{K-1}{K-1-\varepsilon}. \tag{C.1}$$

From (B.9) and (B.5) we obtain that

$$L_5 = \ln \frac{K+1}{K-1} - L_2 \tag{C.2}$$

and

$$\theta_4 = (1+\varepsilon)\frac{K+1}{K-1}e^{-L_2}. \tag{C.3}$$

θ_5 is already expressed in (B.6). In order to simplify reference to the formulas in this section, we repeat that

$$\theta_5 = (1+\varepsilon)e^{-L_2}. \tag{C.4}$$

Using this, (B.7) and (C.1), we calculate that

$$\theta_6 = (1+\varepsilon)\frac{K-1-\varepsilon}{K-1}e^{-L_2} + \frac{K}{\varepsilon}(K-1-\varepsilon)\ln\frac{K-1}{K-1-\varepsilon} - K. \tag{C.5}$$

Note that $\theta_6 + K > 0$. From (B.8) we obtain that

$$L_4 = \ln \frac{K+\theta_6}{K+1}. \tag{C.6}$$

Now we use (H2), (C.1) and (C.6) to express L_1:

$$L_1 = \frac{1}{2} - L_2 + \frac{5}{2}\ln(K-1-\varepsilon) - \frac{3}{2}\ln(K+\theta_6) - \ln(K-1). \tag{C.7}$$

Substituting the last relation into (B.1), we get that

$$\theta_1 = K - \frac{e^{L_2-\frac{1}{2}}(K-1)(K+\theta_6)^{\frac{3}{2}}}{(K-1-\varepsilon)^{\frac{3}{2}}}. \tag{C.8}$$

Then replacing θ_1 by (C.8) in (B.2), we conclude that

$$\theta_2 = Ke^{-L_2} - \frac{e^{-\frac{1}{2}}(K-1)(K+\theta_6)^{\frac{3}{2}}}{(K-1-\varepsilon)^{\frac{3}{2}}} + \frac{K}{\varepsilon}\left((1+\varepsilon)L_2e^{-L_2} + e^{-L_2} - 1\right). \tag{C.9}$$

Parameter θ_3 appeared as a function of $K, \varepsilon, \theta_2, \theta_5$ and L_3 in (B.3). As θ_2, θ_5 and L_3 have already been given as functions of L_2, K and ε, now we see that θ_3 can also be expressed as a function of L_2, K and ε. We will consider θ_3 in the form

$$\begin{aligned}\theta_3 = \theta_2e^{-L_3} &+ \frac{K}{\varepsilon}\left((1+\varepsilon)L_3e^{-L_2-L_3} + e^{-L_3} - 1\right) \\ &- \frac{K^2}{\varepsilon^2}(K-1)\left(1 - \left(1 + L_3 + \frac{L_3^2}{2}\right)e^{-L_3}\right),\end{aligned} \tag{C.10}$$

where θ_2 and L_3 are defined by (C.9) and (C.1), respectively.

Then (B.4) is the only algebraic equation we have not used so far. We substitute (C.3) into the left hand side of (B.4) and then multiply this equation by $e^{L_4} = (K+\theta_6)/(K+1)$. We deduce that

$$(1+\varepsilon)\frac{K+\theta_6}{K-1}e^{-L_2} = \frac{K}{\varepsilon}(K+1)\left(1-(1-L_4)e^{L_4}\right) + \theta_3.$$

Let

$$U = \left\{(L_2, K, \varepsilon) \in \mathbb{R}^3 \colon \ \varepsilon \in (0,1), \ K \in (6.5, 7), \ L_2 \in (-\varepsilon, \varepsilon)\right\}.$$

If we consider θ_3, θ_6 and L_4 as functions on U given by (C.10), (C.5) and (C.6), then we can define a map $F : U \to \mathbb{R}$ by

$$F(L_2, K, \varepsilon) = \frac{K}{\varepsilon}(K+1)\left(1-(1-L_4)e^{L_4}\right) + \theta_3 - (1+\varepsilon)\frac{K+\theta_6}{K-1}e^{-L_2} + L_2$$

for all $(L_2, K, \varepsilon) \in U$. One can easily check that F is well-defined and continuous on U.

The following proposition holds.

Proposition 2.1. *Let* $\varepsilon \in (0,1)$ *and* $K \in (6.5, 7)$*. Suppose that* $p \colon \mathbb{R} \to \mathbb{R}$ *is a* 2ω*-periodic solution of* (1.1)*,* p *is the concatenation of functions* $y_1, y_2, \ldots, y_{10}$ *as given in* (P.1)-(P.2)*, furthermore the functions* $y_1, y_2, \ldots, y_{10}$ *satisfy (H1)-(H5) with some parameters* $L_i > 0$*,* $i \in \{1, 2, ..., 5\}$*, and* θ_i*,* $i \in \{1, \ldots, 6\}$*. Then* $L_2 \in (0, \varepsilon)$ *and* $F(L_2, K, \varepsilon) = L_2$*.*

Proof. Recall from (C.4) that $\theta_5 = (1+\varepsilon)\, e^{-L_2}$, which is greater than 1 by (H3). It follows immediately that $L_2 < \ln(1+\varepsilon) < \varepsilon$. The rest of the statement comes from the above calculations. □

We need to consider F also for $L_2 \in (-\varepsilon, 0]$ because of technical reasons; see Proposition 4.2 in Section 4. We will also use the following remark in the next section.

Remark 2.2. The system of algebraic equations $F(L_2, K, \varepsilon) = L_2$, (C.1)-(C.10) is equivalent to the system of equations (H2), (B.1)-(B.10).

3. The fixed points of F yield periodic solutions

By the previous section, if (H1)-(H5) hold, and $p\colon \mathbb{R} \to \mathbb{R}$ is a 2ω-periodic solution of (1.1) given by (P.1)-(P.2), then $L_2 \mapsto F(L_2, K, \varepsilon)$ has a fixed point. We devote this section to verify the converse statement: if $\varepsilon > 0$ is small enough and $K \in (6.5, 7)$, then all sufficiently small positive fixed points of $L_2 \mapsto F(L_2, K, \varepsilon)$ yield periodic solutions of (1.1).

We need to consider L_1, L_3, L_4, L_5 and θ_i, $1 \le i \le 6$, as functions of L_2, K and ε (and not as parameters given by hypotheses (H1)-(H5)). So assume that

(H6) L_i, $i \in \{1,3,4,5\}$, and θ_i, $1 \le i \le 6$, are defined by (C.1)-(C.10) on

$$U = \left\{ (L_2, K, \varepsilon) \in \mathbb{R}^3 \colon \ \varepsilon \in (0,1),\ K \in (6.5, 7) \text{ and } L_2 \in (-\varepsilon, \varepsilon) \right\}.$$

One easily check that L_i, $i \in \{1,3,4,5\}$, and θ_i, $1 \le i \le 6$, are continuous functions of (L_2, K, ε) on U.

In this section we also need the assumption that

(H7) $y_1, ..., y_{10}$ are the solutions (Y.1)-(Y.10) of the ordinary differential equations (2.3)-(2.12).

Set

$$\theta^*\left(\bar{K}\right) = \bar{K} - \sqrt{\frac{\left(\bar{K}+1\right)^3}{e\left(\bar{K}-1\right)}} \qquad \text{for } \bar{K} \in [6.5, 7]\,.$$

We claim that $\theta^*\left(\bar{K}\right) > 1$ for $\bar{K} \in [6.5, 7]$. As this inequality is equivalent to

$$\left(\bar{K}-1\right)\left(1 - \sqrt{\frac{\left(\bar{K}+1\right)^3}{e\left(\bar{K}-1\right)^3}}\right) > 0,$$

we need to verify that $\left(\bar{K}+1\right)^3 / \left(\bar{K}-1\right)^3 < e$ holds. Indeed, since $\bar{K} \mapsto \left(\bar{K}+1\right) / \left(\bar{K}-1\right)$ is strictly decreasing for $\bar{K} > 1$, we see that

$$\left(\frac{\bar{K}+1}{\bar{K}-1}\right)^3 \le \left(\frac{6.5+1}{6.5-1}\right)^3 = \left(\frac{15}{11}\right)^3 = 2 + \frac{713}{1331} \le 2 + \frac{800}{1200} = 2 + \frac{2}{3} < e \quad (3.1)$$

for $\bar{K} \in [6.5, 7]$.

The first two statements of the subsequent proposition give information on the behavior of F for small positive ε. The third statement examines the limit of $y_2(t)$, $y_3(t)$ and $y_4(t)$ for all t in their domains as $\varepsilon \to 0^+$. Since y_2, y_3 and y_4 are well-defined by (Y.2)-(Y.4) only if $L_i \geq 0$ for $i \in \{2, 3, 4\}$, here we assume that $L_2 \geq 0$ and $L_4 \geq 0$. It is clear that $L_3 = \ln(K-1) - \ln(K-1-\varepsilon)$ is positive.

Proposition 3.1. *The subsequent assertions hold under hypothesis (H6).*
(i) $\theta_6 = 1 + O(\varepsilon)$, $L_4 = O(\varepsilon)$ *and thus*

$$\frac{K}{\varepsilon}(K+1)\left(1 - (1 - L_4)e^{L_4}\right) = O(\varepsilon) \quad \textit{as } \varepsilon \to 0^+.$$

(ii) If $K \to \bar{K} \in [6.5, 7]$ *and* $\varepsilon \to 0^+$, *then* θ_3 *converges to* $\theta^*\left(\bar{K}\right)$.
(iii) Assume in addition that $L_2 \geq 0$ *and* $L_4 \geq 0$. *Define* y_2, y_3 *and* y_4 *by (Y.2)-(Y.4). If* $K \to \bar{K} \in [6.5, 7]$ *and* $\varepsilon \to 0^+$, *then* $y_2(t), y_3(t)$ *and* $y_4(t)$ *converges to* $\theta^*\left(\bar{K}\right)$, *uniformly in* $t \in [0, L_2]$, $t \in [0, L_3]$ *and* $t \in [0, L_4]$, *respectively.*

Before giving the proof of this proposition, let us make a remark on notation O. If g is a function of L_2, K, ε, t (or only some of these variables) on a set D, and k is a positive integer, then the expression $g = O\left(\varepsilon^k\right)$ as $\varepsilon \to 0^+$ (or simply $g = O\left(\varepsilon^k\right)$) means that there exists $M > 0$ such that $|g(L_2, K, \varepsilon, t)| \leq M\varepsilon^k$ if $(L_2, K, \varepsilon, t) \in D$ and $\varepsilon > 0$ is sufficiently close to zero. Constant M is always independent from L_2, K and t in this paper.

Proof. The proof of statement (i). It is well-known that

$$\ln(1+x) = x + O\left(x^2\right) \text{ as } x \to 0. \tag{3.2}$$

If $K \in (6.5, 7)$ and $\varepsilon \to 0^+$, then

$$\frac{\varepsilon}{K-1-\varepsilon} \to 0^+$$

and thus

$$\ln\frac{K-1}{K-1-\varepsilon} = \ln\left(1 + \frac{\varepsilon}{K-1-\varepsilon}\right) = \frac{\varepsilon}{K-1-\varepsilon} + O\left(\varepsilon^2\right) \text{ as } \varepsilon \to 0^+. \tag{3.3}$$

Therefore

$$\frac{K}{\varepsilon}(K-1-\varepsilon)\ln\frac{K-1}{K-1-\varepsilon} = K + O(\varepsilon). \tag{3.4}$$

In addition, since $L_2 \in (-\varepsilon, \varepsilon)$,

$$(1+\varepsilon)\frac{K-1-\varepsilon}{K-1}e^{-L_2} = (1+\varepsilon)\left(1 - \frac{\varepsilon}{K-1}\right)(1 + O(L_2)) = 1 + O(\varepsilon). \tag{3.5}$$

Substituting (3.4) and (3.5) into (C.5), we obtain that $\theta_6 = 1 + O(\varepsilon)$.

Using (C.6), the previous statement regarding θ_6 and (3.2), we immediately get that

$$L_4 = \ln\left(1 + \frac{\theta_6 - 1}{K+1}\right) = O(\varepsilon). \tag{3.6}$$

By the power series expansion of the exponential function,

$$1 - e^{L_4}(1 - L_4) = O\left(L_4^2\right) \text{ as } L_4 \to 0. \tag{3.7}$$

The last statement of Proposition 3.1.(i) then comes from (3.6) and (3.7).

The proof of statement (iii). Let us now prove *(iii)* in three steps. Let $L_2 \geq 0$ and $L_4 \geq 0$.

1. The convergence of $y_2(t)$ for $t \in [0, L_2]$. We see from statement (i) and formula (C.8) that

$$\lim_{\substack{K \to \bar{K} \\ \varepsilon \to 0^+}} \theta_1 = \bar{K} - \sqrt{\frac{\left(\bar{K}+1\right)^3}{e\left(\bar{K}-1\right)}} = \theta^*\left(\bar{K}\right). \tag{3.8}$$

Using that $0 \leq t \leq L_2 < \varepsilon$ and $e^x = 1 + x + O\left(x^2\right)$ as $x \to 0$, we also see that

$$\begin{aligned} \frac{K}{\varepsilon}\left((1+\varepsilon)te^{-t} + e^{-t} - 1\right) &= \frac{K}{\varepsilon}\left((1+\varepsilon)t\left(1 - t + O\left(t^2\right)\right) - t + O\left(t^2\right)\right) \\ &= \frac{K}{\varepsilon}\left(\varepsilon t + O\left(t^2\right)\right) = O(\varepsilon). \end{aligned} \tag{3.9}$$

Substituting (3.8) and (3.9) into (Y.2), we get that $y_2(t)$ converges to $\theta^*\left(\bar{K}\right)$ for all $t \in [0, L_2]$, and this convergence is uniform in t.

