Ferenc Fodor*, Árpád Kurusa and Viktor Vígh

Inequalities for hyperconvex sets

Abstract: An r-hyperconvex body is a set in the d-dimensional Euclidean space $\mathbb{E}^d$ that is the intersection of a family of closed balls of radius r. We prove the analogue of the classical Blaschke–Santaló inequality for r-hyperconvex bodies, and we also establish a stability version of it. The other main result of the paper is an r-hyperconvex version of the reverse isoperimetric inequality in the plane.

Keywords: Blaschke–Santaló inequality, hyperconvex sets, reverse isoperimetric problem, volume product.

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1 Introduction and results

The concept of hyperconvexity may be considered as a generalization of the notion of convexity. Let $r > 0$ be fixed, and let $\mathbf{x}, \mathbf{y}$ be points in the d-dimensional Euclidean space $\mathbb{E}^d$. The closed r-spindle $[\mathbf{x}, \mathbf{y}]_r$ spanned by $\mathbf{x}$ and $\mathbf{y}$ is defined as the intersection of all closed balls of radius r that contain both $\mathbf{x}$ and $\mathbf{y}$, cf. for example [5, Definition 2.1 on page 203]. If the distance of $\mathbf{x}$ and $\mathbf{y}$ is larger than $2r$, then $[\mathbf{x}, \mathbf{y}]_r = \mathbb{E}^d$. A set $H \subseteq \mathbb{E}^d$ is called r-hyperconvex if it contains $[\mathbf{x}, \mathbf{y}]_r$ for every pair of points $\mathbf{x}, \mathbf{y} \in H$. For instance, convex bodies of constant width r are prominent examples of r-hyperconvex sets, cf. [8; 13].

In his 1935 paper, Mayer [21] introduced the term 'Überkonvexität' for this type of convexity in the plane. Following the early literature of the subject, we decided to use the English translation of Mayer's term. However, we note that other expressions such as 'spindle convex' and 'r-convex' have also been used for these sets.

Recently, there has been much renewed interest in r-hyperconvex sets. For details on properties of r-hyperconvex sets, further references and a history of the subject we refer, for example, to Bezdek et al. [5], Bezdek [2; 4], Fejes Tóth and Fodor [14], Lángi et al. [18], and Kupitz et al. [17].

It is a characteristic property of closed convex sets that they are intersections of closed half-spaces. It is known (see e.g. [5, Corollary 3.5 on page 205]) that closed r-hyperconvex sets can be represented as intersections of closed balls of radius r. We use this important property of r-hyperconvex sets throughout the paper. With a slight abuse of notation, if one considers closed balls of radius ∞ as closed half-spaces of $\mathbb{E}^d$, then the closed ∞-convex sets are exactly the closed convex sets of $\mathbb{E}^d$. However, we exclude ∞ from the possible values of r in this paper. Occasionally, we will refer to the classical notion of convexity as *linear convexity* in the text when we want to emphasize its difference from hyperconvexity.

Note that the only unbounded r-hyperconvex set is the whole space $\mathbb{E}^d$, and the only r-hyperconvex sets with no interior points are the one-point sets. We restrict our attention to compact r-hyperconvex sets, which we call r-hyperconvex bodies. For technical reasons, the one-point sets are also considered as r-hyperconvex bodies. We use the term r-hyperconvex disc for a 2-dimensional r-hyperconvex body.

We denote the Euclidean scalar product in $\mathbb{E}^d$ by $\langle \cdot, \cdot \rangle$, the (Euclidean) distance of two points $\mathbf{x}, \mathbf{y} \in \mathbb{E}^d$ by $d(\mathbf{x}, \mathbf{y})$, the d-dimensional volume (Lebesgue measure) of a compact set $H \subset \mathbb{E}^d$ by $\operatorname{vol}(H)$. In the case that $d = 2$, we also use the notation $\operatorname{area}(H)$ for the area of the compact set H in $\mathbb{E}^2$. Let the d-dimensional

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closed unit ball centred at the origin $\mathbf{o}$ be denoted by B^d, its boundary by $\operatorname{bd} B^d = S^{d-1}$, and $\kappa_d = \operatorname{vol}(B^d)$. The interior of a set A is denoted by $\operatorname{int} A$.

The notion of polar duality plays an essential role in the theory of convex bodies. Let $K \subset \mathbb{E}^d$ be a convex body with $\mathbf{z} \in \operatorname{int} K$. The polar of K with respect to $\mathbf{z}$ is defined as

$$K^{\mathbf{z}} = \{\mathbf{x} \in \mathbb{E}^d : \langle \mathbf{x} - \mathbf{z}, \mathbf{y} - \mathbf{z} \rangle \le 1 \text{ for all } \mathbf{y} \in K\}.$$

It is clear that $K^{\mathbf{z}}$ is also a convex body with $\mathbf{z} \in \operatorname{int} K^{\mathbf{z}}$, and $(K^{\mathbf{z}})^{\mathbf{z}} = K$. The latter explains the use of the term 'duality'. For basic properties of polar duality we refer to [25, Section 1.6]. Clearly, $K^{\mathbf{z}}$ depends on the position of $\mathbf{z} \in \operatorname{int} K$.

