

1 **STABLE PERIODIC SOLUTIONS FOR NAZARENKO'S**
2 **EQUATION**

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ABSTRACT. In 1976 Nazarenko proposed studying the delay differential equation

$$\dot{y}(t) = -py(t) + \frac{qy(t)}{r + y^n(t - \tau)}, \quad t > 0,$$

under the assumptions that $p, q, r, \tau \in (0, \infty)$, $n \in \mathbb{N} = \{1, 2, \dots\}$ and $q/p > r$. We show that if τ or n is large enough, then the positive periodic solution oscillating slowly about $K = (q/p - r)^{1/n}$ is unique, and the corresponding periodic orbit is asymptotically stable. We also determine the asymptotic shape of the periodic solution as $n \rightarrow \infty$.

3 **1. Introduction.** We consider the scalar delay differential equation

$$\dot{y}(t) = -py(t) + \frac{qy(t)}{r + y^n(t - \tau)}, \quad t > 0, \tag{1}$$

4 under the assumption that

$$p, q, r, \tau \in (0, \infty), \quad n \in \mathbb{N} = \{1, 2, \dots\} \quad \text{and} \quad \frac{q}{p} > r. \tag{2}$$

5 This equation was proposed by Nazarenko in 1976 to study the control of a
6 single population of cells [9]. The quantity $y(t)$ is the size of the population at time
7 t . The rate of change $y'(t)$ can be given as the difference of the production rate
8 $qy(t)/(r + y^n(t - \tau))$ and the destruction rate $py(t)$. We see that the destruction
9 rate at time t depends only on the present state $y(t)$ of the system, while the
10 production rate also depends on the past of y . This is a typical concept in population
11 dynamics; delay appears due to the fact that organisms need time to mature before
12 reproduction.

13 For further population model equations with delay, see e.g., [14]. One of the most
14 widely studied examples is the Mackey-Glass equation:

$$\dot{y}(t) = -py(t) + \frac{qy(t-1)}{r + y^n(t - \tau)}, \quad t > 0. \tag{3}$$

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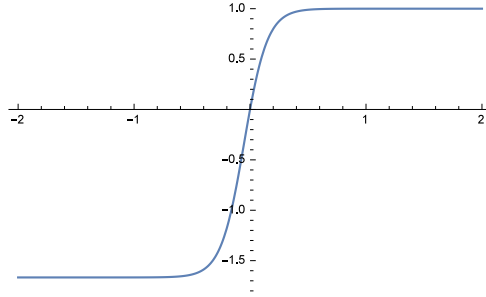


FIGURE 1. The plot of f for $p = 1$, $q = 4$, $r = 1.5$ and $n = 10$

1 In this model the production rate is very similar to the one considered by Nazarenko.

2 The usual phase space for (1) is the Banach space $C = C([- \tau, 0], \mathbb{R})$ with the
 3 supremum norm. A solution of (1) is either a continuously differentiable function
 4 $y: \mathbb{R} \rightarrow \mathbb{R}$ that satisfies (1) for all $t \in \mathbb{R}$, or a continuous function $y: [- \tau, \infty) \rightarrow \mathbb{R}$
 5 that is continuously differentiable for $t > 0$ and satisfies (1) for all $t > 0$. If for
 6 some solution y and $t \in \mathbb{R}$, the interval $[t - \tau, t]$ is in the domain of y , then the
 7 segment $y_t \in C$ is defined by $y_t(s) = y(t + s)$ for $-\tau \leq s \leq 0$. To each $\phi \in C$ there
 8 corresponds a solution $y^\phi: [- \tau, \infty) \rightarrow \mathbb{R}$ with $y_0^\phi = \phi$. Under condition (2), the
 9 functions $\mathbb{R} \ni t \mapsto 0 \in \mathbb{R}$ and $\mathbb{R} \ni t \mapsto K = (q/p - r)^{1/n} \in \mathbb{R}$ are the only constant
 10 solutions, i.e., there exists a unique positive equilibrium besides the trivial one.

11 Several authors have already examined equation (1), see e.g., the works [5, 6, 17,
 12 19]. In this paper we focus on those positive periodic solutions of (1) that oscillate
 13 slowly about K . A solution y is called slowly oscillatory about K if all zeros of
 14 $y - K$ are spaced at distances greater than the delay τ . It is widespread to use the
 15 abbreviation SOP for such periodic solutions.

16 If we restrict our examinations only to positive solutions, then we can apply the
 17 transformation $x = \log y - \log K$. Thereby we obtain the equation

$$x'(t) = -f(x(t - \tau)), \quad (4)$$

18 where the feedback function $f \in C^1(\mathbb{R}, \mathbb{R})$ is defined as

$$f(x) = p - \frac{q}{r + \left(\frac{q}{p} - r\right) e^{nx}} \quad \text{for all } x \in \mathbb{R}, \quad (5)$$

19 see Fig. 1. Note that $f(0) = 0$. Condition (2) implies that f is strictly increasing,
 20 hence we are in the so called “negative feedback” case. Solutions and segments of
 21 solutions of (4) are defined analogously as for equation (1). Note that the positive
 22 equilibrium of (1) given by K is transformed into the trivial equilibrium of (4). In
 23 accordance, an SOP solution of (4) is a periodic solution that have zeros spaced at
 24 distances greater than τ . We know from Theorem 7.1 of Mallet-Paret and Sell in
 25 [8] that if $T_0 < T_1 < T_2$ are three consecutive zeros of an SOP solution of (4), then
 26 $T_2 - T_0$ is its minimal period.

27 Nussbaum verified the global existence of SOP solutions for equations of the form
 28 (4) and for a wide class of feedback functions containing (5), see [11] and also [10].
 29 His proof applies the Browder ejective fixed point principle. By [10, 11], equation

1 (4) has at least one SOP solution for

$$\tau > \tau_0 = \frac{\pi}{2f'(0)} = \frac{q\pi}{2np(q-pr)}.$$

2 Nussbaum also established results on the uniqueness of the SOP solution (up
3 to translation of time) in [13]. However, paper [13] demands f to be odd, thus it
4 cannot be applied for (5). Paper [2] of Cao, a second result on uniqueness, requires
5 $h(x) = xf'(x)/f(x) < 1$ to be monotone decreasing in $x \in (0, b)$ and monotone
6 increasing in $x \in (-a, 0)$ with some $a > 0$ and $b > 0$. One can easily check that
7 this concavity condition does not necessarily hold in our case either. We need to
8 choose a different approach to guarantee the uniqueness of the SOP solution. The
9 monotonicity of f is not sufficient: Cao proved the existence of a monotone f in [1]
10 such that equation (4) has at least two distinct SOP solutions.

11 The stability of the SOP solutions is another central question. A well-known
12 result is due to Kaplan and Yorke [3]: under certain restrictions on f , if the SOP
13 solution is unique, then it is orbitally asymptotically stable. The region of attraction
14 consists of the segments of all eventually slowly oscillatory solutions.

15 For a more detailed summary on SOP solutions of equation (4), we refer to the
16 work [4] of Kennedy and Stumpf.

17 Song, Wei and Han studied the equation in the form (1), and they showed that
18 a series of Hopf bifurcations takes place at the positive equilibrium as τ passes
19 through the critical values

$$\tau_k = \frac{1}{f'(0)} \left(\frac{\pi}{2} + 2k\pi \right) = \frac{q}{np(q-pr)} \left(\frac{\pi}{2} + 2k\pi \right), \quad k \geq 0,$$

20 see [17]. They gave explicit formulae to determine the stability, direction and the
21 period of the bifurcating periodic solutions. Then they verified the global existence
22 of the bifurcating periodic solutions by applying the global Hopf bifurcation theory
23 in [20]. They showed that equation (1) has at least k periodic solutions if $\tau > \tau_k$,
24 $k \geq 1$. Song, Wei and Han could not decide whether equation (1) has a periodic
25 solution if $\tau \in (\tau_0, \tau_1)$. As we have mentioned above, Nussbaum solved this problem
26 in [11].

27 Our work is motivated by the fact that Song and his coauthors could not deter-
28 mine the stability of the periodic orbits for τ far away from the local Hopf bifurcation
29 values. Uniqueness of the SOP solution has not been studied either.

30 The following theorems are the main results of this paper.

Theorem 1.1. *Set p, q, r and n as in (2).*

(i) *If $\tau > 0$ is large enough, then equation (1) has a unique positive periodic so-
lution $\bar{y} : \mathbb{R} \rightarrow \mathbb{R}$ oscillating slowly about K . The corresponding periodic orbit is
asymptotically stable, and it attracts the set*

$$\left\{ \phi : y^\phi(t) > 0 \text{ for } t \geq -\tau, y_t^\phi - K \text{ has at most one sign change for large } t \right\}.$$

31 (ii) *If $\bar{\omega}$ denotes the minimal period of \bar{y} , and*

$$\omega = \left(2 + \frac{q-pr}{pr} + \frac{pr}{q-pr} \right) \tau, \tag{6}$$

32 *then $\lim_{\tau \rightarrow \infty} \bar{\omega}/\omega = 1$.*

33 Uniqueness of the periodic solution is always meant up to time translation.

34 If we fix p, q, r and τ , we can determine the asymptotic shape of the periodic
35 solution as $n \rightarrow \infty$.

- 1 **Theorem 1.2.** Set p, q, r and τ such that (2) and $\tau \min\{p, q/r - p\} > 8$ hold.
 2 (i) Theorem 1.1.(i) is true for all sufficiently large n .
 3 (ii) Define $v : \mathbb{R} \rightarrow \mathbb{R}$ as the ω -periodic extension of the piecewise linear function

$$[0, \omega] \ni t \mapsto \begin{cases} -pt, & 0 \leq t < \tau, \\ \left(\frac{q}{r} - p\right)t - \frac{q}{r}\tau, & \tau \leq t < \left(2 + \frac{pr}{q-pr}\right)\tau, \\ -pt + \left(\frac{q}{r} + p + \frac{p^2r}{q-pr}\right)\tau, & \left(2 + \frac{pr}{q-pr}\right)\tau \leq t < \omega \end{cases} \in \mathbb{R},$$

where ω is given by (6). Let $\eta_1 > 0$ and $\eta_2 > 0$ be arbitrary. If n is large enough, then there exists $T \in \mathbb{R}$ for the $\bar{\omega}$ -periodic SOP solution \bar{y} , such that $|\bar{\omega} - \omega| < \eta_1$, and

$$\left| \log \frac{\bar{y}(t+T)}{K} - v(t) \right| < \eta_2 \quad \text{for all } t \in [0, \bar{\omega}].$$

4 The proofs of these theorems are similar, and they are organized as follows.
 5 Throughout the paper we examine equation (1) in the form (4)-(5). First we calculate an SOP solution v for the "limit equation" $v'(t) = -g(v(t - \tau))$, where
 6 $g : \mathbb{R} \rightarrow \mathbb{R}$ is a piecewise constant function chosen so that (5) is close to g outside
 7 a neighborhood of 0. Then we consider (5) as a perturbation of g and follow the
 8 technique used by Walther in [18] (for a slightly different class of equations) to obtain
 9 information about the solutions of equation (4). We show the existence of a
 10 convex closed subset $\mathcal{A}(\beta) \subseteq C$ such that all solutions of (4) with initial segments in
 11 $\mathcal{A}(\beta)$ return to $\mathcal{A}(\beta)$. Thereby a return map $P : \mathcal{A}(\beta) \rightarrow \mathcal{A}(\beta)$ can be introduced.
 12 Next we explicitly evaluate a Lipschitz constant $L(P)$ for P . If τ or n is large
 13 enough, then $L(P) < 1$, i.e., P is a contraction. The unique fixed point of P is the
 14 initial segment of an SOP solution. Besides this, we need the results of paper [12]
 15 of Nussbaum to show that all SOP solutions have segments in $\mathcal{A}(\beta)$, and hence the
 16 SOP solution is unique. The rest of the theorems will follow easily. In particular,
 17 stability comes from the work [3] of Kaplan and Yorke.

