Polynomials close to 0 resp. 1 on disjoint sets *

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Dedicated to Paul Nevai
for a lifelong collaboration

Abstract

For disjoint compact subsets I, J of a real interval $[A, B]$ a construction is given for polynomials P_n of degree $n = 1, 2, \ldots$ that approximate 0 on I and 1 on J with geometric rate, vanish (in a given order) at finitely many given points of I, take the value 1 (in a given order) at finitely given points of J, and otherwise lie in between 0 and 1 on $[A, B]$. When I and J consist of alternating intervals, then P_n can also be monotone on each subinterval of $[A, B] \setminus (I \cup J)$. Some further consequences (like approximation of piecewise constant functions or the trigonometric variant) are also considered.

1 Introduction

Let I and J be disjoint compact subsets of the real line. In various problems one needs polynomials P_n of degree $n = 1, 2, \ldots$ that are close to 0 on I and to 1 on J. This can easily be achieved by extending the function

$$\chi(x) = \begin{cases} 0 & \text{if } x \in I, \\ 1 & \text{if } x \in J, \end{cases} \tag{1}$$

to a continuous function on an interval containing $I \cup J$, and then use the Weierstrass approximation theorem. In most cases, however, this rate of approximation is not sufficient, and one needs that P_n be exponentially close (with respect to the degree n of P_n) to 0 on I and exponentially close to 1 on J. One situation where this is needed is when creating a global approximant from local ones. In fact, suppose that f is a continuous function on $I \cup J$, $|f| \leq M$ there, and we have polynomials R_m and S_m of degree $m = 1, 2, \ldots$ such that with some $\varepsilon_m < 1$, $m = 1, 2, \ldots$

$$|f - R_m| \leq \varepsilon_m \qquad \text{on } I \tag{2}$$

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and

$$|f - S_m| \le \varepsilon_m \qquad \text{on } J, \tag{3}$$

and the aim is to find polynomials of comparable degree to m that approximate f on the whole $I \cup J$ with a good error. The following is a standard strategy: under week conditions (say I and J have non-empty interiors) (2) implies that R_m is at most exponentially large on J, and (3) implies that S_m is at most exponentially large on I, say

$$|R_m(x)| \le C^m, \quad x \in J \quad \text{and} \quad |S_m(x)| \le C^m, \quad x \in I$$

with some constant C that is independent of m. Now if we have polynomials P_n that exponentially approximate the above function χ, say

$$|\chi - P_n| \le D\theta^n, \qquad n = 1, 2, \dots \tag{4}$$

with some constants $\theta < 1$ and D, then for some fixed k we can set

$$H_{(k+1)m}(x) = (1 - P_{km}(x))R_m(x) + P_{km}(x)S_m(x),$$

which is a polynomial of degree at most $(k+1)m$. If $\rho > 0$ is given and k is such that $\theta^k C \le \rho$, then it is easy to check from (2)–(4) that

$$|f - H_{(k+1)m}| \le \varepsilon_m + (D + M)\rho^m, \qquad m = 1, 2, \dots$$

on $I \cup J$, so $H_{(k+1)m}$ gives a good approximation to f on the whole $I \cup J$ by polynomials the degree of which are comparable to m. The procedure is the same if the local approximants are given on more than one set.

The exponential rate of approximation in (4) is an immediate consequence of a theorem of Bernstein and Walsh (see Theorem 3 in [4, Sec. 3.3] or use [3, Theorem 6.3.1]), according to which if $K \subset \mathbf{R}$ is any compact set and g is an analytic function in a neighborhood of K, then g can be approximated exponentially fast by polynomials of degree $n = 1, 2, \dots$ (the Bernstein-Walsh theorem is more general, it is applicable also to compact subsets K of the complex plane provided the complement of K is connected). Clearly, (4) follows if we extend χ as 0 to a neighborhood of I and as 1 to a neighborhood of J.

It is often required that besides (4) the inequality

$$0 \le P_n(x) \le 1, \qquad x \in I \cup J \tag{5}$$

be also satisfied (often even on a larger set than $I \cup J$), but to achieve that one needs a different construction than what the Bernstein-Walsh theorem provides. Finally, sometimes it is also requested that besides (4) and (5) the polynomial P_n should be equal to 0 at some point(s) of I and it should be equal to 1 at some point(s) of J. This additional property needs a much more careful analysis, see for example the work [2], where, in Theorem 2, the authors prove and later apply the following: suppose that I consists of two intervals I_1 and I_2, and J is an interval lying in between I_1 and I_2, and let $\mathcal{J}$ be an interval containing I and J. If $x_0 \in J$ is given and $a_1, \dots, a_l$ are finitely many points in I, then there is a polynomial Q_n of degree at most $n = 1, 2, \dots$ such that

- $0 \le Q_n \le 1$ on $\mathcal{J}$,
- $Q_n(x_0) = 1$ and $Q_n < 1$ at every other point of $\mathcal{J}$,
- Q_n vanishes at every a_j,
- the derivatives of Q_n vanish in a given order at every a_j and also at x_0, and
- Q_n approximates the function χ exponentially fast on $I \cup J$.

