Hopf bifurcation for Wright-type delay differential equations: the simplest formula, period estimates, and the absence of folds

István Balázs

MTA-SZTE Analysis and Stochastics Research Group,
Bolyai Institute, University of Szeged,
Aradi vértanúk tere 1, Szeged, H-6720, Hungary
balazsi@math.u-szeged.hu

Gergely Röst

Bolyai Institute, University of Szeged,
Aradi vértanúk tere 1, Szeged, H-6720, Hungary,
rost@math.u-szeged.hu

January 14, 2020

Abstract

First we present the simplest criterion to decide that the Hopf bifurcations of the delay differential equation $x'(t) = -\mu f(x(t-1))$ are subcritical or supercritical, as the parameter μ passes through the critical values μ_k. Generally, the first Lyapunov coefficient, that determines the direction of the Hopf bifurcation, is given by a complicated formula. Here we point out that for this class of equations, it can be reduced to a simple inequality that is trivial to check. By comparing the magnitudes of $f''(0)$ and $f'''(0)$, we can immediately tell the direction of all the Hopf bifurcations emerging from zero, saving us from the usual lengthy calculations.

The main result of the paper is that we obtain upper and lower estimates of the periods of the bifurcating limit cycles along the Hopf branches. The proof is based on a complete classification of the possible bifurcation sequences and the Cooke transformation that maps branches onto each other. Applying our result to Wright's equation, we show that the kth Hopf branch has no folds in a neighbourhood of the bifurcation point μ_k with radius $6.83 \times 10^{-3}(4k+1)$.

Finally, we show how our results relate to the often required property that the nonlinearity has negative Schwarzian derivative.

Keywords: period estimates, delay differential equation, Hopf bifurcation, supercritical, normal form

1 Introduction

The appearance of limit cycles around equilibria via Hopf bifurcations is a common phenomenon for delay differential equations, when a parameter of the equation passes through a critical value, and a pair of eigenvalues of the linearized system crosses the imaginary axis on the complex plane. Depending on the nature of the nonlinearity, the Hopf bifurcation can be either supercritical or subcritical, i.e. the bifurcating periodic solution can be stable or unstable on the center manifold. It is well known how to determine the direction of the Hopf bifurcation for delay differential equations at least since the book of Hassard, Kazarinoff and Wan [8]. One can use bilinear forms, center manifold reduction (see [3, 8]), Lyapunov–Schmidt method [7] or alternatively, the theory of normal forms for functional differential equations [4]. Based on these fundamental techniques, the literature of delay differential equations is vast by papers where Hopf bifurcation results are shown to many particular model systems arising from physical, engineering or biological applications. However, most of those articles provide only the complicated formula of the first Lyapunov coefficient, which is typically hard to relate to the original model parameters. In fact, if the reader wants to figure out the direction of the bifurcation in particular cases, it requires almost the same effort as repeating the whole calculation of the general formula. Also, due to the elaborative nature of such calculations, the literature of bifurcation theory is not free of minor mistakes or inaccuracies (some of those are discussed, for example, in [10] or [17]).

To remedy this situation, our first aim here is to derive the simplest criterion for the direction of the bifurcations for the class of scalar delay differential equations of the special form

$$x'(t) = -\mu f(x(t-1)), \tag{1}$$

which will be then trivial to check in any specific situation. Note that the equation

$$z'(s) = -f(z(s-\mu))$$

can be easily rescaled into (1) by the change of variables $s = t\mu$ and $x(t) = z(s)$, hence in the sequel we assume that the delay is one and μ is our bifurcation parameter. This class of equations is frequently studied and includes such notorious examples as the equations of Wright or Ikeda. Equations of the form (1) are often called Wright-type delay differential equations [11]. We characterize all possible sequences of subcritical and supercritical Hopf bifurcations, and provide concrete examples to each one. The calculations are based on the method of Faria and Magalhães [4].

It is well known for a long time, how to calculate the direction of the Hopf bifurcation for delay differential equations. However, these calculations are very lengthy and elaborate. Here we point out that for equations of the form (1), the complicated formula of the first Lyapunov coefficient reduces to a very simple criterion: just by comparing the magnitudes of $f''(0)$ and $f'''(0)$, we can immediately tell the direction of all the Hopf bifurcations emerging from zero. This way we obtain a complete classification of the possible bifurcation sequences, and there is no need for any tedious computations in the future for the bifurcation analysis of such equations. By Theorem 1, the value of the ratio $f'''(0)/f''(0)^2$ tells everything.

To achieve our main result, we use the formulae for the directions of the Hopf bifurcations and combine with the method of Cooke transformation to obtain upper and lower estimates on the periods of the bifurcating solutions. In particular, we find narrow estimates of the period function along branches, and explore the relation between its monotonicity and the directions of the bifurcations. In general, very little is known about how periods change along bifurcation branches of limit cycles for delay differential equations. Our estimates are valid not only locally, but also on large intervals of parameters until the bifurcation branches remain on the same side of the critical parameter.

For Wright's equation with the nonlinearity $f(\xi) = e^{\xi} - 1$, Jaquette and van den Berg [1, 9] have proven by rigorous numerical computation that the Hopf branch starting at $\pi/2$ has no folds, and the period length of the bifurcating solutions is monotone increasing on the interval $(\pi/2, \pi/2 + 6.83 \times 10^{-3}]$. Using the Cooke transform, we show the absence of folds for other branches starting at $\pi/2 + 2k\pi$, $k \geq 1$, on the interval $(\pi/2 + 2k\pi, \pi/2 + 2k\pi + 6.83 \times 10^{-3}(4k+1)]$.

Finally, we explore the connection between our results and the Schwarzian derivative of the nonlinearity, which plays a significant role in many global stability results. We show that by local bifurcations one can not disprove the conjecture that local asymptotic stability implies global asymptotic stability for Wright-type delay differential equations with negative Schwarzian.