18 We give full proofs only when

$$\frac{q}{pr} \quad \text{and} \quad \frac{pr}{q-pr} \tag{7}$$

19 are not integers. In this case we determine the sizes of τ and n exactly. We indicate
 20 the necessary modifications when either $q/(pr)$ or $pr/(q-pr)$ is an integer.

The particular form of f is actually not important. It is possible to show that equation (4) admits an SOP solution if the feedback function f is Lipschitz-continuous, there are constants $A > 0$, $B > 0$ and small $\beta > 0$ such that

$$|f(x) + A| \text{ is small for } x < -\beta \quad \text{and} \quad |f(x) - B| \text{ is small for } x > \beta,$$

21 furthermore, the Lipschitz constant for f restricted to the interval $(-\infty, -\beta] \cup$
 22 $[\beta, \infty)$ is also sufficiently small. The method of Walther in [18] works for all such
 23 nonlinearities. If, in addition, $f(0) = 0$, f is continuously differentiable and $f'(x) >$
 24 0 for all real x , then one can prove uniqueness and stability using papers [12] of
 25 Nussbaum and [3] of Kaplan and Yorke.

Let us also mention that the Schwarzian derivative of (5) equals $-n^2/2$ at all $x \in \mathbb{R}$. Hence Proposition 2 of Liz and Röst in [7] gives bounds for the global attractor: If $\tau f'(0) > 3/2$, then

$$\alpha \leq \liminf_{t \rightarrow \infty} x(t) \leq \limsup_{t \rightarrow \infty} x(t) \leq \beta$$

for all solutions x of (4), where $\{\alpha, \beta\}$ is the unique 2-cycle of $x \mapsto -\tau f(x)$ (that is, $\beta = -\tau f(\alpha)$ and $\alpha = -\tau f(\beta)$). In consequence,

$$Ke^\alpha \leq \liminf_{t \rightarrow \infty} y(t) \leq \limsup_{t \rightarrow \infty} y(t) \leq Ke^\beta$$

1 for all positive solutions y of (1). If $\tau f'(0) \leq 3/2$, then Proposition 2 of [7] states
 2 that all solutions of (4) converge to 0 (and hence all positive solutions of (1) converge
 3 to K) as $t \rightarrow \infty$. This result improves the well-known fact that the trivial equi-
 4 librium of (4) (and hence the positive equilibrium of (1)) is locally asymptotically
 5 stable whenever $\tau f'(0) < \pi/2$.

6 **2. The limit equation.** Consider equation (4) with feedback function (5). Let
 7 $A = q/r - p > 0$ and $B = p > 0$.

8 Note that if p, q, r are fixed according to (2), then $f(x)$ converges to $p - q/r = -A$
 9 as $nx \rightarrow -\infty$ and $f(x)$ tends to $p = B$ as $nx \rightarrow \infty$. Therefore we examine the
 10 "limit equation"

$$v'(t) = -g^{A,B}(v(t - \tau)), \quad (8)$$

11 where $g^{A,B} : \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$g^{A,B}(v) = \begin{cases} -A, & v < 0, \\ 0, & v = 0, \\ B, & v > 0. \end{cases}$$

12 Given any $\phi \in C$, a solution $v^\phi : [-\tau, \infty) \rightarrow \mathbb{R}$ of (8) is an absolutely continuous
 13 function such that $v^\phi|_{[-\tau, 0]} = \phi$ and the integral equation

$$v^\phi(t) = v^\phi(0) - \int_0^t g^{A,B}(v^\phi(s - \tau)) ds \quad (9)$$

14 is satisfied for $t > 0$. Similarly, an absolutely continuous function $v : \mathbb{R} \rightarrow \mathbb{R}$ is a
 15 solution of (8), if the integral equation is satisfied for all $t \in \mathbb{R}$.

16 In this section we evaluate an SOP solution for (8).

Proposition 1. *Equation (8) admits a periodic solution $v : \mathbb{R} \rightarrow \mathbb{R}$ defined as follows:*

$$v(t) = \begin{cases} -Bt, & t \in [0, \tau], \\ At - (A + B)\tau, & t \in [\tau, \sigma + \tau], \\ -Bt + \left(A + 2B + \frac{B^2}{A}\right)\tau, & t \in [\sigma + \tau, \omega], \end{cases}$$

17 where $\sigma = (1 + B/A)\tau$ is the first positive zero, and $\omega = (2 + A/B + B/A)\tau$ is the
 18 second positive zero and the minimal period of v .

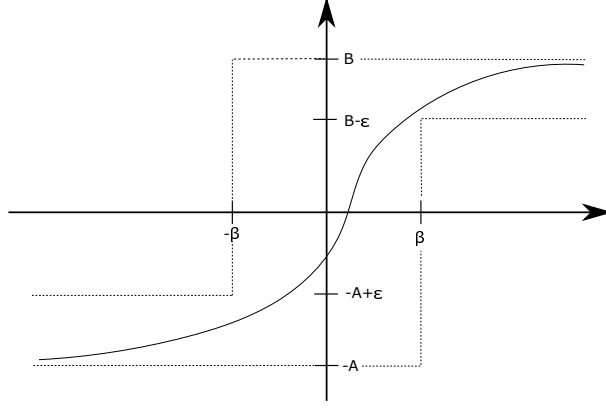
19 *Proof.* Consider any $\phi \in C$ with $\phi(t) > 0$ for all $t \in [-\tau, 0)$ and $\phi(0) = 0$. Consider
 20 the corresponding solution $v = v^\phi$ of (8). We have $v'(t) = -B$ for $t \in [0, \tau)$, hence
 21 $v(t) = -Bt$ for $t \in [0, \tau]$. Necessarily, v is negative on an interval $(0, T)$ where
 22 $T > \tau$. Then $v'(t) = A$ for $t \in (\tau, T + \tau)$ and

$$v(t) = At - (A + B)\tau \text{ for } t \in [\tau, T + \tau]. \quad (10)$$

23 This function has zero at $\sigma = (1 + B/A)\tau$. So formula (10) is valid with $T = \sigma$.
 24 The solution v is positive on $(\sigma, \sigma + \tau]$ and $v(\sigma + \tau) = A\tau$.

25 If $\omega > \sigma + \tau$ is chosen such that v is positive on (σ, ω) , then $v'(t) = -B$ for
 26 $t \in (\sigma + \tau, \omega + \tau)$. We deduce that

$$v(t) = -Bt + \left(A + 2B + \frac{B^2}{A}\right)\tau, \quad t \in [\sigma + \tau, \omega + \tau]. \quad (11)$$

FIGURE 2. An element of $\mathcal{N}(A, B, \beta, \varepsilon)$

From this formula we see that ω can be defined as

$$\omega = \left(2 + \frac{A}{B} + \frac{B}{A}\right) \tau.$$

- 1 Note that $v_\omega(t) > 0$ for all $t \in [-\tau, 0)$ and $v_\omega(0) = 0$. Hence, if we set $\phi = v_\omega$ and
- 2 extend $v|_{[-\tau, \omega]}$ to \mathbb{R} ω -periodically, then we get a periodic solution of (8) on \mathbb{R} . \square

3. Preliminary estimates. For $A > 0$, $B > 0$, $\beta > 0$, $0 < \varepsilon < \min\{A, B\}/2$, let $\mathcal{N}(A, B, \beta, \varepsilon)$ denote the set of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with

$$\begin{aligned} -A \leq f(x) \leq -A + \varepsilon & \quad \text{for } x < -\beta, \\ -A \leq f(x) \leq B & \quad \text{for } -\beta \leq x \leq \beta, \end{aligned}$$

and

$$B - \varepsilon \leq f(x) \leq B \quad \text{for } x > \beta,$$

see Fig. 2. Function (5) is an element of $\mathcal{N}(A, B, \beta, \varepsilon)$ if $A = q/r - p$, $B = p$, $0 < \varepsilon < \min\{A, B\}/2$ and

$$\beta \geq \max\{f^{-1}(B - \varepsilon), -f^{-1}(-A + \varepsilon)\}.$$

- 3 This is true because $f(0) = 0$, $\lim_{x \rightarrow -\infty} f(x) = -A$, $\lim_{x \rightarrow \infty} f(x) = B$ and f is
- 4 strictly increasing.

Let

$$\mathcal{A}(\beta) = \{\phi \in C : \phi(t) \geq \beta \text{ for } -\tau \leq t \leq 0, \phi(0) = \beta\}.$$

In this section we study the solution $x = x^\phi$ of (4) when $f \in \mathcal{N}(A, B, \beta, \varepsilon)$ and $\phi \in \mathcal{A}(\beta)$. Our main goal is to show that if $\phi \in \mathcal{A}(\beta)$, then there exist $q > 0$ and $\tilde{q} > 0$ such that

$$x_q \in -\mathcal{A}(\beta) = \{\phi \in C : \phi(t) \leq -\beta \text{ for } -\tau \leq t \leq 0, \phi(0) = -\beta\},$$

- 5 and $x_{q+\tilde{q}} \in \mathcal{A}(\beta)$.

- 6 Define the integer N by $(N - 1)\tau < \sigma \leq N\tau$, where σ is the first positive zero
- 7 of the periodic function v given by Proposition 1. As $\sigma = (1 + B/A)\tau$, we get that
- 8 $N = \lceil 1 + B/A \rceil$. The next proposition estimates $|x(t) - v(t)|$ for $t \in [0, N\tau]$.

- 1 **Proposition 2.** Let $A > 0$, $B > 0$, $\beta > 0$, $0 < \varepsilon < \min\{A, B\}/2$, $\delta = 2\beta/(B - \varepsilon)$,
 2 $N = \lceil 1 + B/A \rceil$, $f \in \mathcal{N}(A, B, \beta, \varepsilon)$ and $\phi \in \mathcal{A}(\beta)$. Assume that

$$c_1 = \tau - \delta > 0 \quad (\text{C.1})$$

and

$$0 < c_2 = \begin{cases} (B - 2\varepsilon)\tau - (2A + B)\delta - 2\beta & \text{if } N = 2, \\ (A + B)(\tau - \delta) - (A + \varepsilon)(N - 1)\tau - 2\beta & \text{if } N > 2. \end{cases} \quad (\text{C.2})$$

- 3 Then

$$|x(t) - v(t)| \leq \beta + \varepsilon\tau \quad \text{for } t \in [0, \tau] \quad (\text{12})$$

4 and

$$|x(t) - v(t)| \leq \beta + k\varepsilon\tau + (A + B)\delta \quad \text{for } 2 \leq k \leq N \text{ and } t \in [(k - 1)\tau, k\tau]. \quad (\text{13})$$

- 5 *Proof. Estimate (12).* We know that $v(0) = 0$ and $v(t) > 0$ for $t \in [-\tau, 0)$. For
 6 $t \in [0, \tau]$,

$$\begin{aligned} |x(t) - v(t)| &= \left| \beta - \int_0^t f(x(s - \tau)) ds + \int_0^t B ds \right| \\ &\leq \beta + \int_0^t |B - f(x(s - \tau))| ds \\ &\leq \beta + \varepsilon t \leq \beta + \varepsilon\tau. \end{aligned} \quad (\text{14})$$

- 7 So (12) holds.