In this note we settle problem of the existence and construction of similar polynomials once for all by proving

Theorem 1 *Let I, J be non-empty disjoint closed sets lying in an interval $[A, B]$ and let $\mathcal{A} \subset I$, $\mathcal{B} \subset J$ be finite sets in I and J, respectively. Then for given $k \ge 1$ there is a $\delta > 0$ such that for all sufficiently large n, say for $n \ge n_0$, there is a polynomial P_n of degree at most n such that $0 < P_n < 1$ on $[A, B] \setminus (\mathcal{A} \cup \mathcal{B})$,*

$$0 \le P_n(x) \le e^{-\delta n} \prod_{\alpha \in \mathcal{A}} |x - \alpha|^k, \qquad x \in I, \tag{6}$$

and

$$0 \le 1 - P_n(x) \le e^{-\delta n} \prod_{\beta \in \mathcal{B}} |x - \beta|^k, \qquad x \in J. \tag{7}$$

The numbers n_0 and δ in the theorem do not depend on where the points in the sets $\mathcal{A}$, $\mathcal{B}$ are located, they depend only on their number and the sets I, J and $[A, B]$. This follows from the construction. As for how large δ can be, see Section 6.

Note that (6) and (7) imply that $P^{(l)}(x) = 0$ for all $1 \le l < k$ and for all $x \in \mathcal{A} \cup \mathcal{B}$. But more is true, namely the construction in the next section gives that, besides (6)–(7), we also have

$$|P_n^{(l)}(x)| \le e^{-\delta n} \prod_{\alpha \in \mathcal{A}} |x - \alpha|^k \prod_{\beta \in \mathcal{B}} |x - \beta|^k, \qquad x \in I \cup J, \tag{8}$$

for all $1 \le l \le k$.

In the next section we prove the theorem in an elementary manner. The following sections contain further extensions.

2 Proof of Theorem 1

In the construction that follows the degree of P_n will be at most Cn with some constant C, so to get degree at most n apply it to $[n/C]$ instead of n. Also, we shall be multiplying together various polynomials satisfying conditions like in (6) and (7) on some sets and the product will satisfy similar conditions on some other sets, but the δ for the product will have to be smaller than the smallest δ for the various polynomials that were multiplied together. We shall not emphasize this in what follows.

By taking an appropriate neighborhood of I and J we may assume that I and J are unions of finitely many intervals: $I = \cup_i I_i$, $J = \cup_j J_j$, where the intervals I_i, J_j are pairwise disjoint.

We prove the theorem in several steps of increasing generality.

Case I. I and J are intervals, and both $\mathcal{A}$ and $\mathcal{B}$ have one element. We may assume that I lies to the left of J (otherwise make the transformation $x \to -x$). Let α be the only element of $\mathcal{A}$ and β be the only element of $\mathcal{B}$. If τ is the midpoint of the interval in between I and J, then for large n the polynomial

$$P_n(x) = \frac{1}{\gamma_n} \int_\alpha^x \left(1 - \left(\frac{t-\tau}{2(B-A)}\right)^2\right)^n (t-\alpha)^{2k+1}(\beta-t)^{2k+1} dt, \qquad (9)$$

where

$$\gamma_n = \int_\alpha^\beta \left(1 - \left(\frac{t-\tau}{2(B-A)}\right)^2\right)^n (t-\alpha)^{2k+1}(\beta-t)^{2k+1} dt \qquad (10)$$

satisfies all properties.

In fact, it is clear that P_n is decreasing before α, increasing on $[\alpha, \beta]$ and decreasing after β. On I (as well as on $[A, B]$ to the left of I) the absolute value of integrand in the definition of P_n is at most $e^{-n\delta_1}|x-\alpha|^{2k+1}$ with some $\delta_1 > 0$ that depends only on the I, J and $[A, B]$, and since

$$\gamma_n \geq \int_{\tau-1/\sqrt{n}}^{\tau+1\sqrt{n}} \dots \geq \frac{c_1}{\sqrt{n}}$$

with some $c_1 > 0$ (actually, it is easy to see that $\gamma_n \sim 1/\sqrt{n}$ in the sense that the ratio of the two sides lies in between two positive constants), it follows that for large n

$$0 \leq P_n(x) \leq e^{-n\delta_1/2}|x-\alpha|^{2k+2}, \qquad x \in I,$$

which proves (6) with, say, $\delta = \delta_1/4$. Since

$$1 - P_n(x) = \frac{1}{\gamma_n} \int_x^\beta \left(1 - \left(\frac{t-\tau}{2(B-A)}\right)^2\right)^n (t-\alpha)^{2k+1}(\beta-t)^{2k+1} dt,$$

the proof of (7) is the same.

Case II. I and J are intervals, and both $\mathcal{A}$ and $\mathcal{B}$ have at most one element. For example, if $\mathcal{A}$ is empty but $\mathcal{B} = \{\beta\}$, then modify the preceding construction as follows: set

$$P_n(x) = \frac{1}{\gamma_n}\int_{A-1}^{x}\left(1-\left(\frac{t-\tau}{2(B-A)+1}\right)^2\right)^n(\beta-t)^{2k+1}dt \qquad (11)$$

where

$$\gamma_n = \int_{A-1}^{\beta}\left(1-\left(\frac{t-\tau}{2(B-A)+1}\right)^2\right)^n(\beta-t)^{2k+1}dt,$$

and do similar modifications if $\mathcal{B} = \emptyset$ but $\mathcal{A} \neq \emptyset$ or if $\mathcal{A} = \mathcal{B} = \emptyset$ (then the integration should be as in (11), but the integral for γ_n should be on the interval $[A-1, B+1]$).

