Oscillatory behavior of orthogonal polynomials*

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Dedicated to Zoltán Daróczy
for a long and amiable relationship

Abstract

It is shown that under fairly weak conditions on the measure the orthonormal polynomials have almost everywhere oscillatory behavior. A simple lower bound for the amplitude of oscillation is also given in terms of the measure and the equilibrium density of the support. This bound is also shown to be exact in some situations.

1 Introduction and results

Let μ be a measure on the real line of compact support Σ, and consider the orthonormal polynomials $p_n(x) = p_n(\mu, x) = \gamma_n x^n + \cdots$ with respect to μ. Assume that $I \subset \Sigma$ is an open interval, and let w be the Radon-Nikodym derivative of μ with respect to Lebesgue measure, so that $d\mu(x) = w(x)dx + d\mu_s(x)$ on I with the integrable function w and with a singular measure μ_s.

The usual pointwise asymptotic formulas for orthogonal polynomials on a finite interval have the form

$$p_n(x) \approx A(x)\sin(n\rho(x) + B(x)),$$

and the oscillatory nature of the sequence $\{p_n(x)\}_{n=0}^{\infty}$ can easily be deduced from this expression. However, all of the results concerning pointwise asymptotics are rather special, and there are no (probably there can be no) pointwise asymptotics results without imposing some strong smoothness conditions on the measure (like $\mu_s = 0$ and w satisfies some kind of continuity condition). Nevertheless, the oscillatory behavior of the sequence $\{p_n(x)\}_{n=0}^{\infty}$ was established in [7], where it was proven that if the support of μ is $[-1, 1]$ and $w(x) > 0$ almost everywhere, then for almost every $x \in [-1, 1]$ the set of accumulation points of

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the sequence $\{p_n(x)\}_{n=0}^{\infty}$ is an interval $J(x)$, symmetric about the origin, such that

$$|J(x)| \geq 2\sqrt{2/\pi}(w(x))^{-1/2}(1-x^2)^{-1/4}. \tag{1}$$

For the classical Jacobi polynomials (when $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta > -1$) we have equality in (1), so the lower bound in (1) is, in general, tight. Furthermore, oscillation is not necessary at every point, for example the orthonormal Chebyshev polynomials (that are orthonormal with respect to the weight $w(x) = 1/\sqrt{1-x^2}$ on $[-1, 1]$) take only the values $0, \pm\sqrt{2/\pi}$ at $x = 0$.

Two features of this result are as follows.

- The support of μ has to be an interval.
- The condition $w > 0$ has to be assumed on the whole support, and it is known that then it is a pretty strong condition.

The aim of this note is to answer two natural questions that emerge from these features, namely

- what happens if the support of μ consists of several intervals, or even of a general compact subset of the real line?
- Is the result true locally, i.e. on any subinterval I of the support where $w(x) > 0$ almost everywhere?

We shall have a general result that answers these questions but under a somewhat stronger local assumption, namely instead of $w(x) > 0$ almost everywhere we shall assume that $\log w$ is locally integrable on the interval I in question. Some weak global assumption is also necessary, for which we take the condition that μ belongs to the so called **Reg** class consisting of those measures for which the leading coefficients γ_n of p_n satisfy

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{\mathrm{cap}(\Sigma)},$$

where $\mathrm{cap}(\Sigma)$ is the logarithmic capacity of the support Σ of the measure μ (see [8], [10] or [14] for the necessary concepts of logarithmic potential theory that are used in this work). See the book [11] for the **Reg** class, as well as for various criteria for regularity. In particular, if the support Σ of μ consists of several intervals and $w(x) > 0$ almost everywhere on them, then $\mu \in$ **Reg**. As for the necessity of this $\mu \in$ **Reg** global condition, it is relatively easy to construct an example showing the result below does not hold without the $\mu \in$ **Reg** assumption.

To formulate the result in this paper note that if ν_S is the equilibrium measure of Σ, then ν_S is absolutely continuous on any subinterval I of the support S, and we denote its density as ω_Σ, which is the Radon-Nikodym derivative of ν_S with respect to Lebesgue-measure. This ω_Σ is a C^∞ function on I.

Theorem 1 *With the previous notations assume that μ is in the* **Reg** *class, and* $\log w$ *is locally integrable on a subinterval I of the support Σ of the measure μ. Then for almost all $x \in I$ the set of accumulation points of the sequence $\{p_n(\mu, x)\}_{n=0}^{\infty}$ is a closed (possibly infinite) interval, symmetric with respect to the origin, of length*

$$\geq 2\sqrt{2}\sqrt{\frac{\omega_\Sigma(x)}{w(x)}}. \tag{2}$$

When $\Sigma = [-1, 1]$, then

$$\omega_{[-1,1]} = \frac{1}{\pi\sqrt{1-x^2}}, \qquad x \in (-1, 1),$$

so in this case (2) gives back the bound (1).