- 8 It is also clear from the choice of ϕ that $x'(t) < 0$ for $t \in (0, \tau)$.

Proof of (13) for $k = 2$. Condition (C.1) guarantees that $\delta < \tau$, and therefore
 $v(\delta) = -B\delta$ by Proposition 1. Then, by (14) and by the definition of δ ,

$$x(\delta) \leq v(\delta) + \beta + \varepsilon\delta = -B\delta + \beta + \varepsilon\delta = -\beta.$$

- 9 By the monotonicity of x on $[0, \tau]$, we obtain that

$$x(t) < -\beta \quad \text{for } t \in (\delta, \tau]. \quad (\text{15})$$

For $t \in [0, \delta]$, $-A \leq f(x(t)) \leq B$ and thus $|f(x(t)) + A| \leq A + B$. For $t \in [\delta, \tau]$,
 $|f(x(t)) + A| \leq \varepsilon$ because $x(t) \leq -\beta$. These imply that

$$\begin{aligned} \int_0^\tau |f(x(s)) + A| ds &= \int_0^\delta |f(x(s)) + A| ds + \int_\delta^\tau |f(x(s)) + A| ds \\ &\leq (A + B)\delta + \varepsilon(\tau - \delta) \leq (A + B)\delta + \varepsilon\tau. \end{aligned}$$

This observation together with (12) gives that for $t \in [\tau, 2\tau]$,

$$\begin{aligned} |x(t) - v(t)| &\leq |x(\tau) - v(\tau)| + \int_\tau^t |f(x(s - \tau)) + A| ds \\ &\leq \beta + \varepsilon\tau + (A + B)\delta + \varepsilon\tau = \beta + 2\varepsilon\tau + (A + B)\delta. \end{aligned}$$

- 10 We have verified (13) for $k = 2$.

- 11 *Proof of (13) for all $2 < k \leq N$ in case $N > 2$ by induction on k .* Assume that
 12 (13) holds for a given $k \in \{2, 3, \dots, N - 1\}$. We show that it holds for $k + 1$. Recall
 13 that

$$v(t) = At - (A + b)\tau \leq A(N - 1)\tau - (A + B)\tau < 0 \quad \text{for } t \in [\tau, (N - 1)\tau]. \quad (\text{16})$$

Using this, (C.2) and (13) for this k , we obtain that for $t \in [(k - 1)\tau, k\tau]$,

$$x(t) \leq v(t) + \beta + k\varepsilon\tau + (A + B)\delta \leq A(N - 1)\tau - (A + B)\tau + \beta + (N - 1)\varepsilon\tau + (A + B)\delta < -\beta.$$

Then, for $t \in [k\tau, (k+1)\tau]$,

$$\begin{aligned} |x(t) - v(t)| &\leq |x(k\tau) - v(k\tau)| + \int_{k\tau}^t |f(x(s-\tau)) + A| ds \leq |x(k\tau) - v(k\tau)| + \varepsilon\tau \\ &\leq \beta + (k+1)\varepsilon\tau + (A+B)\delta. \end{aligned}$$

1 Summing up, (13) is true for all $2 \leq k \leq N$. □

2 The next proposition shows, among others, that to each $\phi \in \mathcal{A}(\beta)$ there corre-
3 sponds $q \in (0, N\tau)$ with $x_q \in -\mathcal{A}(\beta)$.

4 **Proposition 3.** *In addition to the assumptions of the previous proposition, suppose*
5 *that*

$$c_3 = (A - \varepsilon)N\tau - (A + B)(\tau + \delta) > 0. \quad (\text{C.3})$$

6 Then

$$x(t) < -\beta \text{ for } \delta < t \leq \max\{\tau + \delta, (N-1)\tau\}, \quad (17)$$

$$x(N\tau) > -\beta, \quad (18)$$

$$x'(t) < 0 \text{ for } t \in (0, \tau), \quad (19)$$

$$x'(t) > 0 \text{ for } t \in (\tau + \delta, N\tau), \quad (20)$$

and for the unique solution $q = q(\phi)$ of the equation $x(t) = -\beta$ in $(\tau + \delta, N\tau)$,

$$x_q \in -\mathcal{A}(\beta).$$

10 If $\psi \in \mathcal{A}(\beta)$ with $x_{\delta+\tau}^\psi = x_{\delta+\tau}^\phi$, then $q(\phi) = q(\psi)$. Furthermore, if $\tau B/A > \delta$, then

$$|q(\phi) - \sigma| \leq \frac{1}{A - \varepsilon}(2\beta + N\varepsilon\tau + (A + B)\delta), \quad (21)$$

11 where σ is the smallest positive zero of v .

Proof. Inequality (17). We mentioned in the proof of Proposition 2 that $x(t) < -\beta$ for $t \in (\delta, \tau]$, see (15). In case $N = 2$ we have $\max\{\tau + \delta, (N-1)\tau\} = \tau + \delta$. The formula

$$v(t) = At - (A + B)\tau \leq A\delta - B\tau, \quad t \in [\tau, \tau + \delta],$$

estimate (13) with $k = 2$ and the first line of (C.2) together yield that

$$x(t) < v(t) + \beta + 2\varepsilon\tau + (A + B)\delta \leq A\delta - B\tau + \beta + 2\varepsilon\tau + (A + B)\delta < -\beta$$

12 for all $t \in [\tau, \tau + \delta]$. If $N > 2$, then $\max\{\tau + \delta, (N-1)\tau\} = (N-1)\tau$. A similar
13 argument (combining the second line of (C.2), estimate (13) with $2 \leq k \leq N-1$
14 and also (16)) yields that $x(t) < -\beta$ for $t \in [\tau, (N-1)\tau]$.

15 Inequality (18). We obtain (18) if we apply (13) for $t = N\tau$, $v(N\tau) = AN\tau -$
16 $(A + B)\tau$ and (C.3):

$$x(N\tau) > v(N\tau) - (\beta + N\varepsilon\tau + (A + B)\delta) = c_3 - \beta > -\beta. \quad (22)$$

17 Estimates (19) and (20). It is clear from the choice of ϕ that $x'(t) < 0$ for
18 $t \in (0, \tau)$. By (17), $x(t - \tau) < -\beta$ for $t \in (\delta + \tau, N\tau]$, hence it is also clear that

$$x'(t) = -f(x(t - \tau)) \geq A - \varepsilon > 0 \quad \text{for } t \in (\delta + \tau, N\tau]. \quad (23)$$

19 Statements regarding q . Existence and uniqueness of the solution $q = q(\phi) \in$
20 $(\tau + \delta, N\tau)$ of $x^\phi(t) = -\beta$ are now obvious from (17), (18) and (20). We see that
21 $x_q^\phi \in -\mathcal{A}(\beta)$.

22 It is also easy to see that if $\psi \in \mathcal{A}(\beta)$ with $x_{\delta+\tau}^\psi = x_{\delta+\tau}^\phi$, then $q(\phi) = q(\psi)$.
23 Indeed, this assertion comes from the facts that $x^\phi(t) = x^\psi(t)$ on $[\tau + \delta, \infty)$, $q(\phi) >$
24 $\tau + \delta$ and $q(\psi) > \tau + \delta$.

If in addition $\tau B/A > \delta$, then $\sigma = \tau + \tau B/A > \tau + \delta$. Hence we can apply (23) for all t between $\sigma \in (\delta + \tau, N\tau]$ and $q \in (\delta + \tau, N\tau]$:

$$\begin{aligned} (A - \varepsilon)|\sigma - q| &\leq \left| \int_q^\sigma f(x(t - \tau)) dt \right| = |x(\sigma) - x(q)| \\ &\leq |x(\sigma) - v(\sigma)| + |x(q)| \leq |x(\sigma) - v(\sigma)| + \beta. \end{aligned}$$

1 We obtain (21) from this by using (13) for $t = \sigma \in [(N - 1)\tau, N\tau]$.

2

□

3 We need analogous results for solutions with initial segments in $-\mathcal{A}(\beta)$. The
4 proofs of the subsequent two propositions can be easily brought back to the proofs
5 of Propositions 2 and 3.

6 **Proposition 4.** Let $A > 0$, $B > 0$, $\beta > 0$, $0 < \varepsilon < \min\{A, B\}/2$, $\tilde{\delta} = 2\beta/(A - \varepsilon)$,
7 $\tilde{N} = \lceil 1 + A/B \rceil$, $f \in \mathcal{N}(A, B, \beta, \varepsilon)$ and $\phi \in -\mathcal{A}(\beta)$. Assume that

$$c_4 = \tau - \tilde{\delta} > 0, \quad (\text{C.4})$$

and

$$0 < c_5 = \begin{cases} (A - 2\varepsilon)\tau - (A + 2B)\tilde{\delta} - 2\beta & \text{if } \tilde{N} = 2, \\ (A + B)(\tau - \tilde{\delta}) - (B + \varepsilon)(\tilde{N} - 1)\tau - 2\beta & \text{if } \tilde{N} > 2. \end{cases} \quad (\text{C.5})$$

8 Then

$$|x(t) - v(t + \sigma)| \leq \beta + \varepsilon\tau \quad \text{for } t \in [0, \tau] \quad (\text{24})$$

9 and

$$|x(t) - v(t + \sigma)| \leq \beta + k\varepsilon\tau + (A + B)\tilde{\delta} \quad \text{for } 2 \leq k \leq \tilde{N} \text{ and } t \in [(k - 1)\tau, k\tau]. \quad (\text{25})$$

10 **Proposition 5.** In addition to the assumptions of the previous proposition, suppose
11 that

$$c_6 = (B - \varepsilon)\tilde{N}\tau - (A + B)(\tau + \tilde{\delta}) > 0. \quad (\text{C.6})$$

12 Then

$$x(t) > \beta \quad \text{for } \tilde{\delta} < t \leq \max\{\tau + \tilde{\delta}, (\tilde{N} - 1)\tau\}, \quad (\text{26})$$

13

$$x(\tilde{N}\tau) < \beta, \quad (\text{27})$$

14

$$x'(t) > 0 \quad \text{for } t \in (0, \tau), \quad (\text{28})$$

15

$$x'(t) < 0 \quad \text{for } t \in (\tau + \tilde{\delta}, \tilde{N}\tau), \quad (\text{29})$$

and for the unique solution $\tilde{q} = \tilde{q}(\phi)$ of the equation $x(t) = \beta$ in $(\tau + \tilde{\delta}, \tilde{N}\tau)$,

$$x_{\tilde{q}} \in \mathcal{A}(\beta).$$

If $\psi \in -\mathcal{A}(\beta)$ with

$$x_{\tilde{\delta} + \tau}^\psi = x_{\tilde{\delta} + \tau}^\phi,$$

16 then $\tilde{q}(\phi) = \tilde{q}(\psi)$. Furthermore, if $\tau A/B > \tilde{\delta}$, then

$$|\tilde{q}(\phi) - (\omega - \sigma)| \leq \frac{1}{B - \varepsilon} \left(2\beta + \tilde{N}\varepsilon\tau + (A + B)\tilde{\delta} \right). \quad (\text{30})$$

Proof of Proposition 4 and Proposition 5. Consider $\phi \in -\mathcal{A}(\beta)$ and the solution
 $x = x^\phi: [-\tau, \infty) \rightarrow \mathbb{R}$. For $\tilde{x}: = -x$, we have $\tilde{x}_0 \in \mathcal{A}(\beta)$ and

$$\tilde{x}'(t) = -h(\tilde{x}(t - \tau)) \quad \text{for } t > 0,$$

17 where $h: \mathbb{R} \ni x \mapsto -f(-x) \in \mathbb{R}$. Observe that h is an element of the function class

18 $\mathcal{N}(B, A, \beta, \varepsilon)$.