Case III. I and J are intervals and $\mathcal{B}$ has at most one element. Just multiply together the polynomials constructed in steps I–II for each $\alpha \in \mathcal{A}$.

Case IV. J is an interval, $I = I_1 \cup I_2$ consists of two intervals, say I_1 preceding I_2 on $\mathbf{R}$, and $\mathcal{B}$ has at most one element. We may assume that J lies in between I_1 and I_2 (if not, then we can reduce this situation to Case III by considering the convex hull of I_1 and I_2). Let a be the largest element of I_1, b the smallest element of I_2, and let $J = [c, d]$. Then $a < c < d < b$. Let now $P_{n,1}$ be the polynomial constructed in Case III for the intervals $I' = [A, a]$ and $J' = [c, B]$ and for the point sets $\mathcal{A} \cap I'$ and $\mathcal{B} \cap J'$ lying in them (actually $\mathcal{B} \cap J' = \mathcal{B}$, but $\mathcal{A} \cap I'$ may not contain all points of $\mathcal{A}$). Similarly let $P_{n,2}$ be the polynomial constructed in Case III for the intervals $I^* = [b, B]$ and $J^* = [A, d]$ and for the points sets $\mathcal{A} \cap I^*$ and $\mathcal{B} \cap J^*$ lying in them. Then $P_n = P_{n,1}P_{n,2}$ is suitable in this case.

Case V. I is an interval, J consists of at most two intervals and $\mathcal{A}$ has at most one element. Just take the polynomial from Case IV where I and J, as well as the sets $\mathcal{A}$ and $\mathcal{B}$ are interchanged, and subtract it from 1 (if J is also an intervals, then do the same but refer to Case III).

Case VI. I is an interval, J consists of at most two intervals. Just multiply together the polynomials from Case V constructed for each $\alpha \in \mathcal{A}$ separately.

Case VII. $I = \cup_i I_i$ and $J = \cup_j J_j$ consist of finitely many pairwise disjoint intervals. For each I_i let a_i be the largest element of J that precedes I_i, let b_i be the smallest element of J that follows I_i, and set $I' = I_i$ and $J_1' = [A, a_i]$, $J_2' = [b_i, B]$, $J' = J_1' \cup J_2'$ (with the modification that, say, $[A, a_i]$ is empty if there is no point of J that precedes I_i). If $P_{n,i}$ is the polynomial from Case VI for the sets I' and J' and for the point sets $\mathcal{A} \cap I'$ and $\mathcal{B} \cap J'$ that lie in them (actually $\mathcal{B} \cap J' = \mathcal{B}$), then $P_n = \prod_i P_{n,i}$ is suitable in the theorem.

■

3 Monotonicity

In the special case quoted before Theorem 1 that [2, Theorem 2] dealt with, it was also required that Q_n be monotone on the interval lying in between I_1 and J and on the interval lying in between J and I_2.

Now we have this additional property generally:

Theorem 2 *If I and J consist of finitely many intervals and the intervals in I and J alternate, then P_n in Theorem 1 can also be chosen so that P_n is monotone on any subinterval of* $[A, B] \setminus (I \cup J)$.

The condition that the intervals in I and J alternate is, in general, necessary. Indeed, if this is not the case, say there is no subinterval of J in between $I_1, I_2 \in I$, $I_1 = [a, b]$, $I_2 = [c, d]$, $b < c$, then monotonicity on (b, c) is impossible if both b and c belong to $\mathcal{A}$ (for then P_n has to increase in a right neighborhood of b and has to decrease in a left neighborhood of c because $P_n(b) = P_n(c) = 0$ and otherwise $0 \le P_n \le 1$ on $[b, c]$).

Proof. This theorem does not follow from the construction in the preceding section. However, with the following modification the above construction yields such a P_n, but the details are much more involved.

First of all, we may assume that $[A, B]$ is the smallest interval containing I and J (if this is not the case, just add to I or J the intervals $[A-1, A]$ and $[B, B+1]$ and replace $[A, B]$ by $[A-1, B+1]$). In Case I let (a, b) be the interval in between I and J, and let $a < \tau_1 < \ldots < \tau_m < b$ be finitely many points that divide (a, b) into equal parts. Now modify (9) and (10) so that

$$\left(1 - \left(\frac{t - \tau}{2(B - A)}\right)^2\right)^n$$

is replaced by

$$\frac{1}{m}\sum_{\kappa=1}^{m}\left(1 - \left(\frac{t - \tau_\kappa}{2(B - A)}\right)^2\right)^n \tag{12}$$

to create $P_n = P_{n,a,b}$.

In later steps we have two operations:

A. multiply already constructed polynomials,

B. subtract from 1 already constructed polynomials,

and the final polynomial P_n is obtained by applying repeatedly these operations to the set consisting of the polynomials $P_{n,a,b}$ for all (a, b) that are contiguous to I and J (i.e. connect 1–1 intervals of these sets). Note that in the very last step, namely in Case VII, we multiply together polynomials $P_{n,i}$ that are created for each $I_i \in I$, where $P_{n,i}$ is close to 0 on I_i and close to 1 on $[A, B] \setminus (c, b)$, where c is the largest element of J that precedes I_i (if there is no such element then $c = A$) and b is the smallest element of J that succeeds I_i (if there is no such element, then $d = B$). For large m (which is fixed for all contiguous intervals that appear in the construction) this $P_n = \prod_i P_{n,i}$ will give the desired polynomial.