Santaló proved in [24] that for every convex body K, there exists a unique point $\mathbf{s} \in \operatorname{int} K$ such that $\operatorname{vol}(K^{\mathbf{s}}) \le \operatorname{vol}(K^{\mathbf{z}})$ for all $\mathbf{z} \in \operatorname{int} K$. This unique point $\mathbf{s}$ is called the Santaló point of K. For a convex body K, the quantity $\operatorname{vol}(K)\operatorname{vol}(K^{\mathbf{s}})$ is usually called the volume product of K. The Blaschke–Santaló inequality (see Blaschke [7], Santaló [24], Saint-Raymond [23], Petty [22])

$$\operatorname{vol}(K)\operatorname{vol}(K^{s}) \le \kappa_d^2$$

provides the sharp upper bound κ_d^2 on the volume product for any convex body K in $\mathbb{E}^d$. Equality holds in the Blaschke–Santaló inequality if and only if K is an ellipsoid. On the other hand, it was conjectured by Mahler [20] that the minimum of the volume product is reached by simplices among general convex bodies and by cubes among centrally symmetric convex bodies. Although there are some important partial results in this direction, Mahler's conjecture in its full generality is still unproven. For a discussion and further references on the history of the Blaschke–Santaló inequality and the Mahler conjecture, we refer to the survey paper by Lutwak [19] and to the paper by Böröczky [9].

Recently, Böröczky [9] established a stability version of the Blaschke–Santaló inequality. Note that the volume product of a convex body is invariant with respect to nonsingular affine transformations. Thus it is natural to measure the distance of two convex bodies by the Banach–Mazur distance when dealing with the volume product. Let $\mathrm{GL}(d)$ denote the group of nonsingular linear transformations of $\mathbb{R}^d$. The Banach–Mazur distance of two convex bodies $K_1, K_2 \subset \mathbb{E}^d$ is defined as

$$\delta_{BM}(K_1, K_2) = \min\{\lambda \ge 1 : K_1 - \mathbf{x} \subseteq M(K_2 - \mathbf{y}) \subseteq \lambda(K_1 - \mathbf{x}) \text{ for } M \in \mathrm{GL}(d), \mathbf{x}, \mathbf{y} \in \mathbb{E}^d\}.$$

Theorem (Böröczky [9], Theorem 1.1). *If K is a convex body in $\mathbb{E}^d$, $d \ge 3$, and $\mathbf{s}$ is the Santaló point of K, and*

$$\operatorname{vol}(K)\operatorname{vol}(K^{\mathbf{s}}) > (1 - \varepsilon)\kappa_d^2$$

for $\varepsilon \in (0, 1/2)$, then for some constant γ_0, depending only on the dimension d, it holds that

$$\delta_{BM}(K, B^d) < 1 + \gamma_0 \varepsilon^{\frac{1}{6d}} |\log \varepsilon|^{\frac{1}{6}}.$$

A notion similar to the polar duality of convex sets can be introduced for r-hyperconvex sets following Kupitz et al. [17] and M. Bezdek [6]: the r-hyperconvex dual H^r of a set $H \subseteq \mathbb{E}^d$ consists of the centres of those closed balls of radius r that contain H. In Section 2 we have collected a number of simple properties of r-hyperconvex duality.

Let $S \subset \mathbb{E}^d$ be an r-hyperconvex body. Note that the dual S^r does not depend on the choice of the coordinate system. We define the r-hyperconvex volume product of S as

$$\mathcal{P}(S) := \operatorname{vol}(S)\operatorname{vol}(S^r), \tag{1}$$

and observe immediately that

$$\mathcal{P}\Big(\frac{r}{2}B^d\Big) = \operatorname{vol}^2\Big(\frac{r}{2}B^d\Big). \tag{2}$$

As $\mathcal{P}(rB^d) = 0$, there is no interesting r-hyperconvex version of the Mahler conjecture. However, the r-hyperconvex version of the Blaschke–Santaló inequality can be formulated in the following way.

Theorem 1.1. *If $S \subset \mathbb{E}^d$ is an r-hyperconvex body, then*

$$\mathcal{P}(S) \leq \mathcal{P}\Big(\frac{r}{2}B^d\Big). \tag{3}$$

Equality holds if and only if $S = r/2 \cdot B^d + \mathbf{z}$ for some $\mathbf{z} \in \mathbb{E}^d$.

We establish also a stability version of inequality (3) as follows.

Theorem 1.2. *Let $r > 0$, then there exist constants $c_{d,r} > 0$ and $\varepsilon_{d,r} \in (0, \frac{1}{2})$ depending only on d and r, and a monotonically decreasing positive real function $\mu(\varepsilon)$ with $\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that an r-hyperconvex body $S \subset \mathbb{E}^d$ satisfies*

$$\mathcal{P}(S) \geq (1-\varepsilon)\mathcal{P}\Big(\frac{r}{2}B^d\Big) \tag{4}$$

for some $\varepsilon \in [0, \varepsilon_{d,r}]$ if and only if there exists a vector $\mathbf{z} \in \mathbb{E}^d$ such that

$$\delta_H\Big(S, \frac{r}{2}B^d + \mathbf{z}\Big) \leq c_{d,r}\mu(\varepsilon),$$

where $\delta_H(\,\cdot\,,\,\cdot\,)$ denotes the Hausdorff distance of compact sets.

In Section 4 we prove an r-hyperconvex analogue of the reverse isoperimetric inequality of Ball [1] in the plane. The r-hyperconvex analogue of the reverse isoperimetric problem in the plane is concerned with finding the r-hyperconvex discs of a given perimeter that minimize the area. To the best of our knowledge, this problem was raised by K. Bezdek [3] who communicated it to one of the authors in 2010. K. Bezdek [3] conjectured that among r-hyperconvex bodies of a given surface area, the r-spindle is the unique body that has minimal volume. In our next result, we verify this conjecture in the plane.