In addition, define the ω -periodic function $\tilde{v} : \mathbb{R} \rightarrow \mathbb{R}$ on $[0, \omega]$ as follows:

$$\tilde{v}(t) = -v(t + \sigma) = \begin{cases} -At, & t \in [0, \tau], \\ Bt - (A + B)\tau, & t \in [\tau, (2 + \frac{A}{B})\tau], \\ -At + (B + 2A + \frac{A^2}{B})\tau, & t \in [(2 + \frac{A}{B})\tau, \omega]. \end{cases}$$

1 By Proposition 1, \tilde{v} is a solution of $\tilde{v}'(t) = -g^{B,A}(\tilde{v}(t - \tau))$. Note that $\omega - \sigma$ is the
2 smallest positive zero of \tilde{v} .

3 Exchanging the role of A and B in the proofs of Propositions 2 and 3, we get the
4 desired estimates for $|\tilde{x} - \tilde{v}|$, \tilde{x} , \tilde{x}' and $|\tilde{q} - (\omega - \sigma)|$. \square

Assumptions (C.1)-(C.2) and (C.4)-(C.5) can be satisfied for any $A > 0$ and
 $B > 0$ if $\beta > 0$ and $\varepsilon > 0$ are small enough. However, if B/A is an integer, then
 $N = 1 + B/A$ and thus

$$c_3 = (A - \varepsilon)N\tau - (A + B)(\tau + \delta) = \left(A - \varepsilon + \frac{A - \varepsilon}{A}B\right)\tau - (A + B)\left(\tau + \frac{2\beta}{B - \varepsilon}\right)$$

5 is negative for any $\beta > 0$ and $\varepsilon > 0$. In consequence, estimate (22) in the proof of
6 Proposition 3 is not true in this case, that is, we cannot guarantee inequality (18)
7 (which is a key property in proving the existence of q with $x_q \in -\mathcal{A}(\beta)$).

8 Similarly, (C.6) in Proposition 5 cannot be satisfied with any $\beta > 0$ and $\varepsilon > 0$ if
9 A/B is an integer.

10 Next we discuss how Proposition 3 or Proposition 5 should be modified in case
11 B/A or A/B is an integer.

12 **Remark 1.** One can change Proposition 3 as follows in case B/A is an integer.

The proof of (17) is independent of the value of c_3 , so it is correct even if B/A
is an integer. First, inequality (17) has be extended for a larger interval. Assume
that

$$T = 1 - \frac{N\varepsilon\tau + 2\beta + (A + B)\delta}{A\tau} > 0.$$

It is clear that $T < 1$. Then estimate (13) and the definition of v yield that for
 $t \in [(N - 1)\tau, (N - 1 + T)\tau]$,

$$\begin{aligned} x(t) &\leq v(t) + \beta + N\varepsilon\tau + (A + B)\delta \\ &= At - (A + B)\tau + \beta + N\varepsilon\tau + (A + B)\delta \\ &\leq A(N - 1 + T)\tau - (A + B)\tau + \beta + N\varepsilon\tau + (A + B)\delta = -\beta. \end{aligned}$$

13 This result and (17) together give that

$$x(t) \leq -\beta \text{ for } \delta \leq t \leq \max\{\tau + \delta, (N - 1 + T)\tau\}. \quad (31)$$

As next step, note that if B/A is an integer, then $N\tau = (1 + B/A)\tau = \sigma$ is
the first positive zero of v , i.e., v is negative on $[(N - 1)\tau, (N - 1 + T)\tau]$. This
observation with (31) implies that for $t \in [N\tau, (N + T)\tau]$,

$$\begin{aligned} |x(t) - v(t)| &\leq |x(N\tau) - v(N\tau)| + \int_{N\tau}^t |f(x(s - \tau)) + A|ds \\ &\leq |x(N\tau) - v(N\tau)| + \varepsilon T\tau. \end{aligned}$$

14 By (13), the right hand side is not greater than $\beta + (N + T)\varepsilon\tau + (A + B)\delta$.

It is clear from $N\tau = \sigma$ and $T < 1$ that we need to consider the second line of
the definition of v in Proposition 1 to evaluate $v((N + T)\tau)$:

$$v((N + T)\tau) = A(N + T)\tau - (A + B)\tau = AT\tau.$$

1 We see from the last two results that one can achieve

$$x((N + T)\tau) > -\beta \quad (32)$$

2 if $\beta > 0$ and $\varepsilon > 0$ are small enough.

3 Results (31) and (32) guarantee the existence of q . It is now easy to modify the
4 rest of Proposition 3.

5 Proposition 5 has to be altered in a similar fashion if A/B is an integer. The
6 subsequent proofs of the paper also need to be slightly changed if either B/A or
7 A/B is an integer. We omit the details.

8 **4. Lipschitz continuous return maps.** Set $A, B, \beta, \varepsilon, \delta, \tilde{\delta}, N, \tilde{N}$ as in the previ-
9 ous section. Assume that conditions (C.1)-(C.6) hold. Suppose in addition that
10 $f \in \mathcal{N}(A, B, \beta, \varepsilon)$ is Lipschitz-continuous, and $L(f)$ is a Lipschitz constant for f .
11 Let $L_\beta = L_\beta(f)$ and $L_{-\beta} = L_{-\beta}(f)$ be the Lipschitz constants for the restrictions
12 $f|_{[\beta, \infty)}$ and $f|_{(-\infty, -\beta]}$, respectively.

In this section F denotes the semiflow corresponding to (4):

$$F: [0, \infty) \times C \ni (t, \phi) \mapsto x_t^\phi \in C.$$

Then subset $F(\tau + \delta, \mathcal{A}(\beta)) \subset C$ consists of the $\tau + \delta$ -segments of those solutions
that have initial segments in $\mathcal{A}(\beta)$:

$$F(\tau + \delta, \mathcal{A}(\beta)) = \left\{ x_{\tau+\delta}^\phi : \phi \in \mathcal{A}(\beta) \right\}.$$

We introduce the map

$$s : F(\tau + \delta, \mathcal{A}(\beta)) \ni \psi \mapsto q(\psi) - \tau - \delta \in (0, (N - 1)\tau - \delta),$$

13 where $\psi = F(\tau + \delta, \phi)$. In other words, if $\psi \in F(\tau + \delta, \mathcal{A}(\beta))$, then $s(\psi)$ is the time
14 in $(0, (N - 1)\tau - \delta)$ for which $x_{s(\psi)}^\psi \in -\mathcal{A}(\beta)$. Proposition 3 guarantees that s is
15 well-defined.

Consider the map

$$R : \mathcal{A}(\beta) \ni \phi \mapsto F(q(\phi), \phi) = x_{q(\phi)}^\phi \in -\mathcal{A}(\beta).$$

One can write R in the form $R = F_s \circ F_\delta \circ F_\tau$, where

$$\begin{aligned} F_\tau &= F(\tau, \cdot)|_{\mathcal{A}(\beta)}, \\ F_\delta &= F(\delta, \cdot)|_{F(\tau, \mathcal{A}(\beta))}, \\ F_s &= F(s(\cdot), \cdot)|_{F(\tau+\delta, \mathcal{A}(\beta))}. \end{aligned}$$

16 Our next goal is to determine Lipschitz constants for these maps.

17 **Proposition 6.** τL_β is a Lipschitz constant for F_τ , and $1 + \delta L(f)$ is a Lipschitz
18 constant for F_δ .

Proof. Let $\phi, \bar{\phi}$ in $\mathcal{A}(\beta)$ and let $t \in [-\tau, 0]$. Using that $\phi(0) = \bar{\phi}(0) = \beta$ and
 $\phi(u) \geq \beta, \bar{\phi}(u) \geq \beta$ for $u \in [-\tau, 0]$, we get that

$$|F(\tau, \phi)(t) - F(\tau, \bar{\phi})(t)| \leq \int_0^{\tau+t} |f(\phi(u - \tau)) - f(\bar{\phi}(u - \tau))| du \leq L_\beta \tau \|\phi - \bar{\phi}\|,$$

19 which yields the Lipschitz estimate for F_τ .

Next, consider $x = x^\phi, \bar{x} = x^{\bar{\phi}}$ for arbitrary $\phi, \bar{\phi}$ in C . For $t \in [-\tau, -\delta]$,

$$|F(\delta, \phi)(t) - F(\delta, \bar{\phi})(t)| = |\phi(\delta + t) - \bar{\phi}(\delta + t)| \leq \|\phi - \bar{\phi}\|.$$

20 For $t \in [-\delta, 0]$,

$$\begin{aligned}
|F(\delta, \phi)(t) - F(\delta, \bar{\phi})(t)| &= |x(\delta + t) - \bar{x}(\delta + t)| \\
&\leq |\phi(0) - \bar{\phi}(0)| + \int_0^{\delta+t} |f(\phi(u - \tau)) - f(\bar{\phi}(u - \tau))| du \\
&\leq \|\phi - \bar{\phi}\| + \delta L(f) \|\phi - \bar{\phi}\|.
\end{aligned}$$

1 This proves the Lipschitz constant for F_δ . □

2 In order to determine Lipschitz constants for the maps s and F_s , we estimate
3 $|x^\phi(t) - x^{\bar{\phi}}(t)|$ on $[-\tau, (N-2)\tau]$ if $\phi, \bar{\phi} \in F(\tau + \delta, \mathcal{A}(\beta))$.

Proposition 7. *If $\phi, \bar{\phi} \in F(\tau + \delta, \mathcal{A}(\beta))$, then for the solutions $x = x^\phi$ and $\bar{x} = x^{\bar{\phi}}$ we have*

$$\max_{u \in [-\tau, (N-2)\tau]} |x(u) - \bar{x}(u)| \leq (1 + \tau L_{-\beta})^{N-2} \|\phi - \bar{\phi}\|.$$

4 *Proof.* The assertion is clearly true if $N = 2$, so we may suppose that $N > 2$. We
5 verify that

$$\max_{u \in [(k-1)\tau, k\tau]} |x(u) - \bar{x}(u)| \leq (1 + \tau L_{-\beta})^k \|\phi - \bar{\phi}\| \quad (33)$$

for all $k \in \{0, 1, \dots, N-2\}$ by induction on k . Estimate (33) obviously holds
for $k = 0$. We need to show that it holds for $k+1$ provided it true for some
 $k \in \{0, 1, \dots, N-3\}$. It follows from (17) that $x(t) \leq -\beta$ and $\bar{x}(t) \leq -\beta$ for
 $t \in [-\tau, (N-3)\tau]$. Hence, for $u \in [k\tau, (k+1)\tau]$,

$$\begin{aligned}
|x(u) - \bar{x}(u)| &\leq |x(k\tau) - \bar{x}(k\tau)| + \int_{k\tau}^{(k+1)\tau} |f(x(t-\tau)) - f(\bar{x}(t-\tau))| dt \\
&\leq (1 + \tau L_{-\beta}) \max_{u \in [(k-1)\tau, k\tau]} |x(u) - \bar{x}(u)| \\
&\leq (1 + \tau L_{-\beta})^{k+1} \|\phi - \bar{\phi}\|.
\end{aligned}$$

6 □

7 Now we are in position to evaluate Lipschitz constants for both s and F_s .

Proposition 8. *The map s is Lipschitz continuous with Lipschitz constant*

$$L(s) = \frac{1 + (N-1)\tau L_{-\beta}(1 + \tau L_{-\beta})^{N-2}}{A - \varepsilon}.$$