To prove that, we shall only worry about the monotonicity on the contiguous intervals, for the other properties listed in Theorem 1 follow the same fashion as in the proof of Theorem 1. For simpler discussion we shall also assume that each $I_i \in I$ contains at least one point of $\mathcal{A}$ and each $J_j \in J$ contains at least one point of $\mathcal{B}$.

Let (a, b), $a < b$, be an interval lying in between an interval of I and J (as before, call such intervals contiguous), say a belongs to an I_{i_0} and b belongs to a J_{j_0}. Let also (c, d) be the contiguous interval to the left of I_{i_0}, i.e. $d \in I_{i_0}$ and $c \in J_{j_0-1}$, so the intervals (c, d), $I_{j_0}(\in I)$ and (a, b) follow each other in this order. We assume that this (c, d) exists (i.e. there is a J_j lying to the left of I_{i_0}) – what follows can be easily modified if this is not the case (then things actually become simpler).

We want to show that P_n is monotone (in the situation considered actually increasing) on (a, b), and to this effect it is sufficient to show that

$$\frac{P_n'(x)}{P_n(x)} > 0 \tag{13}$$

on (a, b). Since $P_n = \prod_i P_{n,i}$, we have

$$\frac{P_n'(x)}{P_n(x)} = \sum_i \frac{P_{n,i}'(x)}{P_{n,i}(x)}.$$

For $i \neq i_0$ the polynomial $P_{n,i}$ is exponentially close to 1 on (a, b) and its derivative is exponentially close to 0 there in the sense that there is a $\theta > 0$ independent of (a, b), of $i \neq i_0$, of m (sic!) and n such that $|1 - P_{n,i}| = O(e^{-n\theta})$ and $P_{n,i}' = O(e^{-n\theta})$ on (a, b). This follows from the constructions in Cases I–VII (see also the reasonings below) and the reason for that is that all other subinterval of $[A, B] \setminus (I \cup J)$ are of positive distance from (a, b). Therefore, for (13) it is sufficient to show that

$$\frac{P_{n,i_0}'(x)}{P_{n,i_0}(x)}$$

is positive and it is NOT of the order $O(n^{-n\theta})$ at any point of (a, b).

We shall prove that for $x \in (a, (a + b)/2]$ — when $x \in [(a + b)/2, b)$, can be handled similarly (or by symmetry).

P_{n,i_0} itself was a product (see Case VI) of some polynomials $Q_{n,s}$, one for each element of $\mathcal{A} \cap I_{i_0}$. Then

$$\frac{P_{n,i_0}'(x)}{P_{n,i_0}(x)} = \sum_s \frac{Q_{n,s}'(x)}{Q_{n,s}(x)},$$

and we are going to show that neither of the terms on the right is of the order $O(n^{-n\theta})$ on (a, b) (the terms are of positive sign), and that will complete the proof.

Claim 3 *If m is sufficiently large in (12), then for $x \in (a, (a+b)/2]$ the fractions*

$$\frac{Q'_{n,s}(x)}{Q_{n,s}(x)}$$

are positive and not of the order $O(e^{-n\theta})$.

Proof. It is sufficient to show the claim for $\frac{Q'_{n,1}(x)}{Q_{n,1}(x)}$ (the numbering of the $\alpha_s \in \mathcal{A} \cap I_{i_0}$ was arbitrary). Note that then $Q_{n,1}(\alpha_1) = 0$ for some $\alpha_1 \in \mathcal{A}$. In Case IV we saw that $1-Q_{n,1}(x)$ was the product of two polynomials: $1-Q_{n,1} = \tilde{R}_n R^*_n$, $\tilde{R}_n(\alpha_1) = R^*_n(\alpha_1) = 1$, where the contiguous interval for $\tilde{R}_n$ with respect to its ground sets $\tilde{I} = [b, B], \tilde{J} = [A, a]$ is (a, b), while the contiguous interval for R^*_n with respect to its ground sets $I^* = [A, c], J^* = [d, B]$ is (c, d). Now

$$\frac{Q'_{n,1}(x)}{Q_{n,1}(x)} = -\frac{(\tilde{R}_n)'(x)R^*_n(x)}{1-\tilde{R}_n(x)R^*_n(x)} - \frac{\tilde{R}_n(x)(R^*_n)'(x)}{1-\tilde{R}_n(x)R^*_n(x)}, \tag{14}$$

and here both $\tilde{R}'_n(x)$ and $(R^*_n)'(x)$ are negative on (a, b) (a consequence of the construction even when the modification (12) is used). So $Q'_{n,1}(x)/Q_{n,1}(x)$ is positive, and so is every $Q'_{n,s}(x)/Q_{n,s}(x)$.