Theorem 1.3. *The r-spindle has minimal area among r-hyperconvex discs of equal perimeter.*

Theorem 1.3 is proved in Section 4. We note that our argument does not yield that the r-spindle is the only minimal area r-hyperconvex disc among the r-hyperconvex discs of equal perimeter. Since the unique minimizer of the area is not known, we could not formulate a precise stability statement for the r-hyperconvex reverse isoperimetric problem. However, we have proved with a long and delicate calculation that if the area of an r-hyperconvex triangle is sufficiently close to an r-spindle of the same perimeter, then it is also close to an r-spindle in the Hausdorff metric. Since it is only a partial result, this proof is not included in this paper. However, based on this fact, we formulate the following even stronger conjecture.

Conjecture. *If the volume of an r-hyperconvex body S is sufficiently close to that of an r-spindle S' of the same surface area, then S is close to S' in the Hausdorff metric of compact sets.*

2 Some general properties of *r*-hyperconvex duality

It follows from the definition that the intersection of r-hyperconvex sets is r-hyperconvex. Let S be an r-hyperconvex body, let $\mathbf{x} \in \operatorname{bd} S$ and let $\mathbf{u} \in S^{d-1}$ be an outer unit normal vector to S at $\mathbf{x}$. It is known that $S \subseteq rB^d + \mathbf{x} - r\mathbf{u}$ (see e.g. [5, Corollary 3.4 on page 204]), and we say that the ball $rB^d + \mathbf{x} - r\mathbf{u}$ supports S at $\mathbf{x}$ (see [5, Definition 3.3 on page 205]). The following definition appears in several papers in some form, for example in [5], [6] and [17].

Let $H \subseteq \mathbb{E}^d$ be a point set. We define *the r-hyperconvex dual H^r* of H as

$$H^r = \{\mathbf{y} \in \mathbb{E}^d \mid H \subseteq rB^d + \mathbf{y}\}. \tag{5}$$

Reformulating this as $H^r = \{\mathbf{y} \in \mathbb{E}^d \mid d(\mathbf{x}, \mathbf{y}) \leq r \text{ for every } \mathbf{x} \in H\}$ yields for any set $H \subseteq \mathbb{E}^d$ that

$$H^r = \textstyle\bigcap_{\mathbf{x}\in H}(rB^d + \mathbf{x}). \tag{6}$$

It is immediate from (6) that for any set H the dual H^r is always an r-hyperconvex body (or it is empty).

In the following theorem we summarize certain basic properties of r-hyperconvex duality. We note that Parts (i)–(v) are known (see e.g. [5], [6], [17]). These properties (especially the first one) justify the use of the word 'dual' in view of the corresponding properties of classical polar duality of (linearly) convex bodies; compare Theorems 1.6.1 and 1.6.2 on pages 33–34 in [25]. For a set $H \subseteq \mathbb{E}^d$, let $\operatorname{conv_r} H$ denote the r-hyperconvex hull of H, which is defined as the intersection of all r-hyperconvex sets that contain H.

Theorem 2.1. *For arbitrary sets $H, H_1, H_2 \subseteq \mathbb{E}^d$, and for any r-hyperconvex bodies S, S_1 and S_2 in $\mathbb{E}^d$ we have the following:*

(i) $S^{rr} = S$,

(ii) $H_1 \subseteq H_2$ *implies* $H_1^r \supseteq H_2^r$,

(iii) $(H_1 \cup H_2)^r = H_1^r \cap H_2^r$,

(iv) $H^r = (\operatorname{conv_r} H)^r = (\overline{\operatorname{conv_r} H})^r$,

(v) $(S_1 \cap S_2)^r = \operatorname{conv_r}(S_1^r \cup S_2^r)$.

Furthermore, if $S_1 \cup S_2$ is r-hyperconvex, then $S_1^r \cup S_2^r$ is also r-hyperconvex.

Proof. Part (i) is seen as follows: $S^{rr} = \bigcap_{\mathbf{y} \in S^r}(rB^d + \mathbf{y}) = \bigcap_{\mathbf{y}: S \subseteq rB^d + \mathbf{y}}(rB^d + \mathbf{y}) = S$. The proofs of (ii)–(v) are completely analogous to those of the corresponding statements in linear convexity; for details see e.g. [25, Section 1.6].

It remains to prove the last statement of Theorem 2.1. We claim that if $S_1 \cup S_2$ is r-hyperconvex, then $S_1^r \cup S_2^r = (S_1 \cap S_2)^r$. The relation $S_1^r \cup S_2^r \subseteq (S_1 \cap S_2)^r$ is evident. We need to prove that $S_1^r \cup S_2^r \supseteq (S_1 \cap S_2)^r$. For a set $A \subseteq \mathbb{E}^d$, let A^c denote the complement of A. Suppose, on the contrary, that there exists a point $\mathbf{y} \in (S_1^r \cup S_2^r)^c$ for which $\mathbf{y} \notin ((S_1 \cap S_2)^r)^c$, and seek a contradiction.

