

Poisson-Lie analogues of spin Sutherland models

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Abstract

We present generalizations of the well-known trigonometric spin Sutherland models, which were derived by Hamiltonian reduction of ‘free motion’ on cotangent bundles of compact simple Lie groups based on the conjugation action. Our models result by reducing the corresponding Heisenberg doubles with the aid of a Poisson-Lie analogue of the conjugation action. We describe the reduced symplectic structure and show that the ‘reduced main Hamiltonians’ reproduce the spin Sutherland model by keeping only their leading terms. The solutions of the equations of motion emerge from geodesics on the compact Lie group via the standard projection method and possess many first integrals. Similar hyperbolic spin Ruijsenaars–Schneider type models were obtained previously by L.-C. Li using a different method, based on coboundary dynamical Poisson groupoids, but their relation with spin Sutherland models was not discussed.

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1. Introduction

Integrable systems of particles moving in one dimension have been studied intensively for nearly 50 years, beginning with the pioneering papers of Calogero [6], Sutherland [51] and Moser [35]. Thanks to their fascinating mathematics and diverse applications [11,37–39,44,52], the interest in these models shows no sign of diminishing. New connections to mathematics and new applications are still coming to light in the current literature, see e.g. [4,7,8,22,24,46,53].

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The richness of these models is also due to their many generalizations and deformations. These are associated with different interaction potentials (from rational to elliptic), root systems and extensions with internal degrees of freedom. We call ‘Sutherland models’ the systems defined by trigonometric or hyperbolic potentials. For all these systems, classical and quantum mechanical versions are studied separately, and one needs to pay attention to the distinct features of the systems with real particle positions and their complexifications. The investigations of Ruijsenaars–Schneider (RS) type deformations [44,45] is motivated, for example, by relations to solitons, spin chains, special functions and double affine Hecke algebras.

The internal degrees of freedom are colloquially called ‘spin’, and can be of two rather different kinds. First, the point particles can carry spins varying in a vector space, as is the case for the Gibbons–Hermesen models [20] and their RS type generalizations introduced by Krichever and Zabrodin [28]. Second, the models can involve a collective spin variable that typically belongs to a coadjoint orbit, and is not assigned separately to the particles. An example of this second type is the trigonometric spin Sutherland model defined classically by a Hamiltonian of the following form:

$$\mathcal{H}_{\text{Suth}}(e^{iq}, p, \xi) = \frac{1}{2} \langle p, p \rangle + \frac{1}{2} \sum_{\alpha > 0} \frac{1}{|\alpha|^2} \frac{|\xi_\alpha|^2}{\sin^2 \frac{\alpha(q)}{2}}. \quad (1.1)$$

Here, $\langle \cdot, \cdot \rangle$ is the Killing form of the complexification of the Lie algebra \mathcal{G} of a compact simple Lie group, G , e^{iq} belongs to the interior, \mathbb{T}^o , of a Weyl alcove¹ in the maximal torus $\mathbb{T} < G$, and ip varies in the Lie algebra \mathcal{T} of \mathbb{T} . The spin variable $\xi = \sum_{\alpha > 0} (\xi_\alpha E_\alpha - \xi_\alpha^* E_{-\alpha})$ lies in $\mathcal{O}_0 := \mathcal{O} \cap \mathcal{T}^\perp$, where \mathcal{O} is an arbitrarily chosen coadjoint orbit of G , and α runs over the positive roots. More precisely, the Hamiltonian $\mathcal{H}_{\text{Suth}}$ lives on the phase space $T^*\mathbb{T}^o \times (\mathcal{O}_0/\mathbb{T})$. These spin Sutherland models can be interpreted as Hamiltonian reductions of free motion on G , relying on the cotangent lift of the conjugation action of G on itself. The reduction can be utilized to show their integrability, and to analyze their quantum mechanics with the aid of representation theory [12,17,18,41–43]. Spinless models can be obtained in this way only for $G = \text{SU}(n)$, using a minimal coadjoint orbit, for which the \mathbb{T} -action on \mathcal{O}_0 is transitive.

As was shown by Li and Xu [32], the models (1.1) (and generalizations) result from a different construction as well. Their construction is built on Lie algebroids defined using the solutions of the classical dynamical Yang–Baxter equation. For the connection of these approaches, we refer to [17].

Our original motivation for the present work stems from [14], where it was shown how the Ruijsenaars–Schneider deformation of the standard spinless Sutherland model arises from a Hamiltonian reduction of the Poisson–Lie counterpart of $T^*\text{SU}(n)$, the so-called Heisenberg double. To obtain the spinless model, one has to choose a minimal dressing orbit of $\text{SU}(n)$ in setting up the reduction. It is natural to expect that the analogous reduction of the Heisenberg double of any compact simple Lie group, along an arbitrary dressing orbit, will lead to a generalization of the spin Sutherland model (1.1). Motivated by the recent interest in spin Calogero–Moser and RS models [4,9,22,24,42,43,46], we take up this issue here.

In fact, the purpose of this paper is to describe the spin RS type models that descend from the Heisenberg double of a compact simple Lie group G . The so-called main reduced Hamiltonians, which originate from the characters of the complexification of G , will turn out to have $\mathcal{H}_{\text{Suth}}$ (1.1) as their leading term, up to cubic and higher order terms in p and a spin variable. (Here and

¹ A Weyl alcove is a fundamental domain for the conjugation action of G on itself.

throughout the paper, we refer to the total, or combined, degree in p and the spin variable. For example, p^3 , $p^2\sigma$, $p\sigma^2$ and σ^3 all have degree 3.) The spin variable now belongs to a reduced dressing orbit of the Poisson-Lie group G . The dressing orbits are the Poisson-Lie analogues of the coadjoint orbits, and in the compact case each dressing orbit is diffeomorphic, and is even symplectomorphic [21], to a coadjoint orbit.

In the $SU(n)$ case, using analytic continuation from trigonometric to hyperbolic functions, our models reproduce the spin RS type equations of motion derived by Braden and Hone [5] from the soliton solutions of A_{n-1} affine Toda theory with imaginary coupling. These equations of motion were interpreted previously by L.-C. Li [29,30] as examples of spin RS type Hamiltonian systems obtained by applying (discrete and Hamiltonian) reductions to the coboundary dynamical Poisson groupoids that underlie the geometric interpretation of the classical dynamical Yang-Baxter equation [13]. Remembering also the alternative constructions of spin Sutherland models [17,32], it is clear that there must exist a connection between our systems and corresponding systems of [29,30]. The two approaches are substantially different, but the analytic continuation of our models appears to yield a subclass of those in [30]. This is discussed further in Remark 6.3 and in Section 7, together with other approaches to spin RS type models. The precise connection will be explored in detail in a subsequent publication.

In the trigonometric/hyperbolic case, the papers [4,9] contain two different reduction treatments of the holomorphic spin RS systems of Krichever and Zabrodin [28] associated with $SL(n, \mathbb{C})$. The complexifications of our systems associated with $SU(n)$ do not reproduce those systems. The systems studied in [4,9,28] feature individual spins attached to the particles, while our systems involve only collective spin variables. Hamiltonian reductions leading directly to distinct real forms of the complex trigonometric/hyperbolic spin RS systems of [28], as well as the elliptic systems and the case of general root systems, should be developed in the future.

Now we sketch the organization of the rest of the paper. We start in Section 2 by recalling the reduction treatment of the spin Sutherland models, which can be found in many sources (see e.g. [17,41]). This section puts our generalization in context, and provides motivation for it. In Section 3, we present the rudiments of the standard Heisenberg double of a compact Poisson-Lie group and its ‘natural free system’ that we shall reduce. To our knowledge, this ‘free system’ first appeared in [55], and was utilized previously, for example, in [14,15]. Then, in Section 4, we describe the structure of the reduced phase space in an as complete manner as is known for the spin Sutherland models. In Section 5, we show that the spin Sutherland Hamiltonian (1.1) is recovered as the leading term of the reduced main Hamiltonians associated with the characters of the finite dimensional irreducible representations of $G^{\mathbb{C}}$. In Section 6, we develop the form of the reduced Hamilton equations, detail the projection approach for constructing their solutions, and display a large number of integrals of motion. We also present spectral parameter dependent Lax equations. Section 7 contains an outlook on future studies and open questions.

Finally, let us highlight our main results. The first important result is the description of the symplectic structure on a dense open submanifold of the reduced phase space, given by Theorem 4.3. The second significant result, presented in Section 5.1, is the characterization of the reduced main Hamiltonians from which we can recover the spin Sutherland Hamiltonian (1.1) as the leading term. More precisely, we also recover the Lax matrix of the model (1.1) as a suitable leading term, and explain in Remark 5.1 how our models can be viewed as one-parameter deformations of the spin Sutherland models. In the $SU(n)$ case, we obtain an explicit solution of the moment map constraints, see Section 5.2. Further results are the simple derivation of the reduced equations of motion and their solutions in Section 6.1, and the arguments put forward in Section 6.2 that indicate their integrability.

2. Spin Sutherland model from reduction

First of all, we fix the Lie theoretic [26] notations that will be used throughout the paper. Let $\mathcal{G}^{\mathbb{C}}$ be a complex simple Lie algebra equipped with the normalized Killing form $\langle \cdot, \cdot \rangle$, and a Chevalley basis given by $E_{\pm\alpha}$ ($\alpha \in \Phi^+$) and T_{α_k} ($\alpha_k \in \Delta$), where Φ^+ and Δ denote the sets of positive and simple roots, respectively. The normalization is such that the long roots have length $\sqrt{2}$ and $\langle E_{\alpha}, E_{\beta} \rangle = \frac{2}{|\alpha|^2} \delta_{\alpha, -\beta}$ holds. We let $N := \dim_{\mathbb{C}}(\mathcal{G}^{\mathbb{C}})$ and write $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ for $\mathcal{G}^{\mathbb{C}}$ regarded as a Lie algebra over the real numbers. We then have the real vector space direct sum

$$\mathcal{G}_{\mathbb{R}}^{\mathbb{C}} = \mathcal{G} + \mathcal{B}, \quad (2.1)$$

where

$$\mathcal{G} = \text{span}_{\mathbb{R}}\{(E_{\alpha} - E_{-\alpha}), i(E_{\alpha} + E_{-\alpha}), iT_{\alpha_k} \mid \alpha \in \Phi^+, \alpha_k \in \Delta\} \quad (2.2)$$

is the compact real form of $\mathcal{G}^{\mathbb{C}}$ and

$$\mathcal{B} = \text{span}_{\mathbb{R}}\{E_{\alpha}, iE_{\alpha}, T_{\alpha_k} \mid \alpha \in \Phi^+, \alpha_k \in \Delta\} \quad (2.3)$$

is a ‘Borel’ subalgebra. Consider the connected and simply connected complex Lie group, $G^{\mathbb{C}}$, associated with $\mathcal{G}^{\mathbb{C}}$. When viewed as a real Lie group, we denote it as $G_{\mathbb{R}}^{\mathbb{C}}$, and let G and B stand for the connected Lie subgroups of $G_{\mathbb{R}}^{\mathbb{C}}$ corresponding to the subalgebras \mathcal{G} and \mathcal{B} , respectively. The restriction of $\langle \cdot, \cdot \rangle$ to \mathcal{G} is the negative definite Killing form of \mathcal{G} . The subalgebras \mathcal{G} and \mathcal{B} of $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ are isotropic with respect to the non-degenerate invariant bilinear form on $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ provided by the imaginary part of the complex Killing form, which we denote as

$$(X, Y) := \text{Im}\langle X, Y \rangle, \quad \forall X, Y \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}. \quad (2.4)$$

Notationwise, we shall ‘pretend’ that we are always dealing with matrix Lie groups. For example, the left-invariant Maurer–Cartan form on G will be written as $g^{-1}dg$. If desired, our matrix Lie group notations can be easily converted into more abstract symbolism.