If we write

$$1-\tilde{R}_n(x)R^*_n(x) = 1-\tilde{R}_n(x) + \tilde{R}_n(x)(1-R^*_n(x)),$$

then, depending on $x \in (a, (a+b)/2)$, either

$$1-\tilde{R}_n(x) \geq \tilde{R}_n(x)(1-R^*_n(x)) \tag{15}$$

or

$$\tilde{R}_n(x)(1-R^*_n(x)) \geq 1-\tilde{R}_n(x). \tag{16}$$

In the first case (note that $R^*_n(x)$ is close to 1)

$$\frac{Q'_{n,1}(x)}{Q_{n,1}(x)} \geq -\frac{1}{4}\frac{(\tilde{R}_n)'(x)}{1-\tilde{R}_n(x)}, \tag{17}$$

while in the second case

$$\frac{Q'_{n,1}(x)}{Q_{n,1}(x)} \geq -\frac{1}{2}\frac{(R^*_n)'(x)}{1-R^*_n(x)}. \tag{18}$$

Consider first (17) (i.e. when (15) is true), and let us estimate the right-hand side. $\tilde{R}_n$ is the product of polynomials as in Case III:

$$\tilde{R}_n = \prod_r S_{n,r}$$

where, for each $\beta_r \in \mathcal{B} \cap [b, B]$, $S_{n,r}$ was constructed in Case I with the modification (12) as

$$S_{n,r}(x) = 1 - \frac{1}{\gamma_{n,r}} \int_{\alpha_1}^{x} \left[\frac{1}{m} \sum_{\kappa=1}^{m} \left(1 - \left(\frac{t-\tau_\kappa}{2(B-A)}\right)^2\right)^n \right] (t-\alpha_1)^{2k+1}(\beta_r - t)^{2k+1} dt$$

with

$$\gamma_{n,r} = \int_{\alpha_1}^{\beta_r} \left[\frac{1}{m} \sum_{\kappa=1}^{m} \left(1 - \left(\frac{t-\tau_\kappa}{2(B-A)}\right)^2\right)^n \right] (t-\alpha_1)^{2k+1}(\beta_r - t)^{2k+1} dt,$$

which is again of the order $1/\sqrt{n}$ uniformly in m.

This $S_{n,r}$ is 1 at α_1 and vanishes at $\beta_r \in \mathcal{B} \cap [b, B]$. It follows that for large n

$$-S'_{n,r}(x) \geq \frac{1}{\gamma_{n,r} m} (x-\alpha_1)^{2k+1}(\beta_r - x)^{2k+1} \left(1 - \left(\frac{b-a}{2(B-A)(m+1)}\right)^2\right)^n \tag{19}$$

on (a, b), because each point of (a, b) is of distance $\leq (b-1)/(m+1)$ from one of the τ_κ.

Since for sufficiently large m at least a quarter of the τ_j lie in $[(a+2b)/3, b)$, we also obtain from the definition of $S_{n,r}$ that for $x \in (a, (a+b)/2]$ and large n

$$S_{n,r}(x) = \int_x^{\beta_r} (-S'_{n,r}(t))dt \geq c_1 \tag{20}$$

with some c_1 independent of x and n (use that for each such τ_κ the integral

$$\int_x^{\beta_r} \left(1 - \left(\frac{t-\tau_\kappa}{2(B-A)}\right)^2\right)^n dt \sim \frac{1}{\sqrt{n}} \sim \gamma_{n,r}).$$

If $x \in [a + 1/n, (a+b)/2]$, then (19) yields for large n

$$-S'_{n,r}(x) \geq c_2 \left(\frac{1}{n}\right)^{2k+1} \left(1 - \left(\frac{b-a}{2(B-A)(m+1)}\right)^2\right)^n \tag{21}$$

(note that $\gamma_{n,r} m < 1$ if n is large), which, together with (20) shows that

$$\begin{aligned} -\frac{(\tilde{R}_n)'(x)}{1-\tilde{R}_n(x)} &\geq -(\tilde{R}_n)'(x) = \sum_r (-S'_{n,r}(x)) \prod_{s \neq r} S_{n,s}(x) \\ &\geq c_3 \frac{1}{n^{2k+1}} \left(1 - \left(\frac{b-a}{2(B-A)(m+1)}\right)^2\right)^n, \end{aligned}$$

and for large m this is not of the order $e^{-n\theta}$.

Before turning to $x \in (a, a+1/n)$ let us mention that the just given proof gives also that in the case $\alpha_1 \le a - 1/n$ for all $x \in (a, (a+b)/2]$ we have again $(x-\alpha_1)^{2k+1} \ge (1/n)^{2k+1}$, so (see (14))

$$\begin{aligned}\frac{Q'_{n,1}(x)}{Q_{n,1}(x)} &\ge -\frac{(\tilde{R}_n)'(x)R^*_n(x)}{1-\tilde{R}_n(x)R^*_n(x)} \ge -(\tilde{R}_n)'(x)R^*_n(x) \ge -\frac{1}{2}(\tilde{R}_n)'(x) \\ &\ge c_3\frac{1}{n^{2k+1}}\left(1-\left(\frac{b-a}{2(B-A)(m+1)}\right)^2\right)^n, \end{aligned} \tag{22}$$

hence Claim 3 follows when $\alpha_1 \le a - 1/n$.

