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Scott Induction and Equational Proofs

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Abstract

The equational properties of the iteration operation in Lawvere theories are captured by the notion of iteration theories axiomatized by the Conway identities together with a complicated equation scheme, the "commutative identity". The first result of the paper shows that the commutative identity is implied by the Conway identities and the Scott induction principle formulated to involve only equations. Since the Scott induction principle holds in free iteration theories, we obtain a relatively simple first order axiomatization of the equational properties of iteration theories. We show, by means of an example that a simplified version of the Scott induction principle does not suffice for this purpose: There exists a Conway theory satisfying the scalar Scott induction principle which is not an iteration theory. A second example shows that there exists an iteration theory satisfying the scalar version of the Scott induction principle in which the general form fails. Finally, an example is included to verify the expected fact that there exists an iteration theory violating the scalar Scott induction principle. Interestingly, two of these examples are ordered theories in which the iteration operation is defined via least pre-fixed points.

1 Introduction

Suppose that A is a cpo with a bottom element $\perp$. It is known that for each $n \geq 0$, the poset A^n equipped with the pointwise order is also a cpo with bottom element $\perp_n = (\perp, \ldots, \perp)$. Suppose that f is a continuous function $A^{n+p} \longrightarrow A^p$. Then for each $y \in A^p$, the map $x \mapsto f(x, y)$, $x \in A^n$, has a least fixed point $f^\dagger(y)$. The function $f^\dagger : A^p \longrightarrow A^n$ is also continuous.

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Suppose that $P \subseteq A^{n+p}$ is an inclusive (or inductive) subset, cf. [16,19]. If $y \in A^p$ with $(\perp_n, y) \in P$ and

$$(x,y) \in P \quad \Rightarrow \quad (f(x,y),y) \in P,$$

for all $x \in A^n$, then $(f^\dagger(y), y) \in P$. Indeed, it follows from the assumptions that $(f^{(k)}(y), y) \in P$, for each $k \geq 0$, where $f^{(k)}(y)$ denotes the kth approximation of the least fixed point $f^\dagger(y)$. Thus, since $f^\dagger(y) = \sup_k f^{(k)}(y)$, and since P is inclusive, $(f^\dagger(y), y) \in P$. There are several ways to define inclusive sets and predicates. For example, if u and v are continuous functions $A^{n+p} \longrightarrow A^m$, the predicates

$$\begin{aligned} P(x,y) \quad &\Leftrightarrow \quad u(x,y) = v(x,y) \\ P'(x,y) \quad &\Leftrightarrow \quad u(x,y) \leq v(x,y) \end{aligned}$$

are inclusive.

The above method, called the *Scott induction principle* is frequently used to prove properties of fixed point constructs, see e.g. [14,18,16,19,15,10] and the language LCF. Several questions may be raised about the completeness of the principle and several different answers may be given. In this paper we formulate the Scott induction principle so that it involves only *equations*, sometimes inequations, and study its completeness with respect to the equational theory of fixed point solutions, or iteration. Thus our study is closely related to the logic which was originally proposed by Scott, according to [8].

The equational properties of the iteration operation in Lawvere theories are captured by the equational axioms of *iteration theories*, cf. [1]. We define the Scott induction principle in the language of iteration theories and indicate a few simple facts showing that the iteration theories that arise naturally satisfy this principle.

Iteration theories are axiomatized by the Conway identities and a complicated equation scheme the "commutative identity". In Theorem 3.1, we prove that the commutative identity is implied by the Conway identities and the Scott induction principle. Since the Scott induction principle holds in the free iteration theories, we obtain a relatively simple first order axiomatization of the equational properties of iteration theories, and hence of the equational properties of least fixed point solutions over cpo's, or of the equational properties of unique fixed point solutions in metric structures, etc. We establish a similar result for ordered theories and the form of the Scott induction principle that involves inequations. See Theorem 3.4.

We also consider a simplified form of the Scott induction principle, which only involves *scalar* morphisms and which is of interest from the algebraic point of view. For universal algebra, the importance of taking only scalar morphisms is due to the finiteness or infiniteness of the axiomatics derived from the Scott induction principle. We construct a theory satisfying the Conway identities and the scalar form of the Scott induction principle, but which is not an iteration theory. Thus the Conway identities and the scalar form of the Scott induction principle do not provide a complete axiomatization of the equational theory of iteration. We also give an example of an iteration theory satisfying the scalar version of the Scott induction principle in which the general form fails. Thus, the scalar form does not imply the general form, not even in conjunction with the full strength of iteration theories. Finally, we prove, by means of an example, the expected fact that there exists an iteration theory violating the scalar Scott induction principle. Interestingly, the latter two of these examples are theories which can be ordered so that iteration provides *least (pre-)fixed points*, so that the *Park induction principle* holds.

In the last section we announce a result proved in [7]. A few simple identities and the Park induction principle are complete for the (in)equational theory of iteration.

We have chosen the language of algebraic theories for the presentation. An equivalent treatment is possible using the more general *cartesian categories*, or by using more algebraic languages like that of *clones* or *μ-terms*, see [3].

2 Iteration Theories and the Scott Induction Principle

For basic notions and notations concerning algebraic theories the reader is referred to the book [1]. Thus, the composite of the morphisms $f : n \longrightarrow p$ and $g : p \longrightarrow q$ in a theory T is written $f \cdot g : n \longrightarrow q$. The coproduct injections $1 \longrightarrow n$ are denoted i_n, for $i \in [n] = \{1, \ldots, n\}$. The operation of tupling is determined by the coproduct structure and maps a pair of morphisms $f : n \longrightarrow p$ and $g : m \longrightarrow p$ to the morphism $\langle f, g\rangle : n + m \longrightarrow p$. Separated sum maps $f : n \longrightarrow p$ and $g : m \longrightarrow q$ to the morphism $f \oplus g : n + m \longrightarrow p + q$. These operations can be extended to several arguments. Tuplings of distinguished morphisms i_n are called base morphisms. In particular, 0_p (the unique morphism $0 \longrightarrow p$) and $\mathbf{1}_n$ (the identity morphism $n \longrightarrow n$) are base morphisms. In nontrivial theories, there is a bijection between base morphisms $n \longrightarrow p$ and functions $[n] \longrightarrow [p]$, so that we may call a base morphism surjective or injective if the corresponding function has the appropriate property.

A *preiteration theory* is an algebraic theory T equipped with a *dagger* or *iteration* operation $f : n \longrightarrow n+p \;\mapsto\; f^\dagger : n \longrightarrow p$. A morphism of preiteration theories is a theory morphism which preserves dagger. A *Conway theory* is a preiteration theory such that the dagger operation satisfies the following Conway identities:

Scalar Parameter Identity

$$(f \cdot (\mathbf{1}_1 \oplus g))^\dagger = f^\dagger \cdot g,$$

for all $f : 1 \longrightarrow 1 + p$ and $g : p \longrightarrow q$.

Scalar Double Dagger Identity

$$(f \cdot (\langle \mathbf{1}_1, \mathbf{1}_1\rangle \oplus \mathbf{1}_p))^\dagger = f^{\dagger\dagger},$$

for all $f : 1 \longrightarrow 2 + p$.

Scalar Composition Identity

$$(f \cdot \langle g, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger = f \cdot \langle (g \cdot \langle f, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger,\ \mathbf{1}_p\rangle,$$

for all $f, g : 1 \longrightarrow 1 + p$.

Scalar Pairing Identity

$$\langle f,\ g\rangle^{\dagger} = \langle f^{\dagger}\cdot\langle h^{\dagger}, \mathbf{1}_p\rangle,\ h^{\dagger}\rangle,$$

where

$$h = g\cdot\langle f^{\dagger}, \mathbf{1}_{n+p}\rangle,$$

for all $f : 1 \longrightarrow 1+n+p$ and $g : n \longrightarrow 1+n+p$.

The term "Conway identities" comes from the form these identities take in matrix theories over semirings equipped with a * operation, see [4]. For example, the double dagger identity corresponds to the equation

$$(a+b)^* = (a^*b)^*a^*$$

and the composition identity to the equation

$$(ab)^* = a(ba)^*b + 1.$$

By the scalar pairing identity, the dagger operation in a Conway theory is completely determined by its restriction to the *scalar morphisms* $f : 1 \longrightarrow 1+p$. Thus the essential axioms are the first three, i.e., the scalar parameter, double dagger and composition identities. Several other axiomatizations of Conway theories can be found in [1]. Each Conway theory satisfies *Elgot's fixed point identity* [6]:

$$f^{\dagger} = f\cdot\langle f^{\dagger}, \mathbf{1}_p\rangle,$$

for all $f : n \longrightarrow n+p$.

It can be proved that in Conway theories the 'vector versions' of the defining identities, i.e., the parameter, double dagger and pairing identities, also hold. (E.g., the double dagger identity is obtained from the scalar double dagger identity by taking f to be a morphism $f : n \longrightarrow n+n+p$.) For later use we note that the following identity holds in Conway theories:

(1) $\langle f\cdot(\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p), 0_n \oplus g\rangle^{\dagger} = \langle f^{\dagger}, g^{\dagger}\rangle,$

for all $f : n \longrightarrow n+p$ and $g : m \longrightarrow m+p$.

In any preiteration theory T, we define $\perp := \mathbf{1}_1^{\dagger} : 1 \longrightarrow 0$ and

$$\perp_{n,p} := \langle \perp\cdot 0_p, \ldots, \perp\cdot 0_p\rangle : n \longrightarrow p.$$

We write $\perp_p$ for $\perp_{1,p}$. Note that $\perp_{n+m,p} = \langle \perp_{n,p}, \perp_{m,p}\rangle$. When T is a Conway theory, we can alternatively define

(2) $\perp_{n,p} = (\mathbf{1}_n \oplus 0_p)^{\dagger},$

for all integers $n, p \geq 0$.

An *iteration theory* is a Conway theory which satisfies the following identity:

COMMUTATIVE IDENTITY

$$((\rho\cdot f)\parallel(\rho_1,\ldots,\rho_m))^{\dagger} = \rho\cdot(f\cdot(\rho\oplus\mathbf{1}_p))^{\dagger},$$

for all $f : n \longrightarrow m+p$, surjective base morphism $\rho : m \longrightarrow n$, and for all base morphisms $\rho_i : m \longrightarrow m$, $i \in [m]$, with $\rho_i\cdot\rho = \rho$. Here, $(\rho\cdot f)\parallel(\rho_1,\ldots,\rho_m)$ is an abbreviation for the morphism

$$\langle 1_m\cdot\rho\cdot f\cdot(\rho_1\oplus\mathbf{1}_p), \ldots, m_m\cdot\rho\cdot f\cdot(\rho_m\oplus\mathbf{1}_p)\rangle : m \longrightarrow m+p.$$

In Conway theories, the commutative identity implies the *power identities*:

$$(f^k)^\dagger = f^\dagger,$$

for all $f : n \longrightarrow n + p$ and $k \geq 1$. In any theory T, the *powers* of a morphism $f : n \longrightarrow n + p$ are defined by induction:

(3) $\quad f^0 := \mathbf{1}_n \oplus 0_p$

(4) $\quad f^{k+1} := f \cdot \langle f^k, 0_n \oplus \mathbf{1}_p \rangle.$

Note that $f^1 = f$ since in any theory T, $\langle \mathbf{1}_n \oplus 0_p, 0_n \oplus \mathbf{1}_p \rangle = \mathbf{1}_{n+p}$.