2. The convergence of $y_3(t)$ for $t \in [0, L_3]$, using formula (Y.3). Observe that if θ_1 is given by (C.8), and θ_2 is determined by (C.9), then (Y.2) yields that $y_2(L_2) = \theta_2$. So by our last result, $\lim_{\varepsilon \to 0^+, K \to \bar{K}} \theta_2 = \theta^*\left(\bar{K}\right)$. We also see from (C.4) and from $L_2 \in (-\varepsilon, \varepsilon)$ that $\theta_5 = 1 + O(\varepsilon)$ as $\varepsilon \to 0^+$. Using this and the power series expansion of the exponential function, we get the following for $0 \leq t \leq L_3 = O(\varepsilon)$:

$$(\theta_5 t + 1)e^{-t} - 1 = ((1 + O(\varepsilon))t + 1)\left(1 - t + O\left(t^2\right)\right) - 1 = O\left(\varepsilon^2\right). \tag{3.10}$$

We also observe that

$$\begin{aligned} 1 - e^{-t}\left(1 + t + \frac{t^2}{2}\right) =& 1 - \left(1 - t + \frac{t^2}{2} + O(t^3)\right)\left(1 + t + \frac{t^2}{2}\right) \\ =& O\left(\varepsilon^3\right). \end{aligned} \tag{3.11}$$

Summing up, (Y.3) yields that if $0 \leq t \leq L_3 = O(\varepsilon)$, then

$$\lim_{\substack{K \to \bar{K} \\ \varepsilon \to 0^+}} y_3(t) = \lim_{\substack{K \to \bar{K} \\ \varepsilon \to 0^+}} \theta_2 e^{-t} = \theta^*\left(\bar{K}\right).$$

This convergence is uniform in t.

3. The convergence of $y_4(t)$ for $t \in [0, L_4]$. On the one hand, if y_3 is defined by (Y.3), θ_5 is defined (C.4) and θ_3 is given by (C.10), then $\theta_3 = y_3(L_3)$. Hence, by the previous paragraph, θ_3 converges to $\theta^*\left(\bar{K}\right)$ if $K \to \bar{K} \in [6.5, 7]$ and $\varepsilon \to 0^+$. (Note that we have proved statement (ii) in the case $L_2 \geq 0$). On the other hand, it follows from $\theta_6 = 1 + O(\varepsilon)$ that

$$(K + \theta_6)te^{-t} - (K + 1)\left(1 - e^{-t}\right)$$

equals

$$(K + 1 + O(\varepsilon))\left(t + O\left(t^2\right)\right) - (K + 1)\left(t + O\left(t^2\right)\right) = O\left(\varepsilon^2\right)$$

for $0 \leq t \leq L_4 = O(\varepsilon)$. In consequence, formula (Y.4) shows that $y_4(t)$ converges to $\theta^*\left(\bar{K}\right)$, uniformly in $t \in [0, L_4]$.

The proof of statement (ii). We have already verified (ii) for $L_2 \in [0, \varepsilon)$. Now suppose that $L_2 \in (-\varepsilon, 0)$ and observe that (3.9) holds also with $t = L_2 \in (-\varepsilon, 0)$. Therefore (C.9) and the equality $\theta_6 = 1 + O(\varepsilon)$ together show that θ_2 converges to $\theta^*\left(\bar{K}\right)$ also in the case when $L_2 < 0$. Now we can use $L_3 = O(\varepsilon)$, (C.4) and (3.10)-(3.11) with $t = L_3$ to verify that θ_3 (defined by (C.10)) converges to $\theta^*\left(\bar{K}\right)$ if $K \to \bar{K} \in [6.5, 7]$, $\varepsilon \to 0^+$ and $L_2 \in (-\varepsilon, 0)$. □

Corollary 3.2. *Assume that* $\lim_{n\to\infty} \varepsilon_n = 0^+$,

$$(L_{2,n}, K_n, \varepsilon_n) \in U \quad \textit{and} \quad F\left(L_{2,n}, K_n, \varepsilon_n\right) = L_{2,n} \textit{ for all } n \geq 0.$$

Then $(K_n)_{n=0}^{\infty}$ *is convergent, and* $\lim_{n\to\infty} K_n = K_0$, *where* K_0 *is the unique solution of (1.2) in [6.5,7].*

Proof. We already know from Section 3 of paper [12] that (1.2) has exactly one solution $K_0 \approx 6.87$ in $[6.5, 7]$.

It suffices to prove that each subsequence of $(K_n)_{n=0}^{\infty}$ has a subsequence converging to K_0. As $K_n \in (6.5, 7)$ for all $n \geq 1$, it is clear that any subsequence of $(K_n)_{n=0}^{\infty}$ has a convergent subsequence $(K_{n_l})_{l=0}^{\infty}$. Let

$$\bar{K} = \lim_{l\to\infty} K_{n_l} \in [6.5, 7].$$

Now let l tend to infinity in the equation $F\left(L_{2,n_l}, K_{n_l}, \varepsilon_{n_l}\right) = L_{2,n_l}$. Under the assumptions of the Corollary, $\lim_{l\to\infty} L_{2,n_l} = 0$. This fact, the definition of F and Proposition 3.1.(i)-(ii) together show that $\bar{K} \in [6.5, 7]$ is a solution of

$$K - \sqrt{\frac{(K+1)^3}{e(K-1)}} - \frac{K+1}{K-1} = 0.$$

It is straightforward to show that this equation is equivalent to (1.2), and thus $\bar{K} = K_0$. The proof is complete. □

For $K \in (6.5, 7)$ and $\varepsilon \in (0,1)$, let $\widehat{L}_2$ be that value of L_2 for which $L_4 = 0$, i.e., for which $\theta_6 = 1$. Using (C.5), we can express $\widehat{L}_2$ as a function of K and ε:

$$\begin{aligned}\widehat{L}_2(K,\varepsilon) = &\ln\left((1+\varepsilon)\frac{K-1-\varepsilon}{K-1}\right)\\ &- \ln\left(K+1-\frac{K}{\varepsilon}(K-1-\varepsilon)\ln\frac{K-1}{K-1-\varepsilon}\right).\end{aligned} \tag{3.12}$$

Proposition 3.3. *If $K \in (6.5, 7)$ and $\varepsilon > 0$ is small enough, then $\widehat{L}_2(K,\varepsilon) \in (0,\varepsilon)$.*

Proof. It is well-known that

$$\ln(1+x) = x - \frac{x^2}{2} + O\left(x^3\right) \text{ as } x \to 0.$$

In consequence,

$$\begin{aligned}\ln\frac{K-1}{K-1-\varepsilon} &= \ln\left(1+\frac{\varepsilon}{K-1-\varepsilon}\right)\\ &= \frac{\varepsilon}{K-1-\varepsilon} - \frac{\varepsilon^2}{2(K-1-\varepsilon)^2} + O\left(\varepsilon^3\right),\end{aligned}$$

and

$$\frac{K}{\varepsilon}(K-1-\varepsilon)\ln\frac{K-1}{K-1-\varepsilon} = K - \frac{K\varepsilon}{2(K-1-\varepsilon)} + O\left(\varepsilon^2\right) \text{ as } \varepsilon \to 0^+. \tag{3.13}$$

Applying the geometric series expansion $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$ with the choice $x = \varepsilon/(K-1)$, we easily deduce that

$$\frac{1}{K-1-\varepsilon} = \frac{1}{K-1}\cdot\frac{1}{1-\frac{\varepsilon}{K-1}} = \frac{1}{K-1} + O(\varepsilon),$$

and thus

$$\frac{K}{\varepsilon}(K-1-\varepsilon)\ln\frac{K-1}{K-1-\varepsilon} = K - \frac{K}{2(K-1)}\varepsilon + O\left(\varepsilon^2\right). \tag{3.14}$$

Using this, we get that

$$\ln\left(K+1-\frac{K}{\varepsilon}(K-1-\varepsilon)\ln\frac{K-1}{K-1-\varepsilon}\right) = \frac{K}{2(K-1)}\varepsilon + O\left(\varepsilon^2\right). \tag{3.15}$$

Also note that

$$(1+\varepsilon)\frac{K-1-\varepsilon}{K-1} = 1 + \frac{K-2}{K-1}\varepsilon - \frac{1}{K-1}\varepsilon^2,$$

and thus

$$\ln\left((1+\varepsilon)\frac{K-1-\varepsilon}{K-1}\right) = \frac{K-2}{K-1}\varepsilon + O\left(\varepsilon^2\right). \tag{3.16}$$

Subtracting (3.15) from (3.16), we conclude that

$$\widehat{L}_2 = \left(\frac{K-2}{K-1} - \frac{K}{2(K-1)}\right)\varepsilon + O\left(\varepsilon^2\right) = \frac{K-4}{2(K-1)}\varepsilon + O\left(\varepsilon^2\right). \tag{3.17}$$

Since $(K-4)/(2K-2) \in (0,1)$ for $K \in (6.5, 7)$, we see that $\widehat{L}_2 \in (0, \varepsilon)$ for all sufficiently small positive ε. □

Consider the following subset of U:

$$V = \left\{(L_2, K, \varepsilon): \ \varepsilon \in (0,1), \ K \in (6.5, 7) \text{ and } L_2 \in \left(0, \widehat{L}_2(K,\varepsilon)\right)\right\} \subset U.$$

Remark 3.4. It is clear from (C.5) and (C.6) that θ_6 and L_4 are strictly decreasing functions of L_2. Hence, if $\varepsilon > 0$ is small and $(L_2, K, \varepsilon) \in V$, then $\theta_6 > 1$ and $L_4 > 0$.

Now we are ready to clarify which fixed points of $L_2 \mapsto F(L_2, K, \varepsilon)$ yield periodic solutions.

Proposition 3.5. *Assume that*

- $(L_2, K, \varepsilon) \in V$*,* $F(L_2, K, \varepsilon) = L_2$ *and* $\varepsilon > 0$ *is sufficiently small,*
- L_1, L_3, L_4, L_5 *and* θ_i*,* $1 \le i \le 6$*, are defined as in (H6),*
- y_i*,* $1 \le i \le 10$*, are defined as in (H7).*

Then (H1)-(H5) hold.

Proof. The functions $y_1, ..., y_{10}$ can be defined by (Y.1)-(Y.10) only if L_1, L_2, L_3, L_4 and L_5 are nonnegative. So let us first show (H1). It is clear from (C.1) that $L_3 > 0$, and we know from Remark 3.4 that $L_4 > 0$. Corollary 3.2 and formula (C.2) yield that if ϵ (and hence L_2) tends to 0, then L_5 tends to $\ln((K_0+1)/(K_0-1))$. Similarly, Corollary 3.2 and formula (C.7) give that

$$\lim_{\varepsilon \to 0^+} L_1 = \frac{1}{2} - \frac{1}{2}\ln\left(\frac{K_0+1}{K_0-1}\right)^3$$

This limit is also positive, see (3.1). Thus L_1 and L_5 are positive if $\varepsilon > 0$ is small enough.

By Remark 2.2, equation $F(L_2, K, \varepsilon) = L_2$ and (C.1)-(C.10) imply (H2). Thus (H2) holds under the assumptions of this proposition.

It is clear from (Y.1)-(Y.10) that $y_1, ..., y_{10}$ are continuous. Recall that they are those solutions of the ordinary differential equations (2.3)-(2.12) that satisfy the boundary conditions listed in (H4) for $t = 0$. In addition, by Remark 2.2, equation $F(L_2, K, \varepsilon) = L_2$ and (C.1)-(C.10) imply that the equations (B.1)-(B.10) are true. This means that $y_1, ..., y_{10}$ satisfy the right-end boundary conditions given in (H4). Summing up, (H4) is satisfied.

It remains to show (H3) and (H5).

As $K-1-\varepsilon$ is positive for $(L_2, K, \varepsilon) \in V$, we see from (Y.1) that y_1 strictly monotone increases on $[0, L_1]$. As $y_1(0) = 1+\varepsilon$, it follows that $y_1(t) > 1+\varepsilon$ for $t \in (1, L_1]$. Thus $\theta_1 = y_1(L_1) > 1+\varepsilon$. Using Proposition 3.1.(iii), we obtain that $y_2(t)$, $y_3(t)$ and $y_4(t)$ are greater than $1+\varepsilon$ for all t if $\varepsilon > 0$ is small enough. It follows immediately that $\theta_2 = y_2(L_2)$, $\theta_3 = y_3(L_3)$ and $\theta_4 = y_4(L_4)$ are greater than $1+\varepsilon$ if $\varepsilon > 0$ is small enough. It is clear from (Y.5) that y_5 is strictly monotone decreasing. As $y_5(L_5) = 1+\varepsilon$, we deduce that $y_5(t) > 1+\varepsilon$ for $t \in [0, L_5)$. Summing up the results of this paragraph, $\theta_i > 1+\varepsilon$ for $i \in \{1, 2, 3, 4\}$, and $y_i(t) > 1+\varepsilon$ for all t in the interior of the domain of y_i, where $i \in \{1, 2, \ldots, 5\}$.