Next, let $x \in (a, a+1/n)$. As we have just seen, it is sufficient to consider the situation when $a - 1/n \le \alpha_1 \le a$. Since τ_1 is the smallest of the τ_κ, for all $t \in (\alpha_1, a+1/n)$ and large n

$$-S'_{n,r}(t) \sim \frac{1}{\gamma_{n,r}m}(t-\alpha_1)^{2k+1}\left(1-\left(\frac{t-\tau_1}{2(B-A)}\right)^2\right)^n, \tag{23}$$

where $\sim$ means that the ratio of the two sides lies in between two positive constants independently of t and n. This implies that

$$1-S_{n,r}(x) \sim \frac{1}{\gamma_{n,r}m}\int_{\alpha_1}^{x}(t-\alpha_1)^{2k+1}\left(1-\left(\frac{t-\tau_1}{2(B-A)}\right)^2\right)^n dt, \tag{24}$$

i.e.

$$S_{n,r}(x) = 1-q_{n,r}(x)\frac{1}{\gamma_{n,r}m}\int_{\alpha_1}^{x}(t-\alpha_1)^{2k+1}\left(1-\left(\frac{t-\tau_1}{2(B-A)}\right)^2\right)^n dt =: 1-\Delta_{r,n}(x), \tag{25}$$

where $q_{n,r}(x)$ lies in between two positive constants independently of $x \in (a, a+1/n)$ and of n. Thus,

$$S_{n,r}(x) = 1-\Delta_{r,n}(x) = e^{-\Delta_{r,n}(x)}(1+O(\Delta_{r,n}(x)^2)),$$

$$\begin{aligned}\prod_r S_{n,r}(x) &= \prod_r e^{-\Delta_{r,n}(x)}(1+O(\Delta_{r,n}(x)^2)) = e^{-\sum_r \Delta_{r,n}(x)}(1+O(\sum_r \Delta_{r,n}(x)^2)) \\ &= 1-\sum_r \Delta_{r,n}(x) + O(\sum_r \Delta_{r,n}(x)^2),\end{aligned}$$

which gives

$$1-\tilde{R}_n(x) = 1-\prod_r S_{n,r}(x) = \sum_r \Delta_{r,n}(x) + O(\sum_r \Delta_{r,n}(x)^2).$$

Since here all $\Delta_{n,r}(x)$ are of the same order, we obtain

$$1 - \tilde{R}_n(x) \sim \Delta_{1,n}(x). \tag{26}$$

On the other hand,

$$-(\tilde{R}_n)'(x) = \sum_r (-S'_{n,r}(x)) \prod_{s \neq r} S_{n,s}(x),$$

so

$$-\frac{(\tilde{R}_n)'(x)}{1 - \tilde{R}_n(x)} = \sum_r \frac{(-S'_{n,r}(x)) \prod_{s \neq r} S_{n,s}(x)}{1 - \tilde{R}_n(x)}.$$

Here the products are close to 1 according to the just made calculations and for the denominator we can use (26) to obtain

$$-\frac{(\tilde{R}_n)'(x)}{1 - \tilde{R}_n(x)} \geq c_4 \frac{-S'_{n,1}(x)}{\Delta_{n,1}(x)}$$

with some $c_4 > 0$ independent of x and n. If we also take into account (23), (24) and (25), then it follows that

$$-\frac{(\tilde{R}_n)'(x)}{1 - \tilde{R}_n(x)} \geq c_5 \frac{\Phi_n(x)}{\int_{\alpha_1}^x \Phi_n(t)dt}, \tag{27}$$

where

$$\Phi_n(t) = (t - \alpha_1)^{2k+1} \left(1 - \left(\frac{t - \tau_1}{2(B - A)}\right)^2\right)^n.$$

Since for $t \in [\alpha_1, x]$

$$\Phi_n(t) \leq (x - \alpha_1)^{2k+1}$$

we have

$$\int_{\alpha_1}^x \Phi_n(t)dt \leq (x - \alpha_1)^{2k+1},$$

from which it follows that

$$-\frac{(\tilde{R}_n)'(x)}{1 - \tilde{R}_n(x)} \geq \left(1 - \left(\frac{x - \tau_1}{2(B - A)}\right)^2\right)^n \geq \left(1 - \left(\frac{(b - a)}{2(B - A)(m + 1)}\right)^2\right)^n,$$

and we can conclude again that the left-hand side is not of the order $O(e^{-n\theta})$.

This completes the proof of Claim 3 when (15) holds, and now we turn to the other case, namely when (16), and hence (18) is true. We may assume $a - 1/n \leq \alpha_1 \leq a$, for Claim 3 has been proven in the opposite situation in (22).