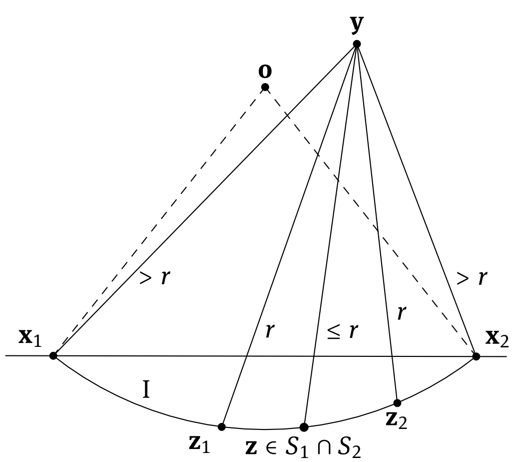


Figure 1. The plane spanned by y, x_1 and x_2

Since $\mathbf{y} \notin S_1^r \cup S_2^r$, there exist $\mathbf{x}_1 \in S_1$ with $d(\mathbf{x}_1, \mathbf{y}) > r$ and $\mathbf{x}_2 \in S_2$ with $d(\mathbf{x}_2, \mathbf{y}) > r$. From the assumption that $\mathbf{y} \notin ((S_1 \cap S_2)^r)^c$ it follows that $\mathbf{x}_1 \notin S_2$ and $\mathbf{x}_2 \notin S_1$. We may clearly assume that the points $\mathbf{y}, \mathbf{x}_1$ and $\mathbf{x}_2$ are not collinear and thus they span a 2-dimensional affine subspace L. We represented L in Figure 1 such that the line through $\mathbf{x}_1$ and $\mathbf{x}_2$ is horizontal and $\mathbf{y}$ is in the upper half-plane. As $S_1 \cup S_2$ is r-hyperconvex, so is $(S_1 \cup S_2) \cap L$. Thus we may join $\mathbf{x}_1$ and $\mathbf{x}_2$ with a shorter circular arc I of radius r and centre $\mathbf{o}$ such that $\mathbf{y}$ and I lie in different half-planes of L determined by the line through $\mathbf{x}_1$ and $\mathbf{x}_2$, as shown in Figure 1.

By continuity, there exists a point $\mathbf{z} \in S_1 \cap S_2$ on the arc I with $d(\mathbf{y}, \mathbf{z}) \leq r$. Furthermore, there is a point $\mathbf{z}_1$ on the the arc between $\mathbf{x}_1$ and $\mathbf{z}$ with $d(\mathbf{y}, \mathbf{z}_1) = r$, and there is another point $\mathbf{z}_2$ on the arc between $\mathbf{z}$ and $\mathbf{x}_2$ such that $d(\mathbf{y}, \mathbf{z}_2) = r$. (Note that $\mathbf{z}_1$ or $\mathbf{z}_2$ (or both) may coincide with $\mathbf{z}$.) Since it is assumed that $\mathbf{y}$ and I

are in different half-planes of L determined by the line through $\mathbf{x}_1$ and $\mathbf{x}_2$, the point $\mathbf{y}$ has to coincide with $\mathbf{o}$, which is a contradiction. This finishes the proof of Theorem 1. □

The support function of a nonempty closed convex set $K \subset \mathbb{E}^d$ is defined as $h_K(\mathbf{x}) := \sup_{\mathbf{y}\in K}\langle \mathbf{x}, \mathbf{y}\rangle$ for $\mathbf{x} \in \mathbb{E}^d$. For basic properties of the support function we refer to [25, Section 1.7].

Note that a supporting hyperplane of an r-hyperconvex body S has exactly one contact point with S. For $\mathbf{u} \in S^{d-1}$, let $\mathbf{x}(\mathbf{u})$ denote the unique point of bd S at which $\mathbf{u}$ is an outer unit normal vector. In the case that S is of constant width, $\mathbf{x}(\mathbf{u})$ and $\mathbf{x}(-\mathbf{u})$ are called opposite points in the literature [8, page 135].

Let $rB^d + \mathbf{y}$ be a supporting ball of S at $\mathbf{x}(\mathbf{u})$. Then, by definition, $\mathbf{y} \in S^r$ and (6) implies that $S^r \subseteq rB^d + \mathbf{x}$. This fact can be summarized in the following well-known statement (see e.g. [8, Section 63]).

Proposition 2.2. *For any $\mathbf{u} \in S^{d-1}$ and any r-hyperconvex body S, we have $h_S(\mathbf{u}) + h_{S^r}(-\mathbf{u}) = r$.*

Proposition 2.2 has a useful consequence, namely that

$$S + (-S^r) = rB^d + \mathbf{x} \tag{7}$$

for some $\mathbf{x} \in \mathbb{E}^d$.

We note that (7) shows that if S is an r-hyperconvex body, then it is a Minkowski summand of the ball rB^d. In fact, using Theorem 3.2.2 in [25] one obtains that for a convex body S the following are equivalent: (i) S is an r-hyperconvex body, (ii) S is a Minkowski summand of rB^d, (iii) S slides freely in rB^d (cf. page 143 in [25]). For more information on Minkowski summands of convex bodies we refer to Sections 3.1 and 3.2 of [25].

If for a set H it holds that $H = H^r$, then we say that H is self-dual with radius r. A self-dual r-hyperconvex body $S \subset \mathbb{E}^d$ with radius r is equal to the intersection of all closed balls of radius r whose centre is contained in S. Eggleston [13] called this the *spherical intersection property* of S. He proved in [13] that a convex body has constant width r if and only if it has the spherical intersection property, that is, it is self-dual with radius r. We state a somewhat similar result that is a direct consequence of Proposition 2.2.