Now, we briefly summarize the reduction that we shall generalize. We start with the master phase space $M := T^*G \times \mathcal{O}$, where T^*G is the cotangent bundle, and \mathcal{O} is a coadjoint orbit of the Lie group G . The phase space is endowed with the Poisson maps

$$J_L : M \rightarrow \mathcal{G}^*, \quad J_R : M \rightarrow \mathcal{G}^*, \quad J_{\mathcal{O}} : M \rightarrow \mathcal{G}^*, \quad (2.5)$$

where J_L (J_R) generates the Hamiltonian left-action of G on T^*G engendered by the left-shifts (right-shifts) and $J_{\mathcal{O}}$ is obtained by combining projection to \mathcal{O} with the tautological embedding of \mathcal{O} into \mathcal{G}^* . One then considers the moment map

$$\mu := J_L + J_R + J_{\mathcal{O}} \quad (2.6)$$

that generates the ‘conjugation action’ of G on M . A dense open subset of the reduced phase space belonging to the zero value of μ can be identified with the (stratified) symplectic space (see [49,50])

$$M_{\text{red}}^{\text{reg}} = T^*\mathbb{T}^{\circ} \times \mathcal{O}_0/\mathbb{T}, \quad (2.7)$$

where \mathbb{T}° is the interior of a Weyl alcove in the maximal torus $\mathbb{T} < G$, and \mathcal{O}_0/\mathbb{T} is the symplectic reduction of \mathcal{O} by \mathbb{T} at the zero value of the respective moment map.

Next, we explain how the above description of the reduced phase space comes about. For this, we let $\pi_G : T^*G \rightarrow G$ denote the bundle projection and use the diffeomorphism

$$(\pi_G, J_R) : T^*G \rightarrow G \times \mathcal{G}^*. \quad (2.8)$$

Together with the identification $\mathcal{G}^* \simeq \mathcal{G}$ defined by the Killing form of \mathcal{G} , this allows us to take

$$G \times \mathcal{G} \times \mathcal{O} = \{(g, J, \xi)\} \quad (2.9)$$

as the model of M . Then the symplectic form ω of M can be written as

$$\omega = -d\langle J, g^{-1}dg \rangle + \omega_{\mathcal{O}}, \quad (2.10)$$

where $\omega_{\mathcal{O}}$ is the canonical symplectic form of \mathcal{O} . The subset of M on which $\mu = 0$ holds is specified by the constraint equation

$$J - gJg^{-1} + \xi = 0. \quad (2.11)$$

We can bring² g^{-1} into its representative $Q \in \mathbb{T}$, which we parametrize as

$$Q = \exp(iq). \quad (2.12)$$

Then the constraint (2.11) becomes

$$e^{-iq} J e^{iq} - J = \xi. \quad (2.13)$$

We assume that q is regular, i.e. e^{iq} belongs to the interior \mathbb{T}^o of a Weyl alcove, which permits us to solve the moment map constraint as follows:

$$J = -ip + \sum_{\alpha \in \Phi^+} (J_{\alpha} E_{\alpha} - J_{\alpha}^* E_{-\alpha}), \quad \xi = \sum_{\alpha \in \Phi^+} (\xi_{\alpha} E_{\alpha} - \xi_{\alpha}^* E_{-\alpha}), \quad (2.14)$$

where $ip \in \mathcal{T}$ is arbitrary and

$$J_{\alpha} = \frac{\xi_{\alpha}}{e^{-i\alpha(q)} - 1}. \quad (2.15)$$

In this way, we obtained a ‘partial gauge fixing’ parametrized by

$$\mathbb{T}^o \times \mathcal{T} \times \mathcal{O}_0 = \{(e^{iq}, ip, \xi)\}. \quad (2.16)$$

We still need to divide this gauge slice by the residual gauge transformations, generated by \mathbb{T} , which act only on \mathcal{O}_0 . This yields the model (2.7) of the reduced phase space, where $T^*\mathbb{T}^o$ is identified with $\mathbb{T}^o \times \mathcal{T}$. The reduced symplectic structure can be displayed as

$$\omega_{\text{red}} = \langle dp \wedge dq \rangle + \omega_{\mathcal{O}}^{\text{red}}. \quad (2.17)$$

Here, $\omega_{\mathcal{O}}^{\text{red}}$ stands for the (stratified) symplectic structure arising from $(\mathcal{O}, \omega_{\mathcal{O}})$, reduced by the \mathbb{T} -action at zero moment map value. That is, $\omega_{\mathcal{O}}^{\text{red}}$ encodes the restriction of the Poisson brackets of the elements of $C^{\infty}(\mathcal{O})^{\mathbb{T}}$ to $\mathcal{O}_0 = \mathcal{O} \cap \mathcal{T}^{\perp}$.

Upon substitution of (2.15), the ‘free’ Hamiltonian

$$\mathcal{H}(g, J, \xi) := -\frac{1}{2} \langle J, J \rangle \quad (2.18)$$

yields the spin Sutherland Hamiltonian $\mathcal{H}_{\text{Suth}}$ given by equation (1.1). The flow generated by \mathcal{H} is called ‘free motion’:

² The inverse is used since g^{-1} is the counterpart of g_R that will appear later.

$$g(t) = g(0) \exp(tJ(0)), \quad J(t) = J(0), \quad \xi(t) = \xi(0), \quad (2.19)$$

and the dynamics governed by $\mathcal{H}_{\text{Suth}}$ results by projecting this to the reduced phase space. The Hamiltonian \mathcal{H} is a member of the Abelian Poisson algebra

$$C_I(M) := J_R^*(C^\infty(\mathcal{G}^*)^G), \quad (2.20)$$

whose functional dimension equals $r = \text{rank}(G)$. The elements of $C_I(M)$ Poisson commute with all elements of the Poisson algebra $C_J(M)$ generated by the components of J_L , J_R and J_O . The functional dimension of the ‘algebra of integrals of motion’ $C_J(M)$ is $\dim(M) - r$, since the functions of J_L and J_R are connected by r independent relations, which express the equality $f \circ J_R = f \circ (-J_L)$ for every $f \in C^\infty(\mathcal{G}^*)^G$. This means [34,36] that the free Hamiltonians $C_I(M)$, with their integrals of motion $C_J(M)$, represent a degenerately integrable (in other words non-commutative integrable or super-integrable) system on M . The various notions of integrability and their relations are reviewed, for example, in [23,56].

All elements of $C_I(M)$ descend to smooth functions on the reduced phase space. Their reduced flows can be found via the projection method, similarly to the case of \mathcal{H} , and all those flows are complete on the full reduced phase space, $M_{\text{red}} = \mu^{-1}(0)/G$. It was shown by Reshetikhin [41–43] that the degenerate integrability of the free Hamiltonian \mathcal{H} (1.1) is inherited at the reduced level with analytic integrals of motion, at least for generic coadjoint orbits and on a dense open subset of M_{red} . Liouville integrability in the same generic case follows from the results of [31]. It would require further work to obtain a full understanding for arbitrary orbits and arbitrary symplectic strata [49,50] of M_{red} . We do not go into this intricate issue, but wish to display a large number of integrals of motion that survive the reduction. Namely, let $\mathcal{P}(J, gJg^{-1})$ be an *arbitrary polynomial* in its non-commutative variables (viewed as elements of the enveloping algebra). Then evaluate the trace of this polynomial in an arbitrary finite dimensional unitary representation ρ of G . It is easy to see that all the functions $\text{tr}_\rho(\mathcal{P}(J, gJg^{-1}))$ Poisson commute with every element of $C_I(M)$ and they are G -invariant with respect to the conjugation action. We suspect that the resulting integrals of motion are sufficient for the integrability of the reduction of $C_I(M)$ in general.

Later we shall derive spin RS type systems, which will be compared to the spin Sutherland systems. Instead of the identification $\mathcal{G}^* \simeq \mathcal{G}$, the comparison will be done using another model of \mathcal{G}^* . This model is defined by realizing any linear functional ϕ on \mathcal{G} in the form $\phi(X) = (\tilde{\xi}, X)$, $\forall X \in \mathcal{G}$, where $\tilde{\xi}$ is from the subalgebra \mathcal{B} of $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$. The two models of \mathcal{G}^* , \mathcal{G} and \mathcal{B} , are in bijection via the equality

$$\phi(X) = \langle \xi, X \rangle = (\tilde{\xi}, X), \quad \xi \in \mathcal{G}, \quad \tilde{\xi} \in \mathcal{B}. \quad (2.21)$$

This implies that

$$\xi = \sum_{k=1}^r i\xi^k T_{\alpha_k} + \sum_{\alpha \in \Phi^+} (\xi_\alpha E_\alpha - \xi_\alpha^* E_{-\alpha}) \quad (2.22)$$

corresponds to

$$\tilde{\xi} = \sum_{k=1}^r \tilde{\xi}^k T_{\alpha_k} + \sum_{\alpha \in \Phi^+} \tilde{\xi}_\alpha E_\alpha \quad \text{with} \quad \tilde{\xi}_\alpha = -2i\xi_\alpha \quad \text{and} \quad \tilde{\xi}^k = -\xi^k. \quad (2.23)$$

Thus, the spin Sutherland Hamiltonian (1.1) can be casted as

$$\mathcal{H}_{\text{Suth}}(e^{iq}, p, \tilde{\xi}) = \frac{1}{2} \langle p, p \rangle + \frac{1}{8} \sum_{\alpha \in \Phi^+} \frac{1}{|\alpha|^2} \frac{|\tilde{\xi}_\alpha|^2}{\sin^2 \frac{\alpha(q)}{2}}. \quad (2.24)$$

3. Unreduced free system on the Heisenberg double

In what follows we freely use basic notions and results from the theory of Poisson-Lie groups, as can be found, e.g., in the reviews [10,27,48]. One may also consult [15], where similar background material as given below is described in more detail.

We start by noting that the Lie algebra $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ and its subalgebras \mathcal{G} and \mathcal{B} form a Manin triple. Consequently, G and B are Poisson-Lie groups in duality. The multiplicative Poisson bracket on $C^\infty(G)$ is given by

$$\{\phi_1, \phi_2\}_G(g) = \left(g^{-1} (d^L \phi_1(g)) g, d^R \phi_2(g) \right), \quad \forall \phi_1, \phi_2 \in C^\infty(G), \forall g \in G, \quad (3.1)$$

and that on $C^\infty(B)$ is given by

$$\{f_1, f_2\}_B(b) = - \left(b^{-1} (d^L f_1(b)) b, d^R f_2(b) \right), \quad \forall f_1, f_2 \in C^\infty(B), \forall b \in B. \quad (3.2)$$

Here, for a real function $\phi \in C^\infty(G)$ the left and right derivatives $d^{L,R} \phi \in C^\infty(G, \mathcal{B})$ are defined by

$$\left. \frac{d}{ds} \right|_{s=0} \phi(e^{sX} g e^{sY}) = \left(X, d^L \phi(g) \right) + \left(Y, d^R \phi(g) \right), \quad \forall X, Y \in \mathcal{G}, \forall g \in G, \quad (3.3)$$

and $d^{L,R} f \in C^\infty(B, \mathcal{G})$ for a real function $f \in C^\infty(B)$ are defined similarly. In the above Poisson bracket formulas conjugation is an informal shorthand for the adjoint action of $G_{\mathbb{R}}^{\mathbb{C}}$ on its Lie algebra.