There is an explicit form for the ω_Σ in (2) if Σ consists of finitely many intervals, and with it (2) becomes more concrete in this case. Indeed, if

$$\Sigma = \cup_{j=1}^{m} [a_j, b_j],$$

where the intervals on the right are disjoint and $a_j < b_j < a_{j+1}$ for all j, then (see [15, Section 14, (14.1)] or [11, Lemma 4.4.1])

$$\omega_\Sigma(t) = \frac{1}{\pi} \frac{\prod_{j=1}^{m-1} |t - \xi_j|}{\prod_{j=1}^{m} \sqrt{|t-a_j||t-b_j|}}, \tag{3}$$

where ξ_j lies in the interval (b_j, a_{j+1}) for all $1 \leq j < m$, and the ξ_j are the unique solutions of the system of equations

$$\int_{b_j}^{a_{j+1}} \frac{\prod_{j=1}^{m-1} (t - \xi_j)}{\prod_{j=1}^{m} \sqrt{|t-a_j||t-b_j|}} = 0, \qquad j = 1, 2, \ldots, m-1. \tag{4}$$

It is beyond the tools of this paper to investigate if the bound (2) can be improved or not, here we shall be content only with the case when Σ consists of two intervals of equal lengths.

Example 2 Let $\Sigma = [-\beta, -\alpha] \cup [\alpha, \beta]$ with some $0 < \alpha < \beta$. There is a large class of measures $d\mu(x) = w(x)dx$ with support Σ for which

$$\limsup_{n\to\infty} |p_n(x)| \leq \sqrt{2}\sqrt{\frac{\omega_\Sigma(x)}{w(x)}}$$

for all $x \in \Sigma$. So in the bound given in (2) for the amplitude of oscillation the equality is attained in this case.

It was informed to us by Peter Yuditskii [13] that, in general situations, the amplitude of oscillation of $\{p_n(x)\}$ is bigger than $\sqrt{2\omega_\Sigma(w)/w(x)}$. Indeed, if Σ consists of several intervals and $\log w$ is integrable on Σ, then this is shown by the L^2 asymptotics for the orthogonal polynomials given in [15, Theorem 12.3]. To determine this amplitude (at least almost everywhere) seems to be a non-trivial problem connected with the structure of the set E and the measure μ. Our theorem gives a simple universal lower bound, and the one-interval case as well as Example 2 show that this universal bound is the best possible in some situations. Furthermore, our result is local, and in such a local form one cannot expect improvement via asymptotic formulae for orthogonal polynomials since so far no local asymptotic result exists in the literature.

We also state the following consequence of Theorem 1.

Corollary 3 *Assume the conditions of Theorem 1. Let φ be a function on I, and suppose that*

$$\limsup_{n\to\infty} |p_n(\mu, x)| \leq \varphi(x), \qquad x \in I.$$

Then

$$w(x) \geq 2\frac{\omega_\Sigma(x)}{\varphi^2(x)} \tag{5}$$

almost everywhere on I.

This corollary gives a quantitative lower bound for the measure on subintervals of the support μ provided we know an upper bound for the orthonormal polynomials. As has been mentioned before, (5) becomes equality for Jacobi polynomials as well as for the polynomials from Example 2.

2 Proof of Theorem 1

By shrinking I if necessary, we may assume that the closure of I lies in the interior of Σ.

A local form of the orthonormal polynomials

We shall use the Christoffel-Darboux kernels

$$K_n(x,y) = \sum_{j=0}^{n} p_j(x)p_j(y), \tag{6}$$

for which it is known that

$$K_n(x,y) = \tau_n \frac{p_{n+1}(y)p_n(x) - p_n(y)p_{n+1}(x)}{y - x}, \tag{7}$$

with some positive constants τ_n. In the simple procedure below we employ an often used idea (see e.g. [4]), namely apply the Christoffel-Darboux formula for values y that are zeros of p_n, in which case the second term in the numerator on the right vanishes and $K_n(x,y)$ becomes

$$\tau_n p_{n+1}(y)\frac{p_n(x)}{y-x}$$

with the fixed value of y.