Proof. Let $\phi, \bar{\phi} \in F(\tau + \delta, \mathcal{A}(\beta))$ and set $\eta = s(\phi)$, $\bar{\eta} = s(\bar{\phi})$. Let $x = x^\phi$ and $\bar{x} = x^{\bar{\phi}}$
denote the corresponding solutions as before. Using $x(\eta) = -\beta$ we get that

$$-\beta = \phi(0) - \int_0^\eta f(x(t-\tau)) dt.$$

We have an analogous equation for $\bar{\phi}$ and $\bar{\eta}$. We conclude that

$$\begin{aligned}
0 &= \left| \phi(0) - \bar{\phi}(0) - \int_{\bar{\eta}}^\eta f(x(t-\tau)) dt - \int_0^{\bar{\eta}} [f(x(t-\tau)) - f(\bar{x}(t-\tau))] dt \right| \\
&\geq \left| \int_{\bar{\eta}}^\eta f(x(t-\tau)) dt \right| - \|\phi - \bar{\phi}\| - \int_0^{\bar{\eta}} |f(x(t-\tau)) - f(\bar{x}(t-\tau))| dt.
\end{aligned}$$

- 1 Recall from (17) in Proposition 3 that $\phi(t) \leq -\beta$ and $\bar{\phi}(t) \leq -\beta$ for all $t \in [-\tau, 0]$.
 2 Hence, if $\eta, \bar{\eta} \in (0, \tau)$, then

$$\left| \int_{\bar{\eta}}^{\eta} f(x(t-\tau)) dt \right| \geq |\eta - \bar{\eta}| (A - \varepsilon) \quad (34)$$

and

$$\begin{aligned} \int_0^{\bar{\eta}} |f(x(t-\tau)) - f(\bar{x}(t-\tau))| dt &= \int_0^{\bar{\eta}} |f(\phi(t-\tau)) - f(\bar{\phi}(t-\tau))| dt \\ &\leq \tau L_{-\beta} \|\phi - \bar{\phi}\|. \end{aligned}$$

- 3 As a result,

$$|\eta - \bar{\eta}| \leq \frac{1 + \tau L_{-\beta}}{A - \varepsilon} \|\phi - \bar{\phi}\|. \quad (35)$$

If $\eta > \tau$ or $\bar{\eta} > \tau$, then the inequalities $\eta, \bar{\eta} < (N-1)\tau - \delta$ imply that $N > 2$. By (17) in Proposition 3, and since $\phi, \bar{\phi} \in F(\tau + \delta, \mathcal{A}(\beta))$, we have $x(t) \leq -\beta$ and $\bar{x}(t) \leq -\beta$ for all $t \in [-\tau, (N-2)\tau - \delta]$. This property with $\bar{\eta} < (N-1)\tau - \delta$ gives that

$$\int_0^{\bar{\eta}} |f(x(t-\tau)) - f(\bar{x}(t-\tau))| dt \leq (N-1)\tau L_{-\beta} \max_{u \in [-\tau, \bar{\eta}-\tau]} |x(u) - \bar{x}(u)|,$$

which is smaller than

$$(N-1)\tau L_{-\beta} (1 + \tau L_{-\beta})^{N-2} \|\phi - \bar{\phi}\|$$

- 4 by Proposition 7. In addition, (34) holds also in this case. In consequence,

$$|\eta - \bar{\eta}| \leq \frac{1 + (N-1)\tau L_{-\beta} (1 + \tau L_{-\beta})^{N-2}}{A - \varepsilon} \|\phi - \bar{\phi}\|. \quad (36)$$

- 5 Estimates (35) and (36) together give the proposition. \square

Proposition 9. F_s is Lipschitz continuous with Lipschitz constant

$$L(F_s) = 3 \left(1 + (N-1)\tau L_{-\beta} (1 + \tau L_{-\beta})^{N-2} \right).$$

Proof. By the definition of F_s ,

$$\begin{aligned} F_s(\phi) - F_s(\bar{\phi}) &= (F(\eta, \phi) - F(\bar{\eta}, \phi)) + (F(\bar{\eta}, \phi) - F(\bar{\eta}, \bar{\phi})) \\ &= (x_\eta - x_{\bar{\eta}}) + (x_{\bar{\eta}} - \bar{x}_{\bar{\eta}}). \end{aligned}$$

As $\eta, \bar{\eta} < (N-1)\tau - \delta$ and $x(u) \leq -\beta$ for $u \in [-\tau, (N-2)\tau - \delta]$, we see that for all $t \in [-\tau, 0]$,

$$\begin{aligned} |x_\eta(t) - x_{\bar{\eta}}(t)| &= \left| \int_{\bar{\eta}+t}^{\eta+t} x'(u) du \right| = \left| \int_{\bar{\eta}+t}^{\eta+t} -f(x(u-\tau)) du \right| \\ &\leq A |\eta - \bar{\eta}| \leq AL(s) \|\phi - \bar{\phi}\|. \end{aligned}$$

Since $\varepsilon < A/2$, we deduce from the formula for $L(s)$ that

$$|x_\eta(t) - x_{\bar{\eta}}(t)| \leq 2 \left(1 + (N-1)\tau L_{-\beta} (1 + \tau L_{-\beta})^{N-2} \right) \|\phi - \bar{\phi}\| \quad \text{for } t \in [-\tau, 0].$$

If $t + \bar{\eta} \geq 0$ for some $t \in [-\tau, 0]$, then

$$\begin{aligned} |x_{\bar{\eta}}(t) - \bar{x}_{\bar{\eta}}(t)| &\leq \left| \phi(0) - \bar{\phi}(0) - \int_0^{\bar{\eta}+t} (f(x(u-\tau)) - f(\bar{x}(u-\tau))) du \right| \\ &\leq \|\phi - \bar{\phi}\| + (N-1)\tau L_{-\beta} \max_{u \in [-\tau, \bar{\eta}-\tau]} |x(u) - \bar{x}(u)| \\ &\leq \left(1 + (N-1)\tau L_{-\beta} (1 + \tau L_{-\beta})^{N-2}\right) \|\phi - \bar{\phi}\|. \end{aligned}$$

If $t + \bar{\eta} \leq 0$ for some $t \in [-\tau, 0]$, then

$$|x_{\bar{\eta}}(t) - \bar{x}_{\bar{\eta}}(t)| = |\phi(\bar{\eta}+t) - \bar{\phi}(\bar{\eta}+t)| \leq \|\phi - \bar{\phi}\|.$$

1 The last three estimates verify the Lipschitz constant for F_s . □

2 We obtain the following corollary.

Corollary 1. *The constant*

$$L(R) = 3\tau L_{-\beta} (1 + \delta L(f)) (1 + (N-1)\tau L_{-\beta} (1 + \tau L_{-\beta})^{N-2})$$

3 *is a Lipschitz constant for R .*

Now consider the map

$$Q : -\mathcal{A}(\beta) \ni \phi \mapsto F(\tilde{q}(\phi), \phi) \in \mathcal{A}(\beta),$$

4 where \tilde{q} is introduced in Proposition 5.

Proposition 10. *The constant*

$$L(Q) = 3\tau L_{-\beta} (1 + \tilde{\delta} L(f)) (1 + (\tilde{N}-1)\tau L_{-\beta} (1 + \tau L_{-\beta})^{\tilde{N}-2})$$

5 *is a Lipschitz constant for Q .*

6 The proof of this proposition is analogous to the reasoning above, thus we leave
7 it to the reader. One needs to use Proposition 5.

8 As a consequence we can state the following.

9 **Proposition 11.** *The Poincaré map $P : \mathcal{A}(\beta) \ni \phi \mapsto Q(R(\phi)) \in \mathcal{A}(\beta)$ is Lipschitz
10 continuous, and*

$$\begin{aligned} L(P) &= L(R)L(Q) \\ &= 3\tau L_{-\beta} (1 + \delta L(f)) (1 + (N-1)\tau L_{-\beta} (1 + \tau L_{-\beta})^{N-2}) \\ &\quad \times 3\tau L_{-\beta} (1 + \tilde{\delta} L(f)) (1 + (\tilde{N}-1)\tau L_{-\beta} (1 + \tau L_{-\beta})^{\tilde{N}-2}). \end{aligned} \tag{37}$$

11 *is a Lipschitz constant for P .*

12 **5. On the ranges of the SOP solutions.** In this section we show that if τ is
13 large enough and β is small enough, then any SOP solution $x : \mathbb{R} \rightarrow \mathbb{R}$ of (4) has
14 segments in $\mathcal{A}(\beta)$.

15 As we are going to apply paper [12] of Nussbaum, we consider equation (4) in
16 form

$$\tilde{x}'(t) = -\tau f(\tilde{x}(t-1)), \tag{38}$$

17 where $\tilde{x}(t) = x(\tau t)$ and f is given in (5). Note that f satisfies hypotheses (H1) and
18 (H2) of [12].

We define $g : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$g(x) = \frac{1}{x} \int_0^x f(u) du \quad \text{if } x \neq 0$$

1 and $g(0) = 0$. By Lemma 1 of [12], g is continuous and nondecreasing on \mathbb{R} , and
 2 there exists $d > 0$ such that $|g(x)| \geq d|x|$ for $|x| \leq 1$ and $|g(x)| \geq d$ for $|x| \geq 1$. We
 3 may choose d as follows.

4 **Proposition 12.** *Set*

$$d = \frac{1}{2} \min\{-f(-1), f(1), f'(0)\}. \quad (39)$$

5 *Then*

6 (i) $|f(x)| \geq 2d|x|$ for $|x| \leq 1$ and $|f(x)| \geq 2d$ for $|x| \geq 1$,

7 (ii) $|g(x)| \geq d|x|$ for $|x| \leq 1$ and $|g(x)| \geq d$ for $|x| \geq 1$.

8 *Proof. Statement (i).*

9 As f is strictly increasing, it is obvious that $|f(x)| \geq 2d$ for $|x| \geq 1$. Next we
 10 prove assertion (i) for $x \in [0, 1]$.

11 Examining the second derivative of f , one can check that $f''(x) > 0$ for $x \in$
 12 $(-\infty, x^*)$ and $f''(x) < 0$ for $x \in (x^*, \infty)$, where

$$x^* = \frac{1}{n} \log \left(\frac{pr}{q - pr} \right) \in \mathbb{R}. \quad (40)$$

13 So f is strictly concave up on $(-\infty, x^*)$, strictly concave down on $[x^*, \infty)$ and x^* is
 14 the unique inflection point of f .

If f is concave down on $[0, 1]$, i.e., $x^* \leq 0$, then – as the graph of f is above the
 straight line joining $(0, 0)$ and $(1, f(1))$ – we see that

$$f(x) \geq f(1)x \geq 2dx \quad \text{for } x \in [0, 1].$$

Now suppose that $x^* > 0$, i.e., x is concave up on $[0, x^*]$. Then, on the interval
 $(0, x^*]$, the graph of f is above the tangent line at $x = 0$:

$$f(x) > f'(0)x \geq 2dx \quad \text{for } x \in (0, x^*].$$

15 If $x^* \geq 1$ or the estimate $f(x) > f'(0)x$ holds for all $x \in (0, 1]$, then we have verified
 16 assertion (i) for all $x \in [0, 1]$.