By (Y.6) and (Y.8), y_6 and y_8 are strictly monotone decreasing on their whole domains. Differentiating (Y.7) with respect to t and using that $\theta_5 > 0$ by (C.4), we conclude that

$$\dot{y}_7(t) = -\theta_5 e^{-t} - (1+K)te^{-t} < 0 \quad \text{for } t \in [0, L_3],$$

i.e., y_7 is also strictly monotone decreasing on $[0, L_3]$. Hence

$$1+\varepsilon = y_6(0) > y_6(L_2) = \theta_5 = y_7(0) > y_7(L_3) = \theta_6 = y_8(0) > y_8(L_4) = 1,$$

and $y_i(s) \in (1, 1+\varepsilon)$ for all s in the interior of the domain of y_i, where $i \in \{6, 7, 8\}$.

From (Y.9) and (Y.10) we conclude that y_9 and y_{10} are strictly monotone decreasing on $[0, L_2 + L_5]$ and on $[0, L_3]$, respectively. As we already know that y_9 and y_{10} fulfill the boundary conditions given in (H4), it is obvious that $y_9(t) \in (-1, 1)$ for all $t \in (0, L_2+L_5)$, and $y_{10}(t) \in (-1-\varepsilon, -1)$ for all $t \in (0, L_3)$.

We have verified (H3) and (H5), and hence the proof is complete. □

Corollary 3.6. *Under the assumptions of the previous proposition, the 2ω-periodic function p, determined by (P.1)-(P.2), satisfies the delay equation (1.1) on $\mathbb{R}$. In addition, the map*

$$V \ni (L_2, K, \varepsilon) \mapsto p_0 \in C$$

is continuous.

Proof. As $p(t) = -p(t-\omega)$ for all real t, and the nonlinearity f_K is odd, it is enough to guarantee (2.2) to prove that p is a solution on $\mathbb{R}$. By Proposition 3.5, the properties listed in (H1)-(H5) are true. We have already pointed out in Section 2 that – under hypotheses (H1)-(H5)– equation (2.2) is equivalent to the ordinary equations

- (2.3)-(2.6) on $[0, \tau_1]$,
- (2.7)-(2.9) on $[\tau_1, \tau_2]$,
- (2.10)-(2.11) on $[\tau_2, \tau_3]$,
- and (2.12) on $[\tau_3, \omega]$.

By assumption (H7), (2.3)-(2.12) hold. So (2.2) is satisfied too.

Recall the observation that under hypothesis (H6), L_i, $i \in \{1,3,4,5\}$, and θ_i, $1 \leq i \leq 6$, are continuous functions of (L_2, K, ε) on V. From this, from the definitions (Y.1)-(Y.10) of $y_1, \ldots, y_{10}$ and from (P.1)-(P.2) one can deduce in a straightforward way that the initial function p_0 varies continuously with (L_2, K, ε). We leave the details to the reader. □

4. The saddle-node bifurcation of F_ε

For $\varepsilon \in (0,1)$, let

$$U_\varepsilon = (-\varepsilon, \varepsilon) \times (6.5, 7)$$

and define

$$F_\varepsilon : U_\varepsilon \ni (L_2, K) \mapsto F\left(L_2, K, \varepsilon\right) \in \mathbb{R}.$$

Appendix A of this paper calculates certain partial derivatives of F_ε and shows that $\partial F_\varepsilon / \partial K$ and $\partial^2 F_\varepsilon / \partial L_2^2$ are both continuous on U_ε. One can actually show that $F_\varepsilon \in C^2(U_\varepsilon)$. We omit the complete proof of this claim.

In this section we consider ε to be a fixed and sufficiently small positive number and show that F_ε undergoes a saddle-node bifurcation as K increases.

The first two propositions show that if $L_2 \in (-\varepsilon, \varepsilon)$, then the equation $F_\varepsilon(L_2, K) = L_2$ can be solved for K.

Proposition 4.1. *If $\varepsilon > 0$ is sufficiently small, then there exists $K_\varepsilon \in (6.5, 7)$ so that $F_\varepsilon\left(0, K_\varepsilon\right) = 0$.*

Proof. Using Proposition 3.1, we obtain that for any $K \in (6.5, 7)$,

$$\lim_{\varepsilon \to 0^+} F_\varepsilon(0, K) = K - \sqrt{\frac{(K+1)^3}{e(K-1)}} - \frac{K+1}{K-1}. \tag{4.1}$$

One can show by elementary calculations that the sign of (4.1) is the same as the sign of

$$w\left(K\right) = e - \frac{(K+1)^3 (K-1)}{(K^2 - 2K - 1)^2},$$

which expression is slightly easier to handle. As

$$w(6.5) = e - \frac{37125}{12769} < e - \frac{36400}{13000} = e - \frac{28}{10} < 0$$

and

$$w(7) = e - \frac{768}{289} > e - \frac{768}{288} = e - \frac{8}{3} > 0,$$

one sees that $\lim_{\varepsilon \to 0^+} F_\varepsilon(0, 6.5) < 0$ and $\lim_{\varepsilon \to 0^+} F_\varepsilon(0, 7) > 0$. Hence, if $\varepsilon > 0$ is small enough, then $F_\varepsilon(0, 6.5) < 0$ and $F_\varepsilon(0, 7) > 0$. As $(6.5, 7) \ni K \mapsto F_\varepsilon(0, K) \in \mathbb{R}$ is continuous, the existence of K_ε follows from the intermediate value theorem. □

Proposition 4.2. *For all sufficiently small positive ε and $L_2 \in (-\varepsilon, \varepsilon)$, the equation $F_\varepsilon(L_2, K) = L_2$ has a unique solution K in $(6.5, 7)$. Furthermore, this solution can be given as $K = \varphi_\varepsilon(L_2)$, where $\varphi_\varepsilon : (-\varepsilon, \varepsilon) \to (6.5, 7)$ is continuous and $\varphi_\varepsilon(0) = K_\varepsilon$.*

Proof. Let $J_\varepsilon = (6.5 - K_\varepsilon, 7 - K_\varepsilon)$ and introduce the map

$$G_\varepsilon : (-\varepsilon, \varepsilon) \times J_\varepsilon \ni (L_2, K) \mapsto F_\varepsilon(L_2, K + K_\varepsilon) - L_2 \in \mathbb{R}.$$

Then $G_\varepsilon(0, 0) = 0$ and G is continuously differentiable (see Propositions A.2 and A.4 in Appendix A). We look for the solution K of $G_\varepsilon(L_2, K) = 0$ for any small $\varepsilon > 0$ and for any $L_2 \in (-\varepsilon, \varepsilon)$.

Let

$$A := \frac{\partial G_\varepsilon(0,0)}{\partial K} = \frac{\partial F_\varepsilon(0, K_\varepsilon)}{\partial K}.$$

By Corollary A.3, we may assume that A is nonzero.

Finding a solution K to $G_\varepsilon(L_2, K) = 0$ is equivalent to finding a fixed point of T_{ε, L_2}, where

$$T_{\varepsilon, L_2} : J_\varepsilon \ni K \mapsto K - A^{-1} G_\varepsilon(L_2, K) \in \mathbb{R}.$$

Choose a constant $q \in (0, 1)$ independent from K, ε and L_2. We claim that T_{ε, L_2} is a uniform contraction on an appropriate subset of J_ε: if $\eta > 0$ is small enough, then $[-\eta, \eta] \subseteq J_\varepsilon$,

$$|T_{\varepsilon, L_2}(K)| \leq \eta \text{ for } K \in [-\eta, \eta], \tag{4.2}$$

and

$$|T_{\varepsilon, L_2}(K_1) - T_{\varepsilon, L_2}(K_2)| < q|K_1 - K_2| \text{ for } K_1, K_2 \in [-\eta, \eta]. \tag{4.3}$$

Set $\eta > 0$ so small that $[-\eta, \eta] \subseteq J_\varepsilon$. Using Lagrange's mean value theorem, we get that for $K_1, K_2 \in [-\eta, \eta]$,

$$|T_{\varepsilon, L_2}(K_1) - T_{\varepsilon, L_2}(K_2)| \leqslant \sup_{|\bar{K}| < \eta} \left| 1 - A^{-1} \frac{\partial G_\varepsilon(L_2, \bar{K})}{\partial K} \right| |K_1 - K_2| \,. \tag{4.4}$$

We see from Proposition A.2 that

$$\begin{aligned} \frac{\partial G_\varepsilon(L_2, \bar{K})}{\partial K} &= 1 - e^{-\frac{1}{2}} \left(\frac{3}{2} \sqrt{\frac{K_\varepsilon + \bar{K} + 1}{K_\varepsilon + \bar{K} - 1}} - \frac{1}{2} \sqrt{\left(\frac{K_\varepsilon + \bar{K} + 1}{K_\varepsilon + \bar{K} - 1} \right)^3} \right) \\ &\quad + \frac{2}{(K_\varepsilon + \bar{K} - 1)^2} + O(\varepsilon). \end{aligned} \tag{4.5}$$

Therefore there exist $\varepsilon_0 > 0$ and $\eta_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, $L_2 \in (-\varepsilon, \varepsilon)$ and $\bar{K} \in (-\eta_0, \eta_0)$, then

$$\left| 1 - A^{-1} \frac{\partial G_\varepsilon(L_2, \bar{K})}{\partial K} \right| < q,$$

that is, (4.3) is satisfied for any $\varepsilon \in (0, \varepsilon_0)$, $L_2 \in (-\varepsilon, \varepsilon)$ and $\eta \in (0, \eta_0)$. Next, using the Taylor expansion of G_ε, we obtain that

$$T_{\varepsilon,L_2}(K) = K - A^{-1}G_\varepsilon(L_2, K) = -A^{-1}\left(\frac{\partial G_\varepsilon(0,0)}{\partial L_2}L_2 + O\left(L_2^2 + K^2\right)\right)$$

as $L_2 \to 0$ and $K \to 0$. In consequence, if $|K| \le \eta < \eta_0$ and $|L_2| < \varepsilon < \varepsilon_0$, then

$$|T_{\varepsilon,L_2}(K)| < |A|^{-1}\left|\frac{\partial G_\varepsilon(0,0)}{\partial L_2}\right|\varepsilon + C\left(\varepsilon^2 + \eta^2\right)$$

with some constant $C > 0$. Fix $\eta_1 < \eta_0$ so small that $C\eta_1^2 < \eta_1/2$. Now set $\varepsilon_1 < \varepsilon_0$ such that

$$|A|^{-1}\left|\frac{\partial G_\varepsilon(0,0)}{\partial L_2}\right|\varepsilon + C\varepsilon^2 < \frac{\eta_1}{2}$$

for all $\varepsilon \in (0, \varepsilon_1)$. Then (4.2) holds for $\eta = \eta_1$, $\varepsilon \in (0, \varepsilon_1)$ and $L_2 \in (-\varepsilon, \varepsilon)$.

Summing up, we conclude that T_{ε,L_2} is a uniform contraction from $[-\eta_1, \eta_1]$ to $[-\eta_1, \eta_1]$ for all $\varepsilon \in (0, \varepsilon_1)$ and $L_2 \in (-\varepsilon, \varepsilon)$. By the Banach fixed point theorem, T_{ε,L_2} has a unique fixed point $\psi_\varepsilon(L_2)$ in $[-\eta_1, \eta_1]$, see Theorem 3.1 of Chapter 0 in [18]. Since $L_2 \mapsto T_{\varepsilon,L_2}(K)$ is continuous for each K, it follows from Theorem 3.2 of Chapter 0 in [18] that ψ_ε is continuous in L_2. It is clear that $\psi_\varepsilon(0) = 0$. Set $\varphi_\varepsilon(K) := K_\varepsilon + \psi_\varepsilon(K)$. □

Now we are ready to verify the saddle-node bifurcation of the fixed points of F_ε.

Proposition 4.3. *For all sufficiently small positive ε, one can give $K^* = K^*(\varepsilon) \in (6.5, 7)$ and $L_2^* = L_2^*(\varepsilon) \in (0, \widehat{L}_2(K, \varepsilon))$ such that F_ε undergoes a saddle-node bifurcation at (L_2^*, K^*): there exist a neighborhood $\mathcal{U}$ of L_2^* in $(0, \widehat{L}_2(K^*, \varepsilon))$ and a constant $\delta_1 > 0$ such that*

- *the map $F_\varepsilon(\cdot, K)$ has no fixed point in $\mathcal{U}$ for $K \in (K^* - \delta_1, K^*)$,*
- *L_2^* is the unique fixed point of $F_\varepsilon(\cdot, K^*)$ in $\mathcal{U}$,*
- *$F_\varepsilon(\cdot, K)$ has exactly two fixed points in $\mathcal{U}$ for $K \in (K^*, K^* + \delta_1)$, and both fixed points converge to L_2^* as $K \to K^*$.*

Proof. By Section 21.1A in [19], F_ε undergoes a saddle-node bifurcation at (L_2^*, K^*) if

$$\begin{aligned} F_\varepsilon(L_2^*, K^*) &= L_2^*, \\ \frac{\partial}{\partial L_2}F_\varepsilon(L_2^*, K^*) &= 1, \\ \frac{\partial}{\partial K}F_\varepsilon(L_2^*, K^*) &\neq 0, \\ \frac{\partial^2}{\partial L_2^2}F_\varepsilon(L_2^*, K^*) &\neq 0. \end{aligned}$$

Furthermore, if

$$\frac{\frac{\partial^2}{\partial L_2^2} F_\varepsilon(L_2^*, K^*)}{\frac{\partial}{\partial K} F_\varepsilon(L_2^*, K^*)} < 0,$$

then the fixed points of F_ε appear for $K \geq K^*$.