The manifold $G_{\mathbb{R}}^{\mathbb{C}}$ carries a natural symplectic structure, Ω_+ , which goes back to Semenov-Tian-Shansky [47] and to Alekseev and Malkin [1]. When equipped with Ω_+ , $G_{\mathbb{R}}^{\mathbb{C}}$ is a Poisson-Lie analogue of the cotangent bundle T^*G , alias the ‘Heisenberg double’ of the Poisson-Lie group G . To present Ω_+ , let us recall that every element $K \in G_{\mathbb{R}}^{\mathbb{C}}$ admits the alternative Iwasawa decompositions

$$K = b_L g_R^{-1} = g_L b_R^{-1}, \quad b_L, b_R \in B, g_L, g_R \in G, \quad (3.4)$$

that define diffeomorphisms between $G_{\mathbb{R}}^{\mathbb{C}}$ and $G \times B$. The pair g_R, b_R or the pair g_L, b_L can be also used as free variables in $G \times B$, utilizing the relation

$$g_L^{-1} b_L = b_R^{-1} g_R. \quad (3.5)$$

By making use of these decompositions, we have

$$\Omega_+ = \frac{1}{2} \left(db_L b_L^{-1} \wedge d g_L g_L^{-1} \right) + \frac{1}{2} \left(db_R b_R^{-1} \wedge d g_R g_R^{-1} \right). \quad (3.6)$$

It is useful to introduce the maps Λ_L, Λ_R from $G_{\mathbb{R}}^{\mathbb{C}}$ to B and the maps Ξ_L, Ξ_R from $G_{\mathbb{R}}^{\mathbb{C}}$ to G by setting

$$\Lambda_L(K) := b_L, \quad \Lambda_R(K) := b_R, \quad \Xi_L(K) := g_L, \quad \Xi_R(K) := g_R. \quad (3.7)$$

These are Poisson maps with respect to the Poisson structure associated with Ω_+ and the multiplicative Poisson structures on B and on G , respectively.

The group G acts on B by the (left) dressing action given by

$$\text{Dress}_\eta(b) = \Lambda_L(\eta b), \quad \forall \eta \in G, b \in B, \quad (3.8)$$

which is a Poisson action. The induced infinitesimal action of \mathcal{G} on B reads

$$\text{dress}_X b = b(b^{-1} X b)_B, \quad \forall X \in \mathcal{G}, \quad (3.9)$$

where on the right-hand side we use projection along \mathcal{G} , by means of (2.1). The ring of invariants $C^\infty(B)^G$ forms the center of the Poisson algebra of B . Thus we obtain an algebra of commuting ‘free Hamiltonians’, $C_I(G_{\mathbb{R}}^{\mathbb{C}})$, by the definition

$$C_I(G_{\mathbb{R}}^{\mathbb{C}}) := \Lambda_R^*(C^\infty(B)^G). \quad (3.10)$$

It is worth remarking that $\Lambda_R^*(C^\infty(B)^G) = \Lambda_L^*(C^\infty(B)^G)$. The flow generated by any Hamiltonian $\Lambda_R^*(h) \in C_I$ can be written down explicitly:

$$g_R(t) = \exp \left[t d^L h(b_R(0)) \right] g_R(0), \quad b_L(t) = b_L(0), \quad b_R(t) = b_R(0). \quad (3.11)$$

Notice the similarity with the corresponding flow³ (2.19) on T^*G . These Hamiltonians Poisson commute with all the elements of $\Lambda_L^*(C^\infty(B))$ and $\Lambda_R^*(C^\infty(B))$, which together generate the Poisson algebra of the integrals of motion, denoted as $C_J(G_{\mathbb{R}}^{\mathbb{C}})$. The functional dimension of C_I is the rank r of $\mathcal{G}^{\mathbb{C}}$, while the functional dimension of C_J is $(2N - r)$. The latter statement follows since for any $f \in C^\infty(B)^G$ we have

$$\Lambda_L^*(f) = \Lambda_R^*(f \circ \text{inv}_B), \quad (3.12)$$

where inv_B is the inversion map on the group B . These identities represent r independent relations between $\Lambda_L^*(C^\infty(B))$ and $\Lambda_R^*(C^\infty(B))$, which otherwise give independent functions. Consequently [23,42,56], the Hamiltonians in C_I (3.10) define a degenerate integrable system.

The following model of the Poisson manifold B is often useful. Let

$$\mathfrak{P} := \exp(i\mathcal{G}) \quad (3.13)$$

denote the closed submanifold of $G_{\mathbb{R}}^{\mathbb{C}}$ diffeomorphic to $i\mathcal{G}$ by the exponential map. Note that \mathcal{G} and G are pointwise fixed by corresponding Cartan involutions [26] θ and Θ of $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ and $G_{\mathbb{R}}^{\mathbb{C}}$. Somewhat colloquially, we write

$$X^\dagger := -\theta(X), \quad K^\dagger := \Theta(K^{-1}) \quad \text{for} \quad X \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}, \quad K \in G_{\mathbb{R}}^{\mathbb{C}}, \quad (3.14)$$

since this anti-involution can be arranged to be the usual matrix adjoint for the classical groups. Then the map

$$m : B \rightarrow \mathfrak{P}, \quad B \ni b \mapsto b b^\dagger \in \mathfrak{P} \quad (3.15)$$

is a diffeomorphism, which converts the dressing action of G on B into the conjugation action of G on \mathfrak{P} . That is, we have

$$m \circ \text{Dress}_\eta = C_\eta \circ m \quad \text{where} \quad C_\eta(P) := \eta P \eta^{-1} \quad \forall P \in \mathfrak{P}. \quad (3.16)$$

³ As was noted before, g in (2.19) corresponds to g_R^{-1} . The analogous eq. (2.38) in [15] contains a typo.

It follows that any dressing orbit, \mathcal{O}_B , is diffeomorphic by m to $\mathcal{O}_{\mathfrak{P}} = m(\mathcal{O}_B)$, and one can parametrize it as

$$\mathcal{O}_{\mathfrak{P}} = m(\mathcal{O}_B) = \{\exp(2iX) \mid X \in \mathcal{O}_{\mathcal{G}}\}, \quad (3.17)$$

where $\mathcal{O}_{\mathcal{G}}$ is an adjoint orbit of G . In terms of this exponential parametrization, the form of the Poisson structure on $\mathfrak{P} = m(B)$ is described in [16].

We end this section by recording another useful feature of the Poisson structure on B . For this, let us consider the decompositions

$$B = B_0 + B_+, \quad \mathcal{G} = \mathcal{T} + \mathcal{T}^\perp \quad (3.18)$$

where B_0 (resp. B_+) is spanned by Cartan elements (resp. root vectors). Choose an arbitrary basis $\{X^\alpha\}$ of \mathcal{T} and a basis $\{Y^i\}$ of \mathcal{T}^\perp . Every element $b \in B$ can be uniquely written in the form

$$b = b_0 b_+ = e^{\beta_0} e^{\beta_+} \quad \text{with} \quad \beta_0 \in B_0, \beta_+ \in B_+, \quad (3.19)$$

and the components

$$\beta_0^\alpha := (\beta_0, X^\alpha), \quad \beta_+^i := (\beta_+, Y^i) \quad (3.20)$$

can be taken as coordinate functions on B . The Poisson brackets of these functions satisfy

$$\{\beta_0^\alpha, \beta_0^\gamma\}_B = 0, \quad \{\beta_+^k, \beta_0^\gamma\}_B = ([Y^k, X^\gamma], \beta_+) \quad (3.21)$$

and

$$\{\beta_+^i, \beta_+^j\}_B = ([Y^i, Y^j], \beta_0 + \beta_+) + o(\beta_+, \beta_0). \quad (3.22)$$

The Poisson brackets $\{\beta_+^i, \beta_+^j\}_B$ are polynomials in β_+ and trigonometric polynomials in β_0 . Equation (3.22), where $o(\beta_+, \beta_0)$ stands for terms whose combined degree in the components of β_+ and β_0 is at least 2, shows that the linear part of the Poisson brackets of the variables $\beta_0^\alpha, \beta_+^i$ is the Lie-Poisson bracket of \mathcal{G} .

4. Reduction along an arbitrary dressing orbit

We recall that the dressing orbits \mathcal{O}_B are the symplectic leaves in B , and let $\Omega_{\mathcal{O}_B}$ stand for the symplectic form on \mathcal{O}_B . Before defining the reduction, we extend the phase space $G_{\mathbb{R}}^{\mathbb{C}}$ by a non-trivial dressing orbit, i.e., we consider the unreduced phase space

$$\mathcal{M} := G_{\mathbb{R}}^{\mathbb{C}} \times \mathcal{O}_B = \{(K, S) \mid K \in G_{\mathbb{R}}^{\mathbb{C}}, S \in \mathcal{O}_B\} \quad (4.1)$$

equipped with the symplectic form

$$\Omega = \Omega_+ + \Omega_{\mathcal{O}_B}. \quad (4.2)$$

The Abelian Poisson algebra (3.10) is trivially extended to yield $C_I(\mathcal{M})$, whose elements do not depend on $S \in \mathcal{O}_B$, and the algebra of the integrals of motion $C_J(G_{\mathbb{R}}^{\mathbb{C}})$ is extended to

$$C_J(\mathcal{M}) = (\Lambda_L, \Lambda_R, \Lambda_{\mathcal{O}_B})^* (C^\infty(B \times B \times \mathcal{O}_B)), \quad (4.3)$$

where $\Lambda_{\mathcal{O}_B}$ is the obvious projection from \mathcal{M} to $\mathcal{O}_B \subset B$, and Λ_L, Λ_R (3.7) are regarded as maps from \mathcal{M} to B . That is, $C_J(\mathcal{M})$ contains all functions of b_L, b_R (3.4) and S . This extension maintains the degenerate integrability.

We shall study Marsden–Weinstein type reduction [33] at the unit value $e \in B$ of a suitable Poisson-Lie moment map $\Lambda : \mathcal{M} \rightarrow B$. Concretely, we introduce the map Λ by taking the product

$$\Lambda = \Lambda_L \Lambda_R \Lambda_{\mathcal{O}_B}, \quad (4.4)$$

i.e.,

$$\Lambda(K, S) = \Lambda_L(K) \Lambda_R(K) S. \quad (4.5)$$

Clearly, the product is a proper generalization of the sum in (2.6). This definition gives a Poisson map because the 3 factors of Λ are Poisson maps into B and they pairwise Poisson commute. We know from general theory [33] that the Poisson map Λ generates an infinitesimal left-action of \mathcal{G} on \mathcal{M} . Namely, the vector field $X_{\mathcal{M}}$ on \mathcal{M} corresponding to $X \in \mathcal{G}$ operates on $f \in C^\infty(\mathcal{M})$ by the following formula:

$$df(X_{\mathcal{M}}) = (X, \{f, \Lambda\}_{\mathcal{M}} \Lambda^{-1}), \quad (4.6)$$

where $\{, \}_{\mathcal{M}}$ is the Poisson bracket on functions on \mathcal{M} , and notationwise we pretend that B is a matrix Lie group. The \mathcal{G} -action (4.6) integrates to a global Poisson-Lie action of G on \mathcal{M} , denoted below $\Psi : G \times \mathcal{M} \rightarrow \mathcal{M}$.