2 Direction of Hopf bifurcation

Consider the scalar delay differential equation

$$x'(t) = -\mu f(x(t-1)) =: g(x_t, \mu), \tag{1}$$

where $\mu \in \mathbb{R}$, f is an $\mathbb{R} \to \mathbb{R}$, C^3-smooth function with $f(0) = 0$, so it can be written as

$$f(\xi) = \xi + B\xi^2 + C\xi^3 + \text{h.o.t.},$$

where $B = f''(0)/2$ and $C = f'''(0)/6$. The solution segment x_t is defined by the relation $x_t(s) = x(t+s)$, $s \in [-1, 0]$. Thus, x_t is an element of the Banach space $C([-1, 0], \mathbb{R})$.

Note that $f'(0) = 1$ can be assumed without the loss of generality, as we can normalize it via the parameter μ. It is known that the direction of the Hopf bifurcation depends on the terms of the Taylor series of the nonlinearity up to order three. In this case, the direction of the Hopf bifurcation around the zero equilibrium is determined by a relation between the coefficients B and C. To our surprise, despite the method is well known for a much more general class of equations, we could not find a derivation of such a simple, readily available criterion for B and C in the literature for (1), only for the first Hopf bifurcation in [19] and in Chapter 6 of the recent book of H. L. Smith [18], and for a different class of equations in [6]. The main result of this Section is the general condition for the stability of the Hopf bifurcation at any critical parameter value.

Many commonly used model equations include a mix of delayed and instantaneous terms, resulting in a very complicated formula for the first Lyapunov coefficient. When we have only a pure delayed term, then the characteristic equation is already significantly simplified. The critical roots and critical parameters can now be expressed easily (see the Appendix). This allows us to derive a simple criterion for the direction of the bifurcation.

Theorem 1. *a) Equation (1) has Hopf bifurcations from the zero equilibrium at the critical parameter values* $\mu_k = \frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$.

b) The kth bifurcation is

- *supercritical if* $C < H(k)B^2$;
- *subcritical if* $C > H(k)B^2$, *where* $H(k) = \frac{22(4k+1)\pi - 8}{15(4k+1)\pi}$.

c) If a Hopf bifurcation of Equation (1) is

- *supercritical, then the bifurcation branch starts to right if* $\mu_k > 0$ *and left if* $\mu_k < 0$;
- *subcritical, then the bifurcation branch starts to left if* $\mu_k > 0$ *and right if* $\mu_k < 0$.

The proof is a straightforward computation following Faria and Magalhães [4]. For the sake of completeness, we included the derivation in the Appendix. The function $H(k)$ is plotted in Figure 1.

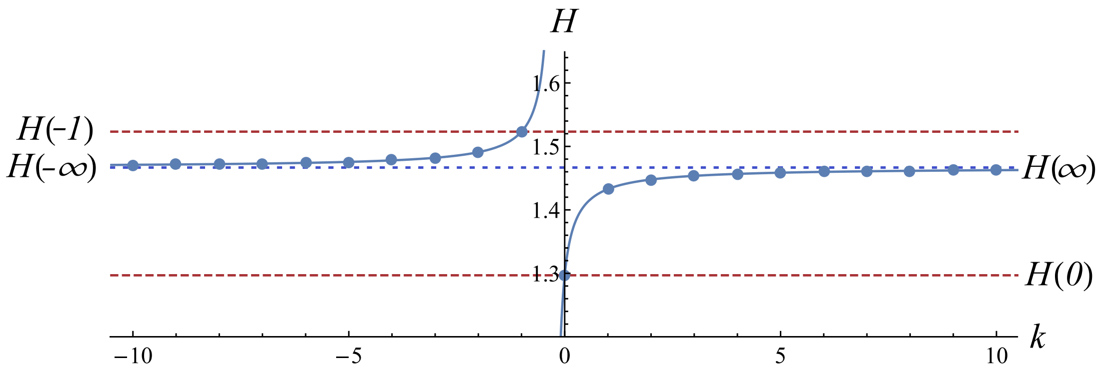


Figure 1: Plot of $H(k) = \frac{22(4k+1)\pi - 8}{15(4k+1)\pi}$. The values are between $H(0) = \frac{22\pi - 8}{15\pi}$ and $H(-1) = \frac{66\pi + 8}{45\pi}$, and they tend to $H(-\infty) = H(\infty) = \frac{22}{15}$ as $k \to \pm\infty$. According to Theorem 1, when the constant C/B^2 is below $H(k)$ for some k, the kth bifurcation is supercritical.

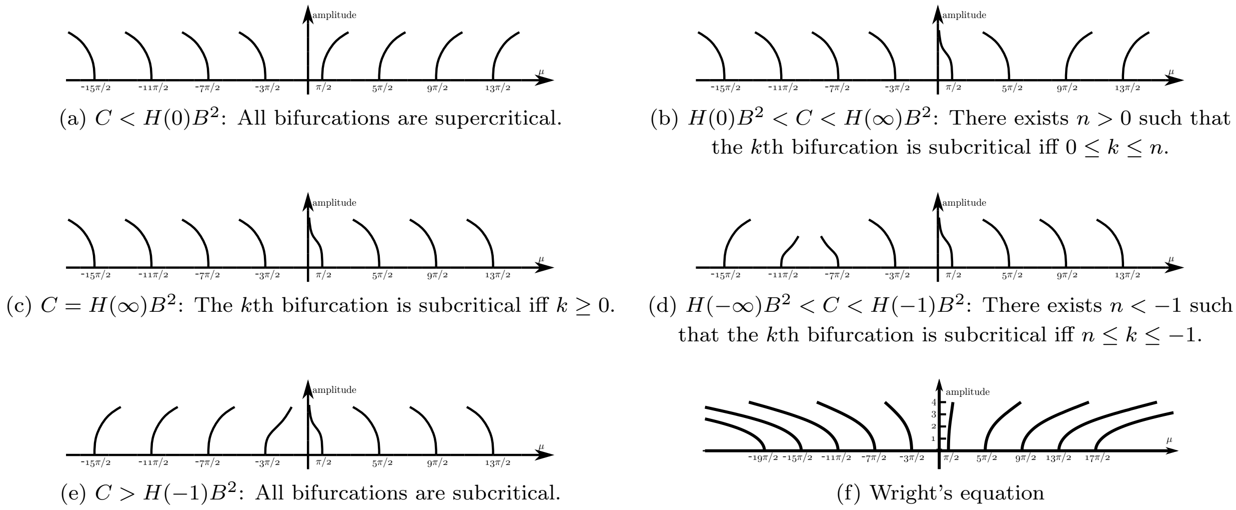
As we mentioned, for the special case $k = 0$, this result can be found in [18], page 97, which we now state as a corollary.