For all small enough positive ε, Proposition 4.2 gives a continuous map $\varphi_\varepsilon : (-\varepsilon, \varepsilon) \to (6.5, 7)$ such that

$$F_\varepsilon\left(L_2, \varphi_\varepsilon(L_2)\right) = L_2 \text{ for all } L_2 \in (-\varepsilon, \varepsilon).$$

It is clear from Corollary A.5 that if $\varepsilon > 0$ is sufficiently small, then

$$\frac{\partial}{\partial L_2} F_\varepsilon(0, \varphi_\varepsilon(0)) > 1 \quad \text{and} \quad \frac{\partial}{\partial L_2} F_\varepsilon\left(\widehat{L}_2, \varphi_\varepsilon\left(\widehat{L}_2\right)\right) < 1.$$

As F_ε is continuously differentiable with respect to L_2, and φ_ϵ is continuous, it is clear that

$$(-\varepsilon, \varepsilon) \ni L_2 \mapsto \frac{\partial}{\partial L_2} F_\varepsilon(L_2, \varphi_\varepsilon(L_2)) \in \mathbb{R}$$

is also continuous. It follows from the intermediate value theorem that there exists $L_2^* \in (0, \widehat{L}_2)$ such that

$$\frac{\partial}{\partial L_2} F_\varepsilon\left(L_2^*, \varphi_\varepsilon\left(L_2^*\right)\right) = 1.$$

Let $K^* := \varphi_\varepsilon\left(L_2^*\right) \in (6.5, 7)$.

We see from Corollary A.3 and from Proposition A.6 that we may assume that

$$\frac{\partial}{\partial K} F_\varepsilon\left(L_2^*, K^*\right) > 0 \quad \text{and} \quad \frac{\partial^2}{\partial L_2^2} F_\varepsilon\left(L_2^*, K^*\right) < 0.$$

Hence F_ε undergoes a saddle-node bifurcation at (L_2^*, K^*), and the fixed points appear for $K \geq K^*$. □

5. The delay equation has no other types of periodic solutions locally

In this section choose $\varepsilon > 0$ so small that Proposition 4.3 holds, i.e., F_ε undergoes a saddle-node bifurcation at $(L_2^*(\varepsilon), K^*(\varepsilon))$, where $(L_2^*(\varepsilon), K^*(\varepsilon), \varepsilon) \in V$.

From now on, let $p \colon \mathbb{R} \to \mathbb{R}$ denote that periodic solution that is given by Corollary 3.6 specially for $(L_2^*(\varepsilon), K^*(\varepsilon), \varepsilon)$. Then p is the concatenation of certain auxiliary functions $y_1, \ldots, y_{10}$ as in (P.1)-(P.2), and its minimal period is 2ω. The functions $y_1, \ldots, y_{10}$ satisfy (H1)-(H5) with some parameters $L_i > 0$, $i \in \{1, 2, ..., 5\}$, and θ_i, $i \in \{1, \ldots, 6\}$.

In order to complete the proof of the main theorem, it remains to verify that all periodic solutions of the delay equation (1.1) derive from fixed points of F - at least locally, in an open ball centered at p_0.

First let us recall the results of Propositions 5.1 and 5.2 in [12].

Proposition 5.1. *Suppose that $\bar{p}:\mathbb{R}\to\mathbb{R}$ is an arbitrary periodic solution of* (1.1) *with minimal period $2\bar{\omega}$.*
(i) If $t_0\in\mathbb{R}$ and $t_1\in(t_0,t_0+2\bar{\omega})$ are chosen so that $\bar{p}(t_0)=\min_{t\in\mathbb{R}}\bar{p}(t)$ and $\bar{p}(t_1)=\max_{t\in\mathbb{R}}\bar{p}(t)$, then $\bar{p}$ is monotone nondecreasing on (t_0,t_1) and monotone nonincreasing on $(t_1,t_0+2\bar{\omega})$.
(ii) If $0\in\bar{p}(\mathbb{R})$, then $\bar{p}(t)=-\bar{p}(t-\bar{\omega})$ for all real t.

The main result of this section is the following.

Proposition 5.2. *Let $\bar{p}:\mathbb{R}\to\mathbb{R}$ be a periodic solution of* (1.1) *for some parameter $\bar{K}$ with minimal period $2\bar{\omega}$. If $\left|\bar{K}-K^*(\varepsilon)\right|$ and $\|\bar{p}_0-p_0\|$ are small enough and $\bar{p}(-1)=1+\varepsilon$, then one can give parameters $\bar{L}_i>0$, $i\in\{1,2,...,5\}$, $\bar{\theta}_i$, $i\in\{1,\dots,6\}$, and continuous functions $\bar{y}_i$, $i\in\{1,\dots,10\}$, such that (H1)-(H5) hold, and $\bar{p}$ is the concatenation of $\bar{y}_1,\dots,\bar{y}_{10}$ as follows:*

$$\begin{aligned}
\bar{p}(t-1)&=\bar{y}_1(t) \quad \text{for } t\in\left[0,\bar{L}_1\right],\\
\bar{p}\left(t-1+\bar{L}_1\right)&=\bar{y}_2(t) \quad \text{for } t\in\left[0,\bar{L}_2\right],\\
\bar{p}\left(t-1+\bar{L}_1+\bar{L}_2\right)&=\bar{y}_3(t) \quad \text{for } t\in\left[0,\bar{L}_3\right],\\
\bar{p}\left(t-1+\bar{L}_1+\bar{L}_2+\bar{L}_3\right)&=\bar{y}_4(t) \quad \text{for } t\in\left[0,\bar{L}_4\right],\\
\bar{p}\left(t-1+\bar{L}_1+\bar{L}_2+\bar{L}_3+\bar{L}_4\right)&=\bar{y}_5(t) \quad \text{for } t\in\left[0,\bar{L}_5\right],\\
\bar{p}\left(t-1+\bar{\tau}_1\right)&=\bar{y}_6(t) \quad \text{for } t\in\left[0,\bar{L}_2\right],\\
\bar{p}\left(t-1+\bar{\tau}_1+\bar{L}_2\right)&=\bar{y}_7(t) \quad \text{for } t\in\left[0,\bar{L}_3\right],\\
\bar{p}\left(t-1+\bar{\tau}_1+\bar{L}_2+\bar{L}_3\right)&=\bar{y}_8(t) \quad \text{for } t\in\left[0,\bar{L}_4\right],\\
\bar{p}\left(t-1+\bar{\tau}_2\right)&=\bar{y}_9(t) \quad \text{for } t\in\left[0,\bar{L}_2+\bar{L}_5\right],\\
\bar{p}\left(t-1+\bar{\tau}_3\right)&=\bar{y}_{10}(t) \quad \text{for } t\in\left[0,\bar{L}_3\right],
\end{aligned} \tag{5.1}$$

where

$$\bar{\tau}_1=\sum_{i=1}^{5}\bar{L}_i,\ \ \bar{\tau}_2=\bar{\tau}_1+\bar{L}_2+\bar{L}_3+\bar{L}_4,\ \ \bar{\tau}_3=\bar{\tau}_2+\bar{L}_2+\bar{L}_5 \text{ and } \bar{\omega}=\bar{\tau}_3+\bar{L}_3. \tag{5.2}$$

In addition,

$$\bar{p}(t)=-\bar{p}(t-\bar{\omega}) \quad \text{for all } t\in\mathbb{R}. \tag{5.3}$$

In consequence, $\bar{L}_2$ is the fixed point of $(0,\varepsilon)\ni L_2\mapsto F(L_2,K,\varepsilon)\in\mathbb{R}$.

Proof. As $\|\bar{p}_0-p_0\|$ is small, we may assume that $0\in\bar{p}(\mathbb{R})$. So the symmetry property (5.3) holds by Proposition 5.1.(ii), and it suffices to investigate $\bar{p}$ on $[-1,-1+\bar{\omega}]$. We prove the theorem by explicitly determining the auxiliary functions $\bar{y}_1,\dots,\bar{y}_{10}$, the parameters $\bar{L}_1,\dots\bar{L}_5$ as the lengths of their domains, and the parameters $\bar{\theta}_1,\dots,\bar{\theta}_6$ as their boundary values.

1. One can easily prove that $|\bar{\omega}-\omega|$ is arbitrary small provided $\left|\bar{K}-K^*(\varepsilon)\right|$ and $\|\bar{p}_0-p_0\|$ are small enough. Hence, by the smallness of $\|\bar{p}_0-p_0\|$ and by the continuity of the solution operator in forward time, one can achieve that $\|\bar{p}_{-1+2\bar{\omega}}-p_{-1+2\omega}\|$ is arbitrary small too. By periodicity, this means that $\|\bar{p}_{-1}-p_{-1}\|$ is arbitrary small. As we shall see, this property is of key role.

It is also straightforward to obtain the subsequent properties of p_{-1} (shown Fig. 5.1) by using (H2), the equations (P.1) and the fact that $p(t) = -p(t-\omega)$ for all $t \in \mathbb{R}$:

$$p(t) > 1+\varepsilon \text{ for } t \in [-2, -2+L_1) \quad \text{and} \quad p(-2+L_1) = 1+\varepsilon, \tag{5.4}$$

$$\begin{aligned} &p(t) \in (-1,1) \text{ for } t \in (-2+\tau_1 - L_5, -2+\tau_1+L_2)\,, \\ &\qquad\qquad p(-2+\tau_1-L_5) = 1 \text{ and } p(-2+\tau_1+L_2) = -1, \end{aligned} \tag{5.5}$$

$$\begin{aligned} &p(t) < -1-\varepsilon \text{ for } t \in (-2+\tau_2 - L_4, -2+\omega+L_1) \\ &\qquad\qquad\qquad\qquad \text{and } p(-2+\tau_2-L_4) = -1-\varepsilon. \end{aligned} \tag{5.6}$$

2. *The parameters* $\bar{L}_1$, $\bar{\theta}_1$ *and the function* $\bar{y}_1$. Since $\|\bar{p}_{-1} - p_{-1}\|$ is arbitrarily small and (5.4) holds, one can give $\bar{L}_1 > 0$ arbitrarily close to L_1 such that

$$\bar{p}(t) > 1+\varepsilon \text{ for } t \in [-2, -2+\bar{L}_1) \quad \text{and} \quad \bar{p}\left(-2+\bar{L}_1\right) = 1+\varepsilon. \tag{5.7}$$

Define $\bar{y}_1 \in C\left([0,\bar{L}_1], \mathbb{R}\right)$ by $\bar{y}_1(t) = \bar{p}(t-1)$ for all $t \in [0, \bar{L}_1]$. As $\bar{p}(-1) = 1+\varepsilon$, it is clear that $\bar{y}_1(0) = 1+\varepsilon$. Set $\bar{\theta}_1 = \bar{y}_1(\bar{L}_1) = \bar{p}(-1+\bar{L}_1)$. Then $\bar{y}_1$ obviously fulfills the conditions in (H4). Also note that

$$\dot{\bar{y}}_1(t) = \dot{\bar{p}}(t-1) = -\bar{p}(t-1) + f_K(\bar{p}(t-2)) = -\bar{y}_1(t) + K \quad \text{for } t \in \left[0, \bar{L}_1\right].$$

Considering the solution of this linear equation, it is clear that $\bar{y}_1$ and $\bar{\theta}_1$ satisfy the conditions in (H3) and (H5): $\bar{\theta}_1 > 1+\varepsilon$ and $\bar{y}_1(t) > 1+\varepsilon$ for all $t \in (0, \bar{L}_1)$.