Lemma 4.1. *The action of $\eta \in G$ on \mathcal{M} is given by the following diffeomorphism Ψ_η ,*

$$\Psi_\eta(K, S) = (\eta K \Xi_R(\eta b_L), \text{Dress}_{\Xi_R(\eta b_L b_R)^{-1}}(S)), \quad (4.7)$$

where we use the notations introduced in (3.4), (3.7) and (3.8). The map

$$\Psi : G \times \mathcal{M} \rightarrow \mathcal{M}, \quad \Psi(\eta, K, S) = \Psi_\eta(K, S) \quad (4.8)$$

is Poisson, and the moment map Λ is equivariant: $\Lambda \circ \Psi_\eta = \text{Dress}_\eta \circ \Lambda$.

Proof. One can verify that this formula defines a group action, and the induced infinitesimal action reproduces the derivations given by the moment map according to (4.6). \square

Remark 4.2. One can check that Λ_L and $\Lambda_L \Lambda_R$ are also equivariant in the sense that

$$\Lambda_L \circ \Psi_\eta = \text{Dress}_\eta \circ \Lambda_L, \quad (\Lambda_L \Lambda_R) \circ \Psi_\eta = \text{Dress}_\eta \circ (\Lambda_L \Lambda_R). \quad (4.9)$$

It follows that all elements of $\Lambda_L^*(C^\infty(B)^G) = \Lambda_R^*(C^\infty(B)^G) := C_I(\mathcal{M})$ are invariant with respect to Ψ_η . Without the extension of the Heisenberg double by the dressing orbit, the action (4.7) was introduced in [25], where it was called ‘quasi-adjoint action’.

Now, we are interested in the reduced phase space

$$\mathcal{M}_{\text{red}} := \Lambda^{-1}(e)/G. \quad (4.10)$$

For certain orbits \mathcal{O}_B this is a smooth symplectic manifold. In general, it is a union of smooth symplectic manifolds of various dimension, a so-called stratified symplectic space [49,50]. Its structure turns out to be quite similar to what occurs in the cotangent bundle case. In particular, a reduction of the orbit \mathcal{O}_B itself will come to fore shortly in our description.

The maximal torus $\mathbb{T} < G$ is a Poisson-Lie subgroup of G , on which the Poisson structure vanishes. Hence the dressing Poisson action of G on \mathcal{O}_B restricts to an ordinary Hamiltonian action of \mathbb{T} . This action operates simply by conjugation. Writing $S \in \mathcal{O}_B$ in the form

$$S = S_0 S_+ \quad \text{with} \quad S_0 \in B_0, S_+ \in B_+, \quad (4.11)$$

the map

$$S \mapsto \log(S_0) \in \mathcal{B}_0 \quad (4.12)$$

is the moment map for the action of \mathbb{T} on \mathcal{O}_B , as follows, for example, from (3.21). Here, \mathcal{B}_0 plays the role of the dual space of \mathcal{T} , via the bilinear form (2.4). By setting this moment map to zero, i.e. setting S_0 equal to the unit element, we obtain the reduced dressing orbit

$$\mathcal{O}_B^{\text{red}} = \{S_+ \in \mathcal{O}_B\} / \mathbb{T}, \quad (4.13)$$

which itself is a stratified symplectic space.

Let $G^{\text{reg}} \subset G$ be the set of regular elements. The space of the conjugacy classes in G^{reg} is a smooth manifold, which can be identified with an open Weyl alcove \mathbb{T}^o , i.e., a connected component of \mathbb{T}^{reg} . In this paper we focus on the reduction of the dense open submanifold of \mathcal{M} given by

$$\mathcal{M}^{\text{reg}} = \Xi_R^{-1}(G^{\text{reg}}), \quad (4.14)$$

that is, we shall assume that in $K = b_L g_R^{-1}$ we have $g_R \in G^{\text{reg}}$. We denote

$$\mathcal{M}_{\text{red}}^{\text{reg}} = \{(K, S) \mid \Lambda(K, S) = e, g_R \in G^{\text{reg}}\} / G. \quad (4.15)$$

Now we state one of the main results of the paper.

Theorem 4.3. *The open dense subset $\mathcal{M}_{\text{red}}^{\text{reg}}$ of the reduced phase space can be identified with*

$$T^* \mathbb{T}^o \times \mathcal{O}_B^{\text{red}}, \quad (4.16)$$

where \mathbb{T}^o is an open Weyl alcove in \mathbb{T} and $\mathcal{O}_B^{\text{red}}$ is the reduced dressing orbit (4.13). The reduced symplectic structure reads

$$\Omega_{\text{red}} = \Omega_{T^* \mathbb{T}^o} + \Omega_{\mathcal{O}_B^{\text{red}}}, \quad (4.17)$$

where the first term is the canonical symplectic form of the cotangent bundle $T^* \mathbb{T}^o$, and the second term refers to the reduced orbit (4.13).

Proof. We wish to parametrize the G -orbits in the regular part of the constraint surface:

$$\Lambda^{-1}(e) \cap \Xi_R^{-1}(G^{\text{reg}}). \quad (4.18)$$

On account of (4.7), the action of $\eta \in G$ works on K according to

$$K = b_L g_R^{-1} \mapsto \Lambda_L(\eta b_L) (\Xi_R(\eta b_L)^{-1} g_R \Xi_R(\eta b_L))^{-1}. \quad (4.19)$$

Since for any $b_L \in B$, the map $\eta \mapsto \Xi_R(\eta b_L)$ is a diffeomorphism on G , we can transform g_R into the maximal torus. More precisely, since we assumed regularity, we see that every gauge orbit has representatives in the set

$$\mathcal{Z} := \{(K, S) \mid \Lambda(K, S) = e, \Xi_R(K) \in \mathbb{T}^o\}. \quad (4.20)$$

In other words, the manifold \mathcal{Z} is the gauge slice of a partial gauge fixing. Now we employ the decomposition

$$b_R = b_0 b_+ \quad \text{with} \quad b_0 \in B_0, b_+ \in B_+, \quad (4.21)$$

and introduce the notation $Q := \Xi_R(K)$. Then the equality $K = b_L g_R^{-1} = g_L b_R^{-1}$ tells us that

$$g_L = g_R^{-1} = Q^{-1} \quad \text{and} \quad b_L = Q^{-1} b_R^{-1} Q. \quad (4.22)$$

Because of the last relation, we can write

$$b_L b_R = Q^{-1} b_R^{-1} Q b_R = Q^{-1} b_+^{-1} b_0^{-1} Q b_0 b_+ = Q^{-1} b_+^{-1} Q b_+. \quad (4.23)$$

Thus, the restriction of the moment map to \mathcal{Z} can be expressed as

$$\Lambda(K, S) = b_L b_R S = Q^{-1} b_+^{-1} Q b_+ S. \quad (4.24)$$

This has the following crucial consequences. First, the B_0 -factor b_0 of b_R is not constrained. Second, we must have $S \in B_+$, i.e., $S = S_+ \in \mathcal{O}_B \cap B_+$. Third, the moment map constraint

$$Q^{-1} b_+^{-1} Q b_+ S_+ = e \quad (4.25)$$

determines b_+ as a function of Q and S_+ . To summarize, we obtain a diffeomorphism

$$\mathcal{Z} \simeq (\mathbb{T}^o \times B_0) \times (\mathcal{O}_B \cap B_+) = \{(Q, b_0, S_+)\} \quad (4.26)$$

by the parametrization

$$K = Q^{-1} b_+^{-1} b_0^{-1}, \quad S = S_+ \quad \text{with} \quad b_+ = b_+(Q, S_+) \quad (4.27)$$

determined by the constraint equation (4.25). We stress that, for any given $Q \in \mathbb{T}^o$ and $S_+ \in \mathcal{O}_B \cap B_+$, equation (4.25) admits a unique solution for b_+ . (See also Section 5.)

Two elements of \mathcal{Z} are gauge equivalent if they are carried into each other by the action of some $\eta \in G$. It follows from the transformation rule of g_R ,

$$g_R \mapsto \Xi_R(\eta b_L)^{-1} g_R \Xi_R(\eta b_L), \quad (4.28)$$

that the ‘residual gauge transformations’ that map elements of \mathcal{Z} to \mathcal{Z} are given by the action of the subgroup $\mathbb{T} < G$. The factors Q and b_0 are invariant under this action, while S_+ and the corresponding b_+ transform according to

$$S_+ \mapsto T S_+ T^{-1}, \quad b_+ \mapsto T b_+ T^{-1}, \quad \forall T \in \mathbb{T}. \quad (4.29)$$

Therefore, recalling that Q and b_0 can be arbitrary, we obtain the identification

$$\mathcal{M}_{\text{red}}^{\text{reg}} \equiv \mathcal{Z}/\mathbb{T} \equiv (\mathbb{T}^o \times B_0) \times \mathcal{O}_B^{\text{red}}. \quad (4.30)$$

By general principles, the reduced (stratified) symplectic structure on $\mathcal{M}_{\text{red}}^{\text{reg}}$ arises from the pull-back of the symplectic form of \mathcal{M} to the submanifold \mathcal{Z} of $\Lambda^{-1}(e)$. Let $\iota_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathcal{M}$ and $\iota_{\mathcal{O}} : (\mathcal{O}_B \cap B_+) \rightarrow \mathcal{O}_B$ denote the tautological injections, and introduce the parametrizations

$$Q = \exp(iq), \quad b_0 = \exp(p), \quad (4.31)$$

where p varies freely in \mathcal{B}_0 . By using these, we find from (4.2) and (3.6) that

$$\iota_{\mathcal{Z}}^*(\Omega) = \langle dp \wedge dq \rangle + \iota_{\mathcal{O}}^*(\Omega_{\mathcal{O}_B}). \quad (4.32)$$

The second term descends to the (stratified) symplectic structure of the reduced dressing orbit (4.13). Together with the identification (4.30), this completes the proof. \square

Remark 4.4. The ring of smooth functions on $\mathcal{O}_B^{\text{red}}$ (4.13) can be identified with the invariants $C^\infty(\mathcal{O}_B \cap B_+)^{\mathbb{T}}$. Such invariants can be constructed as follows. Let us write $S_+ \in \mathcal{O}_B \cap B_+$ in the form

$$S_+ = \exp\left(\sum_{\alpha \in \Phi_+} \sigma_\alpha e_\alpha\right), \quad (4.33)$$

where the σ_α are complex coordinate functions. Consider arbitrary positive roots $\varphi_1, \dots, \varphi_{n_1}$ and $\psi_1, \dots, \psi_{n_2}$ for which

$$\sum_{i=1}^{n_1} \varphi_i = \sum_{j=1}^{n_2} \psi_j. \quad (4.34)$$

Then the following polynomial function is \mathbb{T} -invariant:

$$\prod_{i=1}^{n_1} \sigma_{\varphi_i} \prod_{j=1}^{n_2} \sigma_{\psi_j}^*. \quad (4.35)$$

Here any repetition of the roots is allowed. The real and imaginary parts of these complex polynomials can be regarded as \mathbb{T} -invariant functions on the whole of \mathcal{O}_B , by declaring that they do not depend on S_0 for $S = S_0 S_+ \in \mathcal{O}_B$. If we evaluate their Poisson brackets according to (3.22) and restrict the result to $\mathcal{O}_B \cap B_+$, then we obtain invariant polynomials in the same variables σ_α . In principle, this algorithm leads to the Poisson algebra of smooth functions carried by the reduced dressing orbit. The reduced Poisson bracket closes on the polynomials given by linear combinations of the invariants of the form (4.35).