Corollary 2. *The Hopf bifurcation at* $\mu_0 = \frac{\pi}{2}$ *is supercritical if* $C < H(0)B^2$, *and it is subcritical if* $C > H(0)B^2$.

From the shape of $H(k)$, we easily find the following.

Corollary 3. *If* $C < H(0)B^2$ *then every Hopf bifurcation is supercritical, if* $C > H(-1)B^2$ *then every Hopf bifurcation is subcritical.*

For convenience, note that $H(0) \approx 1.3$ and $H(-1) \approx 1.52$. Theorem 1 and its corollaries allow us to give a complete classification of the possible bifurcation sequences, depicted in Figure 2.



(a) $C < H(0)B^2$: All bifurcations are supercritical.

(b) $H(0)B^2 < C < H(\infty)B^2$: There exists $n > 0$ such that the kth bifurcation is subcritical iff $0 \leq k \leq n$.

(c) $C = H(\infty)B^2$: The kth bifurcation is subcritical iff $k \geq 0$.

(d) $H(-\infty)B^2 < C < H(-1)B^2$: There exists $n < -1$ such that the kth bifurcation is subcritical iff $n \leq k \leq -1$.

(e) $C > H(-1)B^2$: All bifurcations are subcritical.

(f) Wright's equation

Figure 2: All possible configurations of bifurcation branches, based on Theorem 1.

3 Applications

3.1 Wright's equation

The classical Wright–Hutchinson equation (also called delayed logistic equation)

$$y'(t) = -\mu y(t-1)(1+y(t)), \quad \mu > 0,$$

can be transformed into the form

$$x'(t) = -\mu(e^{x(t-1)} - 1) \tag{2}$$

by the change of variable $x(t) = \ln(1 + y(t))$, for solutions $y > -1$. This latter equation is of type (1) with $f(\xi) = e^{\xi} - 1$, $B = \frac{1}{2}$, $C = \frac{1}{6}$. Since $C \approx 0.167 < 0.324 \approx H(0)B^2$, we can apply Corollary 3 to obtain the following fact (which was also derived in [4], page 197, and [2], page 147).

Corollary 4. *In Wright's equation, every Hopf bifurcation is supercritical.*

3.2 Ikeda equation

The equation

$$y'(t) = -\sin(y(t-\mu))$$

arises in the modeling of optical resonator systems. By rescaling, one has the equivalent form

$$x'(t) = -\mu \sin(x(t-1)),$$

which fits into (1) with $f(\xi) = \sin(\xi)$, $B = 0$, $C = -\frac{1}{6}$. Since $C < 0 = H(0)B^2$, Corollary 3 applies.

Corollary 5. *In the Ikeda equation, every Hopf bifurcation is supercritical.*

3.3 A polynomial equation with criticality switching

Consider

$$x'(t) = -\mu(x(t-1) + x^2(t-1) + 1.44x^3(t-1)),$$

which is of the form (1) with $f(\xi) = \xi + \xi^2 + 1.44\xi^3$, $B = 1$, $C = 1.44$. Then $H(1) < \frac{C}{B^2} < H(2)$, so the bifurcations at μ_0 and μ_1 are subcritical, the others are supercritical.

3.4 A totally subcritical polynomial equation

Consider

$$x'(t) = -\mu\left(x(t-1) + x^2(t-1) + \frac{22}{15}x^3(t-1)\right),$$

which is of the form (1) with $f(\xi) = \xi + \xi^2 + \frac{22}{15}\xi^2$, $B = 1$, $C = \frac{22}{15}$. Then $\frac{C}{B^2} = \frac{22}{15} > H(k)$, for all nonnegative integer k, so every Hopf bifurcation is subcritical for positive critical parameter values (and supercritical for negative parameter values).

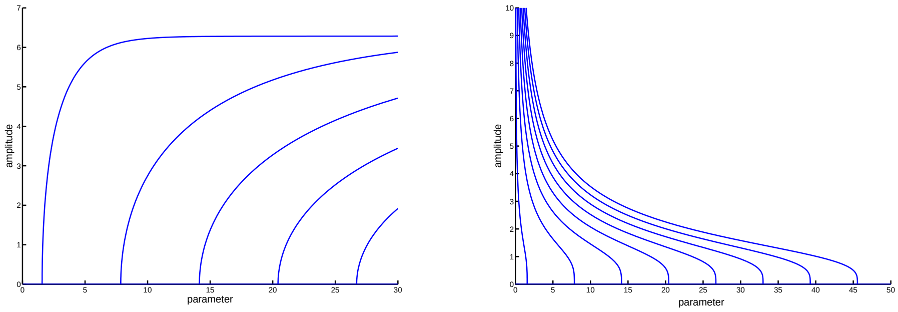


Figure 3: The bifurcation branches of the Ikeda equation (left) and of our totally subcritical example (right).