In Conway theories, the commutative identity is implied by the *weak functorial implication.* Suppose that T is a preiteration theory and $\mathcal{C}$ is a class of morphisms in T. We say that T satisfies the functorial implication for $\mathcal{C}$ if

$$f \cdot (h \oplus \mathbf{1}_p) = h \cdot g \quad \Rightarrow \quad f^\dagger = h \cdot g^\dagger,$$

for all $f : n \longrightarrow n + p$, $g : m \longrightarrow m + p$ and for all $h : n \longrightarrow m$ in $\mathcal{C}$. Two special cases are important here, the case that $\mathcal{C}$ is the class of *pure morphisms* and the case that $\mathcal{C}$ is the class of *surjective base morphisms.* Following [13], we call a morphism $f : n \longrightarrow p$ pure if $f \cdot \perp_{p,0} = \perp_{n,0}$. Note that each base morphism is pure. The weak functorial implication is the functorial implication for the class of surjective base morphisms.

Theorem 2.1 [1] *The class of iteration theories is the smallest variety of preiteration theories containing the Conway theories satisfying the weak functorial implication, or the functorial implication for all pure morphisms.* □

Thus, if T is a Conway theory satisfying the weak functorial implication, then T is an iteration theory. Moreover, an equation holds in all iteration theories iff it is a logical consequence of the Conway theory identities and the weak functorial implication.

It is interesting to note that the pure functorial implication was already used by Eilenberg in the early 70's, in his characterization of the least fixed point operation on continuous functions on cpo's, cf. [5]. See also [16].

We now formulate a form of the Scott induction principle.

Definition 2.2 Suppose that T is a preiteration theory. We say that the *Scott induction principle* holds in T, or that T satisfies the Scott induction principle, if for all $u, v : k \longrightarrow n + p$ and $f : n \longrightarrow n + p$, if

(5) $\quad u \cdot \langle \perp_{n,p}, \mathbf{1}_p \rangle = v \cdot \langle \perp_{n,p}, \mathbf{1}_p \rangle$

and if

(6) $\quad u \cdot \langle z, \mathbf{1}_p \rangle = v \cdot \langle z, \mathbf{1}_p \rangle \quad \Rightarrow \quad u \cdot \langle f \cdot \langle z, \mathbf{1}_p \rangle,\ \mathbf{1}_p \rangle = v \cdot \langle f \cdot \langle z, \mathbf{1}_p \rangle,\ \mathbf{1}_p \rangle,$

for all $z : n \longrightarrow p$, then

(7) $\quad u \cdot \langle f^\dagger, \mathbf{1}_p \rangle = v \cdot \langle f^\dagger, \mathbf{1}_p \rangle.$

Since this is the main definition of the paper we explain it also in diagrammatic form: T satisfies the Scott induction principle if for all morphisms $u, v : k \longrightarrow$

$n+p$ and $f : n \longrightarrow n+p$, if the diagram

$$\begin{array}{ccc} k & \xrightarrow{\quad u \quad} & n+p \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle \langle \bot_{n,p}, \mathbf{1}_p \rangle} \\ n+p & \xrightarrow[\langle \bot_{n,p}, \mathbf{1}_p \rangle]{} & p \end{array}$$

commutes, and if whenever the square

$$\begin{array}{ccc} k & \xrightarrow{\quad u \quad} & n+p \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle \langle z, \mathbf{1}_p \rangle} \\ n+p & \xrightarrow[\langle z, \mathbf{1}_p \rangle]{} & p \end{array}$$

commutes, then so does the square

$$\begin{array}{ccc} k & \xrightarrow{\quad u \quad} & n+p \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle \langle f \cdot \langle z, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle} \\ n+p & \xrightarrow[\langle f \cdot \langle z, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle]{} & p \end{array}$$

for all $z : n \longrightarrow p$, then it follows that the square

$$\begin{array}{ccc} k & \xrightarrow{\quad u \quad} & n+p \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle \langle f^\dagger, \mathbf{1}_p \rangle} \\ n+p & \xrightarrow[\langle f^\dagger, \mathbf{1}_p \rangle]{} & p \end{array}$$

also commutes. When $n = 1$ but k is arbitrary, we call the principle the *scalar Scott induction principle.* When $k = n = 1$, it is called the *weak scalar Scott induction principle.*

When T is a preiteration theory we define the *modified powers* of a morphism $f : n \longrightarrow n+p$ in T by:

(8) $f^{(0)} := \bot_{n,p}$

(9) $f^{(k+1)} := f \cdot \langle f^{(k)}, \mathbf{1}_p \rangle.$

Thus, $f^{(k)} = f^k \cdot \langle \bot_{n,p}, \mathbf{1}_p \rangle.$

The following implications are closely related to the Scott induction principle.

Definition 2.3 Suppose that T is a preiteration theory and $k \geq 0$. We say that T satisfies *the k'th power implication* if for all $u, v : 1 \longrightarrow n+p$ and $f : n \longrightarrow n+p$

$$\left(\bigwedge_{i=0}^{k} u \cdot \langle f^{(i)}, \mathbf{1}_p \rangle = v \cdot \langle f^{(i)}, \mathbf{1}_p \rangle \right) \quad \Rightarrow \quad u \cdot \langle f^\dagger, \mathbf{1}_p \rangle = v \cdot \langle f^\dagger, \mathbf{1}_p \rangle.$$

When $n = 1$, this is called the *scalar k'th power implication.*

For later use we state here a simple lemma whithout proof.

Lemma 2.4 *Suppose that T is a preiteration theory. If T satisfies the (scalar) k'th power implication for some $k \geq 1$, then the (scalar) Scott induction principle holds in T.* □

2.1 Ordered Theories

The Scott induction principle is usually formulated for ordered structures, cpo's and continuous functions. In ordered preiteration theories, it is possible to formulate a form of the Scott induction principle which involves inequations rather than equations.

An *ordered theory* is an algebraic theory T such that each hom-set $T(n,p)$ is a poset and the operations of composition and tupling are monotonic. It then follows that for any two morphisms $f, g : n \longrightarrow p$, $f \leq g$ iff $i_n \cdot f \leq i_n \cdot g$, for all $i \in [n]$.

An *ordered preiteration theory* is an ordered theory which is a preiteration theory, and an ordered Conway theory (iteration theory) an ordered preiteration theory which is a Conway theory (iteration theory). Note that we are not requiring that the dagger operation is monotonic. Nevertheless, this will hold if the operation is defined via least pre-fixed points, see below.

A *strict ordered theory T* is an ordered theory such that each hom-set $T(n,p)$ has a least element, usually denoted $\perp_{n,p}$, and composition is left strict, i.e., $\perp_{n,p} \cdot g = \perp_{n,q}$, for all $g : p \longrightarrow q$. Thus, when $\perp := \perp_{1,0}$, the morphism $\perp_{n,p}$ is $\langle \perp \cdot 0_p, \ldots, \perp \cdot 0_p \rangle$. A *strict ordered preiteration theory* is an ordered preiteration theory in which the equation (2) holds. A strict ordered Conway theory (iteration theory) is a Conway theory (iteration theory) which is a strict ordered preiteration theory.

The Scott induction principle for inequations, or the *ordered Scott induction principle*, is obtained from the above formalization of the Scott induction principle by replacing equations by inequations, i.e., by writing $\leq$ in place of $=$ in the equations (5), (6), (7). The *scalar ordered Scott induction principle* is the inequational version of the scalar Scott induction principle.

If T is a strict ordered preiteration theory satisfying the ordered Scott induction principle, then the *Park induction principle* [14] holds in T: For all $f : n \longrightarrow n + p$ and $\xi : n \longrightarrow p$,

$$f \cdot \langle \xi, \mathbf{1}_p \rangle \leq \xi \quad \Rightarrow \quad f^\dagger \leq \xi,$$

so that $f^\dagger$ is less than or equal to each *pre-fixed point* of f. This fact follows by applying the ordered Scott induction principle for the morphisms $u := \mathbf{1}_n \oplus 0_p$, $v := 0_n \oplus \xi$ and the morphism f. Thus, if the fixed point identity holds in T, or if $f^\dagger$ is a pre-fixed point of f, then $f^\dagger$ is the least pre-fixed point, which is necessarily a fixed point. In this case the dagger operation is also monotonic:

$$f \leq g : n \longrightarrow n + p \quad \Rightarrow \quad f^\dagger \leq g^\dagger : n \longrightarrow p.$$

Indeed, since

$$f \cdot \langle g^\dagger, \mathbf{1}_p \rangle \leq g \cdot \langle g^\dagger, \mathbf{1}_p \rangle \leq g^\dagger,$$

we obtain $f^\dagger \le g^\dagger$ by the Park induction principle.

If T is an ordered preiteration theory in which the equation (2) and the Park induction principle hold, then T is a strict ordered preiteration theory. Indeed, since for each $f : n \longrightarrow p$,

$$(\mathbf{1}_n \oplus 0_p) \cdot \langle f, \mathbf{1}_p \rangle = f,$$

we have $\perp_{n,p} = (\mathbf{1}_n \oplus 0_p)^\dagger \le f$, by the Park induction principle.

A proof of the following theorem can be found in [1], although in a more general form. See also [7].

Theorem 2.5 *Suppose that T is an ordered preiteration theory.*

(i) *If the scalar parameter identity holds and iteration is defined via least pre-fixed points, then T is a Conway theory.*

(ii) *If the scalar parameter and pairing identities hold, and if $f^\dagger$ is the least pre-fixed point of f, for each scalar morphism $f : 1 \longrightarrow 1 + p$, then the same holds for each (vector) morphism $g : n \longrightarrow n + p$.* □

2.2 Some Examples

In this subsection we mention some examples of preiteration theories, in fact iteration theories, satisfying the Scott induction principle.

A strict ordered theory T is called *ω-continuous*, cf. [20,1], if each hom-set $T(n,p)$ is an ω-complete poset and the composition operation is ω-continuous. When T is ω-continuous, each morphism $f : n \longrightarrow n + p$ has a least pre-fixed point. Denoting this morphism by $f^\dagger$, T becomes an iteration theory, cf. [1]. The following fact is well-known, but probably not in this form, see e.g. [16,19].