3. At this point we do not have enough information to determine $\bar{y}_2$, $\bar{y}_3$ or $\bar{y}_4$. Next we define $\bar{y}_5$*,* $\bar{y}_6$ *and those parameters that are related to them (namely,* $\bar{L}_2, \bar{L}_5, \bar{\theta}_4$ *and* $\bar{\theta}_5$*).* Recall that $p(-1+\tau_1) = 1+\varepsilon$. Under the assumptions of the proposition, one can give a minimal $\bar{\tau}_1 > 0$ (arbitrarily close to τ_1) such that $\bar{p}(-1+\bar{\tau}_1) = 1+\varepsilon$. By (5.5) and by the convergence of $\bar{p}_{-1}$ to p_{-1}, we may assume the existence of $\bar{L}_5 > 0$ and $\bar{L}_2 > 0$ (arbitrarily close to L_5 and L_2, respectively) such that

$$\bar{p}(t) \in (-1,1) \quad \text{for } t \in \left(-2+\bar{\tau}_1 - \bar{L}_5, -2+\bar{\tau}_1+\bar{L}_2\right). \tag{5.8}$$

Choose $\bar{L}_2$ and $\bar{L}_5$ such that the time interval in (5.8) is maximal:

$$\bar{p}\left(-2+\bar{\tau}_1-\bar{L}_5\right) = 1 \quad \text{and} \quad \bar{p}\left(-2+\bar{\tau}_1+\bar{L}_2\right) = -1. \tag{5.9}$$

This means that

$$\dot{\bar{p}}(t) = -\bar{p}(t) \text{ for } t \in \left(-1+\bar{\tau}_1-\bar{L}_5, -1+\bar{\tau}_1+\bar{L}_2\right).$$

As p is positive on $(-1+\tau_1-L_5, -1+\tau_1+L_2)$, we may assume that $\bar{p}$ is positive on $(-1+\bar{\tau}_1-\bar{L}_5, -1+\bar{\tau}_1+\bar{L}_2)$, and hence we deduce from the last ordinary differential equation that $\bar{p}$ strictly decreases on this interval. Set $\bar{y}_5 \in C\left([0,\bar{L}_5], \mathbb{R}\right)$ and $\bar{y}_6 \in C\left([0,\bar{L}_2], \mathbb{R}\right)$ by

$$\bar{y}_5(t) = \bar{p}\left(t-1+\bar{\tau}_1-\bar{L}_5\right) \quad \text{for all } t \in \left[0, \bar{L}_5\right],$$

and

$$\bar{y}_6(t) = \bar{p}(t - 1 + \bar{\tau}_1) \quad \text{for all } t \in \left[0, \bar{L}_2\right].$$

Then $\bar{y}_5$ and $\bar{y}_6$ are strictly decreasing functions with $\bar{y}_5(\bar{L}_5) = \bar{y}_6(0) = \bar{p}(-1 + \bar{\tau}_1) = 1 + \varepsilon$. Let

$$\bar{\theta}_4 = \bar{y}_5(0) = \bar{p}\left(-1 + \bar{\tau}_1 - \bar{L}_5\right), \quad \bar{\theta}_5 = \bar{y}_6\left(\bar{L}_2\right) = \bar{p}\left(-1 + \bar{\tau}_1 + \bar{L}_2\right), \tag{5.10}$$

so that $\bar{y}_5$ and $\bar{y}_6$ satisfy the conditions in (H4). With these choices, $\bar{\theta}_4$ and $\bar{\theta}_5$ are arbitrarily close to $\theta_4 = p(-1 + \tau_1 - L_5)$ and $\theta_5 = p(-1 + \tau_1 + L_2)$, respectively, and hence one can achieve that $\bar{\theta}_4 > 1 + \varepsilon$ and $\bar{\theta}_5 \in (1, 1 + \varepsilon)$ – as required by (H3). The monotonicity of $\bar{y}_5$ and $\bar{y}_6$ guarantee that $\bar{y}_5$ and $\bar{y}_6$ also fulfill the next conditions in (H5): $\bar{y}_5(t) > 1 + \varepsilon$ for all $t \in (0, \bar{L}_5)$ and $\bar{y}_6(t) \in (1, 1 + \varepsilon)$ for all $t \in (0, \bar{L}_2)$.

4. *The functions $\bar{y}_8$, $\bar{y}_9$, $\bar{y}_{10}$ and the parameters $\bar{L}_3, \bar{L}_4, \bar{\theta}_6$.* Let $\bar{\tau}_2 \in (-1 + \bar{\tau}_1, -1 + \bar{\omega})$ be minimal with $\bar{p}(-1 + \bar{\tau}_2) = 1$. Such $\bar{\tau}_2$ exists because $\bar{p}_0$, $\bar{\tau}_1$, $\bar{\omega}$ is arbitrarily close to p_0, τ_1, ω, respectively. Then $\bar{\tau}_2$ converges to τ_2 as $\bar{p}_0$ converges to p_0.

The convergence of $\bar{p}_{-1}$ to p_{-1} and property (5.6) ensure the existence of $\bar{L}_4 > 0$ (arbitrarily close to L_4) so that

$$\bar{p}\left(-2 + \bar{\tau}_2 - \bar{L}_4\right) = -1 - \varepsilon. \tag{5.11}$$

and Let $\bar{y}_8$ be the continuous function on $[0, \bar{L}_4]$ given by

$$\bar{y}_8(t) = \bar{p}\left(t - 1 + \bar{\tau}_2 - \bar{L}_4\right) \quad \text{for all } t \in \left[0, \bar{L}_4\right],$$

and let

$$\bar{\theta}_6 = \bar{y}_8(0) = \bar{p}\left(-1 + \bar{\tau}_2 - \bar{L}_4\right). \tag{5.12}$$

Then $\bar{\theta}_6$ is arbitrarily close to $\theta_6 = p(-1+\tau_2-L_4)$, and therefore we may assume that $\bar{\theta}_6 \in (1, 1 + \varepsilon)$ (see again (H3)). At the right-end point of its domain, $\bar{y}_8$ takes the value $\bar{y}_8(\bar{L}_4) = \bar{p}(-1 + \bar{\tau}_2) = 1$.

Note that we have already defined $\bar{L}_2$ and $\bar{L}_5$. Set $\bar{y}_9 \in C([0, \bar{L}_2 + \bar{L}_5], \mathbb{R})$ so that

$$\bar{y}_9(t) = \bar{p}(t - 1 + \bar{\tau}_2) \quad \text{for all } t \in \left[0, \bar{L}_2 + \bar{L}_5\right].$$

It is clear that $\bar{y}_9(0) = \bar{p}(-1 + \bar{\tau}_2) = 1$. Now recall from (5.9) that there exists an interval of length $\bar{L}_2 + \bar{L}_5$ on which $\bar{p}$ decreases from 1 to -1. This fact and Proposition 5.1.(i) together imply that if $\bar{p}$ decreases from 1 to -1 on any subinterval I of $\mathbb{R}$, then the length of I is $\bar{L}_2 + \bar{L}_5$. Thus $\bar{y}_9(\bar{L}_2 + \bar{L}_5) = \bar{p}(-1 + \bar{\tau}_2 + \bar{L}_2 + \bar{L}_5) = -1$.

Set $\bar{\tau}_3 = \bar{\tau}_2 + \bar{L}_2 + \bar{L}_5$. Then, by our last result, $\bar{p}(-1 + \bar{\tau}_3) = -1$.

Fix $\bar{L}_3 > 0$ to be time that $\bar{p}$ needs to decrease from -1 to $-1 - \varepsilon$. As we have mentioned in the previous paragraph, Proposition 5.1.(i) guarantees that $\bar{L}_3$ is well-defined. Choose $\bar{y}_{10}$ to be the continuous function on $[0, \bar{L}_3]$ defined by

$$\bar{y}_{10}(t) = \bar{p}(t - 1 + \bar{\tau}_3) \quad \text{for all } t \in \left[0, \bar{L}_3\right].$$

Then

$$\bar{y}_{10}(0) = \bar{p}\left(-1+\bar{\tau}_3\right) = -1, \quad \bar{y}_{10}\left(\bar{L}_3\right) = \bar{p}\left(-1+\bar{\tau}_3+\bar{L}_3\right) = -1-\varepsilon. \tag{5.13}$$

We have already verified that $\bar{y}_8$, $\bar{y}_9$ and $\bar{y}_{10}$ fulfill the conditions given in (H4). It remains to show the conditions listed in (H5):

$$\bar{y}_8(t) \in (1, 1+\varepsilon) \text{ for } t \in \left(0, \bar{L}_4\right), \quad \bar{y}_9(t) \in (-1, 1) \text{ for } t \in \left(0, \bar{L}_2+\bar{L}_5\right) \tag{5.14}$$

and

$$\bar{y}_{10}(t) \in (-1-\varepsilon, -1) \text{ for } t \in \left(0, \bar{L}_3\right). \tag{5.15}$$

Note that $\bar{\tau}_3 + \bar{L}_3$ is arbitrarily close to $\tau_3 + L_3 = \omega$. Hence, by property (5.6), we may assume that

$$\bar{p}(t) < -1-\varepsilon \text{ for all } t \text{ in } \left(-2+\bar{\tau}_2-\bar{L}_4, -2+\bar{\tau}_3+\bar{L}_3\right]. \tag{5.16}$$

We see from this and from the definitions of $\bar{y}_8$, $\bar{y}_9$ and $\bar{y}_{10}$ that $\bar{y}_i$ is a solution of $\dot{y} = -y - K$ for all $i \in \{8, 9, 10\}$. Hence the functions $\bar{y}_8$, $\bar{y}_9$ and $\bar{y}_{10}$ are strictly decreasing on their domains. Looking at the boundary values of $\bar{y}_8$, $\bar{y}_9$ and $\bar{y}_{10}$, it is clear that (5.14) and (5.15) are satisfied.

5. *The function* $\bar{y}_7 \in C\left([0, \bar{L}_3], \mathbb{R}\right)$. By the last step of the proof, if $\bar{p}$ decreases from -1 to $-1-\varepsilon$ on an interval J, then the length of J is $\bar{L}_3$. Now recall from (5.9) and (5.11) that

$$\bar{p}\left(-2+\bar{\tau}_1+\bar{L}_2\right) = -1 \quad \text{and} \quad \bar{p}\left(-2+\bar{\tau}_2-\bar{L}_4\right) = -1-\varepsilon.$$

Hence necessarily

$$\bar{\tau}_2 = \bar{\tau}_1 + \bar{L}_2 + \bar{L}_3 + \bar{L}_4, \tag{5.17}$$

and the length of $\left[-1+\bar{\tau}_1+\bar{L}_2, -1+\bar{\tau}_2-\bar{L}_4\right]$ is $\bar{L}_3$. Suppose that the function $\bar{y}_7 \in C\left([0, \bar{L}_3], \mathbb{R}\right)$ is defined by

$$\bar{y}_7(t) = \bar{p}\left(t-1+\bar{\tau}_1+\bar{L}_2\right) \quad \text{for } t \in \left[0, \bar{L}_3\right].$$

Then $\bar{y}_7$ satisfies the boundary conditions in (H4):

$$\bar{y}_7(0) = \bar{p}\left(-1+\bar{\tau}_1+\bar{L}_2\right) = \bar{\theta}_5 \quad \text{and} \quad \bar{y}_7\left(\bar{L}_3\right) = \bar{p}\left(-1+\bar{\tau}_2-\bar{L}_4\right) = \bar{\theta}_6,$$

see (5.10) and (5.12). As $\bar{p}_0$ is arbitrarily close to p_0, it is clear that $\bar{y}_7(t) \in (1, 1+\varepsilon)$ for all t in $\left(0, \bar{L}_3\right)$ – as required by (H5).

6. *The functions* $\bar{y}_2$, $\bar{y}_3$, $\bar{y}_4$ *and the parameters* $\bar{\theta}_2, \bar{\theta}_3$. Recall from (5.7) and (5.9) that

$$\bar{p}\left(-2+\bar{L}_1\right) = 1+\varepsilon \quad \text{and} \quad \bar{p}\left(-2+\bar{\tau}_1-\bar{L}_5\right) = 1,$$

that is, $\bar{p}$ decreases from $1+\varepsilon$ to 1 on $\left[-2+\bar{L}_1, -2+\bar{\tau}_1-\bar{L}_5\right]$. Recall from the definition of $\bar{\tau}_1$ and $\bar{\tau}_2$ that $\bar{p}$ decreases from $1+\varepsilon$ to 1 also on $[-1+\bar{\tau}_1, -1+\bar{\tau}_2]$. Necessarily, the length of the interval $\left[-2+\bar{L}_1, -2+\bar{\tau}_1-\bar{L}_5\right]$ equals the length of $[-1+\bar{\tau}_1, -1+\bar{\tau}_2]$, which is $\bar{L}_2+\bar{L}_3+\bar{L}_4$ by (5.17). In consequence, $\bar{\tau}_1 = \sum_{i=1}^{5} \bar{L}_i$ and the length of $\left[-1+\bar{L}_1, -1+\bar{\tau}_1-\bar{L}_5\right]$ is also $\bar{L}_2+\bar{L}_3+\bar{L}_4$. We use this property to introduce $\bar{y}_2 \in C([0,\bar{L}_2],\mathbb{R})$, $\bar{y}_3 \in C([0,\bar{L}_3],\mathbb{R})$ and $\bar{y}_4 \in C([0,\bar{L}_4],\mathbb{R})$ as restrictions of $\bar{p}$ to subintervals of $\left[-1+\bar{L}_1, -1+\bar{\tau}_1-\bar{L}_5\right]$:

$$\begin{aligned}\bar{y}_2(t) &= \bar{p}\left(t-1+\bar{L}_1\right) \quad \text{for } t \in \left[0,\bar{L}_2\right], \\ \bar{y}_3(t) &= \bar{p}\left(t-1+\bar{L}_1+\bar{L}_2\right) \quad \text{for } t \in \left[0,\bar{L}_3\right], \\ \bar{y}_4(t) &= \bar{p}\left(t-1+\bar{L}_1+\bar{L}_2+\bar{L}_3\right) \quad \text{for } t \in \left[0,\bar{L}_4\right].\end{aligned}$$

In addition, let

$$\bar{\theta}_2 = \bar{y}_2\left(\bar{L}_2\right) = \bar{y}_3(0) \quad \text{and} \quad \bar{\theta}_3 = \bar{y}_3\left(\bar{L}_3\right) = \bar{y}_4(0).$$

It is clear that the functions $\bar{y}_2$, $\bar{y}_3$, $\bar{y}_4$ satisfy the boundary conditions required by (H4) because

$$\bar{y}_2(0) = \bar{p}\left(-1+\bar{L}_1\right) = \bar{y}_1\left(\bar{L}_1\right) = \bar{\theta}_1$$

(see Step 2 of this proof) and

$$\bar{y}_4\left(\bar{L}_4\right) = \bar{p}\left(-1+\bar{\tau}_1-\bar{L}_5\right) = \bar{y}_5(0) = \bar{\theta}_4$$

(see 5.10). Note that $\bar{y}_i(t) > 1+\varepsilon$ for all t in the domains of $\bar{y}_i, i \in \{2,3,4\}$, if and only if $\bar{p}(t) > 1+\varepsilon$ for $t \in \left[-1+\bar{L}_1, -1+\bar{\tau}_1-\bar{L}_5\right]$. The last inequality holds if $\|\bar{p}_0 - p_0\|$ is small enough, as $p(t) > 1+\varepsilon$ for $t \in [-1+L_1, -1+\tau_1-L_5]$.