As before, R_n^* is again a product (see Case III in the proof of Theorem 1)

$$R_n^* = \prod_r S_{n,r}^*$$

where, for each $\beta_r^* \in \mathcal{B} \cap [A, c]$ the polynomial $S_{n,r}^*$ was constructed in Case I as

$$S_{n,r}^*(x) = 1 - \frac{1}{\gamma_{n,r}^*} \int_x^{\alpha_1} \left[\frac{1}{m} \sum_{\kappa=1}^{m} \left(1 - \left(\frac{t - \tau_\kappa^*}{2(B-A)} \right)^2 \right)^n \right] (\alpha_1 - t)^{2k+1} (t - \beta_r^*)^{2k+1} dt$$

with

$$\gamma_{n,r}^* = \int_{\beta_r^*}^{\alpha_1} \left[\frac{1}{m} \sum_{\kappa=1}^{m} \left(1 - \left(\frac{t - \tau_\kappa^*}{2(B-A)} \right)^2 \right)^n \right] (\alpha_1 - t)^{2k+1} (t - \beta_r^*)^{2k+1} dt,$$

where now τ_κ^* are the equidistant points that divide the interval (c, d) into $m+1$ equal part, and β_r^* are the points of $\mathcal{B} \cap [A, c]$. The difference from the above discussed $S_{n,r}$ is that now the points τ_κ^* lie of distance $\geq (a-d)$ from $x \in (a, (a+b)/2]$, in particular $(S_{n,r}^*)' = O(e^{-n\theta})$ and $1 - S_{n,r}^* = O(e^{-n\theta})$ on (a, b). This also implies $1 - R_n^*(x) = O(e^{-n\theta})$ on (a, b).

Seeing that we are now discussing the situation when (16) is true, and by (26) and the definition of $\Delta_{n,1}(x)$, for $x \geq a + 1/n$ we have

$$\begin{aligned} 1 - \tilde{R}_n(x) &\geq 1 - \tilde{R}_n(a + 1/n) \geq c_5 \Delta_{n,1}(a + 1/n) \\ &\geq c_6 \left(\frac{1}{n} \right)^{2k+2} \left(1 - \left(\frac{2(b-a)}{2(B-A)(m+1)} \right)^2 \right)^n, \end{aligned}$$

it follows that (for sufficiently large m) x must lie in the interval $(a, a+1/n)$ (for otherwise $1 - \tilde{R}_n(x)$ is much larger than $1 - R_n^*(x)$, which is of the order $O(e^{-n\theta})$, and then (16) cannot hold).

Now if τ_m^* is the largest of the τ_j^*, then

$$\frac{1}{m} \sum_{\kappa=1}^{m} \left(1 - \left(\frac{t - \tau_\kappa^*}{2(B-A)} \right)^2 \right)^n \sim \frac{1}{m} \left(1 - \left(\frac{t - \tau_m^*}{2(B-A)} \right)^2 \right)^n =: \psi_n(t).$$

Here for any $u, v \in (a - 1/n, a + 1/n)$ we have

$$\psi_n(u) \sim \psi_n(v),$$

and we get the analogues

$$-(S_{n,r}^*)'(t) \sim \frac{1}{\gamma_{n,r}^*} (t - \alpha_1)^{2k+1} \psi_n(a), \tag{28}$$

and

$$1 - S_{n,r}^*(x) \sim \frac{1}{\gamma_{n,r}^*} \int_{\alpha_1}^{x} (t-\alpha_1)^{2k+1} \psi_n(a) dt, \tag{29}$$

of (23) and (24). From here the argument that was leading to (27) gives that

$$-\frac{(R_n^*)'(x)}{1-R_n^*(x)} \geq c_7 \frac{\Phi_n^*(x)}{\int_{\alpha_1}^{x} \Phi_n^*(t)dt},$$

where

$$\Phi_n^*(t) = (t-\alpha_1)^{2k+1}\psi_n(a),$$

and

$$-\frac{(R_n^*)'(x)}{1-R_n^*(x)} \geq c_7$$

immediately follows for large n, verifying that the left-hand side of (18) is not of the order $O(e^{-n\theta})$.

With this the proof of Claim 3 is complete.

■

4 Approximation and interpolation of piecewise constant functions

It is also easy to prove the following.

Theorem 4 *Let I_i be finitely many pairwise disjoint compact subsets of $\mathbf{R}$, and for each i let $\mathcal{A}_i \subset I_i$ be a finite subset of I_i. If $k \geq 1$ is given and y_i is a given real number for all i, then there is a $\delta > 0$ such that for all sufficiently large n there are polynomials P_n of degree at most n such that for all i we have*

$$|P_n(x) - y_i| \leq e^{-\delta n} \prod_{\alpha \in \mathcal{A}_i} |x-\alpha|^k, \qquad x \in I_i. \tag{30}$$

Proof. We may again replace each I_i by a set consisting of finitely many intervals that contains I_i, and then, by changing the index set, we may assume that each I_i is an interval. Then the proof proceeds by induction on the number of the intervals I_i, the one-interval case being trivial.

Indeed, let the enumeration be such that the intervals follow each other on the real line in the order I_1, I_2, $\ldots$, and replace I_1 and I_2 by their convex hull I_2^*, and set $I_i^* = I_i$ for all $i > 2$. Let $y_i^* = y_i$ for all $i \geq 2$, and for each I_i^* set $\mathcal{A}_i^* = I_i^* \cap (\cup_i \mathcal{A}_i)$ (in other words, $\mathcal{A}_2^* = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_i^* = \mathcal{A}_i$ for all $i > 2$). Let $P_{n,1}$ be the polynomial guaranteed by the induction hypothesis for these fewer intervals $\{I_i^*\}_{i\geq 2}$ and the given point sets $\mathcal{A}_i^*$ in them.