Proposition 2.6 *Suppose that T is an ω-continuous theory. Then T satisfies the (ordered) Scott induction principle.* □

A concrete example when this proposition can be applied is the case that T is the tree theory $\Sigma_\perp\mathrm{TR}$, for some signature Σ. In the theory $\Sigma_\perp\mathrm{TR}$, a morphism $n \longrightarrow p$ is an n-tuple of finite or infinite trees over the variables $X_p = \{x_1, \ldots, x_p\}$ whose vertices are consistently labeled by some symbol in $\Sigma \cup X_p$. Some leaves may be labeled by the symbol $\perp$, which is not in Σ. Composition is tree substitution. Given trees $f, g : 1 \longrightarrow p$, we define $f \le g$ if g can be obtained from f by replacing some nodes labeled $\perp$ by some trees. When $f, g : n \longrightarrow p$, $n \ne 1$, we define $f \le g$ if $i_n \cdot f \le i_n \cdot g$, all $i \in [n]$. Equipped with this ordering, $\Sigma_\perp\mathrm{TR}$ is an ω-continuous theory, in fact the free ω-continuous theory on the signature Σ.

Another class of theories satisfying the Scott induction principle may be obtained by taking matrix theories over *countably complete semirings*. In such semirings S, in addition to finite sums, all infinite sums of the form $\sum_{i\in I} s_i$ exist, for countable sets I. The infinite sum operation obeys the usual associative and distributive laws. When S is countably complete, so is the semiring $S^{n\times n}$ of all $n \times n$ matrices over S. The matrix theory Mat_S has morphisms $n \longrightarrow p$ all $n \times p$ matrices over S. When $A : n \longrightarrow n$, we define A^* by the usual infinite geometric sum:

$$A^* := \mathbf{1}_n + A + A^2 + \dots.$$

Finally, when $f = [A\ B] : n \longrightarrow n + p$, where A is $n \times n$ and B is $n \times p$, we define

$$f^\dagger := A^* \cdot B : n \longrightarrow p.$$

It is known that Mat_S, equipped with the above dagger operation, is an iteration theory.

Proposition 2.7 *The iteration theory Mat_S over the countably additive semiring S satisfies the Scott induction principle.* □

Proposition 2.6 can be generalized to rational theories, cf. [20]. A *rational theory* T is a strict ordered theory such that for each $f : n \longrightarrow n+p$, the ω-chain $f^{(k)}$, $k \geq 0$, has least upper bound. (The modified powers $f^{(k)}$ were defined by (8) and (9).) Moreover, composition preserves the least upper bound of all such chains:

$$g \cdot \sup f^{(k)} = \sup(g \cdot f^{(k)})$$
$$(\sup f^{(k)}) \cdot h = \sup(f^{(k)} \cdot h),$$

for all $f : n \longrightarrow n + p$, $g : m \longrightarrow n$ and $h : p \longrightarrow q$. Thus, the least upper bound of the ω-chains $g \cdot f^{(k)}$ and $f^{(k)} \cdot h$ also exists. Each rational theory becomes a preiteration theory by defining

$$f^\dagger := \sup f^{(k)},$$

for all $f : n \longrightarrow n + p$. It is well-known that $f^\dagger$ is the least pre-fixed point of f. A concrete rational theory is the theory $\Sigma\mathbf{tr}$ of *regular* trees in $\Sigma_\perp\mathrm{TR}$. (A tree is regular if it has a finite number of subtrees). It is known that $\Sigma\mathbf{tr}$, considered as an unordered preiteration theory, is the free iteration theory on the signature Σ. Each free iteration theory satisfies the (ordered) Scott induction principle.

Another generalization of Proposition 2.6 may be obtained by taking *ω-continuous 2-theories*. In brief, a 2-theory is a 2-category such that the 2-cells form a theory. A 2-theory T is ω-continuous if each vertical category has an initial object and colimits of all ω-chains. Further, horizontal composition is left strict and preserves the colimit of ω-diagrams. It is known that each horizontal morphism $f : n \longrightarrow n + p$ has an *initial* f-algebra, which is a pair $(f^\dagger, \mu_f)$ consisting of a horizontal morphism $f^\dagger : n \longrightarrow p$ and a vertical morphism $\mu_f : f \cdot \langle f^\dagger, \mathbf{1}_p\rangle \longrightarrow f^\dagger$. The initial f-algebra is unique up to isomorphism. It is known, cf. [1], that the operation $f \mapsto f^\dagger$ satisfies all of the iteration theory identities up to isomorphism. It can be seen that it also satisfies the Scott induction principle.

A last class of examples is provided by the *contraction theories*. Suppose that T is a theory such that each hom-set $T(1,p)$ is a complete metric space and $T(1,0)$ is nonempty. We assume that the metrics are related by:

$$d(f,g) = \max_{i \in [n]}\{d(i_n \cdot f,\ i_n \cdot g)\},$$

so that the metric on $T(n,p)$ is determined by that given on $T(1,p)$. Thus all hom-sets are complete metric spaces.

Suppose that $f : n \longrightarrow p$ in T. Then, for each $q \geq 0$, f induces a function

$$L_f^q : T(p,q) \longrightarrow T(n,q)$$
$$g \mapsto f \cdot g.$$

We call T a contraction theory if each morphism $f : 1 \longrightarrow p$ other than one of the distinguished morphisms i_p induces a *proper* contraction. It follows that whenever f is a morphism $n \longrightarrow p$ and none of the morphisms $i_n \cdot f$ is distinguished, the functions L_f^q are proper contractions as well.

A contraction theory gives rise to a preiteration theory. Select a point $\perp : 1 \longrightarrow 0$ and define

$$\perp_{n,p} := \langle \perp \cdot 0_p, \ldots, \perp \cdot 0_p \rangle : n \longrightarrow p.$$

If $f : n \longrightarrow n + p$, define

(10) $f^\dagger := \lim_{k \longrightarrow \infty} f^{(k)},$

where the morphisms $f^{(k)}$ were given in (8) and (9) above. Thus, if the morphisms $i_n \cdot f$, $i \in [n]$, are not distinguished, $f^\dagger$ is the *unique* fixed point of f.

Proposition 2.8 *Suppose that T is a contraction theory and dagger is defined by (10). Then T is an iteration theory satisfying the Scott induction principle.* □

(Each contraction theory is an *iterative theory*, cf. [6], in fact a *pointed iterative theory*. Combining results from this paper with those in [2], it follows that there exists a pointed iterative theory which does not satisfy the Scott induction principle.)

The preiteration theories arising from the ω-continuous, rational, or contraction theories, or from the matrix theories over countably complete semirings are all iteration theories. A more precise result is formulated in the following theorem, proved in [1].

Theorem 2.9 *The class of iteration theories is the variety of preiteration theories generated by each one of the following classes:*

(i) *The preiteration theories defined on ω-continuous or rational theories.*

(ii) *The preiteration theories defined on ω-continuous 2-theories.*

(iii) *The preiteration theories defined on contraction theories.*

The variety generated by the preiteration theories defined on the matrix theories over countably complete semirings consists of the iteration theories with a unique morphism $1 \longrightarrow 0$. □

The above result explains our interest in the completeness of the Scott induction principle for the equations true in all iteration theories.

Some notation. When A is a set, we denote by A^* the set of all finite words, and by A^ω the set of all ω-words over A. We write 1 for the empty word and A^+ for the set $A^* - \{1\}$. When $u \in A^+$, we write u^ω for the infinite word $uuu\ldots$.

A dagger, or iteration congruence on a preiteration theory is a theory congruence which respects the dagger operation.

3 Completeness of the Scott Induction Principle

The results of this section are formulated by Theorems 3.1 and 3.4.

Theorem 3.1 *Suppose that T is a Conway theory satisfying the Scott induction principle. Then T is an iteration theory.*

The proof of Theorem 3.1 is based on two lemmas.

Lemma 3.2 *Suppose that T is a Conway theory or a preiteration theory satisfying the equation (1). Suppose that the Scott induction principle holds in T. Then the following version of the Scott induction principle holds in T. Suppose that $u : k \longrightarrow n+p$, $v : k \longrightarrow m+p$, $f : n \longrightarrow n+p$, $g : m \longrightarrow m+p$. If*

(11) $u \cdot \langle \perp_{n,p}, \mathbf{1}_p \rangle = v \cdot \langle \perp_{m,p}, \mathbf{1}_p \rangle$

and if

(12) $u \cdot \langle r, \mathbf{1}_p \rangle = v \cdot \langle s, \mathbf{1}_p \rangle \quad \Rightarrow \quad u \cdot \langle f \cdot \langle r, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = v \cdot \langle g \cdot \langle s, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle,$

for all $r : n \longrightarrow p$ and $s : m \longrightarrow p$, then

$$u \cdot \langle f^\dagger, \mathbf{1}_p \rangle = v \cdot \langle g^\dagger, \mathbf{1}_p \rangle.$$

Proof. Define

$$\begin{aligned} w &:= u \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p) : k \longrightarrow n+m+p \\ w' &:= 0_n \oplus v : k \longrightarrow n+m+p \\ h &:= \langle f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p), 0_n \oplus g \rangle : n+m \longrightarrow n+m+p. \end{aligned}$$

Then $w \cdot \langle \perp_{n+m,p}, \mathbf{1}_p \rangle = u \cdot \langle \perp_{n,p}, \mathbf{1}_p \rangle = v \cdot \langle \perp_{m,p}, \mathbf{1}_p \rangle = w' \cdot \langle \perp_{n+m,p}, \mathbf{1}_p \rangle$ by (11). Suppose that $u \cdot \langle r, \mathbf{1}_p \rangle = v \cdot \langle s, \mathbf{1}_p \rangle$, i.e., $w \cdot \langle t, \mathbf{1}_p \rangle = w' \cdot \langle t, \mathbf{1}_p \rangle$, for some $t = \langle r, s \rangle : n+m \longrightarrow p$, where $r : n \longrightarrow p$ and $s : m \longrightarrow p$. Then, by (12), $u \cdot \langle f \cdot \langle r, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = v \cdot \langle g \cdot \langle s, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle$, so that

$$\begin{aligned} w \cdot \langle h \cdot \langle t, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle &= u \cdot \langle f \cdot \langle r, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= v \cdot \langle g \cdot \langle s, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= w' \cdot \langle h \cdot \langle t, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle. \end{aligned}$$

Since the Scott induction principle holds in T, $w \cdot \langle h^\dagger, \mathbf{1}_p \rangle = w' \cdot \langle h^\dagger, \mathbf{1}_p \rangle$. But $h^\dagger = \langle f^\dagger, g^\dagger \rangle$, since (1) holds in T. Thus,

$$\begin{aligned} w \cdot \langle h^\dagger, \mathbf{1}_p \rangle &= u \cdot \langle f^\dagger, \mathbf{1}_p \rangle \\ w' \cdot \langle h^\dagger, \mathbf{1}_p \rangle &= v \cdot \langle g^\dagger, \mathbf{1}_p \rangle, \end{aligned}$$

so that $u \cdot \langle f^\dagger, \mathbf{1}_p \rangle = v \cdot \langle g^\dagger, \mathbf{1}_p \rangle$. □

Lemma 3.3 *Suppose that T is a Conway theory or a preiteration theory satisfying the equation (1). If the Scott induction principle holds, then T satisfies the functorial implication for all pure morphisms.*