7. *The proof of the equality $\bar{\omega} = \bar{\tau}_3 + \bar{L}_3$.* As $\bar{\tau}_3$, $\bar{L}_3$ and $\bar{\omega}$ are arbitrarily close to τ_3, L_3 and $\omega = \tau_3 + L_3$, we see that $\bar{\omega}$ is arbitrarily close to $\bar{\tau}_3 + \bar{L}_3$. As $\bar{p}(-1) = 1+\varepsilon$, the symmetry property (5.3) implies that $\bar{p}(-1+\bar{\omega}) = -1-\varepsilon$. We have also mentioned that $\bar{p}\left(-1+\bar{\tau}_3+\bar{L}_3\right) = -1-\varepsilon$, see (5.13). If $\bar{\omega} \neq \bar{\tau}_3 + \bar{L}_3$, then there exists ξ between $\bar{\omega}$ and $\bar{\tau}_3 + \bar{L}_3$ such that

$$\bar{p}(-1+\xi) = -1-\varepsilon \quad \text{and} \quad \dot{\bar{p}}(-1+\xi) = 0$$

(here we use the monotonicity property described in Proposition 5.1.(i)). Then necessarily

$$f_{\bar{K}}(\bar{p}(-2+\xi)) = \dot{\bar{p}}(-1+\xi) + \bar{p}(-1+\xi) = -1-\varepsilon.$$

This result contradicts the fact that $f_{\bar{K}}(\bar{p}(-2+\xi))$ is arbitrarily close to

$$f_{\bar{K}}\left(\bar{p}\left(-2+\bar{\tau}_3+\bar{L}_3\right)\right) = -K$$

(see (5.16)). So $\bar{\omega} = \bar{\tau}_3 + \bar{L}_3$.

Observe that we have verified all the equalities in (5.2).

8. Summing up, $\bar{p}$ is the concatenation of the auxiliary functions $\bar{y}_1, \ldots, \bar{y}_{10}$ as given in (5.1), the equalities (5.2) are satisfied, and all conditions listed in (H1) and (H3)-(H5) hold. *It remains to verify (H2).*

It is clear from above that

$$\bar{L} := 2\bar{L}_1 + 5\bar{L}_2 + 5\bar{L}_3 + 3\bar{L}_4 + 3\bar{L}_5$$

is arbitrarily close to $2L_1 + 5L_2 + 5L_3 + 3L_4 + 3L_5 = 1$ if $\left|\bar{K} - K^*(\varepsilon)\right|$ and $\|\bar{p}_0 - p_0\|$ are small enough. We complete the proof by showing that $\bar{L} = 1$.

Using the symmetry property (5.3) and the equations (5.1)-(5.2), we calculate that

$$
\begin{aligned}
\bar{p}\left(t-1+\bar{L}\right) &= -\bar{y}_8(t) \quad \text{for } t \in \left[0, \bar{L}_4\right], \\
\bar{p}\left(t-1+\bar{L}+\bar{L}_4\right) &= -\bar{y}_9(t) \quad \text{for } t \in \left[0, \bar{L}_2+\bar{L}_5\right], \\
\bar{p}\left(t-1+\bar{L}+\bar{L}_2+\bar{L}_4+\bar{L}_5\right) &= -\bar{y}_{10}(t) \quad \text{for } t \in \left[0, \bar{L}_3\right], \\
\bar{p}\left(t-1+\bar{L}+\bar{L}_2+\bar{L}_3+\bar{L}_4+\bar{L}_5\right) &= \bar{y}_1(t) \quad \text{for } t \in \left[0, \bar{L}_1\right].
\end{aligned}
$$

Recall that $-\bar{y}_8, -\bar{y}_9, -\bar{y}_{10}$ and $\bar{y}_1$ are solutions of $\dot{y} = -y + K$. Hence, by the above equalities,

$$\dot{\bar{p}}(t) = -\bar{p}\,(t) + K \quad \text{for } t \in \left[-1+\bar{L}, -1+\bar{L}+\bar{\tau}_1\right],$$

which is possible only if

$$\bar{p}(t) \geq 1+\varepsilon \quad \text{for } t \in \left[-2+\bar{L}, -2+\bar{L}+\bar{\tau}_1\right].$$

We already know that $\bar{p}(t) < -1-\varepsilon$ for $t \in (-1+\bar{\tau}_1, -1+\bar{\omega})$. Also observe that

$$\bar{p}(t) < -1-\varepsilon \quad \text{for } t \in \left[-1-\bar{L}_3, -1\right)$$

as $\bar{p}(t-1-\bar{L}_3) = -\bar{y}_{10}(t)$ for $t \in [0, \bar{L}_3]$. As $\bar{L}$ is arbitrarily close to 1, necessarily $\bar{L} = 1$.

9. It follows from Proposition 2.1 that $\bar{L}_2$ is the fixed point of $L_2 \mapsto F(L_2, K, \varepsilon)$. □

6. The proof of Theorem 1.1

Proof of Theorem 1.1. Existence. According to Proposition 4.3, if $\varepsilon > 0$ is sufficiently small, then there are $K^* \in (6.5, 7)$ and $L_2^* \in (0, \widehat{L}_2(K^*, \varepsilon))$ such that F_ε undergoes a saddle-node bifurcation at (L_2^*, K^*): one can give a constant $\delta_1 > 0$ such that

- if $K \in (K^* - \delta_1, K^*)$, then $F_\varepsilon(\cdot, K)$ has no fixed points close to L_2^*,
- L_2^* is an isolated fixed point of $F_\varepsilon(\cdot, K^*)$,
- and if $K \in (K^*, K^* + \delta_1)$, then $F_\varepsilon(\cdot, K)$ has exactly two fixed points (converging to L_2^* as $K \to K^*$).

By Corollary 3.6, the fixed points of $F_\varepsilon(\cdot, K)$ yield periodic solutions if $\varepsilon > 0$ is small enough. Thereby we obtain one periodic orbit for $K = K^*$, and two different ones for each $K \in (K^*, K^* + \delta_1)$. Let $p\colon \mathbb{R} \to \mathbb{R}$ denote the periodic solution given for the bifurcation parameter K^*. Corollary 3.6 implies that the initial segments of both periodic solutions corresponding to parameters $K \in (K^*, K^* + \delta_1)$ converge to p_0 as $K \to K^*$.

Uniqueness. Proposition 5.2 gives a constant $\delta_2 > 0$ and an open neighborhood N of p_0 in the hyperplane

$$H = \{\varphi \in C\colon\ \varphi(-1) = 1 + \varepsilon\}$$

such that if $K \in (K^* - \delta_2, K^* + \delta_2)$ and $\bar{p}$ is a periodic solution with $\bar{p}_0 \in N$, then $\bar{p}_0$ derives from a fixed point of $F_\varepsilon(\cdot, K)$ as in Corollary 3.6.

Consider a sufficiently small open neighborhood B of p_0 in the phase space C, and the standard Poincaré map $\mathcal{P}$ from B to H with fixed point p_0. (The existence of such $\mathcal{P}$ can be shown using the implicit function theorem and the fact that $\mathcal{O} = \{p_t : t \in \mathbb{R}\}$ intersects H transversally, see [1, 16] and Appendix I in [7].) As $\mathcal{P}$ depends continuously on $\varphi \in C$ and on the right hand side of (1.1), we may assume that $\mathcal{P}$ maps B into the neighborhood N for all $K \in (K^* - \delta_2, K^* + \delta_2)$. This means that if $K \in (K^* - \delta_2, K^* + \delta_2)$ and $\bar{p}$ is a periodic solution with segments in B, then, by the translation of time, it derives from a fixed point of $F_\varepsilon(\cdot, K)$.

Final step. Choose $\delta \in (0, \min\{\delta_1, \delta_2\})$ so small that for $K \in (K^*, K^* + \delta)$, both periodic solutions given in Step 1 have initial segments in B. (This is possible as the initial functions of these periodic solutions converge to p_0 as $K \to K^*$.) The main theorem of the paper holds with this constant δ and neighborhood B. It is clear from our construction that all periodic solutions in question are of large amplitude, i.e., they oscillate about both unstable fixed points of f_K. □

As the bifurcation of the large-amplitude periodic orbits corresponds to the bifurcation of the fixed points of F_ε, we see from Corollary 3.2 that K^* tends to K_0 as $\varepsilon \to 0^+$.

Appendix A.

In the Appendix we examine the partial derivatives of $F\colon U \to \mathbb{R}$ and work with the assumption that

(H6) L_i, $i \in \{1, 3, 4, 5\}$, and θ_i, $1 \le i \le 6$, are defined by (C.1)-(C.10) on U.

We will use notation O as discussed before the proof of Proposition 3.1.

Remark A.1. Let $\alpha \in \mathbb{R}$. Assume that θ_6 is defined by (C.5) on U. Recall from Proposition 3.1.(i) that $\theta_6 = 1 + O(\varepsilon)$ as $\varepsilon \to 0^+$. Therefore, by the binomial expansion,

$$\begin{aligned}(K + \theta_6)^\alpha &= (K+1)^\alpha \left(1 + \frac{\theta_6 - 1}{K+1}\right)^\alpha \\ &= (K+1)^\alpha \sum_{n=0}^{\infty} \binom{\alpha}{n} \left(\frac{\theta_6 - 1}{K+1}\right)^n \\ &= (K+1)^\alpha + O(\varepsilon) \quad \text{as } \varepsilon \to 0^+.\end{aligned} \tag{A.1}$$

Similarly,

$$(K-1-\varepsilon)^\alpha = (K-1)^\alpha \left(1 - \frac{\varepsilon}{K-1}\right)^\alpha = (K-1)^\alpha + O(\varepsilon) \tag{A.2}$$

if $K \in (6.5, 7)$ and $\varepsilon \to 0^+$.

Proposition A.2. *Assume (H6). Then F is continuously differentiable on U with respect to K. Furthermore,*

$$\begin{aligned}\frac{\partial}{\partial K} F(L_2, K, \varepsilon) =& 1 - e^{-\frac{1}{2}} \left(\frac{3}{2}\sqrt{\frac{K+1}{K-1}} - \frac{1}{2}\sqrt{\left(\frac{K+1}{K-1}\right)^3}\right) \\ &+ \frac{2}{(K-1)^2} + O(\varepsilon)\end{aligned} \tag{A.3}$$

as $\varepsilon \to 0^+$.

Proof. We explicitly calculate the derivatives of the three main terms of F with respect to K in order to see that they are continuous on U. Meanwhile, we show that

(i) $\partial\theta_6/\partial K = O(\varepsilon)$, $\partial L_4/\partial K = O(\varepsilon)$ and thus

$$\frac{\partial}{\partial K}\left(\frac{K(K+1)}{\varepsilon}\left(1-(1-L_4)e^{L_4}\right)\right) = O(\varepsilon),$$

(ii)

$$\frac{\partial \theta_3}{\partial K} = 1 - e^{-\frac{1}{2}} \left(\frac{3}{2}\sqrt{\frac{K+1}{K-1}} - \frac{1}{2}\sqrt{\left(\frac{K+1}{K-1}\right)^3}\right) + O(\varepsilon),$$

(iii) and

$$\frac{\partial}{\partial K}\left((1+\varepsilon)\frac{K+\theta_6}{K-1}e^{-L_2}\right) = \frac{-2}{(K-1)^2} + O(\varepsilon) \text{ as } \varepsilon \to 0^+.$$

For notational simplicity, let $'$ denote differentiation with respect to K in this proof.

(i) The derivative of θ_6 with respect to K is

$$\theta_6' = \frac{\varepsilon(1+\varepsilon)}{(K-1)^2}e^{-L_2} + \frac{2K-1-\varepsilon}{\varepsilon}\ln\frac{K-1}{K-1-\varepsilon} - \frac{2K-1}{K-1}. \tag{A.4}$$

As $L_2 \in (-\varepsilon, \varepsilon)$, it is clear that

$$\frac{\varepsilon(1+\varepsilon)}{(K-1)^2}e^{-L_2} = O(\varepsilon).$$

Using first (3.3) and then (A.2) with $\alpha = -1$, we deduce that

$$\frac{2K-1-\varepsilon}{\varepsilon}\ln\frac{K-1}{K-1-\varepsilon} = \frac{2K-1-\varepsilon}{K-1-\varepsilon} + O(\varepsilon) = \frac{2K-1}{K-1} + O(\varepsilon).$$

It follows from above that $\theta_6' = O(\varepsilon)$.

Next observe that the derivative of L_4 with respect to K is

$$L_4' = \frac{1+\theta_6'}{K+\theta_6} - \frac{1}{K+1}.$$

Applying (A.1) with $\alpha = -1$, we obtain that $(K+\theta_6)^{-1} = (K+1)^{-1} + O(\varepsilon)$. Using $\theta_6' = O(\varepsilon)$ and the last result, we conclude that $L_4' = O(\varepsilon)$.