Remark 4.5. For completeness, it may be worth explaining that the reduced coadjoint orbits $\mathcal{O}_{\text{red}} = \mathcal{O}_0/\mathbb{T}$ and dressing orbits $\mathcal{O}_B^{\text{red}}$ (4.13) are always non-empty. For a coadjoint orbit $\mathcal{O} \subset \mathcal{G}^* = \mathcal{G}$, let us first note that $\mathcal{O} \cap \mathcal{T}$ is an orbit of the Weyl group of the pair (G, \mathbb{T}) . Referring to the famous convexity theorems of Kostant, Atiyah and Guillemin and Sternberg, one knows that the image of the moment map for the \mathbb{T} -action on \mathcal{O} is the convex hull of this Weyl orbit. Now, let $x_i \in \mathcal{T}$, $i = 1, \dots, N$, denote the elements of the Weyl orbit, and form the convex combination $x := \frac{1}{N} \sum_{i=1}^N x_i$. It is clear that x is a fixed point for the action of the Weyl group. But the origin is the unique fixed point, since a fixed point is characterized by the property that it is perpendicular to all the roots that define the Weyl reflections. Thus $x = 0$ is in the image of the moment map, i.e., \mathcal{O}_0 is non-empty.

Essentially the same argument can be applied in the case of the dressing orbits, too.

Remark 4.6. Recall that \mathcal{G}^* equipped with the linear Lie-Poisson bracket and B equipped with the multiplicative Poisson bracket (3.2) are Poisson diffeomorphic [21]. The existence of a \mathbb{T} -equivariant Poisson diffeomorphism implies that every reduced dressing orbit (4.13) is symplectomorphic to a reduced coadjoint orbit \mathcal{O}_0/\mathbb{T} . Such a Ginzburg-Weinstein diffeomorphism has been exhibited in [2] for $G = \text{SU}(n)$. If a \mathbb{T} -equivariant Ginzburg-Weinstein diffeomorphism exists in general, which is believed to be the case, then the phase space $M_{\text{red}}^{\text{reg}}$ in (2.7) is always symplectomorphic to the corresponding phase space $\mathcal{M}_{\text{red}}^{\text{reg}}$ in (4.16).

5. Connection with the spin Sutherland model

We need to recall some group theoretic facts. Let $\rho : G^{\mathbb{C}} \rightarrow \text{GL}(V)$ be a finite dimensional irreducible representation. Then the complex vector space V can be equipped with a Hermitian inner product in such a way that $\rho(K^{\dagger}) = \rho(K)^{\dagger}$ holds $\forall K \in G^{\mathbb{C}}$, that is, the compact subgroup G and \mathfrak{P} (3.13) are represented by unitary and by positive operators, respectively. The character $\chi_{\rho}(K) = \text{tr}(\rho(K))$ restricts to a G -invariant function on \mathfrak{P} , and $\mathbb{C}^{\infty}(\mathfrak{P})^G$ is functionally generated by the characters of the r fundamental highest weight representations.

We shall inspect the so-called *main reduced Hamiltonians*, which descend from the characters. More precisely, we reduce the G -invariant functions $H^{\rho} \in C^{\infty}(\mathcal{M})^G$ of the form

$$H^{\rho}(K, S) := \text{tr}_{\rho}(b_R b_R^{\dagger}) := c_{\rho} \text{tr}(\rho(b_R b_R^{\dagger})). \quad (5.1)$$

Here, $K = g_L b_R^{-1}$ as in (3.4) and c_{ρ} is a normalization constant, chosen so that

$$c_{\rho} \text{tr}(\rho(E_{\alpha})\rho(E_{-\alpha})) = 2/|\alpha|^2. \quad (5.2)$$

The associated representation of $\mathcal{G}^{\mathbb{C}}$ is also denoted by ρ , and below we shall write simply $\text{tr}_{\rho}(XY)$ instead of $c_{\rho} \text{tr}(\rho(X)\rho(Y))$.

We shall demonstrate that, upon evaluation in the diagonal gauge \mathcal{Z} (4.26), H_{red}^{ρ} can be expanded in such a manner that its leading term has the same form as the spin Sutherland Hamiltonian (1.1). Then we shall point out the relationship between the Lax matrix engendered by $b_R b_R^{\dagger}$ and the Lax matrix of the spin Sutherland model. In Remark 5.1, we elucidate the interpretation of these statements in terms of a one-parameter deformation. In Section 5.2, we derive explicit formulas for $G^{\mathbb{C}} = \text{SL}(n, \mathbb{C})$, using its defining representation.

5.1. Reduced main Hamiltonians and Lax matrices

Let us inspect the constraint equation (4.25) by parametrizing the variables as

$$S_{+} = e^{\sigma}, \quad b_{+} = e^{\beta}, \quad \sigma = \sum_{\alpha > 0} \sigma_{\alpha} E_{\alpha}, \quad \beta = \sum_{\alpha > 0} \beta_{\alpha} E_{\alpha} \quad (5.3)$$

using complex expansion coefficients σ_{α} , β_{α} , and $Q = e^{iq}$. The Baker-Campbell-Hausdorff formula permits us to rewrite the constraint equation as

$$\exp(\beta - Q^{-1} \beta Q - \frac{1}{2}[Q^{-1} \beta Q, \beta] + \dots) = \exp(-\sigma), \quad (5.4)$$

where the dots indicate higher commutators. Note that the BCH series is now finite, since \mathcal{B}_{+} is nilpotent. Using that B_{+} is diffeomorphic to its Lie algebra by the exponential map, we see from

$$\beta - Q^{-1} \beta Q - \frac{1}{2}[Q^{-1} \beta Q, \beta] + \dots = -\sigma \quad (5.5)$$

that β_{α} can be expressed in terms of σ and e^{iq} in the following form:

$$\beta_{\alpha} = \frac{\sigma_{\alpha}}{e^{-i\alpha(q)} - 1} + \Gamma_{\alpha}(e^{iq}, \sigma), \quad (5.6)$$

where Γ_{α} contains higher order terms in the components of σ . Namely, we have

$$\Gamma_{\alpha} = \sum_{k \geq 2} \sum_{\varphi_1, \dots, \varphi_k} f_{\varphi_1, \dots, \varphi_k}(e^{iq}) \sigma_{\varphi_1} \dots \sigma_{\varphi_k}, \quad (5.7)$$

where the $\varphi_1, \dots, \varphi_k$ are unordered collections of positive roots, such that $\alpha = \varphi_1 + \dots + \varphi_k$. The $f_{\varphi_1, \dots, \varphi_k}$ are rational functions in e^{iq} , and the sum contains only finitely many terms. These claims are established by inspection of equation (5.5), proceeding iteratively according to the height of the root α labeling σ_α on the right-hand side. It follows that the restriction of H^ρ (5.1) to $\mathcal{Z} \subset \Lambda^{-1}(e)$ (4.26) can be expanded as a polynomial in the components of the ‘spin variable’ σ , with coefficients given by rational functions in the components of e^{iq} multiplied by suitable components of $b_0^2 \equiv \exp(2p)$. Specifically, we see that

$$H_{\text{red}}^\rho(e^{iq}, p, \sigma) = \text{tr}_\rho(e^{2p} b_+ b_+^\dagger) \quad (5.8)$$

can be expanded as

$$H_{\text{red}}^\rho(e^{iq}, p, \sigma) = \text{tr}_\rho \left(e^{2p} \left(\mathbf{1}_\rho + \frac{1}{4} \sum_{\alpha > 0} \frac{|\sigma_\alpha|^2 E_\alpha E_{-\alpha}}{\sin^2(\alpha(q)/2)} + o_2(\sigma, \sigma^*) \right) \right). \quad (5.9)$$

To obtain this from (5.8), we took into account that $E_\alpha^\dagger = E_{-\alpha}$ and that, in consequence of the invariance of tr_ρ with respect to conjugation by the elements of \mathbb{T} , $\text{tr}_\rho(e^{2p} E_\alpha E_{-\gamma}) = 0$ unless $\gamma = \alpha$. Applying (5.2) and expanding e^{2p} , equation (5.9) implies

$$H_{\text{red}}^\rho(e^{iq}, p, \sigma) = \dim_\rho + 2\text{tr}_\rho(p^2) + \frac{1}{2} \sum_{\alpha > 0} \frac{1}{|\alpha|^2} \frac{|\sigma_\alpha|^2}{\sin^2(\alpha(q)/2)} + o_2(\sigma, \sigma^*, p). \quad (5.10)$$

The symbols o_2 indicate terms that are at least cubic in the components of the displayed arguments, and depend rationally on e^{iq} . We used that $\text{tr}_\rho(p) = 0$, because $\mathcal{G}^{\mathbb{C}}$ is a simple Lie algebra. According to (5.10), the leading term of $\frac{1}{4}(H_{\text{red}}^\rho - \dim_\rho)$ matches precisely the spin Sutherland Hamiltonian (2.24).

We have seen that, when expanded in the components p and the spin variable σ , the leading term of the reduced main Hamiltonian (5.10) has the same form as the spin Sutherland Hamiltonian (2.24). It should be noted that the Poisson brackets of the corresponding spin variables, $\tilde{\xi}_\alpha$ and σ_α , are different in the two cases. The relationship between the spin algebra that arises in our Poisson-Lie case and the one that occurs for the spin Sutherland model is given by equation (3.22). It follows that we recover the spin Sutherland model (1.1) from our reduced system if we keep only the leading term of the reduced Hamiltonian H_{red}^ρ and only the leading terms of the Poisson brackets of the spin variables.

All reduced Hamiltonians arising from $C_I(\mathcal{M}) = \Lambda_R^*(C^\infty(B)^G)$ can be expressed as G -invariant functions of the ‘Lax matrix’ $L(e^{iq}, p, \sigma) := b_R b_R^\dagger$, regarded as a function on \mathcal{Z} (4.26). Working in some representation and keeping only linear terms in p and σ , this Lax matrix can be expanded as follows:

$$L(e^{iq}, p, \sigma) = \mathbf{1} + 2p + \sum_{\alpha > 0} \left(\frac{\sigma_\alpha}{e^{-i\alpha(q)} - 1} E_\alpha + \frac{\sigma_\alpha^*}{e^{i\alpha(q)} - 1} E_{-\alpha} \right) + o(\sigma, \sigma^*, p). \quad (5.11)$$

If we replace σ_α by $\tilde{\xi}_\alpha$ and substitute the change of variables (2.23), then (up to irrelevant constants) the linear part of L in σ, σ^* and p reproduces $L_{\text{Suth}} := iJ$ as given by equations (2.14) and (2.15), which can be regarded as a Lax matrix for the spin Sutherland model:

$$L_{\text{Suth}}(e^{iq}, p, \xi) = p + \sum_{\alpha > 0} \left(\frac{i\tilde{\xi}_\alpha}{e^{-i\alpha(q)} - 1} E_\alpha + \frac{(i\tilde{\xi}_\alpha)^*}{e^{i\alpha(q)} - 1} E_{-\alpha} \right). \quad (5.12)$$

It is clear from this that the G -invariant polynomials of L_{Suth} can be recovered as leading terms of suitable invariant polynomials of $(L - \mathbf{1})$.