4 Period estimations

Throughout this section we consider $\mu > 0$. The following idea is known in the delay differential equation community as the Cooke transformation, which has been used for example in [15, 5]. If $p(t)$ is a periodic solution of equation (1) for parameter value $\mu = \mu_* > 0$ with period T, then $q(t) := p((lT+1)t)$ is also a periodic solution of equation (1) for parameter value $\mu = \mu_*(lT+1)$ with period $\frac{T}{lT+1}$, for any $l \in \mathbb{N}$. This can be shown by the straightforward calculations

$$q'(t) = -(\mu(lT+1))f(p((lT+1)t - lT - 1)) = -(\mu(lT+1))f(q(t-1))$$

and

$$q\left(t + \frac{T}{lT+1}\right) = p((lT+1)t + T) = p((lT+1)t) = q(t).$$

Thus we can define a map

$$C_l : (\mu, T, p(t)) \mapsto \left(\mu(lT+1), \frac{T}{lT+1}, p((lT+1)t)\right),$$

where $p(t)$ is a periodic solution of equation (1) with parameter value μ and period T.

Consider the Hopf branch of periodic solutions near the parameter value μ_k, $k \in \mathbb{N}$. Let $\delta_k = 1$ if the kth bifurcation is supercritical, and -1 if the kth bifurcation is subcritical. Then, for any k, there exists a unique local branch of periodic solutions $p_\eta^k(t)$ parametrized by a variable $\mu = \mu_k + \delta_k \eta$ with $\eta \in (0, \eta_k)$ for some $\eta_k > 0$. The minimal period of p_η^k is denoted by T_η^k. As calculated in the Appendix, at the critical parameter value $\mu_k = \frac{\pi}{2} + 2k\pi = \frac{4k+1}{2}\pi$, the critical eigenvalue is $i\omega_k = i\frac{4k+1}{2}\pi$. The linearized equation has a center at critical parameter values, having periodic solutions with minimal period

$$T_0^k := \frac{2\pi}{\omega_k} = \frac{4}{4k+1}.$$

Theorem I. on page 14 of [8] implies that T_η^k is a continuous function of η, and $T_\eta^k \to T_0^k$ as $\eta \to 0$.

Proposition 6. *Let $k, l \in \mathbb{N}$. Then, near the bifurcation points, C_l maps the kth bifurcation branch to the (k+l)th bifurcation branch.*

Proof. Applying the continuous map C_l for the critical parameter value and period as the limits of $\mu_k + \delta_k \eta$ and T_η^k, as $\eta \to 0$, we obtain

$$\mu_k(lT_0^k + 1) = \frac{4k+1}{2}\pi\left(l\frac{4}{4k+1} + 1\right) = \frac{\pi}{2}(4(k+l)+1) = \mu_{k+l}$$

and

$$\frac{T_0^k}{lT_0^k + 1} = \frac{1}{l + \frac{4k+1}{4}} = \frac{4}{4l + 4k + 1} = \frac{2\pi}{\omega_{k+l}} = T_0^{k+l}.$$

By the continuity of the periods T_η^k and T_η^{k+l} as functions of η, and the uniqueness of local branches (see [3, Theorem X.2.7]), we find that the Cooke transformation maps Hopf bifurcation branches to Hopf bifurcation branches. $\square$

Theorem 7. *If $k \geq 0$ and $C < H(k)B^2$ then, for all $l \geq 1$, we have the following estimate on the period of the Hopf solution of equation (1) near μ_k:*

$$T_\eta^k > \frac{4 - \frac{2\eta}{l\pi}}{4k + 1 + \frac{2\eta}{\pi}}.$$

Proof. The assumptions of the theorem imply that the kth bifurcation is supercritical, then $\delta_k = 1$, and by Corollary 2, all the $(k+l)$th bifurcations ($l \in \mathbb{N}$) are supercritical as well. Then, taking Proposition 6 into account, we get

$$(\mu_k + \eta)(lT_\eta^k + 1) > \mu_{k+l}, \tag{3}$$

thus

$$T_\eta^k > \left(\frac{\mu_{k+l}}{\mu_k + \eta} - 1\right) l^{-1} = \frac{4 - \frac{2\eta}{l\pi}}{4k + 1 + \frac{2\eta}{\pi}}.$$

□

Theorem 8. *If $C \geq H(\infty)B^2$ and $k \geq 0$ then, for all $l \geq 1$, we have the following estimate on the period of the Hopf solution of equation (1) near μ_k:*

$$T_\eta^k < \frac{4 + \frac{2\eta}{l\pi}}{4k + 1 - \frac{2\eta}{\pi}}.$$

Proof. Now all the bifurcations are subcritical, so $\delta_k = \delta_{k+l} = -1$ for any $k, l \in \mathbb{N}$, and by Proposition 6,

$$(\mu_k - \eta)(lT_\eta^k + 1) < \mu_{k+l},$$

thus

$$T_\eta^k < \left(\frac{\mu_{k+l}}{\mu_k - \eta} - 1\right) l^{-1} = \frac{4 + \frac{2\eta}{l\pi}}{4k + 1 - \frac{2\eta}{\pi}}.$$

□

Theorem 9. *If $H(0)B^2 \leq C < H(\infty)B^2$ and $k \geq 0$, then define*

$$n := \max\left\{m \in \mathbb{N}_0 : \ C > \frac{22(4m+1)\pi - 8}{15(4m+1)\pi} B^2\right\}.$$

If $k < n$ then near μ_k we have the estimates

$$\frac{4 + \frac{2\eta}{(n-k+1)\pi}}{4k + 1 - \frac{2\eta}{\pi}} < T_\eta^k < \frac{4 + \frac{2\eta}{(n-k)\pi}}{4k + 1 - \frac{2\eta}{\pi}}.$$