In order to complete the proof of (i), let us examine

$$\left(\frac{K(K+1)}{\varepsilon}\left(1-(1-L_4)e^{L_4}\right)\right)',$$

that is

$$\frac{2K+1}{\varepsilon}\left(1-(1-L_4)e^{L_4}\right) + \frac{K(K+1)}{\varepsilon}L_4L_4'e^{L_4}.$$

By Proposition 3.1.(i), the first term of this expression is $O(\varepsilon)$. The second term is also $O(\varepsilon)$ because $L_4 = O(\varepsilon)$ (see again Proposition 3.1.(i)) and $L_4' = O(\varepsilon)$.

(ii) Recall that θ_3 is defined by (C.10). As θ_3 a function of θ_2, first we differentiate θ_2 with respect to K using formula (C.9):

$$\begin{aligned}\theta_2' &= e^{-L_2} + \frac{1}{\varepsilon}\left((1+\varepsilon)L_2e^{-L_2} + e^{-L_2} - 1\right)\\ &\quad - e^{-\frac{1}{2}}\left(\left(\frac{K+\theta_6}{K-1-\varepsilon}\right)^{\frac{3}{2}} - \frac{3}{2}\frac{(K-1)(K+\theta_6)^{\frac{3}{2}}}{(K-1-\varepsilon)^{\frac{5}{2}}}\right)\\ &\quad - \frac{3}{2}e^{-\frac{1}{2}}\frac{(K-1)(K+\theta_6)^{\frac{1}{2}}(1+\theta_6')}{(K-1-\varepsilon)^{\frac{3}{2}}}.\end{aligned}$$

Using the inequality $|L_2| < \varepsilon$, result (3.9) with $t = L_2$, Remark A.1 with various α-s and statement (i) of this proposition, we get that

$$\theta_2' = 1 - e^{-\frac{1}{2}}\left(\frac{3}{2}\sqrt{\frac{K+1}{K-1}} - \frac{1}{2}\sqrt{\left(\frac{K+1}{K-1}\right)^3}\right) + O(\varepsilon). \tag{A.5}$$

Now we differentiate the second term of θ_3 with respect to K. We get from (C.1) that

$$L_3' = -\frac{\varepsilon}{(K-1)(K-1-\varepsilon)} \quad \text{and} \quad \left(e^{-L_3}\right)' = \frac{\varepsilon}{(K-1)^2}. \tag{A.6}$$

Therefore

$$\left(\frac{K}{\varepsilon}\left((1+\varepsilon)L_3e^{-L_2-L_3} + e^{-L_3} - 1\right)\right)'$$

is equivalent to

$$\begin{aligned}&\frac{1}{\varepsilon}\left((1+\varepsilon)L_3e^{-L_2-L_3} + e^{-L_3} - 1\right)\\ &+ \frac{K}{(K-1)^2}\left(1-(1+\varepsilon)e^{-L_2} + L_3(1+\varepsilon)e^{-L_2}\right).\end{aligned}$$

Using that $L_2 = O(\varepsilon)$, $L_3 = O(\varepsilon)$ and applying (3.10) with $t = L_3$, we deduce that

$$\left(\frac{K}{\varepsilon}\left((1+\varepsilon)L_3 e^{-L_2-L_3} + e^{-L_3} - 1\right)\right)' = O(\varepsilon). \tag{A.7}$$

The last term of θ_3 is

$$\frac{K^2}{\varepsilon^2}(K-1)\left(1-\left(1+L_3+\frac{L_3^2}{2}\right)e^{-L_3}\right).$$

Its derivative with respect to K is

$$\frac{3K^2-2K}{\varepsilon^2}\left(1-\left(1+L_3+\frac{L_3^2}{2}\right)e^{-L_3}\right) - \frac{K^2L_3^2}{2\varepsilon(K-1)}.$$

Since $L_3 = O(\varepsilon)$, and (3.11) holds for $t = L_3$, we conclude that

$$\left(\frac{K^2}{\varepsilon^2}(K-1)\left(1-\left(1+L_3+\frac{L_3^2}{2}\right)e^{-L_3}\right)\right)' = O(\varepsilon). \tag{A.8}$$

Statement (ii) follows immediately from (C.10), (A.5)-(A.8), the fact that $L_3 = O(\varepsilon)$, and the boundedness of θ_2 (see the proof of Proposition 3.1).

(iii) It is clear that

$$\left((1+\varepsilon)\frac{K+\theta_6}{K-1}e^{-L_2}\right)' = \frac{1+\varepsilon}{K-1}e^{-L_2}\left(1+\theta_6' - \frac{K+\theta_6}{K-1}\right).$$

Since $L_2 = O(\varepsilon)$, $\theta_6 = 1 + O(\varepsilon)$ and $\theta_6' = O(\varepsilon)$, we get statement (iii).

Looking again at the derivatives of the terms of F calculated above, we see that they are continuous on U. Hence F is continuously differentiable on U with respect to K. Formula (A.3) follows from statements (i)-(iii). □

Corollary A.3. *Assume that* $\lim_{n\to\infty}\varepsilon_n = 0^+$, $(L_{2,n}, K_n, \varepsilon_n) \in U$ *for all* $n \geq 0$ *and* $F(L_{2,n}, K_n, \varepsilon_n) = L_{2,n}$ *for all* $n \geq 0$. *Then*

$$\begin{aligned}\lim_{n\to\infty}\frac{\partial}{\partial K}F(L_{2,n}, K_n, \varepsilon_n) =& 1 - e^{-\frac{1}{2}}\left(\frac{3}{2}\sqrt{\frac{K_0+1}{K_0-1}} - \frac{1}{2}\sqrt{\left(\frac{K_0+1}{K_0-1}\right)^3}\right)\\ &+ \frac{2}{(K_0-1)^2},\end{aligned} \tag{A.9}$$

where K_0 *is the unique solution of (1.2) in [6.5,7]. This limit is positive.*

Proof. It is clear from Corollary 3.2 and from Proposition A.2 that this limit holds. It remains to verify that it is positive. Expressing $e^{-1/2}$ from (1.2), we deduce that the second term on the right hand side of (A.9) is

$$\frac{{K_0}^2 - 2K_0 - 1}{\sqrt{(K_0-1)(K_0+1)^3}}\left(\frac{3}{2}\sqrt{\frac{K_0+1}{K_0-1}} - \frac{1}{2}\sqrt{\left(\frac{K_0+1}{K_0-1}\right)^3}\right),$$

that is

$$\frac{\left(K_0{}^2 - 2K_0 - 1\right)(K_0 - 2)}{(K_0 - 1)^2 (K_0 + 1)}.$$

This expression is smaller than $(K_0 - 2)/(K_0 + 1) < 1$, therefore (A.9) is greater than zero. □

Proposition A.4. *Under assumption (H6), F is continuously differentiable on U with respect to L_2. In addition,*

$$\frac{\partial}{\partial L_2} F(0, K, \varepsilon) = \frac{K^2 + 8K + 2}{2\left(K^2 - 1\right)} + \frac{3}{2} e^{-\frac{1}{2}} \sqrt{\frac{K+1}{K-1}} + O(\varepsilon)$$

and

$$\frac{\partial}{\partial L_2} F\left(\widehat{L}_2, K, \varepsilon\right) = \frac{-K^2 + 6K + 2}{2(K-1)} + \frac{3}{2} e^{-\frac{1}{2}} \sqrt{\frac{K+1}{K-1}} + O(\varepsilon) \text{ as } \varepsilon \to 0^+,$$

where $\widehat{L}_2$ is given by (3.12)*.*

Proof. The proposition will easily follow from the subsequent three claims.

Claim 1. The derivative of the first term of F with respect to L_2 is

$$\frac{\partial}{\partial L_2} \left(\frac{K(K+1)}{\varepsilon} \left(1 - (1 - L_4) e^{L_4}\right) \right) = \frac{K}{\varepsilon} L_4 \frac{\partial \theta_6}{\partial L_2}, \tag{A.10}$$

where

$$\frac{\partial \theta_6}{\partial L_2} = -(1+\varepsilon) \frac{K - 1 - \varepsilon}{K - 1} e^{-L_2}. \tag{A.11}$$

For $L_2 = 0$, (A.10) equals

$$-\frac{K(K-4)}{2(K^2 - 1)} + O(\varepsilon) \text{ as } \varepsilon \to 0^+. \tag{A.12}$$

For $L_2 = \widehat{L}_2$, it equals 0.

Claim 2. The derivative of θ_3 with respect to L_2 is

$$\frac{\partial \theta_3}{\partial L_2} = \frac{3}{2} \frac{(1+\varepsilon)\sqrt{K + \theta_6}}{\sqrt{K - 1 - \varepsilon}} e^{-L_2 - L_3 - \frac{1}{2}} - \frac{K}{\varepsilon}(1+\varepsilon)(L_2 + L_3) e^{-L_2 - L_3}. \tag{A.13}$$

For $L_2 = 0$, this expression is

$$\frac{3}{2} e^{-\frac{1}{2}} \sqrt{\frac{K+1}{K-1}} - \frac{K}{K-1} + O(\varepsilon).$$

For $L_2 = \widehat{L}_2$, the derivative of θ_3 with respect to L_2 is

$$\frac{3}{2} e^{-\frac{1}{2}} \sqrt{\frac{K+1}{K-1}} - \frac{K(K-2)}{2(K-1)} + O(\varepsilon).$$

Claim 3. The derivative of the third term of F with respect to L_2 is

$$\frac{\partial}{\partial L_2}\left((1+\varepsilon)\frac{K+\theta_6}{K-1}e^{-L_2}\right) = -\frac{1+\varepsilon}{K-1}\left(K+\theta_6-\theta_6'\right)e^{-L_2}. \tag{A.14}$$

For both $L_2 = 0$ and $L_2 = \widehat{L}_2$, this expression is

$$-\frac{K+2}{K-1} + O(\varepsilon).$$

Let $'$ now denote differentiation with respect to L_2.

The proof of Claim 1. The derivatives of θ_6 and L_4 with respect to L_2 are (A.11) and

$$L_4' = \frac{\theta_6'}{K+\theta_6}. \tag{A.15}$$

It is clear that

$$\left(\frac{K(K+1)}{\varepsilon}\left(1-(1-L_4)e^{L_4}\right)\right)' = \frac{K(K+1)}{\varepsilon}L_4L_4'e^{L_4}.$$

Replacing e^{L_4} by $(K+\theta_6)/(K+1)$ and using (A.15), we get (A.10).

Let $L_2 = 0$. From (A.11) we obtain that

$$\theta_6' = -1 + O(\varepsilon) \text{ as } \varepsilon \to 0^+. \tag{A.16}$$

Next we calculate L_4 for $L_2 = 0$ using formula (C.6). So let $L_2 = 0$. Then

$$(1+\varepsilon)\frac{K-1-\varepsilon}{K-1}e^{-L_2} = (1+\varepsilon)\left(1-\frac{\varepsilon}{K-1}\right) = 1 + \frac{K-2}{K-1}\varepsilon - \frac{1}{K-1}\varepsilon^2.$$

As a result, by (C.5) and (3.14),

$$\theta_6 = 1 + \frac{K-4}{2(K-1)}\varepsilon + O\left(\varepsilon^2\right).$$

We easily get from this and from (C.6) that

$$L_4 = \ln\left(1+\frac{\theta_6-1}{K+1}\right) = \frac{K-4}{2(K^2-1)}\varepsilon + O(\varepsilon^2). \tag{A.17}$$

Substituting (A.16) and (A.17) into $KL_4\theta_6'/\varepsilon$, we get (A.12).

By the definition of $\widehat{L}_2$, $L_4 = 0$ if $L_2 = \widehat{L}_2$. Hence (A.10) equals 0 in this case, and the proof of Claim 1 is complete.

The proof of Claim 2. By (C.10), the first term of θ_3 is $\theta_2 e^{-L_3}$. Its derivative with respect to L_2 is

$$\left(\theta_2 e^{-L_3}\right)' = \frac{3}{2}\frac{(1+\varepsilon)\sqrt{K+\theta_6}}{\sqrt{K-1-\varepsilon}}e^{-L_2-L_3-\frac{1}{2}} - \frac{K}{\varepsilon}\left(1+\varepsilon\right)L_2e^{-L_2-L_3}.$$

The derivative of the second term of θ_3 with respect to L_2 is

$$\frac{K}{\varepsilon}\left((1+\varepsilon)L_3e^{-L_2-L_3}+e^{-L_3}-1\right)'=-\frac{K}{\varepsilon}(1+\varepsilon)L_3e^{-L_2-L_3}.$$

The third term of θ_3 is independent from L_2. Summing up, (A.13) is verified.

According to Remark A.1,

$$\sqrt{K+\theta_6}=\sqrt{K+1}+O(\varepsilon) \tag{A.18}$$

and

$$\frac{1}{\sqrt{K-1-\varepsilon}}=\frac{1}{\sqrt{K-1}}+O(\varepsilon).$$

Also recall that $L_2=O(\varepsilon)$ and $L_3=O(\varepsilon)$. Hence, for both $L_2=0$ and $L_2=\hat{L}_2$,

$$\frac{3}{2}\frac{(1+\varepsilon)\sqrt{K+\theta_6}}{\sqrt{K-1-\varepsilon}}e^{-L_2-L_3-\frac{1}{2}}=\frac{3}{2}e^{-\frac{1}{2}}\sqrt{\frac{K+1}{K-1}}+O(\varepsilon). \tag{A.19}$$

Using (3.3) for $L_3=\ln(K-1)/(K-1-\varepsilon)$ and then $(K-1-\varepsilon)^{-1}=(K-1)^{-1}+O(\varepsilon)$, we get the following for $L_2=0$:

$$\begin{aligned}\frac{K}{\varepsilon}(1+\varepsilon)\left(L_2+L_3\right)e^{-L_2-L_3}&=\frac{K}{\varepsilon}(1+\varepsilon)\left(\frac{\varepsilon}{K-1-\varepsilon}+O\left(\varepsilon^2\right)\right)(1+O(\varepsilon))\\&=\frac{K}{K-1}+O(\varepsilon).\end{aligned}$$

Subtracting the last result from (A.19), the formula for θ_3' at $L_2=0$ follows.