Let also be $\tilde{J} = I_1$ and let $\tilde{I}$ to be the convex hull of the intervals I_i, $i \geq 2$. Set $\tilde{\mathcal{B}} = \mathcal{A}_1$ and $\tilde{\mathcal{A}} = \cup_{i\geq 2}\mathcal{A}_i$, and let $P_{n,2}$ be the polynomials from Theorem 1 for these two intervals $\tilde{I}$ and $\tilde{J}$ and point sets $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$ lying in them. It is easy to see that then $P_n(x) = (y_1 - y_2)P_{n,2}(x) + P_{n,1}(x)$ is suitable in the theorem.

■

5 The trigonometric case

As a consequence of Theorem 1 we can get the following trigonometric variant ([2] also considered the trigonometric case).

Theorem 5 *Let I, J be non-empty disjoint closed sets lying in $(-\pi, \pi)$, and let $\mathcal{A} \subset I$, $\mathcal{B} \subset J$ be finite sets in I and J, respectively. Then for given $k \geq 1$ there is a $\delta > 0$ such that for all sufficiently large n there is a trigonometric polynomial T_n of degree at most n such that $0 < T_n < 1$ on $[-\pi, \pi] \setminus (\mathcal{A} \cup \mathcal{B})$,*

$$0 \leq T_n(x) \leq e^{-\delta n} \prod_{\alpha \in \mathcal{A}} |x - \alpha|^k, \qquad x \in I, \tag{31}$$

and

$$0 \leq 1 - T_n(x) \leq e^{-\delta n} \prod_{\beta \in \mathcal{B}} |x - \beta|^k, \qquad x \in J. \tag{32}$$

The requirement that I, J lie in $(-\pi, \pi)$ does not restrict generality. Indeed, if $\tilde{I}$, $\tilde{J}$ are non-empty disjoint 2π-periodic sets (with corresponding periodic sets $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$), then select an α such that $\alpha + \pi \notin \tilde{I} \cup \tilde{J}$. Then we can consider $I = (\tilde{I} - \alpha) \cap (-\pi, \pi)$ and $J = (\tilde{J} - \alpha) \cap (-\pi, \pi)$, for which Theorem 5 can already be applied (with $\mathcal{A} = (\tilde{\mathcal{A}} - \alpha) \cap (-\pi, \pi)$ and $\mathcal{B} = (\tilde{\mathcal{B}} - \alpha) \cap (-\pi, \pi)$) giving trigonometric polynomials T_n, and then $T_n(\cdot - \alpha)$ will work for $\mathcal{I}$ and $\mathcal{J}$.

Proof. There is an $a > 0$ such that $I \cup J \subset [-\pi + a, \pi - a]$, and choose a trigonometric polynomial S of some degree N such that S has strictly positive derivative on $[-\pi + a/2, \pi - a/2]$. If now $[A, B]$ is the range of S and P_n is the polynomial from Theorem 1 for the sets $S(I)$, $S(J)$, $S(\mathcal{A})$ and $S(\mathcal{B})$, then $T_n = P_{[n/N]}(S)$ is suitable in the theorem.

(S is easy to find: let f be a continuous 2π-periodic function which is 1 on $[-\pi, \pi - a/2]$ and which has integral 0 on $[-\pi, \pi]$, and take a trigonometric polynomial S_1 that approximates f with error $< 1/10$. Then the constant term a_0 of S_1 is at most $1/10$ in absolute value, and clearly $S(x) = \int_0^x (S_1(t) - a_0)dt$ is suitable).

■

6 How large δ is in Theorem 1?

It is a natural question to ask how large δ can be in Theorem 1. It is clear that δ depends on the distance of the sets I and J: the closer these sets are, the smaller δ must be. Let $d(I,J)$ be the Hausdorff distance of I and J. The construction in the proof of Theorem 1 can be easily traced to verify that $\delta = c\cdot d(I,J)^2$ is suitable with some $c>0$ that depends only on A,B, the number of points in $\mathcal{A}$, $\mathcal{B}$, and the number of sign changes of the function $2\chi(x)-1$ (see (1)), which function is -1 on I and 1 on J. But that is not the correct order regarding $d(I,J)^2$. Indeed, [1, Theorem 1] gives that for large n there are so called fast decreasing polynomials R_n of degree $n=1,2,\ldots$ such that $R_n(0)=1$, $0\le R_n\le 1$ on $[-1,1]$ and

$$0\le R_n(x)\le e^{-nd/30} \qquad \text{for} \quad d/2\le |x|\le 1.$$

Now if instead of (9) one uses these R_n in the proof of Theorem 1, one gets that d can be bigger than a constant times $d(I,J)$, where the constant depends only on A,B, the number of points in $\mathcal{A}$, $\mathcal{B}$, and the number of sign changes of the function $2\chi(x)-1$. On the other hand, [1, Theorem 1] implies (cf. also [1, Theorem 3]) that if $I=[-1,-d/2]$, $J=[d/2,1]$ and $[A,B]=[-1,1]$ (in which case $d(I,J)=d$), then the δ in Theorem 1 must satisfy $\delta\le d/5=d(I,J)/5$. To see that just apply [1, Theorem 1] to the polynomial $1-(P_n(x)-P_n(-x))^2$ of degree $2n$ (where P_n is from Theorem 1) and to the function φ that is 0 on $[-d/2,d/2)$ and equals to $(5/6)n\delta$ on $[-1,-d/2)\cup[d/2,1]$.

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