In fact, taking $G = \text{SU}(n)$ and a minimal dressing orbit our construction reproduces the standard trigonometric RS system [14]. In this case, there are no spin variables since the reduced dressing orbit consists of a single point. The characters of the fundamental representations give rise to $(n - 1)$ independent Hamiltonians, which become the standard RS Hamiltonians [44,45] after a certain canonical transformation. In the general case, the Hamiltonians (5.9) resemble the light-cone Hamiltonians of the RS system. The presence of the leading exponent e^{2p} in (5.9) entails that the reduced main Hamiltonians originating from the r fundamental characters are *functionally independent in general*.

Remark 5.1. The foregoing results justify calling our models generalizations of the spin Sutherland models, but we also would like to view them as *deformations*. However, so far we have not introduced any deformation parameter. In fact, we can not introduce a deformation parameter if we take the spin variables from a fixed reduced dressing orbit. Now we explain that this obstacle can be overcome by placing all our reduced phase spaces (4.16) inside the single Poisson space $T^*\mathbb{T}^o \times (B_+/\mathbb{T})$, where B_+/\mathbb{T} is the Poisson reduction of $(B, \{ , \}_B)$ at the zero value of the moment map for the Hamiltonian \mathbb{T} -action. We can then consider a real, positive scale parameter ϵ and replace the variables $(e^{iq}, p, e^\sigma) \in T^*\mathbb{T}^o \times B_+$ by $(e^{iq}, \epsilon p, e^{\epsilon\sigma})$. At the same time, we use the scaled Poisson structure $\{ , \}^\epsilon = \epsilon \{ , \}$, so that p and q remain canonical conjugates with respect to $\{ , \}^\epsilon$. In this way we obtain the relation

$$\lim_{\epsilon \rightarrow 0} \frac{1}{4\epsilon^2} \left(H_{\text{red}}^\rho(e^{iq}, \epsilon p, \epsilon\sigma) - \dim_\rho \right) = \frac{1}{2} \text{tr}_\rho(p^2) + \frac{1}{8} \sum_{\alpha > 0} \frac{1}{|\alpha|^2} \frac{|\sigma_\alpha|^2}{\sin^2(\alpha(q)/2)}. \quad (5.13)$$

Moreover, it follows from (3.22) that in the $\epsilon = 0$ limit the \mathbb{T} -invariant polynomials of $\sigma \in B_+$ satisfy the Poisson brackets corresponding to the reduction of $\mathcal{G}^* \simeq \mathcal{B}$ at the zero value of the moment map for the respective \mathbb{T} -action. The expansion (5.11) of L gives

$$L(e^{iq}, \epsilon p, \epsilon\sigma) = \mathbf{1} + 2\epsilon L_{\text{Suth}}(e^{iq}, p, \xi) + o(\epsilon), \quad \text{with } 2i\xi_\alpha = \sigma_\alpha, \quad (5.14)$$

and thus we can recover all invariant polynomials of L_{Suth} via suitable scaling limits of invariants of $L(e^{iq}, \epsilon p, \epsilon\sigma)$. For example, if we consider $G^\mathbb{C} = \text{SL}(n, \mathbb{C})$ then we obtain $(n - 1)$ independent Hamiltonians for the spin Sutherland model from

$$\text{tr}(L_{\text{Suth}}(e^{iq}, p, \xi)^k) = \lim_{\epsilon \rightarrow 0} \frac{1}{(2\epsilon)^k} \text{tr} \left((L(e^{iq}, \epsilon p, \epsilon\sigma) - \mathbf{1}_n)^k \right), \quad k = 2, \dots, n. \quad (5.15)$$

The limiting procedure just outlined is similar to the way in which the spinless Sutherland model is obtained as the non-relativistic limit of the standard RS model [44,45]. In that context, the parameter ϵ^{-1} plays the role of the ‘velocity of light’.

Remark 5.2. One-parameter families of dressing orbits of a given type are represented by the G -orbits in \mathfrak{P} that are of the form $\exp(2is\mathcal{O}_\mathcal{G})$ for a fixed coadjoint orbit in $\mathcal{G}^* \equiv \mathcal{G}$ and any non-zero real parameter s . It is tempting to speculate that s can also give rise to a deformation parameter, which should connect the spin RS models associated with these dressing orbits to the spin Sutherland model based on $\mathcal{O}_\mathcal{G}$. This appears an intricate issue because of the complicated mapping from the exponential parametrization $SS^\dagger = \exp(2isX)$ with $X \in \mathcal{O}_\mathcal{G}$ (cf. equation (3.17)) to the parametrization $S = S_0 e^\sigma$ that we found convenient to use. We plan to return to this problem elsewhere.

5.2. Explicit formulas for $G^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$

In this case, it is convenient to parametrize $b \in B$ by using matrix elements. After diagonalizing g_R , we are going to solve the moment map constraint for

$$b_R = e^p b, \quad (5.16)$$

where b is an unknown upper triangular matrix with unit diagonal. The constraint equation is given by

$$Q^{-1} b Q = b S, \quad (5.17)$$

where $Q = \mathrm{diag}(Q_1, \dots, Q_n) \in \mathbb{T}^o$ and $S \in \mathcal{O}_B \cap B_+$ is the constrained ‘spin’ variable. Except for the notation, this is the same as equation (4.25). Applying the principal gradation of $n \times n$ matrices, which is additive under matrix multiplication, we expand b and S according to

$$b = \mathbf{1}_n + b(1) + \dots + b(n-1), \quad S = \mathbf{1}_n + S(1) + \dots + S(n-1). \quad (5.18)$$

For example, $S(j)$ is a linear combination of the $n \times n$ elementary matrices of the form $E_{i,i+j}$. We can spell out the constraint equation as

$$Q^{-1} b(k) Q - b(k) = S(k) + \sum_{i+j=k} b(i) S(j), \quad k = 1, \dots, n-1, \quad (5.19)$$

where $1 \leq i, j \leq n-1$. This can be solved iteratively, proceeding upwards from $k = 1$ until $k = n-1$. To write down the solution, we introduce the shorthand notation

$$\mathcal{I}_{a,a+j} = \frac{1}{Q_{a+j} Q_a^{-1} - 1}, \quad (5.20)$$

which is well defined since Q is regular. It is not difficult to ascertain that the solution for the matrix b is as follows. For the grade 1 matrix entries, we have

$$b_{a,a+1} = \mathcal{I}_{a,a+1} S_{a,a+1}. \quad (5.21)$$

For higher grades, the matrix elements $b_{a,a+k}$, $k = 2, \dots, n-a$, read as follows:

$$b_{a,a+k} = \mathcal{I}_{a,a+k} S_{a,a+k} + \sum_{\substack{m=2,\dots,k \\ (i_1,\dots,i_m) \in \mathbb{N}^m \\ i_1+\dots+i_m=k}} \prod_{\alpha=1}^m \mathcal{I}_{a,a+i_1+\dots+i_{\alpha}} S_{a+i_1+\dots+i_{\alpha-1},a+i_1+\dots+i_{\alpha}}. \quad (5.22)$$

It is understood that $i_0 = 0$ and the S -factor for $\alpha = 1$ is $S_{a,a+i_1}$. We omit the inductive verification of this formula, which can be done using that the form of $b_{a,a+k}$, for $a+k \leq n$, does not depend on n .

By substituting (5.16) and (5.22) into $\mathrm{tr}(b_R b_R^{\dagger})$ we obtain the expansion

$$H_{\mathrm{red}}(e^{iq}, p, S) = \mathrm{tr}(b_R b_R^{\dagger}) = \sum_{a=1}^n e^{2p_a} + \sum_{a=1}^{n-1} e^{2p_a} \sum_{k=1}^{n-a} |\mathcal{I}_{a,a+k} S_{a,a+k}|^2 + \text{higher order terms}, \quad (5.23)$$

where the higher order terms are at-least-cubic polynomials in the matrix elements of S and their complex conjugates. Since $Q_a = e^{iq_a}$, we have

$$|\mathcal{I}_{a,a+k}|^2 = \frac{1}{4 \sin^2((q_{a+k} - q_a)/2)}, \quad (5.24)$$

and therefore

$$H_{\text{red}}(e^{iq}, p, S) = \sum_{a=1}^n e^{2p_a} + \frac{1}{4} \sum_{a=1}^{n-1} e^{2p_a} \sum_{k=1}^{n-a} \frac{|S_{a,a+k}|^2}{\sin^2((q_{a+k} - q_a)/2)} + \text{higher order terms.} \quad (5.25)$$

This generalizes the corresponding spin Sutherland Hamiltonian, since

$$H_{\text{red}}(e^{iq}, p, S) - n = 2 \sum_{a=1}^n p_a^2 + \frac{1}{4} \sum_{a=1}^{n-1} \sum_{k=1}^{n-a} \frac{|S_{a,a+k}|^2}{\sin^2((q_{a+k} - q_a)/2)} + \dots \quad (5.26)$$

with the dots standing for terms whose total degree in p , S and S^* is at least 3.

Since by (5.22) we have solved the constraints, we could write a fully explicit formula for the reduced main Hamiltonian (5.23), represented as a \mathbb{T} -invariant function on \mathcal{Z} (4.26). We present it as an illustration in the simplest $n = 3$ case, for which we have

$$b_{1,2} = \mathcal{I}_{1,2} S_{1,2}, \quad b_{2,3} = \mathcal{I}_{2,3} S_{2,3}, \quad b_{1,3} = \mathcal{I}_{1,3} (S_{1,3} + \mathcal{I}_{1,2} S_{1,2} S_{2,3}) \quad (5.27)$$

and, with $p_1 + p_2 + p_3 = 0$,

$$H_{\text{red}} = (e^{2p_1} + e^{2p_2} + e^{2p_3}) + e^{2p_1} \left(|\mathcal{I}_{1,2} S_{1,2}|^2 + |\mathcal{I}_{1,3} S_{1,3}|^2 \right) + e^{2p_2} |\mathcal{I}_{2,3} S_{2,3}|^2 \\ + 2e^{2p_1} |\mathcal{I}_{1,3}|^2 \text{Re}(\mathcal{I}_{1,2} S_{1,2} S_{2,3} S_{1,3}^*). \quad (5.28)$$

In principle, the structure of the reduced Poisson algebra can be made explicit too, using that the Poisson structure of B can be described in terms of the matrix elements of the freely varying $b \in B$. As one can check, and is well-known, this structure is given by

$$\{b_{m,j}, b_{k,l}\}_B = i b_{k,j} b_{m,l} [\delta_{m,k} + 2\theta(m-k) - \delta_{l,j} - 2\theta(l-j)], \quad (5.29)$$

where θ is the step function (+1 for positive arguments, and 0 for non-positive ones), and

$$\{b_{m,j}, b_{k,l}^*\}_B = i b_{m,j} b_{k,l}^* [\delta_{m,k} - \delta_{j,l}] + 2i \left[\delta_{m,k} \sum_{\beta > m} b_{\beta,j} b_{\beta,l}^* - \delta_{j,l} \sum_{\alpha < j} b_{m,\alpha} b_{k,\alpha}^* \right]. \quad (5.30)$$

This fixes the real Poisson brackets of the real and imaginary parts of the matrix elements, and allows us to find the Poisson algebra of their \mathbb{T} -invariant polynomials.