So assume that $x^* < 1$ and there exists $\hat{x} \in (x^*, 1)$ such that

$$f(x) > f'(0)x \quad \text{for } x \in (0, \hat{x}) \quad \text{and} \quad f(\hat{x}) = f'(0)\hat{x}.$$

Since f' is strictly decreasing on $[x^*, \infty)$, this means that $f(x) < f'(0)x$ for all
 $x \in (\hat{x}, \infty)$. In particular, $f(1) < f'(0)$. We claim that

$$f(x) \geq f(1)x \quad \text{for } x \in [\hat{x}, 1],$$

that is

$$w(x) := f(x) - f(1)x \geq 0 \quad \text{for } x \in [\hat{x}, 1].$$

17 Note that $w(\hat{x}) > 0$ (because $f(\hat{x}) = f'(0)\hat{x} > f(1)\hat{x}$) and $w(1) = 0$. As $w''(x) =$
 18 $f''(x) < 0$ for $x \in [\hat{x}, 1]$, the derivative w' is strictly decreasing on $[\hat{x}, 1]$, and hence
 19 w cannot attain negative values on $(\hat{x}, 1)$.

Summing up,

$$f(x) \geq \min\{f'(0), f(1)\}x \geq 2dx \quad \text{for all } x \in [0, 1].$$

20 One can verify the estimate $|f(x)| \geq 2d|x|$ for $x \in [-1, 0)$ in an analogous manner.

21 *Proof of statement (ii).*

If $-1 < x < 0$, then statement (i) implies that

$$|g(x)| = \frac{1}{-x} \int_x^0 (-f(u))du \geq \frac{1}{-x} \int_x^0 (-2du)du = d|x|.$$

If $x < -1$, then, again by statement (i),

$$\begin{aligned} |g(x)| &= \frac{1}{|x|} \left| \int_0^x f(u) du \right| = \frac{1}{-x} \left(\int_x^{-1} (-f(u)) du + \int_{-1}^0 (-f(u)) du \right) \\ &\geq \frac{1}{-x} \left(\int_x^{-1} (2d) du + \int_{-1}^0 (-2d) du \right) = 2d + \frac{d}{x} \geq d. \end{aligned}$$

1 One can check analogously that $g(x) \geq dx$ if $0 < x \leq 1$, and $g(x) \geq d$ if $x \geq 1$. \square

2 By Lemma 2 of [12], there exists a function $h: [0, \infty) \rightarrow [0, \infty)$ such that

$$\lim_{\tau \rightarrow \infty} h(\tau) = \infty, \quad (41)$$

3 and

$$\text{if } x/f(x) > \tau \text{ for } x \neq 0, \text{ then } |x| > h(\tau). \quad (42)$$

Proposition 13. *The function*

$$h: [0, \infty) \ni \tau \mapsto \begin{cases} 0, & 2d\tau \leq 1, \\ 2d\tau, & 2d\tau > 1 \end{cases} \in [0, \infty)$$

4 *satisfies the properties (41) and (42).*

5 *Proof.* It is clear that (41) holds.

Next we show (42) for $x < 0$. If $x \leq -1$, then $|f(x)| \geq 2d$ by the previous proposition. Hence $x/f(x) > \tau$ implies that

$$|x| > |f(x)|\tau \geq 2d\tau \geq h(\tau).$$

For $x \in (-1, 0)$ the inequality $x/f(x) > \tau$ and the previous proposition give that

$$\tau \leq \frac{|x|}{|f(x)|} \leq \frac{|x|}{2d|x|} = \frac{1}{2d}.$$

6 Then $h(\tau) = 0$ and obviously $|x| > h(\tau)$. One can verify (42) for $x > 0$ in analogous
7 manner. \square

Let $\tilde{x}: \mathbb{R} \rightarrow \mathbb{R}$ be any SOP solution for (38), i.e., a periodic solution which has zeros spaced at distanced greater than 1. We may assume that $\tilde{x}(t) < 0$ for $t \in [-2, -1)$ and $\tilde{x}(-1) = 0$. Then \tilde{x} is strictly increasing on $[-1, 0]$ by (38). Choose $z_2 > z_1 > 0$ minimal such that $\tilde{x}(z_2) = \tilde{x}(z_1) = 0$. Note that paper [12] of Nussbaum studies only those SOP solutions \tilde{x} for which $z_2 + 1$ is the minimal period. In our case, this property follows immediatly from the positivity of f' , see Theorem 7.1 of Mallet-Paret and Sell in [8]. It is also easy to see that

$$\tilde{x}_0 := \tilde{x}(0) = \tilde{x}(z_2 + 1) = \max_{t \in \mathbb{R}} \tilde{x}(t) \quad \text{and} \quad \tilde{x}_1 := \tilde{x}(z_1 + 1) = \min_{t \in \mathbb{R}} \tilde{x}(t).$$

8 Define $z = z_2 - (z_1 + 1)$.

9 Following the proof of Lemma 10 in [12], we get the subsequent estimates for \tilde{x}_0
10 and \tilde{x}_1 .

11 **Proposition 14.** *If $\tau d > 4$, then $\tilde{x}_0 \geq \tau d/2$ and $\tilde{x}_1 \leq -\tau d/2$.*

12 As we are going to apply estimate (24) of [12] in the forthcoming proof, let us
13 note that the second line of (24) contains a typo. With the notations of [12], the
14 correct form of this estimate is the following: if $1 \leq z \leq 3/2$, then

$$x(t) \geq \alpha(2-z)c^{-3}|g((2-z)x_1c^{-1})| \quad \text{for } z_1 + 3 \leq t \leq z_2 + 1. \quad (43)$$

15 There is also a mistype in Lemma 9 of [12]: estimate (33) holds if $z_1 \geq 3/2$.

1 *Proof.* We use the results of [12] with parameters $\alpha = \tau$, $\varepsilon = 1/2$ and $x_0 = \tilde{x}_0$,
 2 $x_1 = \tilde{x}_1$. Constants c and k of [12] both equal 1 in our case. We need to consider
 3 three cases according to the sizes of z and z_1 .

Case $\max\{z_1, z\} \leq 3/2$: By Lemma 6 of [12],

$$|\tilde{x}_1| \geq \tau g(\tilde{x}_0) \quad \text{if } z_1 \leq 1,$$

and

$$\tilde{x}_0 \geq \tau |g(\tilde{x}_1)| \quad \text{if } z \leq 1.$$

From the second line of (23) in Lemma 7 of [12] we see that

$$|\tilde{x}_1| \geq \frac{\tau}{2} g\left(\frac{\tilde{x}_0}{2}\right) \quad \text{if } 1 \leq z_1 \leq 3/2.$$

By (43),

$$\tilde{x}_0 \geq \frac{\tau}{2} \left| g\left(\frac{\tilde{x}_1}{2}\right) \right| \quad \text{if } 1 \leq z \leq 3/2.$$

Case $z_1 \geq 3/2$: As $\tau d > 1$, we get from the first line of estimate (33) in [12] that

$$\tilde{x}_0 \geq h\left(\frac{\tau}{2}\right) = \tau d$$

and from the last line of (33) in [12] that

$$|\tilde{x}_1| \geq \frac{1}{2} \tau f\left(h\left(\frac{\tau}{2}\right)\right) = \frac{1}{2} \tau f(\tau d) \geq \frac{1}{2} \tau f(1) \geq \tau d.$$

4 Here we also used the fact that f is an increasing function.

Case $z \geq 3/2$: Using estimate (34) of [12], the inequality $\tau d > 1$ and the mono-
 tonicity of f , we deduce that

$$|\tilde{x}_1| \geq h\left(\frac{\tau}{2}\right) = \tau d$$

and

$$\tilde{x}_0 \geq \frac{1}{2} \tau f\left(h\left(\frac{\tau}{2}\right)\right) = \frac{1}{2} \tau f(\tau d) \geq \frac{1}{2} \tau f(1) \geq \tau d.$$

5 Summing up the above estimates and using that g is monotone increasing, we
 6 obtain that

$$\tilde{x}_0 \geq \min \left\{ \tau d, \frac{\tau}{2} \left| g\left(\frac{\tilde{x}_1}{2}\right) \right| \right\} \quad (44)$$

7 and

$$|\tilde{x}_1| \geq \min \left\{ \tau d, \frac{\tau}{2} g\left(\frac{\tilde{x}_0}{2}\right) \right\}. \quad (45)$$

Recall that constant d is chosen so that $|g(x)| \geq \min\{d, d|x|\}$ for all $x \in \mathbb{R}$. By
 applying this estimate in (44), we conclude that

$$\tilde{x}_0 \geq \min \left\{ \tau d, \frac{\tau d}{2}, \frac{\tau d}{4} |\tilde{x}_1| \right\} = \min \left\{ \frac{\tau d}{2}, \frac{\tau d}{4} |\tilde{x}_1| \right\}.$$

Estimate (45) and inequality $|g(x)| \geq \min\{d, d|x|\}$ now imply that

$$\tilde{x}_0 \geq \min \left\{ \frac{\tau d}{2}, \frac{\tau^2 d^2}{4}, \frac{\tau^2 d}{8} g\left(\frac{\tilde{x}_0}{2}\right) \right\} \geq \min \left\{ \frac{\tau d}{2}, \frac{\tau^2 d^2}{8}, \frac{\tau^2 d^2}{16} \tilde{x}_0 \right\}.$$

8 It is analogous to show that

$$|\tilde{x}_1| \geq \min \left\{ \frac{\tau d}{2}, \frac{\tau^2 d^2}{8}, \frac{\tau^2 d^2}{16} |\tilde{x}_1| \right\}. \quad (46)$$

9 Since $\tau d > 4$, we must have $\tilde{x}_0 \geq \tau d/2$ and $|\tilde{x}_1| \geq \tau d/2$. \square

1 Now we are able to determine a positive lower bound for \tilde{x} on a specific interval
2 of length 1.

Proposition 15. *If $\tau d > 4$ and B is an upper bound for f , then for each SOP solution $\tilde{x} : \mathbb{R} \rightarrow \mathbb{R}$ of (38), one can give an interval I of length 1 such that*

$$\tilde{x}(t) \geq \frac{\tau(\sqrt{B^2 + d^2} - B)}{2} \quad \text{for } t \in I.$$

3 *Proof.* Actually we prove that for any $\gamma \in (0, 1)$,

$$\tilde{x}(t) \geq \min \left\{ \frac{\tau d(1 - \gamma)}{2}, \frac{\gamma \tau d^2}{2(2B - \gamma d)} \right\} \quad \text{for } t \in \left[-1 + \frac{\gamma d}{2B}, \frac{\gamma d}{2B} \right]. \quad (47)$$

4 We show this again by applying the results of [12].

5 Note that as $d \leq f(1)/2$ and B is an upper bound for f , we have $d \leq B/2$ and
6 thus $0 < \gamma d/(2B) < 1/4$.