Lemma 2.3. *Let S be an r-hyperconvex body and $\varepsilon \geq 0$. If $\delta_H(S, -S^r + \mathbf{y}) \leq \varepsilon$ for some $\mathbf{y} \in \mathbb{E}^d$, then*

$$\delta_H\Big(S, \frac{r}{2}B^d + \mathbf{z}\Big) \leq \frac{\varepsilon}{2}$$

for some $\mathbf{z} \in \mathbb{E}^d$.

Proof. From (7) we have $h_S(\mathbf{u}) + h_{-S^r}(\mathbf{u}) = r + \langle \mathbf{u}, \mathbf{x}\rangle$ for some $\mathbf{x}$, and by [25, Theorem 1.8.11] we know that $\delta_H(S, -S^r + \mathbf{y}) = \sup_{\mathbf{u}\in S^{d-1}} |h_S(\mathbf{u}) - h_{-S^r+\mathbf{y}}(\mathbf{u})|$. Thus the condition of the lemma implies

$$\Big|h_S(\mathbf{u}) - \Big(\frac{r}{2} + \Big\langle \mathbf{u}, \frac{\mathbf{x}+\mathbf{y}}{2}\Big\rangle\Big)\Big| \leq \frac{\varepsilon}{2}$$

for every $\mathbf{u} \in S^{d-1}$, which proves the lemma with $\mathbf{z} = (\mathbf{x} + \mathbf{y})/2$. □

The quermassintegrals $W_i(\cdot)$ with $i = 0, \ldots, d$ are important geometric quantities associated with convex bodies; for the precise definition and basic properties of quermassintegrals see e.g. [25, Section 4.2]. Even though we will not need them in the proof of Theorems 1.1, 1.2 and 1.3, we note that combining Proposition 2.2 with a result of Chakerian [10] (see also [11, Formula (6.7) on page 66]) one can express the quermassintegrals $W_i(S^r)$ with $i = 0, \ldots, d$ of S^r in terms of those of the r-hyperconvex body S as follows:

$$W_i(S^r) = \sum_{j=0}^{d-i}(-1)^j\binom{d-i}{j}W_{d-j}(S)r^{d-i-j}.$$

Let $K \subset \mathbb{E}^d$ be a convex body with C^2 boundary and strictly positive Gaussian curvature. Let $r_1(K, \mathbf{u}) \leq r_2(K, \mathbf{u}) \leq \cdots \leq r_{d-1}(K, \mathbf{u})$ denote the principal radii of curvature of bd K at $\mathbf{x}$. In the case that K is of constant width w, it is known (see [8, page 136]) that for $i = 1, 2, \ldots, d-1$ it holds that

$$r_i(K, \mathbf{u}) + r_{d-i}(K, -\mathbf{u}) = w.$$

Using Proposition 2.2, one can obtain a similar formula for r-hyperconvex bodies as follows:

$$r_i(S, \mathbf{u}) + r_{d-i}(S^r, -\mathbf{u}) = r.$$

3 Proofs of Theorem 1.1 and Theorem 1.2

For the proof we need the classical Brunn–Minkowski inequality that states that if $C, D \subset \mathbb{E}^d$ are compact convex sets, then

$$\operatorname{vol}^{1/d}(C + D) \geq \operatorname{vol}^{1/d}(C) + \operatorname{vol}^{1/d}(D).$$

If C and D are both proper (full-dimensional), then equality holds if and only if C and D are (positive) homothetic copies; see [25, Theorem 6.1.1, page 309].

Proof of Theorem 1.1. Using Proposition 2.2 one obtains $\operatorname{vol}(rB^d) = \operatorname{vol}(S + (-S^r))$, from which the Brunn–Minkowski inequality and the inequality between the arithmetic and geometric means yield

$$\operatorname{vol}^{1/d}(rB^d) = \operatorname{vol}^{1/d}(S + (-S^r)) \geq \operatorname{vol}^{1/d}(S) + \operatorname{vol}^{1/d}(-S^r) \geq 2\sqrt{\operatorname{vol}^{1/d}(S) \cdot \operatorname{vol}^{1/d}(-S^r)}. \tag{8}$$

This implies that $\operatorname{vol}^2(rB^d) \geq 2^{2d} \operatorname{vol}(S) \cdot \operatorname{vol}(S^r)$, hence $\mathcal{P}(\frac{r}{2}B^d) \geq \mathcal{P}(S)$. In this argument equality holds if and only if S and $-S^r$ are positive homothetic copies of each other having the same volume $\operatorname{vol}(r/2 \cdot B^d)$. This means that S and $-S^r$ are congruent, and hence Lemma 2.3 yields with $\varepsilon = 0$ that $S = r/2 \cdot B^d + \mathbf{x}$ for some $\mathbf{x} \in \mathbb{E}^d$. □

For the proof of Theorem 1.2 we need the following stronger version of the inequality between the arithmetic and geometric means (only for two terms). Let $a \geq b$ be two positive numbers and write $\lambda = a/b - 1$. Then

$$\frac{a+b}{2} \geq \sqrt{ab} + \frac{b\lambda^2}{32} \qquad \text{if } 0 \leq \lambda \leq 8, \text{ and} \tag{9}$$

$$\frac{a+b}{2} \geq \frac{a}{2} = \frac{a}{3} + \frac{a}{6} = \sqrt{a \cdot \frac{a}{9}} + \frac{a}{6} \geq \sqrt{ab} + \frac{a}{6} \qquad \text{if } 8 \leq \lambda. \tag{10}$$

Inequality (9) can be verified by a straightforward direct calculation which we leave to the reader.