As usual, we say that $x \in I$ is a Lebesgue-point for w if

$$\lim_{r\to 0}\frac{1}{2r}\int_{-r}^{r}|w(x+t)-w(x)|dt = 0,$$

and for the measure $d\mu(x) = w(x)dx + d\mu_{\text{sing}}(x)$, we call x a Lebesgue-point for μ if it is a Lebesgue-point for w and

$$\lim_{r\to 0}\frac{1}{2r}\mu_{\text{sing}}([x-r,x+r]) = 0.$$

After the fundamental work [5] of D. Lubinsky on universality results on the Christoffel-Darboux kernel a lot of work was devoted to local asymptotics for K_n. What follows is the one of them that we need to prove our theorem. Let E be the set of all $x \in I$ which are Lebesgue-points for both μ and $\log w$. Then E has full measure in I. For each $x \in E$ the following relations are known:

$$\frac{K_n\left(x+\frac{a}{w(x)K_n(x,x)}, x+\frac{b}{w(x)K_n(x,x)}\right)}{K_n(x,x)} = (1+o_n(1))\frac{\sin\pi(a-b)}{\pi(a-b)} \tag{8}$$

(see Theorem [12, Theorem 1]) and

$$\frac{1}{n}K_n(x+a/n, x+a/n) = (1+o_n(1))\frac{\omega_\Sigma(x)}{w(x)} \tag{9}$$

(see [12, Theorem 3]), where $o_n(1)$ tends to 0 as $n\to\infty$, and its convergence to 0 is uniform in a, b lying on any fixed subinterval of $\mathbf{R}$.

We fix $x \in E$. In what follows let $A \geq 2$ be a large number, and for a given $a \in [-A, A]$ and a given n let a_n be defined by the relation

$$\frac{a}{n\omega_\Sigma(x)} = \frac{a_n}{w(x)K_n(x,x)}.$$

In view of (9) we have

$$a - a_n = a\left(1 - \frac{w(x)K_n(x,x)}{n\omega_\Sigma(x)}\right) = o_n(1).$$

Thus, (8) and (9) imply

$$\begin{aligned}
\frac{K_n\left(x+\frac{a}{n\omega_\Sigma(x)}, x+\frac{b}{n\omega_\Sigma(x)}\right)}{n\omega_\Sigma(x)/w(x)} &= \frac{K_n\left(x+\frac{a_n}{w(x)K_n(x,x)}, x+\frac{b_n}{w(x)K_n(x,x)}\right)}{n\omega_\Sigma(x)/w(x)} \\
&= (1+o_n(1))\frac{\sin\pi(a_n-b_n)}{\pi(a_n-b_n)} \\
&= (1+o_n(1))\frac{\sin\pi(a-b)}{\pi(a-b)}.
\end{aligned}$$

Now if on the left we use (7), then we obtain

$$\begin{aligned}
&\tau_n \frac{p_{n+1}\left(x+\frac{b}{n\omega_\Sigma(x)}\right)p_n\left(x+\frac{a}{n\omega_\Sigma(x)}\right) - p_{n+1}\left(x+\frac{a}{n\omega_\Sigma(x)}\right)p_n\left(x+\frac{b}{n\omega_\Sigma(x)}\right)}{(b-a)/w(x)} \\
&\qquad = (1+o_n(1))\frac{\sin\pi(a-b)}{\pi(a-b)}. \qquad (10)
\end{aligned}$$

The spacing of the zeros $z_{n,k}$ of p_n about x (more precisely in any $O(1/n)$ neighborhood of x) obeys the law

$$\lim_{n\to\infty} n(z_{n,k+1}-z_{n,k})\omega_\Sigma(x) = 1$$

([12, Theorem 2]), so, for large n, p_n has a zero which lies closer to x than $2/n\omega_\Sigma(x)$. Thus, if $A \geq 2$, then we can select a $b = b_n \in [-A, A]$ (that depends on n and x) such that

$$p_n\left(x+\frac{b_n}{n\omega_\Sigma(x)}\right) = 0.$$

But then, with this choice

$$\tau_n w(x) p_{n+1}\left(x+\frac{b}{n\omega_\Sigma(x)}\right)p_n\left(x+\frac{a}{n\omega_\Sigma(x)}\right) = (1+o_n(1))\frac{\sin\pi(a-b)}{\pi}, \quad (11)$$

and so

$$p_{n+1}\left(x+\frac{b}{n\omega_\Sigma(x)}\right) \neq 0$$

(a well-known fact that p_n and p_{n+1} do not have common zeros), and since this factor is independent of t, we can divide with it, and the rearrangement of (11) shows that uniformly in $a \in [-A, A]$

$$p_n\left(x+\frac{a}{n\omega_\Sigma(x)}\right) = (1+o_n(1))\alpha_n(x)\sin\pi(a-b_n)$$

with some non-zero $\alpha_n(x)$. Finally, if we set here $t = a/n\omega_\Sigma(x)$, then we obtain

$$p_n(x+t) = (1+o_n(1))\alpha_n(x)\sin\left(nt\rho(x)+\beta_n(x)\right), \qquad (12)$$

uniformly in $t \in [-A/n, A/n]$ where $\beta_n(x) = -\pi b_n$ and $\rho(x) = \pi\omega_\Sigma(x)$ (recall that A can be any fixed number, so, for simplicity, we wrote here A instead $A/\omega_\Sigma(x)$). By replacing $\beta_n(x)$ by $\beta_n(x) + \pi$ if necessary, we may assume that $\alpha_n(x) > 0$. This is the local form of $p_n(x+t)$ we shall be working with. In what follows shall often omit the argument x from $\alpha_n(x)$, $\beta_n(x)$ and $\rho(x)$.

