Coefficient estimates on general compact sets*

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Dedicated to Ferenc Schipp
for the many enjoyable discussions of the past

Abstract

This paper deals with best possible estimates for the coefficients of polynomials in terms of the supremum norm of the polynomials on a given compact subset K of the plane. The results solve a problem of D. Dauvergne.

Let K be a compact set on the plane of positive logarithmic capacity $\mathrm{cap}(K)$ (for the notions of logarithmic potential theory see the book [3]). A classical result of Fekete and Szegő implies that if $p_n(z) = z^n + \cdots$ is a monic polynomial, then (see e.g. [3, Theorem 5.5.4])

$$\|p_n\|_K \geq \mathrm{cap}(K)^n, \tag{1}$$

where $\|\cdot\|_K$ denotes the supremum norm on K. Hence, for monic polynomials

$$\liminf_{n\to\infty} \frac{1}{n} \log ||p_n||_K \geq \log \mathrm{cap}(K).$$

D. Dauvergne asked ([1]) if one has the same conclusion if, instead of monic, one only has one of the coefficients, say a_m (the coefficient of z^m), in p_n, satisfies $|a_m| \geq 1$ for some m with $n - b\log n \leq m \leq n$. Here b is a fixed constant.

This is a natural problem related to the classical (1). The present simple note grew out of this problem — it will follow that the answer is YES even for the larger range of coefficients a_m, $n - o(n) \leq m \leq n$.

In general, we can ask if $p_n(z) = \sum_j a_j z^j$ is a polynomial of degree n, then what upper estimates are true on the coefficients a_j in terms of K and the norm $\|p_n\|_K$. Our first result provides such a simple estimate.

Theorem 1 *For a compact set $K \subset \mathbf{C}$ of positive capacity set*

$$R_K = \sup_{z\in K} |z|.$$

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Then for any polynomial $p_n(z) = \sum_{k=0}^n a_k z^k$ *of degree* n *and for any* $0 \le k \le n$ *we have*

$$|a_k| \le \binom{n}{k} (R_K)^{n-k} \|p_n\|_K \frac{1}{\mathrm{cap}(K)^n} \tag{2}$$

As an immediate consequence we obtain the first half of the following corollary that gives a positive answer to the problem mentioned in the beginning of this paper.

Corollary 2 *Let* $K \subset \mathbf{C}$ *be a compact set of positive capacity, and let* $p_n(z) = \sum_{k=0}^n a_{k,n} z^k$ *be polynomials of degree* $n = 1, 2, \ldots$. *Then for any sequence* $\{i_n\}$ *with* $i_n = o(n)$ *we have*

$$\limsup_{n\to\infty} \left(\frac{|a_{n-i_n,n}|}{\|p_n\|_K} \right)^{1/n} \le \frac{1}{\mathrm{cap}(K)}. \tag{3}$$

This is best possible:

- *for any sequence* $\{i_n\}$ *of integers with* $i_n = o(n)$ *there are* p_n *for which*

$$\lim_{n\to\infty} \left(\frac{|a_{n-i_n,n}|}{\|p_n\|_K} \right)^{1/n} = \frac{1}{\mathrm{cap}(K)}, \tag{4}$$

- *if* $i_n \ne o(n)$, *then there is a* K *and a sequence of polynomials* p_n *such that*

$$\limsup_{n\to\infty} \left(\frac{|a_{n-i_n,n}|}{\|p_n\|_K} \right)^{1/n} > \frac{1}{\mathrm{cap}(K)}. \tag{5}$$

Proof of Theorem 1. For $k = n$ the claim is immediate from (1), so in what follows we assume $k < n$. Below we set $k = n - i$, and then $i \ge 1$.

We write

$$p_n(z) = a_n \prod_{j=1}^n (z - z_j).$$

If μ_K is the equilibrium measure of K, then

$$\log|a_n| + \sum_{j=1}^n \int \log|z - z_j| d\mu_K(z) = \int \log|p_n(z)| d\mu_K(z) \le \log\|p_n\|_K,$$

and hence

$$\log|a_n| + \sum_{j=1}^n \Big(\int \log|t - z_j| d\mu_K(t) - \log \mathrm{cap}(K) \Big) \le \log\|p_n\|_K - n \log \mathrm{cap}(K).$$

On the left

$$\int \log|t - \xi| d\mu_K(z) - \log \mathrm{cap}(K) = g_{\overline{\mathbf{C}} \setminus K}(\xi)$$

is the Green's function of the unbounded component of the complement of K with pole at infinity (see e.g. [3, Sec. 4.4]), therefore we have

$$\log|a_n| + \sum_{j=1}^{n} g_{\overline{\mathbf{C}}\setminus K}(z_j) \leq \log\|p_n\|_K - n\log\mathrm{cap}(K),$$

which automatically implies

$$\log|a_n| + \sum_{j=1}^{i} g_{\overline{\mathbf{C}}\setminus K}(z_j) \leq \log\|p_n\|_K - n\log\mathrm{cap}(K)$$

for any $1 \leq i \leq n$.