In order to prove Theorem 1.2, we use the stability version of the Brunn–Minkowski inequality proved by Groemer [15, Theorem 3 on page 367]; see also [16, pages 134–135]. We do not state Groemer's theorem in its most general form, we only formulate the following consequence of it which we use in our proof.

Let K_1 and K_2 be proper convex bodies in $\mathbb{E}^d$ and let $\varrho > 0$ be a real number with $\operatorname{diam}(K_i) \leq \varrho \operatorname{vol}^{1/d}(K_i)$ for $i = 1, 2$, where $\operatorname{diam}(\cdot)$ denotes the diameter of a set. Let M denote the maximum and m the minimum of $\operatorname{vol}^{1/d}(K_1)$ and $\operatorname{vol}^{1/d}(K_2)$. Furthermore, let K_1' and K_2' be homothetic copies of K_1 and K_2, respectively, that share the same centroid and have unit volume. Then it holds that

$$\operatorname{vol}^{1/d}\Big(\frac{K_1 + K_2}{2}\Big) \geq \frac{1}{2}\operatorname{vol}^{1/d}(K_1) + \frac{1}{2}\operatorname{vol}^{1/d}(K_2) + \omega\delta_H^{d+1}(K_1', K_2'), \tag{11}$$

where

$$\omega = \frac{m}{2^{d+2}}\Big(\frac{d(3 + 2^{-13})}{3^{1/d}}\Big(\frac{2M}{m} + 2\Big)\varrho\Big)^{-d-1}. \tag{12}$$

Proof of Theorem 1.2. Since the 'if' part of the statement is evident, we only prove the 'only if' part. Without loss of generality, we may assume that $\operatorname{vol}(S) \geq \operatorname{vol}(S^r)$. Then using (2) we obtain from (4) and (3) that

$$\operatorname{vol}(S) \geq \sqrt{\mathcal{P}(S)} \geq (1 - \varepsilon)\operatorname{vol}\Big(\frac{r}{2}B^d\Big), \tag{13}$$

$$\operatorname{vol}(S^r) \leq \sqrt{\mathcal{P}(S)} \leq \operatorname{vol}\Big(\frac{r}{2}B^d\Big). \tag{14}$$

Let $a = \operatorname{vol}^{1/d}(S)$ and $b = \operatorname{vol}^{1/d}(S^r) = \operatorname{vol}^{1/d}(-S^r)$.

Assume that $\lambda \geq 8$. Similarly as before, inequality (10) yields

$$\begin{aligned}\operatorname{vol}^{1/d}\Big(\frac{r}{2}B^d\Big) &= \frac{\operatorname{vol}^{1/d}(rB^d)}{2} = \frac{\operatorname{vol}^{1/d}(S + (-S^r))}{2} \geq \frac{\operatorname{vol}^{1/d}(S) + \operatorname{vol}^{1/d}(-S^r)}{2}\\ &\geq \sqrt{ab} + \frac{a}{6} \geq \sqrt{(\mathcal{P}(S))^{1/d}} + \frac{\operatorname{vol}^{1/d}(S)}{6}.\end{aligned}$$

Raising both sides to the power $2d$, then using (2) and (4), we obtain

$$\mathcal{P}\Big(\frac{r}{2}B^d\Big) \geq \mathcal{P}(S) + \frac{\mathrm{vol}^2(S)}{6^{2d}} \geq (1-\varepsilon)\mathcal{P}\Big(\frac{r}{2}B^d\Big) + \frac{\mathrm{vol}^2(S)}{6^{2d}},$$

which can be reformulated by (2) and (13) as

$$6^{2d}\varepsilon \geq \frac{\mathrm{vol}^2(S)}{\mathcal{P}(\frac{r}{2}B^d)} \geq (1-\varepsilon)^2.$$

Therefore there is an $\varepsilon_{d,r} \in (0, 1/2)$ such that the above inequality cannot hold for any $\varepsilon \in (0, \varepsilon_{d,r})$.

From now on we assume that $\varepsilon \in (0, \varepsilon_{d,r})$, whence we have $\lambda \in (0, 8)$.

First we show that the volume of S^r is close to that of S. From the condition on λ we get $b \geq a/9$, thus

$$\mathrm{vol}(S^r) = b^d \geq \frac{a^d}{9^d} \geq \frac{1-\varepsilon}{9^d}\,\mathrm{vol}\Big(\frac{r}{2}B^d\Big).$$

Inequality (9) yields that

$$\begin{aligned}\mathrm{vol}^{1/d}\Big(\frac{r}{2}B^d\Big) &= \frac{\mathrm{vol}^{1/d}(rB^d)}{2} = \frac{\mathrm{vol}^{1/d}(S+(-S^r))}{2} \geq \frac{\mathrm{vol}^{1/d}(S)+\mathrm{vol}^{1/d}(-S^r)}{2}\\ &\geq \sqrt{ab} + \frac{b\lambda^2}{32} \geq \sqrt{(\mathcal{P}(S))^{1/d}} + \frac{(1-\varepsilon)^{1/d}}{9}\cdot\frac{\mathrm{vol}^{1/d}\big(\frac{r}{2}B^d\big)\lambda^2}{32}.\end{aligned}$$