We show (47) first on interval $[0, \gamma d/(2B)]$. Proposition 14 gives us a lower bound for \tilde{x} on $[0, 1]$:

$$\tilde{x}(t) = \tilde{x}(0) - \tau \int_0^t f(\tilde{x}(s-1)) ds \geq \frac{\tau d}{2} - \tau B t \quad \text{for } t \in [0, 1].$$

Hence \tilde{x} is positive on $[0, d/(2B))$ and $z_1 \geq d/(2B)$. In addition, if $\gamma \in (0, 1)$, then

$$\tilde{x} \left(\frac{\gamma d}{2B} \right) \geq \frac{\tau d(1 - \gamma)}{2}.$$

7 As $\tilde{x}'(t) = -\tau f(\tilde{x}(t-1)) < 0$ for $t \in (0, z_1]$, \tilde{x} is strictly decreasing on $[0, z_1]$.

8 Therefore

$$\tilde{x}(t) \geq \frac{\tau d(1 - \gamma)}{2} \quad \text{for } t \in \left[0, \frac{\gamma d}{2B} \right]. \quad (48)$$

9 Next we estimate \tilde{x} on $[z_2 + \gamma d/(2B), z_2 + 1]$. We consider four cases according
10 to the size of z .

Case $0 \leq z \leq 1 - \gamma d/(2B)$: By the first line of estimate (10) in [12] and by Proposition 14,

$$\tilde{x}(t) \geq (t - z_2) \frac{1}{z} |\tilde{x}_1| \geq (t - z_2) \frac{2B}{2B - \gamma d} |\tilde{x}_1| \geq (t - z_2) \frac{B\tau d}{2B - \gamma d}, \quad t \in [z_2, z_1 + 2].$$

With $t = z_2 + \gamma d/(2B) \in [z_2, z_1 + 2]$ we obtain that

$$\tilde{x} \left(z_2 + \frac{\gamma d}{2B} \right) \geq \frac{\gamma \tau d^2}{2(2B - \gamma d)}.$$

Case $1 - \gamma d/(2B) \leq z \leq 1$: By the last estimate of Lemma 5 in [12],

$$\tilde{x}(t) \geq \tau(t - z_2) |g(\tilde{x}_1)| \quad \text{for } t \in [z_1 + 2, z_2 + 1].$$

As $\tilde{x}_1 \leq -\tau d/2 < -1$, we have $|g(\tilde{x}_1)| \geq d$. Choosing $t = z_2 + \gamma d/(2B) \in [z_1 + 2, z_2 + 1]$, we conclude that

$$\tilde{x} \left(z_2 + \frac{\gamma d}{2B} \right) \geq \frac{\gamma \tau d^2}{2B}.$$

Case $1 \leq z \leq 3/2$: By the first line of estimate (24) in [12],

$$\tilde{x}(t) \geq \tau(t - z_2) |g((2 - z)\tilde{x}_1)| \quad \text{for } t \in [z_2, z_1 + 3].$$

As $\tilde{x}_1 \leq -\tau d/2$ and $\tau d \geq 4$,

$$|g((2-z)\tilde{x}_1)| \geq \left| g\left(-\frac{\tau d}{4}\right) \right| \geq d.$$

Observe that $z_2 \leq z_2 + \gamma d/(2B) \leq z_1 + 3$. Thus

$$\tilde{x}\left(z_2 + \frac{\gamma d}{2B}\right) \geq \frac{\gamma \tau d^2}{2B}.$$

Case $z \geq 3/2$: By the second line of (34) in [12],

$$\tilde{x}(t) \geq \tau(t - z_2)f\left(h\left(\frac{\tau}{2}\right)\right) \quad \text{for } t \in \left[z_2, z_2 + \frac{1}{2}\right].$$

Using $\tau d > 4$, the definition of h and Proposition 12, we get that

$$\tilde{x}(t) \geq \tau(t - z_2)f(\tau d) \geq 2\tau d(t - z_2), \quad t \in \left[z_2, z_2 + \frac{1}{2}\right].$$

In particular,

$$\tilde{x}\left(z_2 + \frac{\gamma d}{2B}\right) \geq \frac{\gamma \tau d^2}{B}.$$

Summing up the four cases and using that $B < 2B - \gamma d$, we conclude that

$$\tilde{x}\left(z_2 + \frac{\gamma d}{2B}\right) \geq \frac{\gamma \tau d^2}{2(2B - \gamma d)}.$$

As $\tilde{x}'(t) = -\tau f(\tilde{x}(t-1)) > 0$ for $t \in (z_2, z_2+1)$, \tilde{x} is strictly increasing on $[z_2, z_2+1]$. Thus

$$\tilde{x}(t) \geq \frac{\gamma \tau d^2}{2(2B - \gamma d)} \quad \text{for } t \in \left[z_2 + \frac{\gamma d}{2B}, z_2 + 1\right].$$

1 The $z_2 + 1$ -periodicity of solution \tilde{x} gives that the same estimate holds for $t \in$
 2 $[-1 + \gamma d/(2B), 0]$.

3 The last result and (48) together give (47).

Observe that $\tau d(1 - \gamma)/2$ is decreasing in γ , and $\gamma \tau d^2/(4B - 2\gamma d)$ is increasing in γ . The lower bound given for \tilde{x} in (47) is maximal if we choose γ such that the two terms are equal, i.e., if

$$\gamma = \frac{B + d - \sqrt{B^2 + d^2}}{d} \in (0, 1).$$

4 We obtain the statement of the proposition with this choice of γ . □

5 The main result of this section follows easily from the last proposition.

6 **Corollary 2.** *If $\tau d > 4$ and $\beta \leq \tau(\sqrt{B^2 + d^2} - B)/2$, where B is an upper bound*
 7 *for f , then any SOP solution of (4) has a segment in $\mathcal{A}(\beta)$.*

8 *Proof.* Proposition 15 guarantees that for any SOP solution $x: \mathbb{R} \rightarrow \mathbb{R}$ of (4), there
 9 exists an interval J of length τ such that $x(t) \geq \beta$ for $t \in J$. Let $q^* \geq \sup J$
 10 be minimal with $x(q^*) = \beta$. It is clear that q^* exists (as x is continuous and has
 11 arbitrary large zeros), and then $x_{q^*} \in \mathcal{A}(\beta)$. □

1 **6. Proofs of the main theorems.** The main theorems follow from the partial
2 results of the previous sections.

Proof of Theorem 1.1. Consider (4)-(5), where p, q, r, n are fixed according to (2). We prove that for all sufficiently large $\tau > 0$, equation (4) has a unique SOP solution $\bar{x} : \mathbb{R} \rightarrow \mathbb{R}$. The corresponding periodic orbit is asymptotically stable and its region of attraction is

$$\left\{ \phi : x_t^\phi \text{ has at most one sign change for sufficiently large } t \right\}.$$

3 In addition, we show that if $\bar{\omega}$ denotes the minimal period of \bar{x} , then $\bar{\omega}/\omega$ tends to
4 1, where ω is defined by (6). Theorem 1.1 will follow by setting $\bar{y} = Ke^{\bar{x}}$.

5 Set $A = q/r - p > 0, B = p > 0, N = \lceil 1 + B/A \rceil$ and $\tilde{N} = \lceil 1 + A/B \rceil$. We
6 consider only the case when

$$\frac{A}{B} = \frac{q - pr}{pr} \notin \mathbb{N} \quad \text{and} \quad \frac{B}{A} = \frac{pr}{q - pr} \notin \mathbb{N}, \quad (49)$$

7 and thus

$$1 + \frac{B}{A} < N < 2 + \frac{B}{A} \quad \text{and} \quad 1 + \frac{A}{B} < \tilde{N} < 2 + \frac{A}{B}. \quad (50)$$

8 *Existence of an SOP solution \bar{x} .* Recall $c_i, i \in \{1, \dots, 6\}$, from (C.1)-(C.6). Using
9 the definitions of δ and $\tilde{\delta}$, we can write c_i in the form $c_i = a_i\tau + b_i\beta$ for all $i \in$
10 $\{1, \dots, 6\}$, where $a_i \neq 0$ and $b_i \neq 0$ are functions of A, B and ε . We emphasize that
11 a_i and b_i are independent of τ and β . Fix $\varepsilon > 0$ such that

$$\varepsilon < \min \left\{ \frac{B}{2}, \frac{A+B}{N-1} - A, A - \frac{A+B}{N}, \frac{A}{2}, \frac{A+B}{\tilde{N}-1} - B, B - \frac{A+B}{\tilde{N}} \right\}. \quad (51)$$

12 Inequalities (50) guarantee that the minimum on the right hand side is positive, so
13 this choice of ε is possible. One can easily check that for such ε , the coefficient a_i
14 is positive for all $i \in \{1, \dots, 6\}$. In consequence, if τ is an arbitrary positive number
15 and $\beta = \alpha\tau$, where

$$0 < \alpha < \min \left\{ \frac{a_1}{|b_1|}, \frac{a_2}{|b_2|}, \dots, \frac{a_6}{|b_6|} \right\}, \quad (52)$$

16 then c_i is positive for all $i \in \{1, \dots, 6\}$, that is, (C.1)-(C.6) are satisfied.

17 In the following, we fix ε as above, and use $\beta = \alpha\tau$ with α set as above.

As nonlinearity f defined in (5) is strictly increasing with $\lim_{x \rightarrow -\infty} f(x) = -A$
and $\lim_{x \rightarrow \infty} f(x) = B$, it is clear that $f \in \mathcal{N}(A, B, \alpha\tau, \varepsilon)$ if

$$\alpha\tau \geq \max \left\{ f^{-1}(B - \varepsilon), -f^{-1}(-A + \varepsilon) \right\}.$$

18 This inequality holds if $\tau \geq \tau_1$, where $\tau_1 = \max \left\{ f^{-1}(B - \varepsilon), -f^{-1}(-A + \varepsilon) \right\} / \alpha$.

19 In addition, recall that f admits a unique inflection point $x^* \in \mathbb{R}$ (given in (40)),
20 f' is strictly increasing on $(-\infty, x^*]$ and strictly decreasing on $[x^*, \infty)$. Hence f is
21 Lipschitz continuous with Lipschitz constant

$$L(f) = \sup_{x \in \mathbb{R}} f'(x) = f'(x^*) = \frac{qn}{4r}. \quad (53)$$

In consequence, we can use the results of Sections 3 and 4 for $\tau \geq \tau_1$. We
conclude that

$$\begin{aligned} L(P) = L(R)L(Q) &= 3\tau L_{\alpha\tau}(1 + \delta L(f)) \left(1 + (N-1)\tau L_{-\alpha\tau}(1 + \tau L_{-\alpha\tau})^{N-2} \right) \\ &\quad \times 3\tau L_{-\alpha\tau} \left(1 + \tilde{\delta} L(f) \right) \left(1 + (\tilde{N}-1)\tau L_{\alpha\tau}(1 + \tau L_{\alpha\tau})^{\tilde{N}-2} \right) \end{aligned}$$

22 is a Lipschitz constant for the Poincaré map P .

| $p =$ | $q =$ | $r =$ | $n =$ | $\tau \geq$ |
|-------|-------|-------|-------|-------------|
| 2.8 | 6 | 1.3 | 19 | 5.16 |
| 2.8 | 6.9 | 0.9 | 25 | 2.41 |
| 2.8 | 6.9 | 0.9 | 2 | 23.68 |
| 1.9 | 4.2 | 0.8 | 20 | 3.88 |
| 0.7 | 1.3 | 0.7 | 30 | 8.84 |
| 1.9 | 6.9 | 0.8 | 15 | 8.16 |
| 6.6 | 9.3 | 0.4 | 10 | 2.63 |
| 3 | 5.3 | 1.3 | 15 | 9.71 |
| 8.8 | 5.9 | 0.5 | 20 | 8.52 |
| 9 | 6.4 | 0.4 | 5 | 6.62 |
| 9 | 6.4 | 0.4 | 2 | 16.54 |

TABLE 1. A few parameters for which Theorem 1.1 holds.

If $\tau \geq \tau_2 = x^*/\alpha$, then $\alpha\tau \geq x^*$. Since f' is decreasing on $[x^*, \infty)$, we see that $L_{\alpha\tau}$ can be chosen as

$$L_{\alpha\tau} = \sup_{x \in [\alpha\tau, \infty)} f'(x) = f'(\alpha\tau) = \frac{q \left(\frac{q}{p} - r \right) n}{r^2 e^{-n\alpha\tau} + 2r \left(\frac{q}{p} - r \right) + \left(\frac{q}{p} - r \right)^2 e^{n\alpha\tau}}.$$

1 This formula shows that $\lim_{\tau \rightarrow \infty} \tau^k L_{\alpha\tau} = 0$ for any positive integer k . Similarly,
 2 $\lim_{\tau \rightarrow \infty} \tau^k L_{-\alpha\tau} = 0$ for any positive integer k .