Let $\Delta_R(z_0)$ be the closed disk of radius R about the point z_0, and let $\Delta_R = \Delta_R(0)$. Since $K \subset \Delta_{R_K}$, and the Green's function is a monotone decreasing function of its domain, we have for all $|z_j| > R_K$ the inequality

$$g_{\overline{\mathbf{C}}\setminus K}(z_j) \geq g_{\overline{\mathbf{C}}\setminus \Delta_{R_K}}(z_j) = \log(|z_j|/R_K),$$

where we have used that

$$g_{\overline{\mathbf{C}}\setminus \Delta_R}(z) = \log(|z|/R)$$

(as easily follows from the defining properties of Green's functions). The inequality

$$g_{\overline{\mathbf{C}}\setminus K}(z_j) \geq \log(|z_j|/R_K),$$

also holds if $|z_j| \leq R_K$ (the right-hand side is then non-positive), therefore we can conclude

$$\log|a_n| + \sum_{j=1}^{i} \log(|z_j|/R_K) \leq \log\|p_n\|_K - n\log\mathrm{cap}(K),$$

i.e.

$$|a_n||z_1||z_2|\cdots|z_i| \leq R_K^i\|p_n\|_K\frac{1}{\mathrm{cap}(K)^n}.$$

But the labelling of the zeros was arbitrary, therefore we have the same inequality with any i different indices:

$$|a_n||z_{j_1}||z_{j_2}|\cdots|z_{j_i}| \leq R_K^i\|p_n\|_K\frac{1}{\mathrm{cap}(K)^n}.$$

Now $a_{n-i} = \pm a_n\sigma_i$, where σ_i is the i-th elementary symmetric polynomial of the zeros z_j, and, by summing up the previous inequalities, we obtain (2) for $k = n - i$.

■

The next proposition shows that, in general, one cannot have a better estimate than what was given in Theorem 1.

Proposition 3 *For every R and every $\varepsilon > 0$ there is a $K \subseteq \Delta_R$ and there are polynomials $p_n(z) = \sum_{k=0}^n a_k z^k$ of degree $n = 1, 2, \ldots$ such that for all k*

$$|a_k| \geq \binom{n}{k} (R - \varepsilon)^{n-k} \|p_n\|_K \frac{1}{\mathrm{cap}(K)^n}.$$

Proof. Just set $K = \Delta_\varepsilon(R - \varepsilon)$ (the disk of radius ε about the point $R - \varepsilon$) and

$$p_n(z) = (z - (R - \varepsilon))^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (R - \varepsilon)^{n-k} z^k,$$

for which $\|p_n\|_K = \varepsilon^n$ and $\mathrm{cap}(K) = \varepsilon$, so the claim is obvious.

■

Now we are ready to prove Corollary 2.

Proof of Corollary 2. In view of (2) for $k = n - i_n$ it is sufficient to show that if $i_n = o(n)$, then

$$\binom{n}{i_n}^{1/n} \to 1.$$

But this follows from

$$\binom{n}{i_n} \leq \frac{n^{i_n}}{i_n!}$$

and the fact that (see [2], formulae (1) and (2))

$$m! \geq \frac{\sqrt{2\pi} m^{m+1/2}}{e^{m-1/(12m+1)}}, \qquad m = 1, 2, \ldots.$$

Indeed, then

$$\binom{n}{i_n}^{1/n} \leq \left(\frac{n}{i_n}\right)^{i_n/n} \left(\frac{e^{i_n - 1/(12 i_n + 1)}}{\sqrt{2\pi i_n}}\right)^{1/n},$$

and for $i_n/n \to 0$ both factors on the right tend to 0 because $(i_n/n)\log(n/i_n) \to 0$ (the function $x \log 1/x$ has zero right limit at 0). This proves (3).

To prove (4) let $T_m(z) = z^m + \cdots$ be the Chebyshev polynomial of the set K of degree m. We have (see [3, Corollary 5.5.5])

$$\|T_m\|_K^{1/m} \to \mathrm{cap}(K), \qquad m \to \infty,$$

so

$$\|T_{n-i_n}\|_K^{1/(n-i_n)} \to \mathrm{cap}(K), \qquad n \to \infty,$$

which implies, in view of $i_n = o(n)$,

$$\|T_{n-i_n}\|_K^{1/n} \to \mathrm{cap}(K), \qquad n \to \infty. \tag{6}$$

Add to T_{n-i_n} (which is of degree $n - i_n$) some term $\varepsilon_n z^n$, where $\varepsilon_n \to 0$ so fast that along with (6) we also have for $p_n(z) = \varepsilon_n z^n + T_{n-i_n}(z)$

$$\|p_n\|_K^{1/n} \to \mathrm{cap}(K).$$

Since $a_{n-i_n} = 1$ for p_n, (4) follows.