Proof. Suppose that the square

$$(13)\qquad \begin{array}{ccc} n & \xrightarrow{\ f\ } & n+p \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h \oplus \mathbf{1}_p} \\ m & \xrightarrow[\ g\ ]{} & m+p \end{array}$$

commutes, where h is a pure morphism. We have

$$(\mathbf{1}_n \oplus 0_p) \cdot \langle \bot_{n,p}, \mathbf{1}_p \rangle = \mathbf{1}_n \cdot \bot_{n,p} = \bot_{n,p} = h \cdot \bot_{m,p} = (h \oplus 0_p) \cdot \langle \bot_{m,p}, \mathbf{1}_p \rangle.$$

Further, if $(\mathbf{1}_n \oplus 0_p) \cdot \langle r, \mathbf{1}_p \rangle = (h \oplus 0_p) \cdot \langle s, \mathbf{1}_p \rangle$, i.e., $r = h \cdot s$, for some $r : n \longrightarrow p$ and $s : m \longrightarrow p$, then

$$\begin{aligned} f \cdot \langle r, \mathbf{1}_p \rangle &= f \cdot \langle h \cdot s, \mathbf{1}_p \rangle \\ &= f \cdot (h \oplus \mathbf{1}_p) \cdot \langle s, \mathbf{1}_p \rangle \\ &= h \cdot g \cdot \langle s, \mathbf{1}_p \rangle, \end{aligned}$$

so that $(\mathbf{1}_n \oplus 0_p) \cdot \langle f \cdot \langle r, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = (h \oplus 0_p) \cdot \langle g \cdot \langle s, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle$. By Lemma 3.2,

$$(\mathbf{1}_n \oplus 0_p) \cdot \langle f^\dagger, \mathbf{1}_p \rangle = (h \oplus 0_p) \cdot \langle g^\dagger, \mathbf{1}_p \rangle,$$

proving $f^\dagger = h \cdot g^\dagger$. □

Proof of Theorem 3.1. Suppose that T is a Conway theory satisfying the Scott induction principle. Then, by Lemma 3.3, T satisfies the functorial implication for pure morphisms, and thus, by Theorem 2.1, T is an iteration theory. □

Since in (strict) ordered preiteration theories, the ordered Scott induction principle apparently does not imply the Scott induction principle, Theorem 3.1 does not answer the question if a (strict) ordered Conway theory satisfying the ordered Scott induction principle is an iteration theory. We have two arguments to establish this fact. The first follows from the proof of Theorem 3.1. It is easy to modify the proof of Lemma 3.2 in order to obtain the following fact. Suppose that T is an ordered Conway theory or an ordered preiteration theory satisfying (1). Suppose the ordered Scott induction principle holds in T. Let $u : k \longrightarrow n+p$, $v : k \longrightarrow m+p$, $f : n \longrightarrow n+p$, $g : m \longrightarrow m+p$ in T. If

$$u \cdot \langle \bot_{n,p}, \mathbf{1}_p \rangle \le v \cdot \langle \bot_{m,p}, \mathbf{1}_p \rangle$$

and if

$$u \cdot \langle r, \mathbf{1}_p \rangle \le v \cdot \langle s, \mathbf{1}_p \rangle \quad \Rightarrow \quad u \cdot \langle f \cdot \langle r, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \le v \cdot \langle g \cdot \langle s, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle,$$

for all $r : n \longrightarrow p$ and $s : m \longrightarrow p$, then

$$u \cdot \langle f^\dagger, \mathbf{1}_p \rangle \le v \cdot \langle g^\dagger, \mathbf{1}_p \rangle.$$

Now, using this fact, one proves that if the square (13) commutes, where h is a pure morphism, then $f^\dagger \le h \cdot g^\dagger$ and $h \cdot g^\dagger \le f^\dagger$. The details are similar to the proof of Lemma 3.3.

The second argument is more direct. Suppose now that T is a strict ordered preiteration theory such that $f^\dagger$ is a pre-fixed point of f, for each $f : n \longrightarrow n+p$. Suppose the ordered Scott induction principle holds in T. If the square (13) commutes and h is pure, then $h \cdot g^\dagger$ is a pre-fixed point of f, so that $f^\dagger \leq h \cdot g^\dagger$ by the Park induction principle. To show that $h \cdot g^\dagger \leq f^\dagger$, we apply the ordered Scott induction principle. Since h is pure, we have

$$(h \oplus 0_p) \cdot \langle \perp_{m,p}, \mathbf{1}_p \rangle = \perp_{n,p} \leq f^\dagger = (0_m \oplus f^\dagger) \cdot \langle \perp_{m,p}, \mathbf{1}_p \rangle.$$

Further, if $(h \oplus 0_p) \cdot \langle z, \mathbf{1}_p \rangle \leq (0_m \oplus f^\dagger) \cdot \langle z, \mathbf{1}_p \rangle$, i.e., $h \cdot z \leq f^\dagger$, for some $z : m \longrightarrow p$, then

$$\begin{aligned} (h \oplus 0_p) \cdot \langle g \cdot \langle z, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle &= h \cdot g \cdot \langle z, \mathbf{1}_p \rangle \\ &= f \cdot (h \oplus \mathbf{1}_p) \cdot \langle z, \mathbf{1}_p \rangle \\ &= f \cdot \langle h \cdot z, \mathbf{1}_p \rangle \\ &\leq f \cdot \langle f^\dagger, \mathbf{1}_p \rangle \\ &= f^\dagger \\ &= (0_m \oplus f^\dagger) \cdot \langle g \cdot \langle z, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} h \cdot g^\dagger &= (h \oplus 0_p) \cdot \langle g^\dagger, \mathbf{1}_p \rangle \\ &\leq (0_m \oplus f^\dagger) \cdot \langle g^\dagger, \mathbf{1}_p \rangle \\ &= f^\dagger, \end{aligned}$$

by the ordered Scott induction principle.

The previous argument was known to Gordon Plotkin [17] and was found independently by Stephen Bloom. We have proved the following result.

Theorem 3.4 *Let T be an ordered preiteration theory. Suppose that either the equation (1) holds, or T is a strict ordered preiteration theory in which the dagger operation provides pre-fixed points. If T satisfies the ordered Scott induction principle then T satisfies the functorial implication for pure morphisms. Thus if T is a strict ordered Conway theory in which the ordered Scott induction principle holds, then T is an iteration theory.* □

Corollary 3.5 *The class of iteration theories is the variety of preiteration theories generated by those Conway theories satisfying the Scott induction principle.*

Proof. Let us denote by $\mathcal{V}$ the variety generated by those Conway theories in which the Scott induction principle holds. Then, by Theorem 3.1, $\mathcal{V}$ is included in the class of iteration theories. But each free iteration theory satisfies the Scott induction principle. Thus, each iteration theory is in the variety $\mathcal{V}$. □

Corollary 3.5 can be reformulated in the following way: An equation between "iteration terms" holds in all iteration theories iff it is a logical consequence of the Conway identities and the Scott induction principle.

Corollary 3.6 *The class of iteration theories is the least variety of preiteration theories containing all theories satisfying the scalar parameter identity and the fixed point identity which can be turned into a strict ordered preiteration theory such that the ordered Scott induction principle holds.*

Proof. Let $\mathcal{V}$ denote the variety generated by those preiteration theories satisfying the scalar parameter and fixed point identities which can be turned into a strict ordered theory in which the ordered Scott induction principle (and hence the Park induction principle) holds. Then, by Theorem 2.5, each theory in $\mathcal{V}$ is a Conway theory, and hence an iteration theory, by Theorem 3.4. Since the free iteration theories are in $\mathcal{V}$, it follows that $\mathcal{V}$ is the class of all iteration theories. □

4 Incompleteness of the Scalar Scott Induction Principle

In this section we construct a Conway theory T satisfying the scalar Scott induction principle which is not an iteration theory. Thus, the scalar Scott induction principle and the Conway identities are not complete for the equational theory of iteration. A natural source to look for such a theory is the class of free Conway theories. Although we don't have a concrete representation of the free Conway theory over any signature Σ, below we give a concrete representation in the case that Σ consists of symbols of rank one.

We denote by X a countably infinite set $\{x_1, x_2, \ldots\}$. When $p \geq 0$, we write X_p for the set $\{x_1, \ldots, x_p\}$.

Suppose that A is a nonempty set. The theory T_0 has morphisms $1 \longrightarrow p$ all ordered pairs $(u, v) = u \bullet v$, where $u \in A^*$ and $v \in A^+ \cup X_p \cup \{\perp\}$. The morphisms $n \longrightarrow p$, $n \neq 1$, are all n-tuples of morphisms $1 \longrightarrow p$. Suppose that $f = u \bullet v : 1 \longrightarrow p$ and $g_i = u_i \bullet v_i : 1 \longrightarrow q$, $i \in [p]$, so that $g = (g_1, \ldots, g_p) : p \longrightarrow q$. When $v = x_i$, for some $i \in [p]$, we define $f \cdot g := uu_i \bullet v_i : 1 \longrightarrow q$. When $v \notin X_p$, we define $f \cdot g := u \bullet v : 1 \longrightarrow q$. The distinguished morphism i_p, where $i \in [p]$, is the ordered pair $1 \bullet x_i : 1 \longrightarrow p$. It is easy to see that T_0 is a theory. When f_i, $i \in [n]$, are morphisms $1 \longrightarrow p$, the unique morphism $\langle f_1, \ldots, f_n \rangle : n \longrightarrow p$ with $i_n \cdot \langle f_1, \ldots, f_n \rangle = f_i$, for all $i \in [n]$, is the n-tuple $(f_1, \ldots, f_n)$.

We turn T_0 into a preiteration theory. Suppose that $f = u \bullet v : 1 \longrightarrow 1 + p$ in T_0. We define:

$$f^\dagger := \begin{cases} 1 \bullet u & \text{if } v = x_1 \text{ and } u \neq 1; \\ 1 \bullet \perp & \text{if } v = x_1 \text{ and } u = 1; \\ u \bullet x_i & \text{if } v = x_{i+1}, \text{ for some } i \in [p]; \\ u \bullet v & \text{if } v \in A^+ \cup \{\perp\}. \end{cases}$$

When $f : n \longrightarrow n + p$, where $n \geq 2$, we define $f^\dagger$ by induction on n so that the scalar pairing identity holds:

$$f^\dagger := \langle g^\dagger \cdot \langle k^\dagger, \mathbf{1}_p \rangle, k^\dagger \rangle,$$

where $f = \langle g, h \rangle$, $g : 1 \longrightarrow n + p$, $h : n - 1 \longrightarrow n + p$, and $k := h \cdot \langle g^\dagger, \mathbf{1}_{n-1+p} \rangle$. Finally, when $f = 0_p$, we define $f^\dagger := 0_p$.

Claim 4.1 *The scalar parameter, double dagger and pairing identities hold in* T_0. □

Thus, except for the scalar composition identity, all of the defining equations of Conway theories hold in T_0. However, the scalar composition identity, or even, the scalar fixed point identity, fails in T_0. Indeed, when $f = u \bullet x_1 : 1 \longrightarrow 1$, for some $u \in A^+$, we have $f^\dagger = 1 \bullet u \neq u \bullet u = f \cdot f^\dagger$. We will construct the Conway theory T as a quotient of the theory T_0.