We shall also need that $\alpha_n(x)$ and $\beta_n(x)$ can be selected as measurable — actually continuous — functions. Indeed, (12) shows that we may choose

$$\alpha_n(x) = \max_{t\in[-2/n\omega_\Sigma(x),2/n\omega_\Sigma(x)]} p_n(x+t),$$

and then $\beta_n(x)$ can be

$$\beta_n(x) = \arcsin(p_n(x)/\alpha_n(x))$$

(set $t = 0$ in (12)).

(12) is closely related to a beautiful recent result of D. Lubinsky [6] on local (relative) asymptotics on orthogonal polynomials in terms of their local maximal values. In [6] t was also allowed to be a complex number, which has the advantage that then the result can be differentiated, so [6] also contains local asymptotics in the same spirit for the derivatives.

The largest and smallest accumulation points

Recall now (9), which implies

$$\frac{1}{n}\sum_{k=n+1}^{2n} p_k^2(x+t) = (1+o_n(1))\frac{\omega_\Sigma(x)}{w(x)}$$

uniformly in $t \in [-A/n, A/n]$. If we substitute here (12), then we obtain

$$\frac{1}{n}\sum_{k=n+1}^{2n}(1+o_k(1))\alpha_k^2\sin^2(kt\rho+\beta_k) = (1+o_n(1))\frac{\omega_\Sigma(x)}{w(x)}, \qquad t \in [-A/n, A/n].$$

Write $\sin^2(\cdot) = (1-\cos(2\cdot))/2$ and integrate the preceding relation with respect to t over $[0, A/n]$ to obtain

$$\frac{1}{n}\sum_{k=n+1}^{2n}(1+o_k(1))\alpha_k^2\left(\frac{A}{2n} - \int_0^{A/n}\cos(2kt\rho - 2\beta_k)dt\right) = (1+o_n(1))\frac{\omega_\Sigma(x)}{w(x)}\frac{A}{n}.$$

Here

$$\int_0^{A/n}\cos(2kt\rho-2\beta_k)dt = \frac{1}{n}\int_0^{A}\cos(2(k/n)t\rho-2\beta_k)dt = O(1)\frac{1}{n}$$

uniformly in A, $n < k \le 2n$ and n, hence

$$\frac{1}{n}\sum_{k=n+1}^{2n}(1+o_k(1))\alpha_k^2\left(\frac{A}{2n}+O\left(\frac{1}{n}\right)\right)=(1+o_n(1))\frac{\omega_\Sigma(x)}{w(x)}\frac{A}{n}.$$

Since here A can be any large number, this relation implies

$$\frac{1}{n}\sum_{k=n+1}^{2n}\alpha_k^2=2(1+o_n(1))\frac{\omega_\Sigma(x)}{w(x)}, \tag{13}$$

and hence

$$\alpha(x):=\limsup_{n\to\infty}\alpha_n(x)\ge\sqrt{2\frac{\omega_\Sigma(x)}{w(x)}}. \tag{14}$$

In what follows we shall use several times that ($|\cdot|$ denoting Lebesgue measure on the real line) if $J\subset[-1,1]$ is an interval, then

$$\left|\left\{t\in\left[0,\frac{2\pi}{\rho}\right]\,\middle|\,\sin(t\rho+\beta_n)\in J\right\}\right|\ge\frac{|J|}{\rho}. \tag{15}$$

Let $\{n_s\}=\{n_s(x)\}$ be a sequence such that

$$\lim_{n_s\to\infty}\alpha_{n_s}(x)=\alpha(x)$$

(see (14)). There are two possibilities: $\alpha(x)<\infty$ or $\alpha(x)=\infty$. In the first case if $\varepsilon>0$ is given, then, in view of (15) (apply it with $J=[1-\frac{\varepsilon}{2},1]$), there is a set $H_{n_s}(x)\subseteq[0,2\pi/n_s\rho(x)]$ of measure $\ge\varepsilon/2n_s\rho(x)$ such that for large n_s and for $t\in H_{n_s,\varepsilon}(x)$ we have

$$p_{n_s}(x+t)=(1+o_{n_s}(1))\alpha_{n_s}(x)\sin(n_st\delta(x)+\beta_{n_s}(x))>(1-\varepsilon)\alpha(x). \tag{16}$$