Finally, for $R - \varepsilon > 1$ Proposition 3 proves (5). Indeed, if $i_n \geq cn$ for infinitely many n with some $c > 0$, then for $cn \leq i_n \leq n/2$ we have with $m = [cn]$

$$\binom{n}{i_n} \geq \binom{n}{m} = \frac{n(n-1)\cdots(n-m+1)}{m(m-1)\cdots 1} \geq \left(\frac{n}{m}\right)^m,$$

and so

$$\binom{n}{i_n}^{1/n} \geq (\frac{1}{c})^c > 1,$$

while for $i_n \geq n/2$

$$((R-\varepsilon)^{i_n})^{1/n} \geq \sqrt{R-\varepsilon} > 1.$$

In either case, if we set $k = n - i_n$ in the polynomials in Proposition 3, the inequality (5) follows.

■

For coefficients of low order (2) yields, in the spirit of Corollary 2, that if $j_n = o(n)$, then

$$\limsup_{n\to\infty} \left(\frac{|a_{j_n,n}|}{\|p_n\|_K}\right)^{1/n} \leq \frac{R_K}{\mathrm{cap}(K)}. \tag{7}$$

Our last result shows that while, as we have seen, among all sets this cannot be improved, for individual K's it can.

We say that K is a regular set if the Green function $g_{\mathbf{C}\setminus K}(z)$ (considered to be extended to 0 outside the unbounded component of $\mathbf{C} \setminus K$) is continuous, i.e. $g_{\mathbf{C}\setminus K}(z) = 0$ for all $z \in K$.

Theorem 4 *Let K be regular, and let $L_K = g_{\mathbf{C}\setminus K}(0)$ be the value of the Green's function of the complement of K at the origin. Then for $j_n = o(n)$ and for any polynomials $p_n(z) = \sum_{j=0}^n a_{j,n} z^j$ of degree $n = 1, 2, \ldots$ we have*

$$\limsup_{n\to\infty} \left(\frac{|a_{j_n,n}|}{\|p_n\|_K}\right)^{1/n} \leq e^{L_K}. \tag{8}$$

Furthermore, this estimate is sharp:

- *for any $j_n = o(n)$ there are polynomials p_n with*

$$\lim_{n\to\infty} \left(\frac{|a_{j_n,n}|}{\|p_n\|_K}\right)^{1/n} = e^{L_K}, \tag{9}$$

- *for any $j_n \neq o(n)$ there is a K and there are polynomials p_n with*

$$\limsup_{n\to\infty} \left(\frac{|a_{j_n,n}|}{\|p_n\|_K} \right)^{1/n} > e^{L_K}. \tag{10}$$

In particular, if 0 belongs to a bounded component of $\mathbf{C} \setminus K$ or if $0 \in K$, then

$$\limsup_{n\to\infty} \left(\frac{|a_{j_n,n}|}{\|p_n\|_K} \right)^{1/n} \leq 1.$$

To compare (7) and (8) consider the segment K_α connecting the points $e^{\pm i\alpha}$ for an $0 < \alpha \leq \pi/2$. In this case $R_{K_\alpha} = 1$, $\mathrm{cap}(K_\alpha) = \frac{1}{2}\sin\alpha$ (because the capacity of a line segment is one quarter of its length, see [3, Table 5.1]), while using that the Green's function of the complement of the segment $[-1,1]$ is $\log|z + \sqrt{z^2-1}|$, simple computation shows that $L_{K_\alpha} = \log\cot(\alpha/2)$. Hence, the right-hand side of (7) is $2/\sin\alpha = 1/\sin(\alpha/2)\cos(\alpha/2)$, while the right-hand side of (8) is $\cot(\alpha/2) = \cos(\alpha/2)/\sin(\alpha/2)$. For example, the latter one is 1 for $\alpha = \pi/2$, while the former one is 2.

Proof. Let $\varepsilon > 0$, and consider the level set

$$G = \{z \,|\, g_{\overline{\mathbf{C}}\setminus K}(z) < L_K + \varepsilon\}.$$

The Green's function is subharmonic, hence upper semi-continuous. Therefore G is an open set that contains K and contains the origin, say it contains the closed disk Δ_δ, $\delta > 0$. By the Bernstein-Walsh lemma ([3, Theorem 5.5.7(a)])

$$|p_n(z)| \leq \|p_n\|_K e^{n(L_K+\varepsilon)}, \qquad z \in G.$$

Cauchy's formula written for the circle about the origin and of radius δ yields

$$|a_{j_n,n}| \leq \left| \frac{1}{2\pi i} \int_{|\xi|=\delta} \frac{p_n(\xi)}{\xi^{j_n+1}} d\xi \right| \leq \frac{\|p_n\|_K e^{n(L_K+\varepsilon)}}{\delta^{j_n}},$$

and if we take here n-th root we obtain

$$\limsup_{n\to\infty} \left(\frac{|a_{j_n,n}|}{\|p_n\|_K} \right)^{1/n} \leq e^{L_K+\varepsilon},$$

which proves (8) since $\varepsilon > 0$ is arbitrary.