Definition 4.2 Suppose that $f = ua \bullet va : 1 \longrightarrow p$ in T_0, where $u, v \in A^*$, $a \in A$. Let $g := u \bullet av$, so that $g : 1 \longrightarrow p$. Then we write $f \rightarrow g$.

The reflexive-transitive closure of the relation $\rightarrow$, defined on the set of T_0-morphisms $1 \longrightarrow p$, is denoted $\xrightarrow{*}$.

Claim 4.3 *Suppose that $f, g, h : 1 \longrightarrow p$ in T_0. If $f \rightarrow h$ and $g \rightarrow h$, or if $h \rightarrow f$ and $h \rightarrow g$, then $f = g$. Thus, if $f \xrightarrow{*} h$ and $g \xrightarrow{*} h$, then either $f \xrightarrow{*} g$ or $g \xrightarrow{*} f$. Similarly, if $h \xrightarrow{*} f$ and $h \xrightarrow{*} g$, then either $f \xrightarrow{*} g$ or $g \xrightarrow{*} f$.* □

Claim 4.4 (i) *Suppose that $f, f' : 1 \longrightarrow p$ in T_0. If $f \rightarrow f'$ then $f \cdot g \rightarrow f' \cdot g$, for all $g : p \longrightarrow q$.*

(ii) *Suppose that $g_1, \ldots, g_p$ and g_i' are morphisms $1 \longrightarrow q$ in T_0, where $i \in [p]$, $p > 0$. If $g_i \rightarrow g_i'$, then for each $f : 1 \longrightarrow p$, either*

$$f \cdot \langle g_1, \ldots, g_p \rangle = f \cdot \langle g_1, \ldots, g_i', \ldots, g_p \rangle,$$

or

$$f \cdot \langle g_1, \ldots, g_p \rangle \rightarrow f \cdot \langle g_1, \ldots, g_i', \ldots, g_p \rangle.$$

(iii) *Suppose that $f, g : 1 \longrightarrow 1 + p$ in T_0. If $f \rightarrow g$ then $f^\dagger \rightarrow g^\dagger$.* □

Definition 4.5 Suppose that $f, g : n \longrightarrow p$ in T_0. We define $f \approx g$ if for each $i \in [n]$, either $i_n \cdot f \xrightarrow{*} i_n \cdot g$ or $i_n \cdot g \xrightarrow{*} i_n \cdot f$.

Corollary 4.6 *The relation $\approx$ is a dagger congruence on T_0. The quotient theory $T = T_0/\approx$ is a Conway theory.*

Proof. That $\approx$ is a dagger congruence on T_0 follows from the previous two facts. Since the scalar parameter, double dagger and pairing identities hold in T_0, they hold in T also. Thus, to complete the proof that T is a Conway theory, we only need to show that T satisfies the scalar composition identity

(14) $(f \cdot \langle g, 0_1 \oplus \mathbf{1}_p \rangle)^\dagger = f \cdot \langle (g \cdot \langle f, 0_1 \oplus \mathbf{1}_p \rangle)^\dagger, \mathbf{1}_p \rangle,$

for all $f, g : 1 \longrightarrow 1 + p$. The only nontrivial case is that $f = [u \bullet x_1]$ and $g = [v \bullet x_1]$, i.e., f is the $\approx$-congruence class containing the ordered pair $u \bullet x_1$ and g is the $\approx$-congruence class containing the ordered pair $v \bullet x_1$, for some $u, v \in A^*$ such that either $u \neq 1$ or $v \neq 1$. In this case the l.h.s. of (14) is $[1 \bullet uv]$ and the r.h.s. is $[u \bullet vu]$. But $u \bullet vu \xrightarrow{*} 1 \bullet uv$, so that the two sides of (14) are equal. □

Remark 4.7 Each letter $a \in A$ may be identified with the T-morphism $[a \bullet x_1] : 1 \longrightarrow 1$. It can be seen that T is freely generated by these morphisms in the class of Conway theories.

It is possible to select a morphism from each $\approx$-congruence class in a canonical way.

Definition 4.8 A T_0-morphism $f : 1 \longrightarrow p$ is *reduced* if there exists no T_0-morphism $g : 1 \longrightarrow p$ with $f \to g$. A T_0-morphism $f : n \longrightarrow p$, $n \neq 1$, is reduced if each morphism $i_n \cdot f$, $i \in [n]$, is reduced.

Claim 4.9 *A T_0-morphism $f : 1 \longrightarrow p$ is reduced iff*

(i) $f = u \bullet x_i$ *or* $f = u \bullet \bot$, *for some* $u \in A^*$, *or*

(ii) $f = u \bullet v$ *for some* $u \in A^*$ *and* $v \in A^+$, *and if* $u \neq 1$ *then the last letter of* u *is different from the last letter of* v. □

Claim 4.10 *Each* $\approx$*-congruence class contains a unique reduced morphism. When* $f, g : n \longrightarrow p$ *in* T_0 *and* g *is reduced,* $f \approx g$ *iff* $i_n \cdot f \xrightarrow{*} i_n \cdot g$, *for all* $i \in [n]$. *In particular, when* $f = u \bullet v : 1 \longrightarrow p$ *in* T_0 *and* $v \in X_p \cup \{\bot\}$, *the only morphism in the* $\approx$*-congruence class of* f *is the morphism* f *itself.* □

Lemma 4.11 *Suppose that* $f, g : 1 \longrightarrow 1 + p$ *in* T. *If* $f \cdot \langle \bot_p, \mathbf{1}_p \rangle = g \cdot \langle \bot_p, \mathbf{1}_p \rangle$, *then one of the following three cases holds:*

(i) $f = g$.

(ii) $f = [u \bullet x_1]$ *and* $g = [u \bullet \bot]$, *for some* $u \in A^*$.

(iii) $f = [u \bullet \bot]$ *and* $g = [u \bullet x_1]$, *for some* $u \in A^*$. □

Lemma 4.12 *Suppose that* $f = [u \bullet x_1]$ *and* $g = [u \bullet \bot]$ *are* $1 \longrightarrow 1 + p$ *morphisms in* T. *Suppose that* $t : 1 \longrightarrow 1 + p$ *and*

(15) $f \cdot \langle t \cdot \langle \bot_p, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = g \cdot \langle t \cdot \langle \bot_p, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle$.

Then either $t = 1_{1+p} = [1 \bullet x_1]$, *or* $t = \bot_{1+p} = [1 \bullet \bot]$.

Proof. Suppose that $t = [v \bullet w]$, say, where $v \bullet w$ is reduced. Then,

$$f \cdot \langle t \cdot \langle \bot_p, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = \begin{cases} [uv \bullet \bot] & \text{if } w = x_1; \\ [uv \bullet x_i] & \text{if } w = x_{1+i}, i \in [p]; \\ [uv \bullet w] & \text{if } w \in A^+ \cup \{\bot\}. \end{cases}$$

On the other hand, the r.h.s. of (15) is $[u \bullet \bot]$, considered as a morphism $1 \longrightarrow p$. It follows that the two sides of (15) are equal iff $v = 1$ and $w \in \{x_1, \bot\}$. □

Proposition 4.13 *The scalar Scott induction principle holds in the Conway theory* T.

Proof. We prove that the scalar first power implication holds in T, which in turn implies that T staisfies the scalar Scott induction principle, by Lemma 2.4. Suppose that $f, g, t : 1 \longrightarrow 1 + p$ in T such that

$$f \cdot \langle \bot_p, \mathbf{1}_p \rangle = g \cdot \langle \bot_p, \mathbf{1}_p \rangle$$

and

$$f \cdot \langle t \cdot \langle \bot_p, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = g \cdot \langle t \cdot \langle \bot_p, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle.$$

When $f = g$ we obviously have $f \cdot \langle t^\dagger, \mathbf{1}_p \rangle = g \cdot \langle t^\dagger, \mathbf{1}_p \rangle$. Thus, by Lemma 4.11, we may assume that $f = [u \bullet x_1]$ and $g = [u \bullet \bot]$, for some $u \in A^*$. By Lemma

4.12, either $t = [1 \bullet x_1]$ or $t = [1 \bullet \perp]$. But in either case, $t^\dagger = [1 \bullet \perp] = \perp_p$, so that

$$f \cdot \langle t^\dagger, \mathbf{1}_p \rangle = f \cdot \langle \perp_p, \mathbf{1}_p \rangle = [u \bullet \perp] = g \cdot \langle t^\dagger, \mathbf{1}_p \rangle. \qquad \square$$

Proposition 4.14 *The theory T is not an iteration theory.*

Proof. Indeed, when f is the morphism $[a \bullet 1] : 1 \longrightarrow 1$, where a is a letter in A, we have $(f^2)^\dagger = [1 \bullet aa]$ and $f^\dagger = [1 \bullet a]$. Thus $(f^2)^\dagger \neq f^\dagger$, so that T is not an iteration theory. □

We summarize the results of this section.

Theorem 4.15 *There exists a Conway theory which satisfies the scalar Scott induction principle but which is not an iteration theory.* □

Remark 4.16 In the Conway theory T, a modified version of the scalar Scott induction principle holds also. Suppose that $f, g, t, s : 1 \longrightarrow 1+p$ in T. Suppose that

(16) $f \cdot \langle \perp_p, \mathbf{1}_p \rangle = g \cdot \langle \perp_p, \mathbf{1}_p \rangle$,

and

(17) $f \cdot \langle x, \mathbf{1}_p \rangle = g \cdot \langle y, \mathbf{1}_p \rangle \Rightarrow f \cdot \langle t \cdot \langle x, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = g \cdot \langle s \cdot \langle y, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle$,

for all $x, y : 1 \longrightarrow p$. Then

(18) $f \cdot \langle t^\dagger, \mathbf{1}_p \rangle = g \cdot \langle s^\dagger, \mathbf{1}_p \rangle$.

Indeed, if both f and g factor through 0, i.e., $f = f' \cdot 0_{1+p}$ and $g = g' \cdot 0_{1+p}$, for some $f', g' : 1 \longrightarrow 0$, and if (16) holds, then (18) holds obviously. If $f = 0_1 \oplus f'$ and $g = 0_1 \oplus g'$, then (16) implies (18). So we are left with two nontrivial cases.

(i) $f = [u \bullet x_1]$, $g = [u \bullet x_1]$, so that $f = g$. Then $t = s$, by (17). Thus (18) holds.

(ii) $f = [u \bullet x_1]$, $g = [u \bullet \perp]$. Then $t = [1 \bullet \perp] = \perp_{1+p}$ or $t = [1 \bullet x_1] = 1_{1+p}$, so that $t^\dagger = \perp_p$.

5 An Iteration Theory Satisfying the Scalar Scott Induction Principle in which the Scott Induction Principle Fails

In this section we give an iteration theory T which satisfies the scalar Scott induction principle but such that the general form of the Scott induction principle fails in T.