6. Reduced Hamiltonian flows and integrability

Here, we derive the equations of motion of the reduced system and present an algorithm for obtaining its solutions. We then point out the connection between our construction and the work of L.-C. Li [29,30]. Finally, we briefly discuss the integrability of the system.

6.1. Equations of motion and solutions by projection

For any G -invariant function H on \mathcal{M} , the reduced Hamiltonian flow can be obtained by first restricting the original flow to the moment map constraint surface, and then projecting it to \mathcal{M}_{red} . The reduced Hamiltonian vector field can be obtained in an analogous manner. If the reduced phase space is realized via gauge fixing, this means that in general we have to add to the Hamiltonian vector field of H a point dependent infinitesimal gauge transformation⁴ that ensures that the additional constraints are preserved. We can apply the same procedure to partial gauge fixings as well, like the ‘diagonal gauge’ \mathcal{Z} (4.20), where g_R belongs to \mathbb{T}^o .

The determination of the ‘compensating gauge transformation’ relies on the following. Let $X_{\mathcal{M}}$ be the vector field on \mathcal{M} that gives the infinitesimal action of $X \in \mathcal{G}$ generated by the moment map Λ . Regarding b_R , g_R , S etc as evaluation functions on \mathcal{M} , their Lie-derivatives with respect to this vector field can be easily evaluated. We obtain

$$\mathcal{L}_{X_{\mathcal{M}}}(b_R) = \text{dress}_{(b_L^{-1} X b_L)_{\mathcal{G}}}(b_R), \quad \mathcal{L}_{X_{\mathcal{M}}}(g_R) = [(b_L^{-1} X b_L)_{\mathcal{G}}, g_R]. \quad (6.1)$$

The first formula and (3.9) imply

$$\mathcal{L}_{X_{\mathcal{M}}}(b_R b_R^\dagger) = [(b_L^{-1} X b_L)_{\mathcal{G}}, b_R b_R^\dagger], \quad (6.2)$$

and for completeness we also record

$$\mathcal{L}_{X_{\mathcal{M}}}(S) = \text{dress}_{(b_R^{-1} b_L^{-1} X b_L b_R)_{\mathcal{G}}}(S), \quad \mathcal{L}_{X_{\mathcal{M}}}(b_R S) = \text{dress}_{(b_L^{-1} X b_L)_{\mathcal{G}}}(b_R S). \quad (6.3)$$

Note that $X \mapsto (b_L^{-1} X b_L)_{\mathcal{G}}$ is an invertible map on \mathcal{G} . Therefore, any point dependent infinitesimal gauge transformation can be represented as a vector field of the above form, where $(b_L^{-1} X b_L)_{\mathcal{G}}$ is replaced by some \mathcal{G} -valued function, called Y below, on the phase space.

Now, consider a function $H \in C^\infty(\mathcal{M})^G$ of the form $H(K, S) = h(b_R)$, and denote

$$(d^L h)(b_R) =: \mathcal{V}(L) \quad \text{with} \quad L := b_R b_R^\dagger. \quad (6.4)$$

Before reduction, the Hamiltonian vector field of H can be symbolically written as

$$\dot{g}_R = \mathcal{V}(L) g_R, \quad \dot{b}_R = 0, \quad \dot{S} = 0. \quad (6.5)$$

For the description of the reduced symplectic structure, it was appropriate to use the variables $Q = e^{iq}$, p and S , but the equations of motion are more conveniently expressed in terms of Q and L . Since we work in the diagonal gauge, p can be uniquely recovered from b_R , which itself is uniquely determined by L , via either of the unique decompositions

$$b_R = e^p b_+ = n_+ e^p \quad \text{with} \quad b_+, n_+ \in B_+. \quad (6.6)$$

Taking advantage of the moment map constraint (4.25), $S = S_+$ is determined by the formula

$$S = b_R^{-1} Q^{-1} b_R Q. \quad (6.7)$$

Thus, it is enough to find the time development of Q and L , and then recover p and S by the above algebraic relations.

Corresponding to $\mathcal{G} = \mathcal{T} + \mathcal{T}^\perp$, we decompose the \mathcal{G} -valued functions \mathcal{V} and Y as

$$\mathcal{V} = \mathcal{V}_{\mathcal{T}} + \mathcal{V}_{\perp} \quad \text{and} \quad Y = Y_{\mathcal{T}} + Y_{\perp}. \quad (6.8)$$

Then we can state

⁴ This ‘compensation’ is needed if the restricted Hamiltonian vector field is not tangent to the gauge slice.

Proposition 6.1. *The projection of the Hamiltonian vector field (6.5) to the diagonal gauge \mathcal{Z} (4.20) gives*

$$\dot{Q} = \mathcal{V}_{\mathcal{T}}(L)Q \quad \text{and} \quad \dot{L} = [Y_{\mathcal{T}} + Y_{\perp}(Q, L), L], \quad (6.9)$$

where

$$Y_{\perp}(Q, L) = (\text{Ad}_Q - \text{id})^{-1}(\mathcal{V}_{\perp}(L)) \quad (6.10)$$

and $Y_{\mathcal{T}}$ is arbitrary, reflecting the residual gauge transformations.

Proof. By restriction to \mathcal{Z} , where $g_R = Q$, and addition of a vector field tangent to the gauge orbits, we obtain from (6.5) and (6.1), (6.2) the ‘corrected’ derivatives

$$\dot{Q} = \mathcal{V}(L)Q + [Y, Q], \quad \dot{L} = [Y, L]. \quad (6.11)$$

The tangency to \mathcal{Z} means that $\dot{Q}Q^{-1}$ must belong to \mathcal{T} , which is equivalent to

$$QY_{\perp}Q^{-1} - Y_{\perp} = \mathcal{V}_{\perp}(L). \quad (6.12)$$

Since $Q \in \mathbb{T}^o$ is regular, $(\text{Ad}_Q - \text{id})$ is invertible on \mathcal{T}^{\perp} , and therefore Y_{\perp} is uniquely found to be given by (6.10). The constituent $Y_{\mathcal{T}}$ of Y is left undetermined, since we have not fixed the residual gauge transformations that map \mathcal{Z} to \mathcal{Z} . \square

In principle, one can construct the solutions of the equations of motion by projecting the unreduced flow (3.11) as follows. Pick any initial values $Q(0), L(0)$ defined by a point of \mathcal{Z} . Then, there exists $\eta(t) \in G$ and unique $Q(t)$, in general from the closure of the Weyl alcove \mathbb{T}^o , verifying

$$Q(t) = \eta(t) \exp(t\mathcal{V}(L(0)))Q(0)\eta(t)^{-1}. \quad (6.13)$$

If $\exp(t\mathcal{V}(L(0)))Q(0)$ belongs to G^{reg} , which certainly holds for small t , then one can choose $\eta(t)$ in such a way to depend analytically on t and on the initial values. The resulting $Q(t) \in \mathbb{T}^o$ and $L(t) = \eta(t)L(0)\eta(t)^{-1}$ solve the equations of motion (6.9). Along this solution, one has

$$Y(t) = \dot{\eta}(t)\eta(t)^{-1}. \quad (6.14)$$

We note that $\mathcal{V}(L(t)) = \eta(t)\mathcal{V}(L(0))\eta(t)^{-1}$ follows, since $h \in C^{\infty}(B)^G$ (see (6.4)). As $\eta(t)$ can be replaced by $T(t)\eta(t)$ with an arbitrary function $T(t) \in \mathbb{T}$, one can eliminate $Y_{\mathcal{T}}$ for any given solution.

According to (6.6), the time development of p can be recovered from the decomposition

$$L(t) = b_R(t)b_R(t)^{\dagger} = n_+(t)e^{2p(t)}n_+(t)^{\dagger}, \quad \text{with} \quad n_+(t) \in B_+. \quad (6.15)$$

After finding $Q(t)$, and $b_R(t)$ from $L(t)$, the time development of S follows from (6.7).

It requires further delicate analysis to see whether and for which dressing orbits can the trajectory $g_R(t) = \exp(t\mathcal{V}(L(0)))Q(0)$ leave the set of regular elements. Then some aspects of our analysis break down, although Q in the closure of \mathbb{T}^o still remains a well-defined gauge invariant object. Moreover, it follows from general results [49,50] that the reduced Hamiltonian flows are all complete, and they stay inside the symplectic strata of the full reduced phase space. We plan to investigate this issue, first in examples, in a future work.

Remark 6.2. For regular $Q = e^{iq}$, let $R(Q) \in \text{End}(\mathcal{G})$ be the following linear operator:

$$R(Q)|_{\mathcal{T}} := 0, \quad R(Q)|_{\mathcal{T}^\perp} := \frac{1}{2}(\text{Ad}_Q + \text{id})((\text{Ad}_Q - \text{id})|_{\mathcal{T}^\perp})^{-1} = \frac{1}{2} \coth\left(\frac{1}{2}(\text{ad}_{iq})|_{\mathcal{T}^\perp}\right). \quad (6.16)$$

Viewing it as a function of $iq \in \mathcal{T}_{\text{reg}}$, this is a solution of the modified classical dynamical Yang–Baxter equation [13]. The equations of motion (6.9) admit the equivalent form

$$\dot{Q} = \mathcal{V}_{\mathcal{T}}(L)Q \quad \text{and} \quad \dot{L} = [Y'_{\mathcal{T}} + R(Q)(\mathcal{V}(L)), L]. \quad (6.17)$$

Here, $Y'_{\mathcal{T}} = Y_{\mathcal{T}} + \frac{1}{2}\mathcal{V}_{\mathcal{T}}(L)$, which is as arbitrary as $Y_{\mathcal{T}}$ is, and we used that, in consequence of the definitions, $[\mathcal{V}(L), L] = 0$. We recall [17] that the same dynamical R -matrix features in the equations of motion of the spin Sutherland models of Section 2. However, for those models the Lax matrix belongs to \mathfrak{ig} , while our L lies in $\exp(\mathfrak{ig})$.

As an illustration, let us consider

$$h := \text{tr}(b_R b_R^\dagger) \quad (6.18)$$

for $G_{\mathbb{R}}^{\mathbb{C}} = \text{SL}(n, \mathbb{C})$. The form of the corresponding reduced Hamiltonian was presented in the preceding section. To display the reduced equations of motion, note that in this case

$$\mathcal{V}(L) = 2iL - \frac{2i}{n} \text{tr}(L) \mathbf{1}_n. \quad (6.19)$$

Writing $Q = \exp(\text{diag}(iq_1, \dots, iq_n))$, (6.10) gives

$$Y_{jk} = \left(\cot \frac{q_j - q_k}{2} - i \right) L_{jk} \quad \text{for } 1 \leq j \neq k \leq n. \quad (6.20)$$

Substitution into (6.9) yields the equations of motion:

$$\dot{q}_j = 2L_{jj} - \frac{2}{n} \text{tr}(L), \quad \dot{L}_{jj} = 2 \sum_{\ell \neq j} |L_{j\ell}|^2 \cot \frac{q_j - q_\ell}{2}, \quad (6.21)$$

and, for $1 \leq j \neq k \leq n$,

$$\begin{aligned} \dot{L}_{jk} = & \sum_{\ell \neq j} L_{j\ell} L_{\ell k} \cot \frac{q_j - q_\ell}{2} - \sum_{\ell \neq k} L_{j\ell} L_{\ell k} \cot \frac{q_\ell - q_k}{2} \\ & + (Y_{jj} - Y_{kk} + iL_{jj} - iL_{kk})L_{jk}. \end{aligned} \quad (6.22)$$

Since the diagonal entries of Y are arbitrary, one can choose them in such a way to get rid of the second line in \dot{L}_{jk} . After analytic continuation from trigonometric to hyperbolic functions, these equations coincide with Eq. (15) in [5] as well as with Eq. (5.15) in [29]. They are also special cases of Eq. (3.36) in [3] if the ‘potential’ that appears there is taken to be simply $\cot(q)$ instead of $\cot(q) - \cot(q + \gamma)$, i.e., if we omit the term containing the coupling constant γ .