On the other hand, if $\alpha(x)=\infty$, then, for every M and for large n_s there is a set $H^*_{n_s,M}(x)\subseteq[0,2\pi/n_s\rho(x)]$ of measure $>1/2n_s\rho(x)$ such that for large n_s and for $t\in H^*_{n_s,M}(x)$ we have

$$p_{n_s}(x+t)=(1+o_{n_s}(1))\alpha_{n_s}(x)\sin(n_st\delta+\beta_{n_s}(x))>M. \tag{17}$$

Now we are ready to prove that for almost all $x\in I$ we have

$$\limsup_{n\to\infty}p_n(x)\ge\sqrt{2\frac{\omega_\Sigma(x)}{w(x)}}. \tag{18}$$

We prove more (cf. (14)), namely that for almost all $x\in I$

$$\limsup_{n\to\infty}p_n(x)=\alpha(x). \tag{19}$$

That the left hand side is at most as large as the right hand side is clear from (12) and from the definition of $\alpha(x)$ in (14), so we only need to show that the left hand side is at least as large as $\alpha(x)$ almost everywhere in I. Suppose to the contrary that this is not the case, and we have

$$\limsup_{n\to\infty} p_n(x) < \alpha(x)$$

on a set $F \subset I$ of positive measure. Without loss of generality we may assume that $\alpha(x) < \infty$ on F or that $\alpha(x) = \infty$ on F (the set of those x with one of these properties must be of positive measure, and then just replace F with that set).

Case I: $\alpha(x) < \infty$ on F. In this case, by reducing F somewhat, we may assume that for some M we have $\alpha(x) < M$ for all $x \in F$, and that

$$\limsup_{n\to\infty} p_n(x) < (1-\varepsilon)^2\alpha(x)$$

for some $\varepsilon > 0$. Redefine $\alpha(x)$ outside F to be 0, and let $x \in E$ be a density point of F and at the same time a Lebesgue-point for this $\alpha(x)$ (which is an integrable function after the redefinition). Then for large $n_s = n_s(x)$ as above, the set

$$\mathcal{K}_{n_s} := \left\{ t \in [0, 2\pi/n_s\rho(x)] \,\middle|\, x+t \in F,\ \alpha(x+t) < \frac{1}{1-\varepsilon}\alpha(x) \right\}$$

has measure $> 2\pi/n_s\rho(x) - \varepsilon/2n_s\rho(x)$ (we used here the Lebesgue-point property of x for α as well as the fact that x is a point of density of the set F). Since the measure of $H_{n_s,\varepsilon}$ in (16) is bigger than $\varepsilon/2n_s\rho(x)$, the intersection $H_{n_s,\varepsilon} \cap \mathcal{K}_{n_s}$ cannot be empty. Now if t is a point in this intersection, then

$$p_{n_s}(x+t) < (1-\varepsilon)^2\alpha(x+t) < (1-\varepsilon)\alpha(x)$$

by the fact that $x+t \in F$ and $t \in \mathcal{K}_{n_s}$, but this contradicts (16) which must also be true since $t \in H_{n_s,\varepsilon}$.

Case II: $\alpha(x) = \infty$ on F. In this case, by reducing F somewhat, we may assume that for all $x \in F$

$$\limsup_{n\to\infty} p_n(x) < M$$

for some M. Let $x \in E$ be a density point of F. Then for large $n_s = n_s(x)$ as above, the set

$$\mathcal{K}^*_{n_s} := \{t \in [0, 2\pi/n_s\rho(x)] \,|\, x+t \in F\}$$

has measure $> 2\pi/n_s\rho(x) - 1/2n_s\rho(x)$. Since the measure of $H^*_{n_s,M}$ in (17) is bigger than $1/2n_s\rho(x)$, there is a $t \in H^*_{n_s,\varepsilon} \cap \mathcal{K}^*_{n_s}$ for which we have $p_{n_s}(x+t) < M$ because $x+t \in F$, and at the same time $p_{n_s}(x+t) > M$ because $t \in H_{n_s,M}$ (see (17)).

Either way we get a contradiction, and this contradiction proves the claim that (19) is true almost everywhere on I.

In a completely similar manner can one prove that

$$\liminf_{n\to\infty} p_n(x) = -\alpha(x) \tag{20}$$

almost everywhere on I.

The set of accumulation points is an interval

Let $E^* \subseteq E$ be the set of points $x \in E$ for which (12), (13), (19) and (20) are true. Then E^* has full measure in I.

In view of (19) and (20) the proof of the theorem will be complete if we show that the set Λ_x of the accumulation points of $\{p_n(x)\}_{n=0}^{\infty}$ is an interval for almost all $x \in E^*$.