Let Σ be the ranked alphabet with $\Sigma_1 = A = \{a, b, c\}$ and $\Sigma_k = \emptyset$ for $k \neq 1$. A morphism $f : n \longrightarrow p$ in $\Sigma_\perp$TR may be identified with an n-tuple of finite or infinite words in $A^*(X_p \cup \{\perp\}) \cup A^\omega$, where X_p is the set $\{x_1, \ldots, x_p\}$. Using this identification, the composition and dagger operations take the following forms. Suppose that $f : 1 \longrightarrow p$ and $g = (g_1, \ldots, g_p) : p \longrightarrow q$. If f is the word

ux_i, for some $u \in A^*$ and $i \in [p]$, then $f \cdot g$ is the word ug_i. If f is a word in $A^*\{\perp\} \cup A^\omega$, then $f \cdot g$ is the same word being a morphism $1 \longrightarrow q$.

Suppose that f is a morphism $1 \longrightarrow 1+p$. When f is a word in $A^*\{\perp\} \cup A^\omega$, then $f^\dagger$ is the same word being considered as a morphism $1 \longrightarrow p$. If $f = ux_{1+i}$, for some $i \in [p]$ and $u \in A^*$, then $f^\dagger$ is ux_i. Finally, when $f = ux_1$, for some $u \in A^*$, $f^\dagger$ is the word u^ω or the symbol $\perp$ depending on whether u is nonempty or the empty word 1.

We shall denote the length of a word $u \in A^*(X \cup \{\perp, 1\}) \cup A^\omega$ by $|u|$. When u is infinite, its length is ω.

Definition 5.1 Let $u, v \in A^*(X \cup \{\perp, 1\}) \cup A^\omega$. We define $u \to v$ if one of the following conditions holds:

(i)

$$u = w_0 a^{k_1} c w_1 a^{k_2} c w_2 \dots w_{n-1} a^{k_n} c w_n$$
$$v = w_0 c b^{k_1} w_1 c b^{k_2} w_2 \dots w_{n-1} c b^{k_n} w_n,$$

for some $n > 0$, positive integers $k_1, \dots, k_n$, finite words $w_0, \dots, w_{n-1}$, and a possibly infinite word w_n.

(ii)

$$u = w_0 a^{k_1} c w_1 a^{k_2} c w_2 \dots$$
$$v = w_0 c b^{k_1} w_1 c b^{k_2} w_2 \dots,$$

for some finite words w_i, $i \geq 0$, and positive integers k_j, $j > 0$.

(iii) $u = wcb^k\perp$ and $v = wa^k\perp$, for some finite word w and $k \geq 0$.

(iv) $u = wc^\omega$ and $v = w\perp$, for some finite word w.

Let $\xrightarrow{*}$ and $\xleftrightarrow{*}$ denote the usual closures of $\to$. Lastly, we extend the above relations to n-tuples of words componentwise.

Definition 5.2 Suppose that u is a word in $A^*(X \cup \{\perp, 1\}) \cup A^\omega$. We call u a *reduced word*, if there exists no word v with $u \to v$. An n-tuple of words is reduced if its components are reduced.

Thus a word u is reduced iff it contains no subword of the form ac, $cb^k\perp$, or c^ω, where $k \geq 0$.

It is easy to prove that for any word $u \in A^*(X \cup \{\perp, 1\}) \cup A^\omega$ there exists a unique reduced word $\mathrm{red}(u)$ such that $u \xrightarrow{*} \mathrm{red}(u)$. The word $\mathrm{red}(u)$ can be constructed from u by moving the c's occurring in u as left as possible through sequences of a's, replacing each a with a b. If the resulting word ends in a maximal subword of the form $cb^kc^n\perp$, or cb^kc^ω, where $k, n \geq 0$, this subword is replaced by $a^k\perp$.

Lemma 5.3 *For words $u, v \in A^*(X \cup \{\perp, 1\}) \cup A^\omega$, the conditions $u \xleftrightarrow{*} v$ and $\mathrm{red}(u) = \mathrm{red}(v)$ are equivalent.* □

Lemma 5.4 *Suppose $u \in A^*$, $v \in A^*(X \cup \{\perp, 1\}) \cup A^\omega$. Then*

$$\mathrm{red}(uv) = \mathrm{red}(\mathrm{red}(u)\mathrm{red}(v)).$$ □

When $u = (u_1, \ldots, u_n)$ is an n-tuple of words, $\mathrm{red}(u)$ is an abbreviation for $(\mathrm{red}(u_1), \ldots, \mathrm{red}(u_n))$.

Definition 5.5 Suppose that $f, g : n \longrightarrow p$ in $\Sigma_\perp\mathrm{TR}$. We define $f \approx g$ if $\mathrm{red}(f) = \mathrm{red}(g)$.

Proposition 5.6 *$\approx$ is a dagger congruence on $\Sigma_\perp\mathrm{TR}$.*

Proof. The fact that $\approx$ is a theory congruence is immediate from Lemma 5.4. In order to prove that it is a dagger congruence, one needs to check that for all $f, g : 1 \longrightarrow 1 + p$, if $f \longrightarrow g$ then $f^\dagger \longrightarrow g^\dagger$. We omit the details. □

Let T denote the quotient iteration theory $\Sigma_\perp\mathrm{TR}/\approx$.

Proposition 5.7 *The iteration theory T satisfies the scalar Scott induction principle, but the general form of the Scott induction principle fails in T.*

Proof. Let us consider the morphisms $f = ax_1, g = bx_1, h = cx_1 : 1 \longrightarrow 1$ in $\Sigma_\perp\mathrm{TR}$. Then $f \cdot h \approx h \cdot g$ and $h \cdot \perp \approx \perp$, but

$$f^\dagger = a^\omega \not\approx cb^\omega = h \cdot g^\dagger.$$

This shows the functorial dagger implication for pure morphisms doesn't hold in T. Hence the Scott induction principle fails, by Lemma 3.3.

We show that T staisfies the scalar first power implication. Thus, the scalar Scott induction principle also holds in T, by Lemma 2.4. Suppose that $u, v, f : 1 \longrightarrow 1 + p$ in $\Sigma_\perp\mathrm{TR}$ such that

(19) $u \cdot \langle \perp_p, \mathbf{1}_p \rangle \approx v \cdot \langle \perp_p, \mathbf{1}_p \rangle$

and

(20) $u \cdot \langle f \cdot \langle \perp_p, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \approx v \cdot \langle f \cdot \langle \perp_p, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle$.

We need to show that

(21) $u \cdot \langle f^\dagger, \mathbf{1}_p \rangle \approx v \cdot \langle f^\dagger, \mathbf{1}_p \rangle$.

In our argument, we will make use of the fact that if $w\perp \overset{*}{\leftrightarrow} ww'\perp$, for some words w and w', then $w' = c^k$, for some $k \geq 0$. Moreover, if $w\perp \overset{*}{\leftrightarrow} w'\perp$ and $ww_1\perp \overset{*}{\leftrightarrow} w'w_1\perp$ for the words w, w' and w_1, then either $w \overset{*}{\leftrightarrow} w'$ or $w_1 = c^k$, for some integer $k \geq 0$.

If f does not end in x_1, then $f^\dagger = f \cdot \langle \perp_p, \mathbf{1}_p \rangle$. Thus, (21) follows from (20). In the rest of the proof we consider several cases. We assume that $f = f'x_1$, for some finite word f'.

(i) Neither u nor v ends in x_1. Then (19) implies (21).

(ii) u ends in x_1 and v does not end in x_1. Then $u = u'x_1$, for some finite word u', and v is infinite or ends in $\perp$, since otherwise (19) would fail. Since $f = f'x_1$, from conditions (19) and (20) we get $u'\perp \overset{*}{\leftrightarrow} v$ and $u'f'\perp \overset{*}{\leftrightarrow} v$. Thus, $u'\perp \overset{*}{\leftrightarrow} u'f'\perp$ and f' must be c^k, for some $k \geq 0$. Since $f^\dagger \approx \perp_p$, (21) holds.

(iii) v ends in x_1 and u does not end in x_1. This case follows by symmetry from the previous one.

(iv) Both u and v end in x_1, $u = u'x_1$ and $v = v'x_1$, say. By (19) and (20), we have $u'\bot \overset{*}{\leftrightarrow} v'\bot$ and

(22) $u'f \cdot \langle \bot_p, \mathbf{1}_p \rangle \overset{*}{\leftrightarrow} v'f \cdot \langle \bot_p, \mathbf{1}_p \rangle$.

Since $f = f'x_1$, (22) can be written $u'f'\bot \overset{*}{\leftrightarrow} v'f'\bot$. If $u' \overset{*}{\leftrightarrow} v'$ then (21) is obviously true. Otherwise f' must be c^k, for some $k \geq 0$. Thus we have

$$u \cdot \langle f^\dagger, \mathbf{1}_p \rangle \overset{*}{\leftrightarrow} u'\bot \overset{*}{\leftrightarrow} v'\bot \overset{*}{\leftrightarrow} v \cdot \langle f^\dagger, \mathbf{1}_p \rangle. \qquad \Box$$

We prove that it is possible to define a partial order on T such that for each $f : n \longrightarrow n + p$, $f^\dagger$ is the least pre-fixed point of f.

Recall that $\Sigma_\bot\mathrm{TR}$ is an ω-continuous theory under the partial order $\leq$ on $\Sigma_\bot\mathrm{TR}$. When $f, g : 1 \longrightarrow p$, we have $f \leq g$ if $f = g$ or $f = u\bot$ and $g = uv$, for some words u and v.

Definition 5.8 Suppose that $f, g : 1 \longrightarrow p$ in $\Sigma_\bot\mathrm{TR}$. We write $f \preceq g$ if one of the following conditions holds:

(i) $f \leq g$.

(ii) $f = ua^k\bot$ and $g = ucb^kv$, for some $k > 0$, some finite word u, and for a possibly infinite word v.

(iii) $f = ua^\omega$ and $g = ucb^\omega$, for some finite word u.

Observe that $f \preceq g$ implies $|f| \leq |g|$. When $n \neq 1$ and $f, g : n \longrightarrow p$ in $\Sigma_\bot\mathrm{TR}$, we define $f \preceq g$ if $i_n \cdot f \preceq i_n \cdot g$, for all $i \in [n]$.

Lemma 5.9 *The relation $\preceq$ is a partial order on each hom-set $\Sigma_\bot\mathrm{TR}(n, p)$. Moreover, equipped with the ordering $\preceq$, $\Sigma_\bot\mathrm{TR}$ is a strict ordered preiteration theory.* $\Box$

The proof of the following lemma is given in the Appendix.

Lemma 5.10 *Suppose that $f, g : n \longrightarrow p$ in $\Sigma_\bot\mathrm{TR}$. If $f \preceq g$, then* $\mathrm{red}(f) \preceq \mathrm{red}(g)$. $\Box$

When α is a $\approx$-equivalence class in the theory $T = \Sigma_\bot\mathrm{TR}/\approx$, we write $\mathrm{red}(\alpha)$ for the unique reduced morphism in α.

Corollary 5.11 *The following two conditions are equivalent for any two morphisms $\alpha, \beta : n \longrightarrow p$ in T:*

(i) *There exist $f \in \alpha$ and $g \in \beta$ with $f \preceq g$.*

(ii) $\mathrm{red}(\alpha) \preceq \mathrm{red}(\beta)$.