Remark 6.3. In the papers [29,30], L.-C. Li constructed certain hyperbolic spin RS type models. The construction begins with Hamiltonian systems on $\mathcal{T}_{\text{reg}}^{\mathbb{C}} \times G_{\mathbb{R}}^{\mathbb{C}} \times \mathcal{T}_{\text{reg}}^{\mathbb{C}}$, which is a coboundary dynamical Poisson groupoid based on a classical dynamical R -matrix. The R -matrix in question is a map from $\mathcal{T}_{\text{reg}}^{\mathbb{C}}$ to $\text{End}(\mathcal{G}_{\mathbb{R}}^{\mathbb{C}})$ that can be chosen to be a natural extension of the R -matrix in

(6.16). The next step involves restriction to the fixed point set of a suitable Poisson involution, and subsequent Hamiltonian reduction. In the case described in most detail in [30], the fixed point set has the form $i\mathcal{T}_{\text{reg}} \times \mathfrak{H}$, where $\mathfrak{H} \subset G_{\mathbb{R}}^{\mathbb{C}}$ is the fixed point set of the inverse of the Cartan involution, i.e., the set of Hermitian elements. Upon analytic continuation from hyperbolic to trigonometric functions, the equations of motion, worked out in [29,30] for $G^{\mathbb{C}} = \text{SL}(n, \mathbb{C})$, coincide with the equations presented above. Moreover, commuting Hamiltonians are obtained in [30] from the G -invariant functions on \mathfrak{H} , similarly to our usage of $C^{\infty}(\mathfrak{P})^G$. However, because the constructions are very different, it is not clear at present whether the reduced Poisson structures are also related by analytic continuation. We expect that this is the case, but it requires further effort to show it. We note that neither the distribution of the variables as q , p and S , nor the connection between the spin RS type models and spin Sutherland models appear in [29,30].

6.2. Integrals of motion and Lax equations

We noted in Section 4 that the unreduced free system is degenerately integrable. The unreduced free Hamiltonians form the Abelian Poisson algebra

$$C_I(\mathcal{M}) = \Lambda_R^* \left(C^{\infty}(B)^G \right). \quad (6.23)$$

The elements of $C_I(\mathcal{M})$ (4.3) represent integrals of motion, since they Poisson commute with every element of $C_I(\mathcal{M}) \subset C_J(\mathcal{M})$. It is known that, generically, degenerate integrability is inherited under Hamiltonian reduction based on a compact symmetry group. In the C^{∞} category, it is also known that degenerately integrable systems are integrable in the usual Liouville sense, too. For precise statements, see the reviews [23,56] and references therein. The arguments of Reshetikhin [41,42], which show degenerate integrability by proving the existence of enough analytic integrals of motion on a dense open subset of the reduced phase space, can be also adapted to our case.

However, for a concrete integrable system one is not content with existence statements, but would like to obtain the required integrals of motion in explicit form. The elements of $C_I(\mathcal{M})$ are the G -invariant functions of the ‘unreduced Lax matrix’ $b_R b_R^{\dagger}$, and they descend to the reduced phase space, since $C_I(\mathcal{M}) \subset C^{\infty}(\mathcal{M})^G$. Now we exhibit a large class of G -invariant elements of $C_J(\mathcal{M})$. To this end, let us consider an arbitrary polynomial

$$\mathcal{P}(b_R b_R^{\dagger}, g_R^{-1} b_R b_R^{\dagger} g_R) \quad (6.24)$$

in the two non-commutative variables that belong to \mathfrak{P} (3.13). Take an arbitrary finite dimensional representation ρ of $G_{\mathbb{R}}^{\mathbb{C}}$, whose restriction to G is unitary. Then the real and imaginary parts of the function \mathcal{F} , given on \mathcal{M} by

$$\mathcal{F}(K, S) := \text{tr}_{\rho}(\mathcal{P}(b_R b_R^{\dagger}, g_R^{-1} b_R b_R^{\dagger} g_R)), \quad (6.25)$$

are G -invariant elements of $C_J(\mathcal{M})$. In order to see this, one may use the relation

$$g_R^{-1} b_R b_R^{\dagger} g_R = b_L^{-1} (b_L^{-1})^{\dagger}, \quad (6.26)$$

which follows from (3.4), and the fact that $b_R b_R^{\dagger}$ transforms in the same way as g_R (4.28). These G -invariant integrals of motion descend to smooth (even real analytic, and in a certain sense algebraic) functions on the full reduced phase space. They are generalizations of the conserved quantities $\text{tr}_{\rho}(\mathcal{P}(J, g J g^{-1}))$ displayed in Section 2.

Upon imposing the moment map constraint and working in the diagonal gauge, where $g_R = Q \in \mathbb{T}^o$, we obtain

$$g_R^{-1} b_R b_R^\dagger g_R = Q^{-1} L Q \quad \text{with} \quad L \equiv b_R b_R^\dagger. \quad (6.27)$$

Thus the above constants of motion take the form $\text{tr}_\rho(\mathcal{P}(L, Q^{-1} L Q))$. Special cases of these conserved quantities can be generated by evaluation of the trace on powers of the following spectral parameter (denoted by λ) dependent Lax matrix:

$$\mathcal{L} := L + \lambda Q^{-1} L Q. \quad (6.28)$$

Working in the diagonal gauge, the Hamiltonian vector field of $H \in C_I(\mathcal{M})$, for which $H(K, S) = h(b_R)$, implies the Lax equation

$$\dot{\mathcal{L}} = [Y, \mathcal{L}]. \quad (6.29)$$

Here, Y is the \mathcal{G} -valued function responsible for consistency with the (partial) gauge fixing. Equation (6.29) follows directly from (6.11) using that $[\mathcal{V}(L), L] = 0$ because $h \in C^\infty(B)^G$.

It is an open problem if the above exhibited integrals of motion are sufficient for the degenerate integrability of the reduced system. The question of Liouville integrability with real analytic integrals of motion also requires further study.

7. Discussion and outlook

In this paper we applied Hamiltonian reduction to ‘free systems’ on Heisenberg doubles, generalizing the derivation of the trigonometric spin Sutherland models (1.1) based on the corresponding cotangent bundles. The reduced systems that we obtained appear to be related by analytic continuation to certain hyperbolic spin RS type models introduced earlier by L.-C. Li in [30].⁵ The different approaches have various advantages with respect to each other.

Our approach can, in principle, accommodate action-angle duals of all of our systems. Indeed, the unreduced phase space \mathcal{M} supports two natural Abelian Poisson algebras, namely $\Lambda_R^*(C^\infty(B)^G)$ and $\Xi_R^*(C^\infty(G)^G)$, which define two degenerately integrable systems. We worked out a model of the reduced phase space whereby the reduction of $\Lambda_R^*(C^\infty(B)^G)$ represents many-body Hamiltonians of spin RS type, and $\Xi_R^*(C^\infty(G)^G)$ reduces to their position variables. There should exist another model of the same reduced phase space, making it possible to view the reduction of $\Xi_R^*(C^\infty(G)^G)$ as many-body Hamiltonians with their position variables descending from $\Lambda_R^*(C^\infty(B)^G)$. This is the standard way to produce pairs of many-body systems in action-angle duality via reduction, see e.g. [3, 15, 37, 41]. Another potential advantage of our method is that it automatically leads to complete flows of the interesting Hamiltonians on the full reduced phase space, simply since the unreduced flows are complete. This may not hold in the framework of [29, 30], since it relies on dynamical R -matrices, whose domain of definition requires a restriction of the variables. In our present work, the restriction to regular elements of G was merely a technical convenience, and we shall investigate the global features of our reduced systems in the future.

Here, it is proper to stress that the method of [29, 30] is applicable for a large class of dynamical R -matrices, not only for the standard trigonometric R -matrix, which is related to our

⁵ Incidentally, we noticed the connection with [30] only during the completion of this manuscript.

case. We plan to report the precise relationship between our systems and those in [30] in a future publication.

In passing, we wish to mention that other approaches are also available for constructing trigonometric/hyperbolic spin RS type systems. In particular, the moduli space of flat $G^{\mathbb{C}}$ connections on a torus with a hole is believed to support such systems at arbitrary monodromy. These moduli spaces can be described, e.g., by quasi-Hamiltonian reduction or by the approach of Fock and Rosly [19]. See also Section 6 in [42], and references therein. Our systems should be related to real forms of these holomorphic systems.

We gave a linear algebraic algorithm for obtaining the solutions of our spin RS type models, and advanced some arguments in favor of their degenerate integrability. It could be interesting to further explore their integrability properties. The ultimate aim would be to exhibit the required number of (real analytic) integrals of motion in explicit form on every symplectic strata [49,50] of \mathcal{M}_{red} .

The present work can be generalized in such a way to obtain Poisson-Lie analogues of the spin Sutherland models based on the cotangent lifts of twisted conjugations acting on G [17]. Moreover, it should be possible to use our approach as the starting point for developing the quantum mechanics of the spin RS type systems. For this, one should generalize the quantization of the spin Sutherland models that proceeds via quantum Hamiltonian reduction and related methods of harmonic analysis [12,18].

Another organic continuation of the present work, whose implementation is in progress in collaboration with I. Marshall, consists in deriving Poisson-Lie analogues of the real, trigonometric Gibbons–Hermesen model [20]. For this purpose, one needs to reduce a phase space of the form

$$\mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^n \times \cdots \times \mathbb{C}^n \quad (7.1)$$

with an arbitrary number, $k \geq 2$, of copies of \mathbb{C}^n . Here, $\mathrm{GL}(n, \mathbb{C})$ is the Heisenberg double of the Poisson-Lie group $U(n)$, and \mathbb{C}^n carries a symplectic structure that enjoys $U(n)$ Poisson-Lie symmetry [54]. The pertinent moment map constraint can be written as

$$\Lambda_L \Lambda_R \Lambda_1^{\mathbb{C}^n} \Lambda_2^{\mathbb{C}^n} \cdots \Lambda_k^{\mathbb{C}^n} = e^\gamma \mathbf{1}_n, \quad (7.2)$$

where γ is a non-zero real constant, and the $\Lambda_i^{\mathbb{C}^n}$ ($i = 1, \dots, k$) are Poisson-Lie moment maps on the independent copies of \mathbb{C}^n . At the level of the equations of motion, we have proved that this construction yields the trigonometric real form of the spin RS systems studied earlier in [3, 28,40], but still have not completed the description of the resulting reduced Poisson brackets. For recent derivations of the Poisson structure of the Krichever–Zabrodin type trigonometric spin RS models in the complex holomorphic case, we refer to the papers [9,4]. Our construction should lead to a real form of the Poisson structure obtained in these papers.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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