3 As $L(f)$, N , \tilde{N} are independent of τ , and $\delta, \tilde{\delta}$ are linear functions of $\beta = \alpha\tau$, we
 4 obtain that $\lim_{\tau \rightarrow \infty} L(P) = 0$. Therefore there exists $\tau_3 \geq \max\{\tau_1, \tau_2\}$ such that
 5 $L(P) < 1$ for $\tau > \tau_3$, and hence P is a contraction on $\mathcal{A}(\alpha\tau)$. The unique fixed
 6 point of P in $\mathcal{A}(\alpha\tau)$ is the initial segment of a periodic solution \bar{x} . It is clear from
 7 the construction that \bar{x} is an SOP solution.

8 *Uniqueness.* We may assume that the parameter α was fixed so small above that
 9 $\alpha \leq (\sqrt{B^2 + d^2} - B)/2$. If $\tau d > 4$, where d is set in Proposition 12, then Corollary 2
 10 gives that all SOP solutions of (4) have segments in $\mathcal{A}(\alpha\tau)$. Hence all SOP solutions
 11 arise as fixed points of P in $\mathcal{A}(\alpha\tau)$. The uniqueness of the fixed point of P yields
 12 the uniqueness of the SOP solution for $\tau > \max\{\tau_3, 4/d\}$.

13 *Stability.* Kaplan and Yorke proved that the uniqueness of the SOP orbit gives its
 14 asymptotic stability if $\tau > \pi/(2f'(0))$, see Theorem 2.1 and Remark 2.5 of [3]. Note
 15 that our previous assumption $\tau > 4/d$ and the definition of d together guarantee
 16 that $\tau > \pi/(2f'(0))$. The region of attraction is also determined in [3].

17 *Minimal period.* The statement regarding the limit of the minimal period of \bar{x}
 18 follows at once from Theorem 1 of [12].

19 One can modify the proof of Theorem 1.1 to cover the case when either A/B or
 20 B/A is an integer using Remark 1.

21 □

22 Table 1 presents some examples when Theorem 1.1 is true.

23 Only slight modifications are needed to verify Theorem 1.2.(i).

24 *Proof of Theorem 1.2.(i).* Consider again (4)-(5), and now fix parameters p, q, r, τ
 25 according to (2) such that the inequality $\tau \min\{p, q/r - p\} > 8$ also holds. Set
 26 A, B, N, \tilde{N} as before, and assume (49).

1 *Existence of an SOP solution \bar{x} .* Choose ε and β as in the previous proof: let
 2 $\varepsilon > 0$ be so small such that (51) holds, and set $\beta = \alpha\tau$, where

$$\alpha < \frac{1}{2} \left(\sqrt{B^2 + \frac{\min\{A, B\}^2}{4}} - B \right), \quad (54)$$

3 and α satisfies (52). Then (C.1)-(C.6) hold. We emphasize that not only parameter
 4 ε but also β is fixed now (as τ is fixed too).

5 Note that $\lim_{n \rightarrow \infty} f(-\beta) = -A$ and $\lim_{n \rightarrow \infty} f(\beta) = B$. This observation and
 6 the monotonicity of f together imply that $f \in \mathcal{N}(A, B, \beta, \varepsilon)$ if n is large enough.

By Sections 3 and 4, the Poincaré map P is Lipschitz continuous with Lipschitz
 constant $L(P)$ defined in (37). We claim that $\lim_{n \rightarrow \infty} L(P) = 0$. First, recall from
 (53) that $L(f)$ can be chosen to be linear in n . In addition, if n is large enough,
 then $\beta > x^*$ (see (40)), and hence

$$L_\beta = \sup_{x \in [\beta, \infty)} f'(x) = f'(\beta) = \frac{q \left(\frac{q}{p} - r \right) n}{r^2 e^{-n\beta} + 2r \left(\frac{q}{p} - r \right) + \left(\frac{q}{p} - r \right)^2 e^{n\beta}}.$$

7 This formula shows that $\lim_{n \rightarrow \infty} n^k L_\beta = 0$ for any positive integer k . Analogously,
 8 $\lim_{n \rightarrow \infty} n^k L_{-\beta} = 0$ for any positive integer k . As $\tau, N, \tilde{N}, \delta$ and $\tilde{\delta}$ are independent
 9 of n , we see from formula (37) for $L(P)$ that $\lim_{n \rightarrow \infty} L(P) = 0$, and $L(P) < 1$ if n
 10 is large enough.

11 The unique fixed point of P in $\mathcal{A}(\beta)$ is the initial segment of an SOP solution \bar{x} .
Uniqueness. Note that

$$\begin{aligned} d &= \frac{1}{2} \min\{-f(-1), f(1), f'(0)\} \\ &= \frac{1}{2} \min \left\{ -p + \frac{q}{r + \left(\frac{q}{p} - r \right) e^{-n}}, p - \frac{q}{r + \left(\frac{q}{p} - r \right) e^n}, \frac{pn(q - pr)}{q} \right\} \end{aligned}$$

12 converges to $\min\{p, q/r - p\}/2 = \min\{A, B\}/2$ as $n \rightarrow \infty$. Hence our initial as-
 13 sumption $\tau \min\{p, q/r - p\} > 8$ ensures that $\tau d > 4$ provided n is large enough. In
 14 addition, (54) guarantees that $\beta = \alpha\tau \leq \tau(\sqrt{B^2 + d^2} - B)/2$ if n is large enough.
 15 In consequence, Corollary 2 confirms the uniqueness of the SOP solution for all
 16 sufficiently large n .

17 *Stability* follows as in the proof of Theorem 1.1.

18 Set $\bar{y} = Ke^{\bar{x}}$. Summing up the reasoning above, we conclude that there exists a
 19 threshold parameter n_0 so that Theorem 1.1.(i) is true for all $n > n_0$. As previously,
 20 it is possible to modify the reasoning above to handle the case when (49) is violated.
 21 □

22 It remains to prove the statements regarding the minimal period and the asymp-
 23 totic shape of the periodic solution as $n \rightarrow \infty$.

24 *Proof of Theorem 1.2.(ii).* In the proof of Theorem 1.2.(i), we have set β, ε and n_0
 25 such that for $n > n_0$, $f \in \mathcal{N}(A, B, \beta, \varepsilon)$ and equation (4) admits a unique SOP
 26 solution $\bar{x} : \mathbb{R} \rightarrow \mathbb{R}$.

Now let $\eta_1 > 0$ and $\eta_2 > 0$ be arbitrary. Choose $\beta' \in (0, \beta]$ and $\varepsilon' \in (0, \varepsilon]$ so that

$$\delta = \frac{2\beta'}{B - \varepsilon'} < \frac{B}{A} \tau \quad \text{and} \quad \tilde{\delta} = \frac{2\beta'}{A - \varepsilon'} < \frac{A}{B} \tau, \quad (55)$$

$$\frac{1}{A - \varepsilon'}(2\beta' + N\varepsilon'\tau + (A + B)\delta) + \frac{1}{B - \varepsilon'}(2\beta' + \tilde{N}\varepsilon'\tau + (A + B)\tilde{\delta}) < \eta_1 \quad (56)$$

and

$$\left(1 + \frac{\max\{A, B\}}{A - \varepsilon'}\right)(2\beta' + N\varepsilon'\tau + (A + B)\delta) < \eta_2. \quad (57)$$

We intend to use Propositions 2, 3, 4 and 5 with this β' and ε' . Recall that

$$\lim_{n \rightarrow \infty} f(x) = -A \quad \text{for any } x < 0,$$

and

$$\lim_{n \rightarrow \infty} f(x) = B \quad \text{for any } x > 0.$$

- 1 By the monotonicity of f , this yields that there exists $n_1 = n_1(\beta', \varepsilon') > n_0$ so that
 2 $f \in \mathcal{N}(A, B, \beta', \varepsilon')$ for all $n > n_1$. In addition, as the initial function of the SOP
 3 solution \bar{x} (corresponding to any $n > n_1$) belongs to $\mathcal{A}(\beta)$, and \bar{x} is a continuous
 4 function oscillating about 0, there exists $T = T(\beta')$ such that $\bar{x}_T \in \mathcal{A}(\beta')$. Therefore
 5 we can indeed apply Propositions 2, 3, 4 and 5 for any $n > n_1$ with β' and ε' chosen
 6 above.

It comes from the construction that the minimal period of \bar{x} is $\bar{\omega} = q + \tilde{q}$. Condition (55) ensures that we can apply estimates (21) and (30) from Propositions 3 and 5. These results together with assumption (56) imply that for $n > n_1$,

$$|\bar{\omega} - \omega| = |q + \tilde{q} - \omega| \leq |q - \sigma| + |\tilde{q} - (\omega - \sigma)| < \eta_1.$$

- 7 By (13) in Propositions 2 and by (57),

$$|\bar{x}(t + T) - v(t)| \leq \beta' + N\varepsilon'\tau + (A + B)\delta < \eta_2 \quad \text{for } t \in [0, N\tau], \quad (58)$$

where v is the ω -periodic function from Proposition 1. We see from Proposition 1 that v is Lipschitz continuous with Lipschitz constant $\max\{A, B\}$. Hence by (25) in Proposition 4,

$$\begin{aligned} |\bar{x}(t + q + T) - v(t + q)| &\leq |\bar{x}(t + q + T) - v(t + \sigma)| + |v(t + q) - v(t + \sigma)| \\ &\leq \beta' + \tilde{N}\varepsilon'\tau + (A + B)\tilde{\delta} + \max\{A, B\}|q - \sigma| \end{aligned}$$

for $t \in [0, \tilde{q}] \subset [0, \tilde{N}\tau]$. By (21) and (57), this is smaller than

$$\beta' + \tilde{N}\varepsilon'\tau + (A + B)\tilde{\delta} + \frac{\max\{A, B\}}{A - \varepsilon'}(2\beta' + N\varepsilon'\tau + (A + B)\delta) < \eta_2.$$

- 8 The last two estimates yield that $|\bar{x}(t + T) - v(t)| \leq \eta_2$ for all $t \in [0, \bar{\omega}]$. As
 9 $\bar{y} = Ke^{\bar{x}}$, the proof of Theorem 1.2.(ii) is complete. \square

- 10 For an application of Theorem 1.2.(ii), see Fig. 3.

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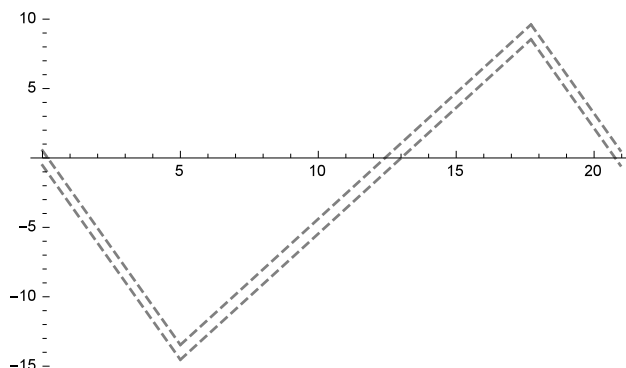


FIGURE 3. Upper and lower estimates for the SOP solution \bar{x} of (4) if $p = 2.8$, $q = 6$, $r = 1.3$, $\tau = 5$ and $n = 350$. For these parameters, $|\bar{x}(t) - v(t)| < 0.54$ for all $t \in [0, \bar{\omega}]$.

1

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