Suppose this is not the case. Then there is a set $F \subset E^*$ of positive measure and for each $x \in F$ a closed interval $J_x \subset (-\alpha(x), \alpha(x))$, $|J_x| > 0$, with rational endpoints such that $J_x \cap \Lambda_x = \emptyset$. Then for all such x there is an N_x such that $p_n(x) \notin J_x$ for $n \geq n_x$. Since the possible J_x, N_x form a countable set, we may assume that $J_x = J = [a, b]$ and $N_x = N$ for all $x \in F$. We are going to show that this assumption (existence of F) leads to a contradiction.

We may also assume that $2\omega_\Sigma(x)/w(x) < M$ on F for some M (the set of points $x \in F$ with $2\omega_\Sigma(x)/w(x) < M$ must be of positive measure for some M).

There are now again two cases: $\alpha(x) < \infty$ for almost all $x \in F$, or $\alpha(x) = \infty$ on a subset of F of positive measure.

Case I. $\alpha(x) < \infty$ for almost all $x \in F$. By decreasing F somewhat as before, then we may assume that $\alpha(x) < M^*$ on F with some M^*. For $x \in F$ we can choose $n_s = n_s(x)$ so that $p_{n_s}(x) \to \alpha(x)$. Since $b < \alpha(x)$, we get that $\alpha_{n_s}(x) \in (b, 2\alpha(x)) \subseteq (b, 2M^*)$ is true for all large n_s. But then, for all such n_s, we obtain from (12) and (15) that the set

$$H_{n_s,J} = \{t \in [0, 2\pi/n_s\rho(x)] \,|\, \alpha_{n_s}(x)\sin(n_s t\rho(x) + \beta_{n_s}(x)) \in J/2\},$$

where $J/2$ is the interval J shrunk by factor 2 from its center, has measure $\geq |J|/4M^* n_s\rho(x)$.

Let x be a density point of F. For large $n_s > N$ the measure of the set

$$\{t \in [0, 2\pi/n_s\rho(x)] \,|\, x + t \in F\}$$

has measure $> 2\pi/n_s\rho(x) - |J|/4M^* n_s\rho(x)$, so there is a $t \in H_{n_s,J}$ that lies also in that set, as well. But then on the one hand $p_{n_s}(x+t) \notin J$ by the choice of J and F, and on the other hand $p_{n_s}(x+t) = (1+o(1))\alpha_{n_s}(x)\sin(n_s t\rho(x)+\beta_{n_s}(x))$ must lie in J because $\alpha_{n_s}(x)\sin(n_s t\rho(x)+\beta_{n_s}(x))$ lies in the middle half interval $J/2$. This is a contradiction.

Case II. $\alpha(x) = \infty$ *on a subset of* F *of positive measure.* By decreasing F we may assume $\alpha(x) = \infty$ for all $x \in F$.

Let $x \in F$ be arbitrary. Since $2\omega_\Sigma(x)/w(x) < M$ on F, (13) shows that for large n there can be at most $n/4$ indices $k \in [n+1, 2n]$ for which $\alpha_k(x) > 3\sqrt{M}$. So for all large n there is necessarily a $k \in [n+1, 2n]$ such that $\alpha_k(x), \alpha_{k-1}(x) \le 3\sqrt{M}$. In particular, if Λ_x is the set of accumulation points of $\{\alpha_n(x)\}_{n=1}^\infty$, then Λ_x has a point in $[0, 3\sqrt{M}]$. Next we show, that there is an $L > 1$ (actually independent of x) such that Λ_x has an element in each of the intervals $[(2L)^l 3\sqrt{M}, (2L)^{l+1} 3\sqrt{M}]$, $l = 0, 1, \ldots$.

The orthonormal polynomials $p_n(x)$ obey a three-term recurrence

$$xp_n(x) = a_n p_{n+1}(x) + \beta_n p_n(x) + a_{n-1} p_{n-1}(x).$$

Since the support of the generating measure is compact, the recurrence coefficients a_n, b_n are bounded. Also, by a result of Dombrowski [2] $\liminf a_n > 0$ unless μ is a singular measure, which is certain not the case. Thus, if the support Σ of μ lies in the interval $[-S, S]$, then there is an L such that

$$|p_{n+1}(x)| \le \frac{L}{2}(|p_n(x)| + |p_{n-1}(x)|)$$

for all $x \in [-2S, 2S]$. In particular, for $x \in F$ and $|t| < S$

$$|p_{n+1}(x+t)| \le \frac{L}{2}(|p_n(x+t)| + |p_{n-1}(x+t)|),$$

and if we use here the formula (12), then we obtain for large n and $|t| \le A/n$ (with any fixed $A > 0$)

$$\begin{aligned} \alpha_{n+1}(x)|\sin((n+1)t\rho(x) + \beta_{n+1}(x))| &\le L(\alpha_n(x)|\sin(nt\rho(x) + \beta_n(x))| \\ &+ \alpha_{n-1}(x)|\sin((n-1)t\rho(x) + \beta_{n-1}(x))|). \end{aligned}$$