If $k = n$ then we only have the lower estimate

$$T_\eta^k > \frac{4 + \frac{2\eta}{\pi}}{4k + 1 - \frac{2\eta}{\pi}}.$$

Proof. Assume that $k \leq n$. Then the kth bifurcation is subcritical and $\delta_k = -1$. First, suppose that $l_1 > 0$ and the $(k + l_1)$th bifurcation is supercritical. Then

$$(\mu_k - \eta)(l_1 T_\eta^k + 1) > \mu_{k+l_1},$$

thus

$$T_\eta^k > \frac{4 + \frac{2\eta}{l_1\pi}}{4k + 1 - \frac{2\eta}{\pi}}.$$

Now we choose l_1 to be the minimal index which still gives a supercritical bifurcation, that is $l_1 := n - k + 1$. Next, suppose that the $(k + l_2)$th bifurcation is subcritical. This is only possible if $k < n$. Then

$$(\mu_k - \eta)(l_2 T_\eta^k + 1) < \mu_{k+l_2},$$

thus

$$T_\eta^k < \frac{4 + \frac{2\eta}{l_2\pi}}{4k + 1 - \frac{2\eta}{\pi}}.$$

Finally, choose l_2 to be the maximal index that still gives subcritical bifurcation, which is $l_2 := n - k$. □

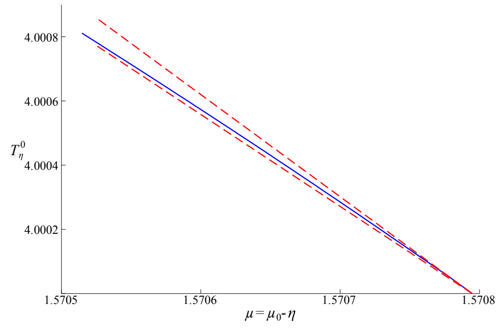


Figure 4: Narrow estimates on the period of the 0th branch of Example 3.3 by Theorem 9 (red dashed curves) compared to numerically obtained periods (blue solid curve).

In some situations this theorem provides very sharp estimations of the period function, which is illustrated in Figure 4.

Corollary 10. *If in equation (1) the kth Hopf bifurcation is subcritical for some $k \geq 0$, and the periods satisfy $T^k_\eta < T^k_0$ near μ_k, then for all $l \geq 1$ the $(k+l)$th Hopf bifurcation is also subcritical.*

Proof. If the kth Hopf bifurcation is subcritical, then

$$(\mu_k - \eta)(lT^k_\eta + 1) < \mu_k(lT^k_0 + 1) = \mu_{k+l}.$$

This means that the Cooke transformation maps the kth branch to the left side of μ_{k+l}, thus the $(k+l)$th bifurcation is also subcritical. □

In the situation $H(0)B^2 < C < H(\infty)B^2$ (see Figure 2.b.), we can infer the monotonicity of the period functions at the subcritical bifurcations, as the next corollary shows.

Corollary 11. *If in equation (1) the kth Hopf bifurcation is subcritical for some $k \geq 0$, but the $(k+l)$th Hopf bifurcation is supercritical for any $l \geq 1$, then $T^k_\eta > 0$ is monotone increasing for small η.*

Proof. If the kth Hopf bifurcation is subcritical, but the $(k+l)$th is supercritical, then for $\eta_1 < \eta_2$ we have

$$(\mu_k - \eta_1)(lT^k_{\eta_1} + 1) < (\mu_k - \eta_2)(lT^k_{\eta_2} + 1).$$

This is possible only if $T^k_{\eta_1} < T^k_{\eta_2}$. □

4.1 Application to Wright's equation

There is a huge literature about the dynamics of equation (1) when $f(\xi) = e^\xi - 1$, and the related conjectures of Wright and Jones (now proven theorems, see [1, 9]). In particular, Jaquette [9] has shown the following.

Theorem 12 (Jones' conjecture)**.** *For every $\mu > \pi/2$ there exists a unique slowly oscillating periodic solution to* (2)*.*

The 0th branch of (2) is formed by these unique slowly oscillating periodic solutions. The following conjecture is stated in [9, Conjecture 7.1].

Conjecture 13. *The period length of slowly oscillating periodic solutions of* (2) *increases monotonically in μ.*

The monotonicity was shown for $\mu \in (\pi/2, \pi/2 + 6.83 \times 10^{-3}]$ by Jaquette and van den Berg [1]. Applying our result, we can formulate the following theorem.

Theorem 14. *The kth Hopf bifurcation branch of Wright's equation* (2) *has no folds on the parameter interval $(\pi/2 + 2k\pi, \pi/2 + 2k\pi + 6.83 \times 10^{-3}(4k+1)]$, and the period length increases monotonically on this interval.*

Proof. For $k = 0$, we have the results of [1, 9], so there exists a unique periodic orbit on the 0th branch for each parameter value $\pi/2 + \eta$, $\eta \in (0, 6.83 \times 10^{-3}]$. Moreover, its period T_η^0 is increasing in η, and $T_\eta^0 \to 4$ as $\eta \to 0$.