Raising both sides to the power $2d$ we obtain $\mathcal{P}(\frac{r}{2}B^d) - \mathcal{P}(S) \geq \gamma_1\lambda^{4d}$, where γ_1 is a strictly positive constant depending only on d and r. Thus according to (4) we have

$$\varepsilon\mathcal{P}\Big(\frac{r}{2}B^d\Big) \geq \mathcal{P}\Big(\frac{r}{2}B^d\Big) - \mathcal{P}(S) \geq \gamma_1\lambda^{4d}.$$

Since $\mathcal{P}(\frac{r}{2}B^d)$ is bounded from above, we get $\lambda \leq \gamma_2\varepsilon^{1/4d}$, where γ_2 is a constant depending on d and r. As $b\lambda + b = a$, this gives

$$\mathrm{vol}^{1/d}(S) - \mathrm{vol}^{1/d}(S^r) \leq \gamma_3\varepsilon^{1/4d}. \tag{15}$$

for some positive constant γ_3 that depends on d and r only.

Equations (13) and (14) with (15) give

$$\begin{aligned}&\mathrm{vol}^{1/d}(S) - \mathrm{vol}^{1/d}\Big(\frac{r}{2}B^d\Big) \leq \mathrm{vol}^{1/d}(S) - \mathrm{vol}^{1/d}(S^r) \leq \gamma_3\varepsilon^{1/4d} \quad\text{and}\\ &\mathrm{vol}^{1/d}\Big(\frac{r}{2}B^d\Big) - \mathrm{vol}^{1/d}(S) \leq \varepsilon^{1/d}\,\mathrm{vol}^{1/d}\Big(\frac{r}{2}B^d\Big) \leq \varepsilon^{1/4d}\,\mathrm{vol}^{1/d}\Big(\frac{r}{2}B^d\Big).\end{aligned}$$

Thus, using (15), we can choose a positive constant γ_4 that depends only on d and r and satisfies

$$\max\Big\{\Big|\mathrm{vol}^{1/d}(S) - \mathrm{vol}^{1/d}\Big(\frac{r}{2}B^d\Big)\Big|,\ \Big|\mathrm{vol}^{1/d}(S^r) - \mathrm{vol}^{1/d}\Big(\frac{r}{2}B^d\Big)\Big|\Big\} \leq \gamma_4\varepsilon^{1/4d}. \tag{16}$$

Having established (16), now we are ready to complete the proof using (11). Let $\hat{S}$ and $-\hat{S}^r$ be (positive) homothetic copies of S and $-S^r$, respectively, that share a common centroid and have unit volume. Applying (11) to S and $-S^r$, we get

$$\mathrm{vol}^{1/d}\Big(\frac{r}{2}B^d\Big) \geq \frac{1}{2}\,\mathrm{vol}^{1/d}(S) + \frac{1}{2}\,\mathrm{vol}^{1/d}(-S^r) + \omega\delta_H^{d+1}(\hat{S}, -\hat{S}^r),$$

where ω is defined in (12).

Inequality (16) implies that there exists $v_0 > 0$ with the property that $\mathrm{vol}(-S^r) \geq v_0$ for all S that satisfy the conditions of Theorem 1.2. Thus m is bounded away from 0, and M/m is bounded from above (as usual, the constants depend on r and d). Moreover, there exists a $\rho > 0$ with $\varrho < \rho$ for every S that satisfies the conditions of Theorem 1.2. Thus, it follows from (12) that there exists an $\omega_0 > 0$, that depends only on d and r, such that $\omega > \omega_0$.

Finally, comparing this to (16) leads to

$$\gamma_4\varepsilon^{1/4d} \geq \mathrm{vol}^{1/d}\Big(\frac{r}{2}B^d\Big) - \frac{1}{2}\,\mathrm{vol}^{1/d}(S) - \frac{1}{2}\,\mathrm{vol}^{1/d}(-S^r) \geq \omega_0\delta_H^{d+1}(\hat{S}, -\hat{S}^r).$$

This implies the statement of Theorem 1.2 by Lemma 2.3. □

4 Proof of Theorem 1.3

Authors' note. *We note that the following proof of Theorem* 1.3*, and especially the proof of Lemma* 4.1*, is very similar to the one presented in Section 4 of Csikós, Lángi and Naszódi* [12] *on pages 125–126. We have learned of this similarity only after the manuscript had been accepted for publication.*

Clearly, it is sufficient to prove Theorem 1.3 in the case that $r = 1$. We recall that the intersection of a finite number of closed unit radius circular discs is called a (convex) disc-polygon. The notion of side and vertex are self-explanatory, for more details we refer to [6, Definition 1.1]. First, we prove Theorem 1.3 for disc-triangles.

Let **xyz** be a disc-triangle with vertices **x**, **y** and **z**, and with edge-lengths (central angles) α, β and γ, and let **xyz**△ be the corresponding Euclidean triangle with vertices **x**, **y** and **z**, and (Euclidean) edge-lengths a, b and c, as shown in Figure 2. We will show that if one keeps **x** and **y** fixed and moves **z** such that the perimeter of **xyz** remains constant, then area(**xyz**) becomes minimal precisely when **xyz** degenerates into a spindle. We will prove this fact using a combination of elementary geometry and basic calculus. Although the proof does not contain any deep tools, it is quite intricate.