Proof. Suppose that $f \preceq g$ holds for some $f \in \alpha$ and $g \in \beta$. Then, by Lemma 5.10, $\mathrm{red}(f) \preceq \mathrm{red}(g)$. Thus, the first condition implies the second one. The converse direction is trivial. $\Box$

Definition 5.12 Suppose that $\alpha, \beta : n \longrightarrow p$ in T. We write $\alpha \preceq \beta$ if $\mathrm{red}(\alpha) \preceq \mathrm{red}(\beta)$.

Corollary 5.13 *Equipped with the relation $\preceq$, T is a strict ordered preiteration theory.*

Proof. This follows from Lemma 5.9 and Corollary 5.11. □

A proof of the following lemma is given in the Appendix.

Lemma 5.14 *Suppose that* $u \in A^+$ *and* $v \in A^*(X \cup \{\bot, 1\}) \cup A^\omega$ *are reduced words. If* $\mathrm{red}(uv) \preceq v$ *then* $\mathrm{red}(u^\omega) \preceq v$. □

Proposition 5.15 *The Park induction principle holds in the strict ordered preiteration theory* T*.*

Proof. By Theorem 2.5, we need to show that for all $\alpha : 1 \longrightarrow 1 + p$ and $\beta : 1 \longrightarrow p$ in T,

$$\alpha \cdot \langle \beta, \mathbf{1}_p \rangle \preceq \beta \quad \Rightarrow \quad \alpha^\dagger \preceq \beta.$$

So suppose that

(23) $\alpha \cdot \langle \beta, \mathbf{1}_p \rangle \preceq \beta$.

Let $w = \mathrm{red}(\alpha)$ and $v = \mathrm{red}(\beta)$, say. If $w \in A^*\{\bot\} \cup A^\omega$, then (23) can be rephrased as $w \preceq v$. But $w \in \alpha^\dagger$, so that $\alpha^\dagger \preceq \beta$. If $w = ux_{1+i}$, for some $i \in [p]$, then (23) can be written as $ux_i \preceq v$. But since $ux_i \in \alpha^\dagger$, we have $\alpha^\dagger \preceq \beta$.

Lastly, suppose that $w = ux_1$, for some u in A^*. In this case $uv \preceq v$, so that $\mathrm{red}(uv) \preceq \mathrm{red}(v) = v$ by Lemma 5.10. If $u = 1$ then $\bot \in \alpha^\dagger$, so that $\alpha^\dagger \preceq \beta$. If $u \neq 1$, then $\mathrm{red}(u^\omega) \preceq v$, by Lemma 5.14. But since $u^\omega \in \alpha^\dagger$, we have $\alpha^\dagger \preceq \beta$. □

The results of this section are summarized in the following theorem.

Theorem 5.16 *The iteration theory* T *satisfies the scalar, but not the general form of the Scott induction principle. Further, as a strict ordered preiteration theory,* T *satisfies the Park induction principle, so that iteration is defined by least pre-fixed points.* □

Remark 5.17 The (weak) scalar ordered Scott induction principle does not hold in the iteration theory T. Indeed, consider the morphisms cx_1, a^ω and bx_1 in $\Sigma_\bot\mathrm{TR}$. Let $\alpha, \beta, \gamma : 1 \longrightarrow 1$ in T be defined by $\mathrm{red}(\alpha) = cx_1$, $\mathrm{red}(\beta) = a^\omega$ and $\mathrm{red}(\gamma) = bx_1$. Then, $\alpha \cdot \bot \preceq \beta \cdot \bot$, and if $\alpha \cdot \delta \preceq \beta \cdot \delta$ holds for some morphism $\delta : 1 \longrightarrow 0$ in T, then $\mathrm{red}(\delta) = b^k\bot$, for some $k \geq 0$. Thus, $\alpha \cdot \gamma \cdot \delta \preceq \beta \cdot \gamma \cdot \delta$. But

$$\mathrm{red}(\alpha \cdot \gamma^\dagger) = cb^\omega \succ a^\omega = \mathrm{red}(\beta \cdot \gamma^\dagger).$$

6 An Iteration Theory Not Satisfying the Scalar Scott Induction Principle

Recall that a cpo C is a poset such that each directed set $X \subseteq C$ has a least upper bound $\sup X$. When C is a cpo, so is the poset C^n, ordered pointwise, for each $n \geq 0$. Suppose that C is a cpo with a bottom element $\bot$. We denote the strict ordered theory of monotonic functions on C by $Th(C)$. A morphism $n \longrightarrow p$ in $Th(C)$ is a monotonic function $C^p \longrightarrow C^n$. Composition is function composition written in the opposite order, so that if $f : n \longrightarrow p$ and $g : p \longrightarrow q$ in $Th(C)$, then $f \cdot g$ is the function

$$C^q \xrightarrow{g} C^p \xrightarrow{f} C^n.$$

The distinguished morphism i_n is the ith projection function $C^n \longrightarrow C$. Suppose that $f : n \longrightarrow n+p$ in $Th(C)$, i.e., f is a monotonic function $C^{n+p} \longrightarrow C^n$. It is well-known that f has a least pre-fixed point $f^\dagger : n \longrightarrow p$ in $Th(C)$. The function $f^\dagger$ can be constructed pointwise, so that for each $y \in C^p$, the value of $yf^\dagger$ is the least $x \in C^n$ such that $(x, y)f \leq x$. Due to the pointwise construction of $f^\dagger$, it follows that the dagger operation, defined on the theory $Th(C)$, satisfies the (scalar) parameter identity and hence $Th(C)$ is a Conway theory.

A function $C \longrightarrow D$ from a cpo C to a cpo D is continuous if it preserves the sup of each directed set.

Proposition 6.1 *$Th(C)$ satisfies the functorial implication for pure continuous functions.*

Proof. Suppose that the square

$$\begin{array}{ccc} n & \xrightarrow{\quad f \quad} & n+p \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h \oplus \mathbf{1}_p} \\ m & \xrightarrow[\quad g \quad]{} & m+p \end{array} \qquad (24)$$

commutes, where h is a pure continuous function $C^m \longrightarrow C^n$. Given $y \in C^p$, let f_y denote the function

$$\begin{aligned} &C^n \longrightarrow C^n \\ &\quad x \mapsto (x, y)f. \end{aligned}$$

The function $g_y : C^m \longrightarrow C^m$ is defined similarly. The commutativity of the square (24) can be restated as

(25) $f_y \cdot h = h \cdot g_y$,

for all $y \in C^p$.

Let us define the sequences (a_α) and (b_α) by transfinite induction on the ordinal α as follows:

$$\begin{aligned} a_0 &= \bot_n \\ b_0 &= \bot_m \\ a_\alpha &= \begin{cases} a_\beta f_y & \text{if } \alpha = \beta + 1; \\ \sup\limits_{\beta<\alpha} a_\beta & \text{if } \alpha \neq 0 \text{ is a limit ordinal;} \end{cases} \\ b_\alpha &= \begin{cases} b_\beta g_y & \text{if } \alpha = \beta + 1; \\ \sup\limits_{\beta<\alpha} b_\beta & \text{if } \alpha \neq 0 \text{ is a limit ordinal.} \end{cases} \end{aligned}$$

(Here, $\bot_n = (\bot, \ldots, \bot)$ is the least element of C^n and $\bot_m$ is defined similarly.) It is well-known that $yf^\dagger = a_\gamma$ and $yg^\dagger = b_\gamma$, where γ is the least infinite ordinal with $|\gamma| > |C|$. But, using the assumption (25) and that h is a pure continuous function, it follows by a straightforward transfinite induction argument that $a_\alpha = b_\alpha h$, for all ordinals α. In particular, $yf^\dagger = a_\gamma = b_\gamma h = yg^\dagger h$, proving $f^\dagger = h \cdot g^\dagger$. □

Corollary 6.2 *$Th(C)$ is an iteration theory.* □

Suppose now that C is the well-ordered chain $\omega + 2$, so that

$$C = \{0 < 1 < \ldots < \omega < \omega + 1\}.$$

Proposition 6.3 *When $C = \omega{+}2$, the iteration theory $Th(C)$ does not satisfy the (weak) scalar Scott induction principle.*

Proof. Let $u, f : C \longrightarrow C$ be defined as follows:

$$xu = \begin{cases} x & \text{if } x \neq \omega; \\ \omega + 1 & \text{if } x = \omega; \end{cases}$$

$$xf = \begin{cases} x + 1 & \text{if } x < \omega; \\ x & \text{if } x \geq \omega. \end{cases}$$

Thus u and f are morphisms in $Th(C)$. Now, if v denotes the identity function $C \longrightarrow C$, we have $u \cdot \bot = 0u = 0 = 0v = v \cdot \bot$, and if $u \cdot x = v \cdot x$, so that $x \neq \omega$, then $u \cdot f \cdot x = v \cdot f \cdot x$. Yet, $u \cdot f^\dagger = \omega + 1 \neq \omega = v \cdot f^\dagger$. □

The same proof shows that T does not satisfy the (weak) scalar ordered Scott induction principle. We have proved:

Theorem 6.4 *There exists a strict ordered iteration theory in which iteration is defined via least pre-fixed points (so that the Park induction principle holds), but which does not satisfy the (weak) scalar (ordered) Scott induction principle.* □

7 Discussion and Further Results

Our main positive result was that the Scott induction principle, in conjunction with the Conway identities, is complete for the equational theory of iteration. This was a direct consequence of Theorem 2.1 and the simple calculations involved in the proofs of Lemmas 3.2 and 3.3.

If one considers ordered preiteration theories and the ordered Scott induction principle, then it is natural to ask for completeness with respect to the inequations, not just the equations. By Theorem 2.9, the equations true in iteration theories may be regarded the standard equational theory of iteration. But what is the standard theory for inequations? The best candidate is the collection of the inequations which hold in the preiteration theories defined on ω-continuous theories. Another candidate would be the inequations which hold in all ordered preiteration theories of monotonic functions over cpo's, or complete lattices, where iteration is defined by least fixed points. But these two sets of inequations are the same, justifying our choice. Using the representation of the free iteration theories, see [1] on pages 191-193, it is not hard to prove the following completeness result: An inequation holds in all ω-continuous theories iff it follows (in the ordered setting) from the iteration theory axioms and the inequation $\perp_{n,p} \leq f$, all $f : n \longrightarrow p$, (or $\perp_{1,p} \leq f$, for all $f : 1 \longrightarrow p$). Thus, by the results of this paper, the scalar parameter identity and the fixed point identity, together with the inequation $\perp_{1,p} \leq f$, all $1 \longrightarrow p$, and the ordered Scott induction principle are also complete.

A stronger result is proved in [7].