Choose now a $t \in [0, 2\pi/(n+1)\rho(x)]$ for which $\sin((n+1)t\rho(x) + \beta_{n+1}(x)) = 1$ and conclude the inequality

$$\alpha_{n+1}(x) \le L(\alpha_n(x) + \alpha_{n-1}(x)). \tag{21}$$

Let $l \ge 0$ be an integer. We have seen that there are arbitrarily large k such that $\alpha_k(x), \alpha_{k-1}(x) \le 3\sqrt{M}$, and, since $\lambda(x) = \infty$, after every such k there is an n for which $\alpha_n(x) > (2L)^l 3\sqrt{M}$. Now if n is the first such n following k, then (21) shows that we must have $\alpha_n(x) \in [(2L)^l 3\sqrt{M}, (2L)^{l+1} 3\sqrt{M}]$. Since this happens infinitely often, $\{\alpha_n(x)\}$ has, indeed, an accumulation point in the interval $[(2L)^l 3\sqrt{M}, (2L)^{l+1} 3\sqrt{M}]$.

Let x be a density point for F, and choose the smallest l such that $(2L)^l 3\sqrt{M} > b$ (recall that $J = [a, b]$ is the interval such that $p_n(x) \notin J$ for $x \in F$ and $n > N$). According to what we have just proven, the sequence $\{\alpha_n(x)\}$ has infinitely many elements in the interval $(b, 3(2L)^{l+1}\sqrt{M}]$, say $\alpha_{n_s}(x)$ are such elements. For all such n_s, we obtain from (12) and (15) that the set

$$H_{n_s,J} = \{t \in [0, 2\pi/n_s\rho(x)] \,|\, \alpha_{n_s}(x)\sin(n_s t\rho(x) + \beta_{n_s}(x)) \in J/2\},$$

where $J/2$ is the interval J shrunk by factor 2 from its center, has measure $\geq |J|/6n_s(2L)^{l+1}\sqrt{M}\rho(x)$. For large $n_s > N$ the set

$$\mathcal{K}^*_{n_s} := \{t \in [0, 2\pi/n_s\rho(x)] \,|\, x + t \in F\}$$

has measure $> 2\pi/n_s\rho(x) - |J|/12n_s(2L)^{l+1}\sqrt{M}\rho(x)$, so there is a $t \in H_{n_s,J} \cap \mathcal{K}^*_{n_s}$. But then we obtain a contradiction as before: on the one hand $p_{n_s}(x+t) \notin J$ by the choice of J and F, and on the other hand $p_{n_s}(x+t) = (1+o(1))\alpha_{n_s}(x)\sin(n_s t\rho(x) + \beta_{n_s}(x))$ must lie in J because $\alpha_{n_s}(x)\sin(n_s t\rho(x) + \beta_{n_s}(x))$ lies in the middle half interval $J/2$.

With this the proof of Theorem 1 is complete.

∎

3 Details on Example 2

In view of (3)–(4) the equilibrium density of $\Sigma = [-\beta, -\alpha] \cup [\alpha, \beta]$ is

$$\omega_\Sigma(t) = \frac{1}{\pi}\frac{|t|}{\sqrt{|t^2-\alpha^2||t^2-\beta^2|}}, \qquad t \in \Sigma.$$

Let the weight function $w(t) = d\mu(t)/dt$ be defined on Σ by

$$w(t) = |t|\frac{\varphi(t^2)}{\sqrt{|t^2-\alpha^2||t^2-\beta^2|}},$$

where φ is any continuously differentiable positive function on $[\alpha^2, \beta^2]$. Consider also

$$W(u) = \frac{\varphi(u)}{\sqrt{|u-\alpha^2||u-\beta^2|}}$$

on the interval $[\alpha^2, \beta^2]$, and let P_n be the orthonormal polynomials with respect to W. The function $\xi = 2(u-\alpha^2)/(\beta^2-\alpha^2) - 1$ maps $[\alpha^2, \beta^2]$ into $[-1, 1]$, and set

$$\rho(\xi) = W(u)\frac{2}{\beta^2-\alpha^2} =: \frac{\theta(\xi)}{\sqrt{1-\xi^2}},$$

where θ is a continuously differentiable positive function. Then, by a result of Bernstein [1] and Szegő [9, Chap. XII], if Φ_m are the orthonormal polynomials with respect to ρ, we have uniformly in t

$$\Phi_m(\cos t) = \sqrt{\frac{2}{\pi}}\Re\left(e^{-im\varphi}\pi(e^{it})\right) + O(m^{1/2}),$$

where

$$\pi(z) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\theta(\cos t)dt\right\},$$

for which $\theta(\cos t) = 1/|\pi(e^{it})|^2$. Since the equilibrium density of $[-1,1]$ is $1/\pi\sqrt{1-\xi^2}$, this implies

$$\limsup_{m\to\infty}|\Phi_m(\xi)| \le \sqrt{2}\sqrt{\frac{\omega_{[-1,1]}(\xi)}{\rho(\xi)}},$$

which translates into

$$\limsup_{m\to\infty}|P_m(u)| \le \sqrt{2}\sqrt{\frac{\omega_{[\alpha^2,\beta^2]}(u)}{W(u)}}. \tag{22}$$