Let $k \geq 1$. Using the Cooke transform C_k, as the 0th branch is mapped onto the kth branch, the parameter $\pi/2 + \eta$ is mapped to $\mu(\eta) = (\pi/2 + \eta)(kT_\eta^0 + 1)$. This is an increasing function of η with limit

$$\lim_{\eta \to 0} \mu(\eta) = \frac{\pi}{2}(4k+1) = \frac{\pi}{2} + 2k\pi.$$

Using $T^0_{6.83\times 10^{-3}} > 4$ (see [16]), we find from the Cooke transform that

$$\mu\left(6.83 \times 10^{-3}\right) = \left(\frac{\pi}{2} + 6.83 \times 10^{-3}\right)\left(kT^0_{6.83\times 10^{-3}} + 1\right) \geq \frac{\pi}{2} + 2k\pi + 6.83 \times 10^{-3}(4k+1).$$

Since the 0th branch has no fold on $(\pi/2, \pi/2 + 6.83 \times 10^{-3}]$, and the Cooke transform maps this parameter interval monotone increasingly onto $(\pi/2 + 2k\pi, \pi/2 + 2k\pi + 6.83 \times 10^{-3}(4k+1)]$, the kth Hopf branch has no folds on this interval as well. The period on the kth branch for parameter $\mu(\eta)$ is $T_\eta^0/(kT_\eta^0 + 1)$. Choosing η_1 and η_2 such that $\mu(\eta_1), \mu(\eta_2) \in (\pi/2 + 2k\pi, \pi/2 + 2k\pi + 6.83 \times 10^{-3}(4k+1)]$ and $\eta_1 < \eta_2$, we have $T_{\eta_1}^0 < T_{\eta_2}^0$ and

$$\frac{T^0_{\eta_2}}{kT^0_{\eta_2} + 1} - \frac{T^0_{\eta_1}}{kT^0_{\eta_1} + 1} = \frac{kT^0_{\eta_1}T^0_{\eta_2} + T^0_{\eta_2} - kT^0_{\eta_1}T^0_{\eta_2} - T^0_{\eta_1}}{(kT^0_{\eta_1} + 1)(kT^0_{\eta_2} + 1)} = \frac{T^0_{\eta_2} - T^0_{\eta_1}}{(kT^0_{\eta_1} + 1)(kT^0_{\eta_2} + 1)} > 0,$$

hence the period is increasing along the kth branch as well. □

Since Jones' conjecture has been proven, it is now known that there are no folds on the 0th Hopf branch of slowly oscillating periodic solutions of Wright's equation. Furthermore, isolas of slowly oscillating periodic solutions are excluded as well. By the Cooke transform, we can also exclude isolas of rapidly oscillatory periodic solutions. However, this is not sufficient to show there are no folds in the branches of rapidly oscillating periodic solutions [9]. Our Theorem 14 above shows the non-existence of folds on some intervals of the parameter on the Hopf branches of rapidly oscillatory solutions, using the monotonicity of the period established in [9] for a small interval of the 0th branch. Conjecture 13 would imply the absence of folds along all branches.

5 Schwarzian derivative and the direction of the Hopf bifurcation

The Schwarzian derivative of a C^3 function f is defined as

$$(Sf)(\xi) = \frac{f'''(\xi)}{f'(\xi)} - \frac{3}{2}\left(\frac{f''(\xi)}{f'(\xi)}\right)^2$$

at points ξ where $f'(\xi) \neq 0$. This quantity plays an important role in many results regarding the global dynamics of difference equations, which can be extended to delay differential equations in various cases (see [11, 12, 13, 14] and references thereof). A global stability conjecture was formulated in [11], stating that the zero solution of (1) is globally asymptotically stable whenever it is locally asymptotically stable, $Sf < 0$, and some other technical conditions hold (for related conjectures, see [12]). An obvious way to disprove this global stability conjecture would be the following: find a nonlinearity f with $Sf < 0$, where the Hopf bifurcation of (1) is subcritical at μ_0. This would provide a counterexample. Since both the directions of the bifurcation and the sign of the Schwarzian are determined by the derivatives of the nonlinearity up to order three, in view of the results of the previous sections, it is most natural to make a comparison to check whether the existence of such a counterexample is possible.

Corollary 15. *If $Sf < 0$, then all Hopf bifurcations are supercritical. Furthermore, if $f''(0) = 0$, then for any k, the kth bifurcation is supercritical if and only if $Sf(0) < 0$.*

Proof. From the definition, it is easy to evaluate $Sf(0) = 6(C - B^2)$, thus $Sf < 0$ implies $Sf(0) < 0$ and $C < B^2$. By Corollary 3, all Hopf bifurcations are supercritical. In the special case $f''(0) = 0$, we have $B = 0$ and $Sf(0) = 6C$, thus both the sign of the Schwarzian and the direction of the bifurcation are determined by the sign of C. □

We found that it is not possible to construct a counterexample to the conjecture of Liz et al. by means of a subcritical Hopf bifurcation.

Appendix

Proof of Theorem 1. a) The linearization of equation (1) is

$$x'(t) = -\mu x(t-1). \tag{4}$$

From the exponential Ansatz $x(t) = e^{\lambda t}$, we get

$$\lambda e^{\lambda t} = -\mu e^{\lambda(t-1)},$$

and the characteristic equation is

$$\lambda = -\mu e^{-\lambda}.$$

To find the Hopf bifurcation points, we substitute $\lambda = i\omega$, $\omega \in \mathbb{R} \setminus \{0\}$, and write

$$i\omega = -\mu e^{-i\omega} = -\mu \cos\omega + i\mu \sin\omega.$$

Taking real and imaginary parts, we obtain the following system of real equations

$$0 = \mu \cos\omega, \qquad \omega = \mu \sin\omega.$$

From the first equation, we get $\omega = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$. Substituting this into the second equation we have

$$\frac{\pi}{2} + n\pi = \mu \sin\left(\frac{\pi}{2} + n\pi\right).$$

We distinguish two cases:

- $n = 2k$, $k \in \mathbb{Z}$, in which case we find

$$\frac{\pi}{2} + 2k\pi = \mu \sin\left(\frac{\pi}{2} + 2k\pi\right) = \mu;$$

- $n = 2l + 1$, $l \in \mathbb{Z}$, in which case we find

$$\frac{\pi}{2} + (2l+1)\pi = \mu \sin\left(\frac{\pi}{2} + (2l+1)\pi\right) = -\mu,$$

  which is equivalent to

$$\frac{\pi}{2} - (2l+2)\pi = \mu.$$

We find that the via $k = -(l+1)$, the two cases can be treated together, and for the critical values we may just write $\mu_k = \frac{\pi}{2} + 2k\pi = \frac{4k+1}{2}\pi$, $k \in \mathbb{Z}$. For each μ_k there is a pair of critical eigenvalues $\pm i\omega_k$, where $\omega_k = \frac{4k+1}{2}\pi$.

b) We follow the procedure developed in [4], using the parametrization $\mu = \mu_k + \delta_k\eta$ as in Section 4. Let L and F be defined by the relation

$$L(\delta_k\eta)x_t + F(x_t, \delta_k\eta) = -(\mu_k + \delta_k\eta) f(x(t-1)),$$

where x_t is the solution segment defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-1, 0]$. Here, $L(\delta_k\eta)$ is a linear operator from $C([-1,0], \mathbb{R})$ to $\mathbb{R}$, F is an operator from $C([-1,0], \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$ with $F(0, \delta_k\eta) = 0$ and $D_1F(0, \delta_k\eta) = 0$. As equation (1) depends on the parameter linearly, we write

$$L(\delta_k\eta) = L_0 + \delta_k\eta L_1,$$

and, for $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$,

$$\begin{aligned} &F(x_1 e^{i\omega_k\theta} + x_2 e^{-i\omega_k\theta} + x_3 \cdot 1 + x_4 e^{2i\omega_k\theta}, 0) \\ &\quad = B_{(2,0,0,0)}x_1^2 + B_{(1,1,0,0)}x_1x_2 + B_{(1,0,1,0)}x_1x_3 + B_{(0,1,0,1)}x_2x_4 + B_{(2,1,0,0)}x_1^2x_2 + \ldots \end{aligned}$$

Since $\mu_k = \omega_k = \frac{4k+1}{2}\pi$, the equality $e^{i\omega_k} = i$ holds. For $\phi \in C([-1,0], \mathbb{R})$ we have

$$L_0(\phi) = -\mu_k\phi(-1) = -\frac{4k+1}{2}\pi\phi(-1).$$

Hence,

$$\begin{aligned}
L_0(1) &= -\frac{4k+1}{2}\pi, \\
L_0(\theta e^{i\omega_k\theta}) &= -\frac{4k+1}{2}\pi\left(-e^{-i\frac{4k+1}{2}\pi}\right) = -i\frac{4k+1}{2}\pi, \\
L_0(e^{2i\omega_k\theta}) &= -\frac{4k+1}{2}\pi\left(e^{-2i\frac{4k+1}{2}\pi}\right) = \frac{4k+1}{2}\pi,
\end{aligned}$$

and the expansion of F can be written as

$$\begin{aligned}
&F(x_1e^{i\omega_k\theta} + x_2e^{-i\omega_k\theta} + x_3\cdot 1 + x_4e^{2i\omega_k\theta}, 0) \\
&= -\frac{4k+1}{2}\pi\left(B(x_1(-i) + x_2i + x_3 - x_4)^2 + C(x_1(-i) + x_2i + x_3 - x_4)^3 + \text{h.o.t.}\right).
\end{aligned}$$

Then the $B_{(a,b,c,d)}$ coefficients are

$$\begin{aligned}
&B_{(2,0,0,0)} = \frac{4k+1}{2}\pi B, \quad B_{(1,1,0,0)} = -(4k+1)\pi B, \quad B_{(1,0,1,0)} = (4k+1)\pi Bi, \\
&B_{(0,1,0,1)} = (4k+1)\pi Bi, \quad B_{(2,1,0,0)} = \frac{3}{2}(4k+1)\pi Ci.
\end{aligned}$$

According to [4] (see formula (3.18) and Theorem 3.20), the direction of the bifurcation is determined by the sign of

$$K = \mathrm{Re}\left[\frac{1}{1 - L_0(\theta e^{i\omega_k\theta})}\left(B_{(2,1,0,0)} - \frac{B_{(1,1,0,0)}B_{(1,0,1,0)}}{L_0(1)} + \frac{B_{(2,0,0,0)}B_{(0,1,0,1)}}{2i\omega_k - L_0(e^{2i\omega_k})}\right)\right].$$

We shall use the notation $a \sim b$ whenever $a = qb$ for some $q > 0$. Substituting all terms into K, we need to find the sign of the real part of

$$\begin{aligned}
&\frac{1}{1 + i\frac{4k+1}{2}\pi}\left(\frac{3}{2}(4k+1)\pi Ci - \frac{-(4k+1)\pi B(4k+1)\pi Bi}{-\frac{4k+1}{2}\pi} + \frac{\frac{4k+1}{2}\pi B(4k+1)\pi Bi}{i(4k+1)\pi - \frac{4k+1}{2}\pi}\right) \\
&= \frac{(2 - (4k+1)\pi i)}{2(1 + \frac{(4k+1)^2}{2^2})}(4k+1)\pi\left(\frac{3}{2}Ci - 2B^2i + \frac{(-2i-1)B^2i}{5}\right) \\
&\sim (2 - (4k+1)\pi i)(4k+1)\left(\frac{3}{2}Ci + \frac{2}{5}B^2 - \frac{11}{5}B^2i\right).
\end{aligned}$$