(9) is trivial for $p_n(z) = \varepsilon_n z^n + x^{j_n}$ with sufficiently small $\varepsilon_n > 0$ if 0 belongs to K or to a bounded connected component of $\mathbf{C} \setminus K$. Hence in proving (9) we may assume that 0 belongs to the unbounded component of $\mathbf{C} \setminus K$. Consider $1/z$ and its best approximant Q_m of degree m on K. Since 0 belongs to the unbounded component of $\mathbf{C} \setminus K$, the function $1/z$ is analytic on the so-called polynomial convex hull $\mathrm{Pc}(K)$ of K (which is the union of K with all the

bounded components of $\mathbf{C}\setminus K$), and this latter set has connected complement, so the Bernstein-Walsh theorem (see Theorem 3 in [4, Sec. 3.3] or use [3, Theorem 6.3.1]) gives (note that $L_K = L_{\mathrm{Pc}(K)}$)

$$\lim_{n\to\infty} \left\| \frac{1}{z} - Q_{n-1-j_n}(z) \right\|_K^{1/(n-1-j_n)} = e^{-L_K},$$

which implies first

$$\lim_{n\to\infty} \left\| \frac{1}{z} - Q_{n-1-j_n}(z) \right\|_K^{1/n} = e^{-L_K},$$

then

$$\limsup_{n\to\infty} \|1 - zQ_{n-1-j_n}(z)\|_K^{1/n} \le e^{-L_K},$$

and finally

$$\limsup_{n\to\infty} \left\| z^{j_n} - z^{j_n+1}Q_{n-1-j_n}(z) \right\|_K^{1/n} \le e^{-L_K},$$

because

$$\left(\max_{z\in K} |z|^{j_n} \right)^{1/n} \to 1.$$

Since for $p_n(z) = z^{j_n} - z^{j_n+1}Q_{n-1-j_n}(z)$ we have $a_{j_n} = 1$, we can conclude

$$\limsup_{n\to\infty} \left(\frac{|a_{j_n,n}|}{\|p_n\|_K} \right)^{1/n} \ge e^{L_K},$$

and (9) follows from here and from (8) (if $z^{j_n+1}Q_{n-1-j_n}(z)$ happens to have smaller than n degree, just add to it $\varepsilon_n z^n$ with some very small ε_n). This proves (9).

Now let $j_n \ne o(n)$, say $j_n > cn$ for infinitely many n with some $c > 0$. (7) and (9) shows that

$$\frac{R_K}{\mathrm{cap}(K)} \ge e^{L_K}$$

if 0 belongs to the unbounded component of $\mathbf{C} \setminus K$. Choose now an R and small ε such that $R - \varepsilon = 1$ and $(1/c)^c > R$, and consider the set K and the polynomials p_n from the proof of Proposition 3. For $cn < j_n < n - cn$ we have

$$\frac{|a_{j_n,n}|}{\|p_n\|_K} = \binom{n}{j_n}(R-\varepsilon)^{n-j_n}\frac{1}{\varepsilon^n} = \binom{n}{j_n}\frac{1}{\varepsilon^n},$$

and if we take here n-th root and follow the argument in the proof of (5) we get that

$$\limsup_{n\to\infty} \left(\frac{|a_{j_n,n}|}{\|p_n\|_K} \right)^{1/n} \ge \frac{(1/c)^c}{\varepsilon} > \frac{R}{\varepsilon} = \frac{R_K}{\mathrm{cap}(K)} \ge e^{L_K}.$$

This settles (10) when there are infinitely many j_n with $cn < j_n < 1 - cn$ for some c. If this is not the case, then there is a subsequence of the natural numbers along which $n_m - j_{n_m} = o(n_m)$, and then consider any K with $1/\mathrm{cap}(K) > e^{L_K}$ (say $K = [0,1]$) and the polynomials p_n from (4), for which (4) with $i_{n_m} = n_m - j_{n_m}$ yields

$$\lim_{m\to\infty} \left(\frac{|a_{j_{n_m}, n_m}|}{\|p_{n_m}\|_K} \right)^{1/n_m} = \frac{1}{\mathrm{cap}(K)} > e^{L_K}.$$

■

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