The same argument gives the same conclusion for the weight function

$$\tilde{W}(u) = uW(u), \qquad u \in [\alpha^2,\beta^2]$$

and for the corresponding orthonormal polynomials $\tilde{P}_n$:

$$\limsup_{m\to\infty}|\tilde{P}_m(u)| \le \sqrt{2}\sqrt{\frac{\omega_{[\alpha^2,\beta^2]}(u)}{\tilde{W}(u)}} = \sqrt{2}\sqrt{\frac{\omega_{[\alpha^2,\beta^2]}(u)}{uW(u)}}. \tag{23}$$

After these let us turn to the orthonormal polynomials $p_n(x) = \gamma_n x^n + \cdots$ with respect to w on $\Sigma = [-\beta,-\alpha]\cup[\alpha,\beta]$. Because of the symmetry of w and Σ, these p_n are even for even n and odd for odd n. Let first n be even, say $n = 2m$. Using that the monic orthogonal polynomial is the one that minimizes the L^2 integral with the given weight (see e.g. [11, (3.10)]) we have

$$\int_\Sigma \left(\frac{1}{\gamma_n}p_n(x)\right)^2 w(x)dx = \min_{h_n(x)=x^n+\cdots}\int_\Sigma h_n^2(x)w(x)dx,$$

and by simple symmetrization (i.e. considering $(h_n(x)+h_n(-x))/2$ instead of h_n), in the minimum on the right the polynomials h_n can be taken to be even. But then the substitution $u = x^2$ shows that

$$\int_{\alpha^2}^{\beta^2}\left(\frac{1}{\gamma_n}p_n(\sqrt{u})\right)^2 W(u)du = \min_{h_n(\sqrt{u})=u^m+\cdots}\int_{\alpha^2}^{\beta^2} h_n^2(\sqrt{u})W(u)du,$$

so $(1/\gamma_n)p_n(\sqrt{u})$ is the m-th monic orthogonal polynomial with respect to W. The $u = x^2$ substitution also shows that the L^2 norm of $p_n(\sqrt{u})$ with respect to W is 1, so we have $p_n(x) = P_m(x^2)$. Since

$$\omega_\Sigma(x) = |x|\omega_{[\alpha^2,\beta^2]}(x^2),$$

we can see that (22) is the same as

$$\limsup_{n=2m\to\infty} |p_n(x)| \leq \sqrt{2}\sqrt{\frac{\omega_\Sigma(x)}{w(x)}}. \tag{24}$$

In a similar fashion, if $n = 2m+1$ is odd, then p_n is odd, and

$$\int_{\alpha^2}^{\beta^2} \left(\frac{1}{\gamma_n} p_n(\sqrt{u})\right)^2 W(u) = \int_{\alpha^2}^{\beta^2} \left(\frac{1}{\gamma_n} \frac{p_{2m+1}(\sqrt{u})}{\sqrt{u}}\right)^2 uW(u)du$$

is the minimal value of the L^2 integrals for $\tilde{W}(u) = uW(u)$ among all monic polynomials of degree m, and we can conclude as before that $p_{2m+1}(\sqrt{u})/\sqrt{u} = \tilde{P}_m(u)$, i.e. $p_{2m+1}(x) = x\tilde{P}_m(x^2)$. But then (23) gives

$$\begin{aligned} \limsup_{n=(2m+1)\to\infty} |p_n(x)| &= |x| \limsup_{m\to\infty} |\tilde{P}_n(x^2)| \leq |x|\sqrt{2}\sqrt{\frac{\omega_{[\alpha^2,\beta^2]}(x^2)}{\tilde{W}(x^2)}} \\ &= \sqrt{2}\sqrt{\frac{x^2\omega_{[\alpha^2,\beta^2]}(x^2)}{x^2W(x^2)}} = \sqrt{2}\sqrt{\frac{|x|\omega_{[\alpha^2,\beta^2]}(x^2)}{|x|W(x^2)}} = \sqrt{2}\sqrt{\frac{\omega_\Sigma(x)}{w(x)}}. \end{aligned}$$

This and (24) prove the claim in Example 2.

■

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