The latter expression has real part

$$(4k+1)\left[(4k+1)\pi\frac{3}{2}C + \frac{4}{5}B^2 - \frac{11}{5}(4k+1)\pi B^2\right] \sim C - H(k)B^2.$$

c) To determine the direction a pair of characteristic roots crosses the imaginary axis at a bifurcation point, we differentiate the real part with respect to the parameter. Let us consider a parameter dependent solution of the characteristic equation $\lambda = -\mu e^{-\lambda}$, written as $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$, where α and ω are the real and imaginary parts. Then we have

$$\alpha + i\omega = -\mu e^{-\alpha - i\omega} = -\mu e^{-\alpha}(\cos\omega - i\sin\omega).$$

Separating real and imaginary parts, we get

$$\alpha = -\mu e^{-\alpha}\cos\omega, \qquad \omega = \mu e^{-\alpha}\sin\omega.$$

Differentiating these equations with respect to μ, we find

$$\begin{aligned}
\alpha' &= -e^{-\alpha}\cos\omega - \mu e^{-\alpha}(-\alpha')\cos\omega - \mu e^{-\alpha}(-\sin\omega)\omega', \\
\omega' &= e^{-\alpha}\sin\omega + \mu e^{-\alpha}(-\alpha')\sin\omega + \mu e^{-\alpha}\cos(\omega)\omega'.
\end{aligned}$$

Assuming that the root is critical, $\alpha(\mu_k) = 0$ and $\omega(\mu_k) = \omega_k$. As we have seen in part a), in the critical case $\cos\omega_k = 0$ and $\sin\omega_k = 1$. Then, evaluating the derivatives at μ_k, we obtain

$$\alpha' = \mu_k \omega', \qquad \omega' = 1 - \mu_k \alpha'.$$

Now we substitute ω' into the first equation, and express α' as

$$\alpha'(\mu_k) = \frac{\mu_k}{1 + \mu_k^2}.$$

Hence, $\alpha'(\mu_k)$ and μ_k has the same sign. This means that, at a Hopf bifurcation, a pair of characteristic roots crosses the imaginary axis from left to right if and only if $\mu_k > 0$. Hence the branch of a supercritical Hopf bifurcation starts to the right if and only if $\mu_k > 0$, and the subcritical case is the opposite.

□

Acknowledgements

IB was supported by the EU funded Hungarian project EFOP-3.6.2-16-2017-00015, 20391-3/2018/FEKUSTRAT and NKFI K129322. GR was supported by NKFI FK124016 and Marie Skłodowska-Curie Grant No. 748193. The authors thank Maria Vittoria Barbarossa and Jan Sieber for helping with DDE-BifTool.

Compliance with ethical standards All authors declare that they have no conflict of interest.

References

[1] J. B. van den Berg, J. Jaquette: *A proof of Wright's conjecture.* Journal of Differential Equations 264.12 (2018), pp. 7412–7462.

[2] S.-N. Chow, J. Mallet-Paret: *Integral averaging and bifurcation.* Journal of Differential Equations 26.1 (1977), pp. 112–159.

[3] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, H.-O. Walther: *Delay Equations.* Springer (1995).

[4] T. Faria, L. T. Magalhães: *Normal Forms for Retarded Functional Differential Equations with Parameters and Applications to Hopf Bifurcation.* Journal of Differential Equations 122.2 (1995), pp. 201–224.

[5] Á. Garab, T. Krisztin: *The period function of a delay differential equation and an applicatio.* Periodica Mathetmatica Hungarica 63.2 (2011), pp. 173–190.

[6] F. Giannakopoulos, A. Zapp: *Local and Global Hopf Bifurcation in a Scalar Delay Differential Equation.* Journal of Mathematical Analysis and Applications 237.2 (1999), pp. 425–450.

[7] S. Guo, J. Wu: *Bifurcation Theory of Functional Differential Equations.* Springer (2013).

[8] B. D. Hassard, N. D. Kazarinoff, Y-H. Wan: *Theory and Applications of Hopf Bifurcation.* Cambridge University Press (1981).

[9] J. Jaquette: *A proof of Jones' conjecture.* Journal of Differential Equations 266.6 (2019), pp. 3818–3859.

[10] B. Lani-Wayda: *Hopf Bifurcation for Retarded Functional Differential Equations and for Semiflows in Banach Spaces.* Journal of Dynamics and Differential Equations 25.4 (2013), pp. 1159–1199.

[11] E. Liz, M. Pinto, G. Robledo, S. Trofimchuk, V. Tkachenko: *Wright Type Delay Differential Equations with Negative Schwarzian.* Discrete and Continuous Dynamical Systems 9.2 (2003), pp. 309–321.

[12] E. Liz, G. Röst: *Dichotomy results for delay differential equations with negative Schwarzian derivative.* Nonlinear Analysis: Real World Applications 11.3 (2010), pp. 1422–1430.

[13] E. Liz, G. Röst: *On the Global Attractor of Delay Differential Equations with Unimodal Feedback.* Discrete and Continuous Dynamical Systems 24.4 (2009), pp. 1215–1224.

[14] E. Liz, G. Röst: *Global dynamics in a commodity market model.* Journal of Mathematical Analysis and Applications 398.2 (2013), pp. 707–714.

[15] J. Mallet-Paret, R. Nussbaum: *Global continuation and asymptotic behaviour for periodic solutions of a differential-delay equation.* Annali di Matematica Pura ed Applicata 145.1 (1986), pp. 33–128.

[16] R. Nussbaum: *Wright's equation has no solutions of period four.* Proceedings of the Royal Society of Edinburgh 113.3–4 (1989), pp. 81–288.

[17] G. Röst: *Bifurcation of Periodic Delay Differential Equations at Points of 1:4 Resonance.* Functional Differential Equations 13.3–4 (2006), pp. 519–536.

[18] H. L. Smith: *An Introduction to Delay Differential Equations with Applications to the Life Sciences.* Springer (2011).

[19] H. W. Stech: *Hopf Bifurcation Calculations for Functional Differential Equations.* Journal of Mathematical Analysis and Applications 109.2 (1985), pp. 472–491.