Let $L_2=\widehat{L}_2$. Then by (3.17) and by (3.3),

$$\frac{K}{\varepsilon}(1+\varepsilon)\left(\widehat{L}_2+L_3\right)e^{-\widehat{L}_2-L_3}$$

equals

$$\begin{aligned}&\frac{K}{\varepsilon}(1+\varepsilon)\left(\left(\frac{K-4}{2(K-1)}+\frac{1}{K-1}\right)\varepsilon+O\left(\varepsilon^2\right)\right)(1+O(\varepsilon))\\&=\frac{K\left(K-2\right)}{2(K-1)}+O\left(\varepsilon\right).\end{aligned}$$

Subtracting this from (A.19), we complete the proof of Claim 2.

The proof of Claim 3. The proof of Claim 3 is similar and easy, so we leave it to the reader.

The continuous differentiability of F with respect to L_2 is obvious from (A.10)-(A.11), (A.13)-(A.14) and from the definition of F. The formulas for $F'(0,K,\varepsilon)$ and $F'(\widehat{L}_2,K,\varepsilon)$ immediately follow from Claims 1–3. □

Corollary A.5. *Under hypothesis (H6),*

$$\lim_{\varepsilon\to 0^+}\frac{\partial}{\partial L_2}F(0,\varphi_\varepsilon(0),\varepsilon)>1 \tag{A.20}$$

and

$$\lim_{\varepsilon\to 0^+}\frac{\partial}{\partial L_2}F\left(\widehat{L}_2,\varphi_\varepsilon\left(\widehat{L}_2\right),\varepsilon\right)<1, \tag{A.21}$$

where φ_ε *is the map given by Proposition 4.2.*

Proof. Recall from Corollary 3.2 that if $\varepsilon\to 0^+$, then $\varphi_\varepsilon(0)\to K_0\in(6.5,7)$. We know from (1.2) that

$$e^{-\frac{1}{2}}=\frac{K_0^2-2K_0-1}{\sqrt{K_0-1}\sqrt{(K_0+1)^3}}.$$

Therefore, by the results of the previous proposition,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{\partial}{\partial L_2}F(0,\varphi_\varepsilon(0),\varepsilon)&=\frac{K_0^2+8K_0+2}{2\left(K_0^2-1\right)}+\frac{3\left(K_0^2-2K_0-1\right)}{2\left(K_0^2-1\right)}\\&=1+\frac{2K_0^2+2K_0+1}{2\left(K_0^2-1\right)}.\end{aligned}$$

As $K_0\in(6.5,7)$, the last quotient is clearly positive, and the limit above is greater than 1.

Similarly,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{\partial}{\partial L_2}F\left(\widehat{L}_2,\varphi_\varepsilon\left(\widehat{L}_2\right),\varepsilon\right)&=\frac{-K_0^2+6K_0+2}{2\left(K_0-1\right)}+\frac{3\left(K_0^2-2K_0-1\right)}{2\left(K_0^2-1\right)}\\&=1+\frac{-K_0^3+6K_0^2+2K_0+1}{2K_0^2-2}.\end{aligned} \tag{A.22}$$

Since $0.5K^2>2K+1$ for $K>2+\sqrt{6}$, we deduce that

$$K^3>6.5K^2>6K^2+2K+1 \qquad \text{for} \quad K>6.5.$$

As $K_0>6.5$, this means that the last quotient in (A.22) is negative, and hence the limit for $L_2=\widehat{L}_2$ is smaller than 1. □

In the next proposition we write that $u(L_2,K,\varepsilon)\sim v(K,\varepsilon)$ as $\varepsilon\to 0^+$ for functions u and v defined on U if

$$\lim_{K\to\bar{K},\,\varepsilon\to 0^+,\,L_2\in(-\varepsilon,\varepsilon)}\frac{u(L_2,K,\varepsilon)}{v(K,\varepsilon)}=1.$$

Proposition A.6. *Under assumption (H6),* $\partial^2F/\partial L_2^2$ *is continuous on* U*, and*

$$\frac{\partial^2}{\partial L_2^2}F(L_2,K,\varepsilon)\sim-\frac{K^2}{(K+1)\varepsilon}\ \text{ as } \varepsilon\to 0^+. \tag{A.23}$$

Proof. We explicitly calculate the second partial derivatives of the three main terms of F with respect to L_2. Meanwhile we verify that

(i)
$$\frac{\partial^2}{\partial L_2^2}\left(\frac{K(K+1)}{\varepsilon}\left(1-(1-L_4)e^{L_4}\right)\right) \sim \frac{K}{(K+1)\varepsilon}$$

(ii) and
$$\frac{\partial^2}{\partial L_2^2}\theta_3 \sim -\frac{K}{\varepsilon} \text{ as } \varepsilon \to 0^+.$$

(iii) In addition, we show that
$$\frac{\partial^2}{\partial L_2^2}\left((1+\varepsilon)\frac{K+1}{K-1}e^{L_4-L_2}\right)$$
is bounded for small positive ε.

Let us again use the symbol $'$ for differentiation with respect to L_2.

(i) It is clear from (A.11) that $\theta_6'' = -\theta_6'$. Also recall from (A.15) that $L_4' = \theta_6'/(K+\theta_6)$. Therefore, by formula (A.10), we get that
$$\left(\frac{K(K+1)}{\varepsilon}\left(1-(1-L_4)e^{L_4}\right)\right)'' = \left(\frac{K}{\varepsilon}L_4\theta_6'\right)' = \frac{K}{\varepsilon}\theta_6'\left(\frac{\theta_6'}{K+\theta_6} - L_4\right)$$
which is continuous on U. Since $\theta_6' = -1 + O(\varepsilon)$, $L_4 = O(\varepsilon)$ and $(K+\theta_6)^{-1} = (K+1)^{-1} + O(\varepsilon)$ as $\varepsilon \to 0^+$, statement (i) follows.

(ii) The second derivative of θ_3 with respect to L_2 can be calculated from (A.13). It is also continuous on U:
$$\begin{aligned}\theta_3'' = &\frac{3(1+\varepsilon)}{2\sqrt{K-1-\varepsilon}}e^{-L_2-L_3-\frac{1}{2}}\left(\frac{\theta_6'}{2\sqrt{K+\theta_6}} - \sqrt{K+\theta_6}\right)\\ &-\frac{K}{\varepsilon}(1+\varepsilon)\left(1-L_2-L_3\right)e^{-L_2-L_3}.\end{aligned}$$

Using the same series expansions as before, we can easily see that statement (ii) is true.

(iii) One can calculate from (A.14) and from $\theta_6'' = -\theta_6'$ that the second derivative of third term of F with respect to L_2 is the next continuous function:
$$\left((1+\varepsilon)\frac{K+1}{K-1}e^{L_4-L_2}\right)'' = \frac{1+\varepsilon}{K-1}\left(K+\theta_6-3\theta_6'\right)e^{-L_2}.$$

Since $\theta_6 \to 1$, $\theta_6' \to -1$ and $L_2 \to 0$ as $\varepsilon \to 0^+$, this expression is bounded for small $\varepsilon > 0$.

The continuity of $\partial^2 F/\partial L_2^2$ and (A.23) comes from above and from the definition of F. □

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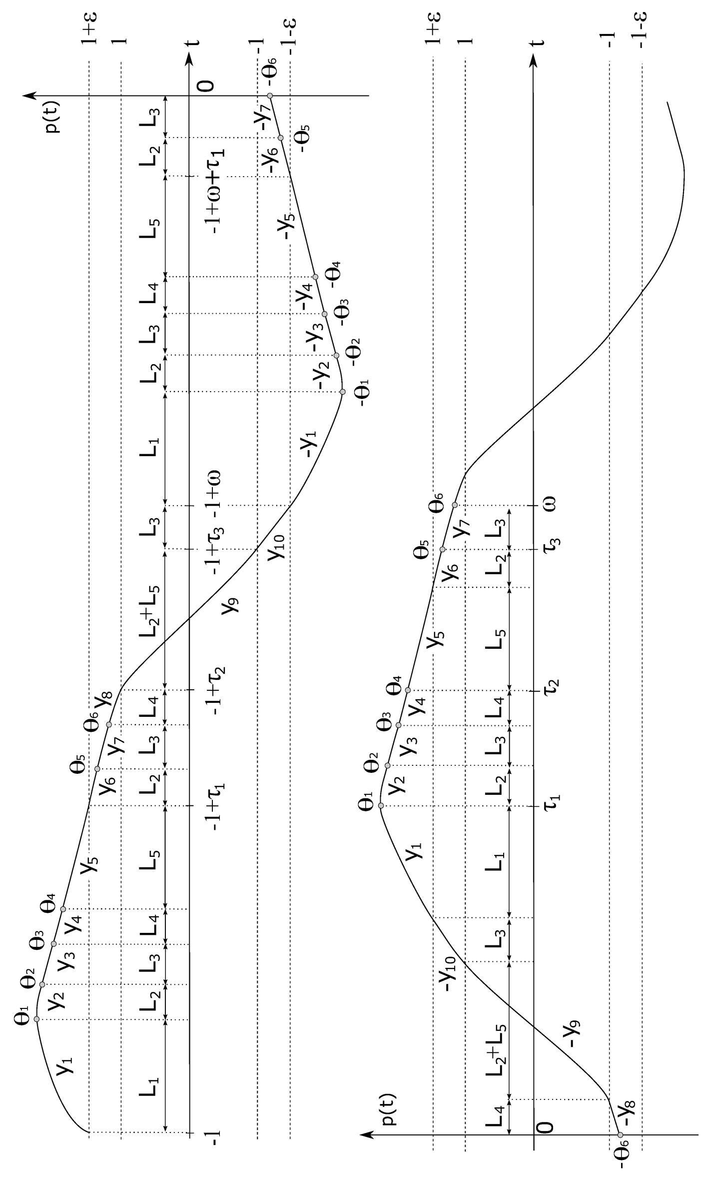

Figure 2.1: The plot of p on [-1,0] and on [0,1].

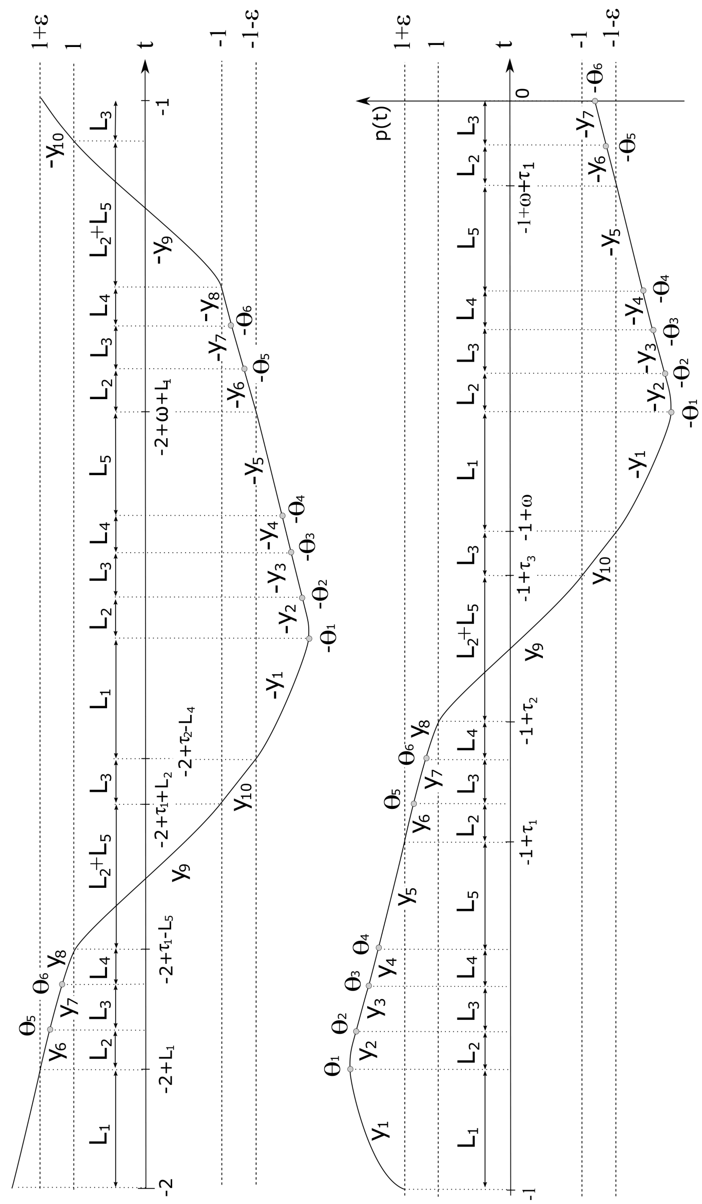

Figure 5.1: The plot of p on [-2,-1] and on [-1,0].