Theorem 7.1 *An (in)equation between preiteration theory terms holds in all ω-continuous theories iff it is a logical consequence (in the ordered setting) of the scalar parameter identity, the fixed point identity, and the Park induction principle:*

$$f \cdot \langle \xi, \mathbf{1}_p \rangle \leq \xi \quad \Rightarrow \quad f^\dagger \leq \xi,$$

for all $f : n \longrightarrow n + p$ and $\xi : n \longrightarrow p$. □

Thus, the scalar parameter identity, the fixed point identity and the Park induction principle form a complete axiomatization of the inequational theory of iteration. It then follows that the system consisting of the scalar parameter, fixed point and pairing identities and the *scalar Park induction principle* obtained by taking $n = 1$ above, or the *scalar inequational Scott induction principle* are also complete. Thus, Theorem 7.1 is a significant improvement on the results formulated in Theorem 3.4.

The above theorem is also of interest in connection with a result of Dexter Kozen [9]. He showed that a complete axiomatization of the equational theory of the regular sets consists of analogues of the Conway identities, formulated for * semirings, and the (scalar) Park induction principle and its dual, which take the following form:

(26) $ax + b \leq x \quad \Rightarrow \quad a^*b \leq x$

(27) $xa + b \leq x \quad \Rightarrow \quad ba^* \leq x.$

A refinement of this result is due to Krob, cf. [11]. His axioms consist of the Conway semiring identities and the *self dual* implication

(28) $a^2 = a \quad \Rightarrow \quad a^* = 1 + a.$

Note that the axiom (28) is a particular instance of each of (26) and (27).

We briefly outline the completeness proof. Let us call an ordered preiteration theory a *Park theory* if it satisfies the axioms involved in Theorem 7.1. By Theorem 2.5, the unordered reduct of a Park theory is a Conway theory.

First we establish a variant of the weak functorial implication in all Park theories. Then we associate an identity $C(S)$ with every finite semigroup S and establish the identities $C(G)$ for finite groups G in Park theories, and the identity $C(U)$ in all Conway theories, where U is the 3-element unit semigroup of the Krohn-Rhodes decomposition theorem for finite semigroups. Then we prove that if $C(S')$ and the "vector form" of $C(S)$ hold in a Conway theory, where the semigroup S' acts on S, then so does $C(S \star S')$, where $S \star S'$ is the semidirect product determined by the action. Then we prove that in Park theories, if $C(S)$ holds then $C(S')$ also holds, where S' is a subsemigroup or a quotient of S. In conclusion, by the Krohn-Rhodes decomposition of finite semigroups and by a general metatheorem on the vector forms of first order sentences, all of the semigroup identities $C(S)$ hold in Park theories. Next we associate an identity with each transformation semigroup and each automaton, and show it holds in Park theories. Then we prove that in Conway theories any instance of the commutative identity is equivalent with the vector form of the identity associated with an automaton. Finally, the commutative identity is shown to hold in Park theories using the metatheorem mentioned above: Since the Park theory axioms imply their own vector forms, if an identity holds in all Park theories, then so does its vector form.

By taking the above line of the proof we have benefited from Krob's excellent paper [11]. Krob proved the conjecture of Conway [4] that the "group identities" associated with the finite (simple) groups and a small set of simple equational axioms including the Conway identities form a complete axiomatization of the equational theory of the regular sets. In a large part of the proof, he gave a translation of a standard proof of the Krohn-Rhodes decomposition theorem [12] for finite semigroups and finite automata into equational logic. In contrast, our completeness argument uses of the Krohn-Rhodes decomposition in a direct way. No part of its proof is reproduced. The way we assign an identity to each finite semigroup and finite automaton is new in the general setting of theories, but it essentially coincides with the one used in [4,11] when translated to star form in the language of matrix theories over idempotent semirings. Nevertheless, matrix theories do not appear in an explicit way in [4,11]. Several arguments in [11] make use of the fact that the additive structure is idempotent. This condition is not available in the general setting.

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8 Appendix

In this appendix we give the promised proofs of Lemmas 5.10 and 5.14, restated here as Lemma 8.4 and Lemma 8.5.

Lemma 8.1 *Suppose that $u \in A^*$ and $v \in A^*(X \cup \{\bot, 1\}) \cup A^\omega$ are reduced. Then* $\mathrm{red}(uv) = uv$*, except for the following cases:*

(i) *If $u = u_0 a^k$ and $v = cv_0$, for some words u_0 and v_0 and for some integer $k > 0$ such that u_0 does not end in a, then* $\mathrm{red}(uv) = u_0 c b^k v_0$.

(ii) *If $u = u_0 c b^m c^k$ and $v = \bot$, for some word u_0 and some integers $m > 0$ and $k \geq 0$, then* $\mathrm{red}(uv) = u_0 a^m \bot$.

(iii) *If $u = u_0 c^k$ and $v = \bot$, for some word u_0 and for some integer $k > 0$, where u_0 does not end in a word of the form cb^m, $m \geq 0$, then* $\mathrm{red}(uv) = u_0 \bot$.

(iv) *If $u = u_0 c b^m$ and $v = b^k \bot$, where $k > 0$ and $m \geq 0$, then* $\mathrm{red}(uv) = u_0 a^{m+k} \bot$. □

Lemma 8.2 *Suppose that $u \in A^*$ and $v \in A^*(X \cup \{\bot, 1\}) \cup A^\omega$ are reduced. Then* $\mathrm{red}(u\bot) \preceq \mathrm{red}(uv)$.

Proof. We may assume that $v \neq \bot$.

(i) $u = u_0 c b^m c^k$, $m > 0$, $k \geq 0$.

$$\mathrm{red}(u\bot) = u_0 a^m \bot$$

$$\mathrm{red}(uv) = \begin{cases} u_0 c b^m c^{k-1} a^n \bot & \text{if } k > 0 \text{ and } v = b^n \bot \text{ for some } n > 0; \\ u_0 a^{m+n} \bot & \text{if } k = 0 \text{ and } v = b^n \bot \text{ for some } n > 0; \\ uv & \text{otherwise.} \end{cases}$$

(ii) $u = u_0 c^k$, $k > 0$, and u_0 does not end in a word of the form cb^m, $m \geq 0$.

$$\mathrm{red}(u\bot) = u_0 \bot$$

$$\mathrm{red}(uv) = \begin{cases} u_0 c^{k-1} a^n \bot & \text{if } v = b^n \bot \text{ for some } n > 0; \\ uv & \text{otherwise.} \end{cases}$$

(iii) In all other cases,

$$\mathrm{red}(u\bot) = u\bot$$

$$\mathrm{red}(uv) = \begin{cases} u_0 c b^k v_0 & \text{if } u = u_0 a^k,\ v = cv_0, \text{ for some words } u_0 \text{ and } v_0 \text{ and some } k > 0 \text{ such that } u_0 \text{ does not end in } a; \\ uv & \text{otherwise.} \end{cases}$$

In either case, $\mathrm{red}(u\bot) \preceq \mathrm{red}(uv)$. □

Corollary 8.3 *Suppose that $u \in A^*$ and $v \in A^*(X \cup \{\bot, 1\}) \cup A^\omega$. Then* $\mathrm{red}(u\bot) \preceq \mathrm{red}(uv)$. □

Lemma 8.4 *Suppose that $f \preceq g : 1 \longrightarrow p$ in $\Sigma_\bot \mathrm{TR}$. Then* $\mathrm{red}(f) \preceq \mathrm{red}(g)$.

Proof. We prove the lemma by cases.

(i) $f \leq g$. The nontrivial case is that $f = u\bot$ and $g = uv$, for some words u and v. But then $\mathrm{red}(f) = \mathrm{red}(u\bot) \preceq \mathrm{red}(uv) = \mathrm{red}(g)$, by Corollary 8.3.

(ii) $f = ua^k\bot$ and $g = ucb^kv$, for some words u and v and some integer $k > 0$. Let us write $u = u_0a^m$, where u_0 does not end in a. Then $\mathrm{red}(f) = \mathrm{red}(u_0)a^{m+k}\bot$. If $\mathrm{red}(v) = b^n\bot$, for some $n \geq 0$, then $\mathrm{red}(g) = \mathrm{red}(u_0)a^{m+k+n}\bot$, so that $\mathrm{red}(f) \leq \mathrm{red}(g)$. If $\mathrm{red}(v)$ is not of the form $b^n\bot$, then $\mathrm{red}(g) = \mathrm{red}(u_0)cb^{m+k}\mathrm{red}(v)$, so that $\mathrm{red}(f) \preceq \mathrm{red}(g)$.

(iii) $f = ua^\omega$ and $g = ucb^\omega$. Then $\mathrm{red}(f) = \mathrm{red}(u)a^\omega$, $\mathrm{red}(g) = \mathrm{red}(uc)b^\omega$. We can write $u = u_0a^k$, where $k \geq 0$ and u_0 does not end in a. Then $\mathrm{red}(u) = \mathrm{red}(u_0)a^k$ and $\mathrm{red}(uc) = \mathrm{red}(u_0)cb^k$, so that

$$\mathrm{red}(f) = \mathrm{red}(u_0)a^\omega \preceq \mathrm{red}(u_0)cb^\omega = \mathrm{red}(g).$$

□

Lemma 8.5 *Suppose that* $u \in A^+$ *and* $v \in A^*(X \cup \{\bot, 1\}) \cup A^\omega$ *are reduced words. If* $\mathrm{red}(uv) \preceq v$ *then* $\mathrm{red}(u^\omega) \preceq v$.

Proof. If uv is reduced then v is infinite and $uv = v$ by Definition 5.8. Thus $v = u^\omega$.

Supposing that uv is not reduced, one of the four cases of Lemma 8.1 applies. The second and fourth cases may be ruled out. We consider the remaining cases.

(i) $u = u_0a^k$ and $v = cv_0$, for some words u_0 and v_0 and for some integer $k > 0$ such that u_0 does not end in a. Then $\mathrm{red}(uv) = u_0cb^kv_0$, so that $u_0cb^kv_0 \preceq cv_0$. Since $|u_0cb^k| > |c|$, v_0 must be infinite. Thus, by Definition 5.8, we must have $u_0cb^kv_0 = cv_0$.

  (a) $u_0 = 1$. Then $b^kv_0 = v_0$, so that $v_0 = b^\omega$. Thus, $v = cv_0 = cb^\omega$ and $u^\omega = (a^k)^\omega = a^\omega$. But $a^\omega \preceq cb^\omega$, by Definition 5.8.

  (b) $u_0 \neq 1$. Then $u_0 = cw$ and $wcb^kv_0 = v_0$, for some w. Thus, $v_0 = (wcb^k)^\omega$ and $u^\omega = (cwa^k)^\omega = c(wa^kc)^\omega$, proving

$$\mathrm{red}(u^\omega) = c(wcb^k)^\omega = cv_0 = v.$$

(ii) $u = u_0c^k$ and $v = \bot$, for some word u_0 and for some integer $k > 0$, where u_0 does not end in in a word of the form cb^m, $m \geq 0$. Then $\mathrm{red}(uv) = u_0\bot$, so that $u_0\bot \preceq \bot$. This is possible only if $u_0 = 1$ in which case $u = c^k$. But then $u^\omega = (c^k)^\omega = c^\omega$ and $\mathrm{red}(u^\omega) = \bot = v$.

□