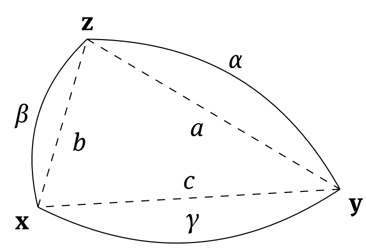


Figure 2. The disc-triangle **xyz** with edge-lenghts α, β and γ, where $\alpha + \beta + \gamma = \kappa$ is the perimeter.

Denote the perimeter of **xyz** by $\kappa = \alpha + \beta + \gamma$. Let $\mu = (\alpha + \beta)/2 = (\kappa - \gamma)/2$, and let ξ be such that $\alpha = \mu + \xi$ and $\beta = \mu - \xi$. Clearly, the variable ξ parametrizes the vertex **z** and makes area(**xyz**) a function of ξ. By symmetry, we may assume that $\alpha \geq \beta$, so it is enough to consider $\xi \in [0, \gamma/2]$. Then we have

$$\operatorname{area}(\mathbf{xyz}) = \frac{\alpha - \sin\alpha}{2} + \frac{\beta - \sin\beta}{2} + \frac{\gamma - \sin\gamma}{2} + \operatorname{area}(\mathbf{xyz}\triangle). \tag{17}$$

For the edges of **xyz**△ we have $a = 2\sin\frac{\alpha}{2}$, $b = 2\sin\frac{\beta}{2}$ and $c = 2\sin\frac{\gamma}{2}$, whence the half perimeter is $s = \frac{a+b+c}{2} = \sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}$. Heron's formula yields

$$\operatorname{area}(\mathbf{xyz}\triangle) = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\Big(-\sin^2\frac{\gamma}{2} + \Big(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2}\Big)^2\Big)\Big(\sin^2\frac{\gamma}{2} - \Big(\sin\frac{\alpha}{2} - \sin\frac{\beta}{2}\Big)^2\Big)}.$$

Since $\sin\alpha + \sin\beta = 2\sin\mu\cos\xi$, $\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} = 2\sin\frac{\mu}{2}\cos\frac{\xi}{2}$ and $\sin\frac{\alpha}{2} - \sin\frac{\beta}{2} = 2\sin\frac{\xi}{2}\cos\frac{\mu}{2}$, equation (17) and the above formula imply

$$\begin{aligned}\operatorname{area}(\mathbf{xyz}) &= \mu - \sin\mu\cos\xi + \frac{\gamma - \sin\gamma}{2} + \operatorname{area}(\mathbf{xyz}\triangle),\\ \operatorname{area}(\mathbf{xyz}\triangle) &= \sqrt{\Big(4\sin^2\frac{\mu}{2}\cos^2\frac{\xi}{2} - \sin^2\frac{\gamma}{2}\Big)\Big(\sin^2\frac{\gamma}{2} - 4\sin^2\frac{\xi}{2}\cos^2\frac{\mu}{2}\Big)}.\end{aligned} \tag{18}$$

To reduce clutter in the calculations below, we introduce the functions $\hat{A}(\xi) = \operatorname{area}(\mathbf{xyz})$ and $A(\xi) = \operatorname{area}(\mathbf{xyz}\triangle)$.

Lemma 4.1. *We have* $\frac{d\hat{A}(\xi)}{d\xi} \leq 0$ *for* $0 \leq \xi < \gamma/2$, *and* $\frac{d\hat{A}(\xi)}{d\xi} = 0$ *if and only if* $\xi = 0$.

Proof. Differentiation of $\hat{A}(\xi)$ with respect to ξ yields

$$\begin{aligned}\frac{d\hat{A}(\xi)}{d\xi} &= \sin\mu\sin\xi + \frac{1}{2A(\xi)}\Big(-4\sin^2\frac{\mu}{2}\cos\frac{\xi}{2}\sin\frac{\xi}{2}\Big(\sin^2\frac{\gamma}{2} - 4\sin^2\frac{\xi}{2}\cos^2\frac{\mu}{2}\Big)\\ &\quad - 4\sin\frac{\xi}{2}\cos\frac{\xi}{2}\cos^2\frac{\mu}{2}\Big(4\sin^2\frac{\mu}{2}\cos^2\frac{\xi}{2} - \sin^2\frac{\gamma}{2}\Big)\Big)\\ &= \sin\mu\sin\xi - \frac{\sin\xi}{A(\xi)}\Big(\sin^2\frac{\mu}{2}\Big(\sin^2\frac{\gamma}{2} - 4\sin^2\frac{\xi}{2}\cos^2\frac{\mu}{2}\Big) + \cos^2\frac{\mu}{2}\Big(4\sin^2\frac{\mu}{2}\cos^2\frac{\xi}{2} - \sin^2\frac{\gamma}{2}\Big)\Big)\\ &= \sin\mu\sin\xi - \frac{\sin\xi}{A(\xi)}\Big(\sin^2\mu\Big(\cos^2\frac{\xi}{2} - \sin^2\frac{\xi}{2}\Big) - \sin^2\frac{\gamma}{2}\Big(\cos^2\frac{\mu}{2} - \sin^2\frac{\mu}{2}\Big)\Big)\\ &= \frac{\sin\xi\sin\mu}{A(\xi)}\Big(A(\xi) - \sin\mu\cos\xi + \sin^2\frac{\gamma}{2}\cot\mu\Big).\end{aligned}$$

Thus $\frac{d\hat{A}(\xi)}{d\xi} \le 0$ if and only if

$$A(\xi) \le \sin\mu\cos\xi - \sin^2\frac{\gamma}{2}\cot\mu, \tag{19}$$

because $\xi \in [0, \frac{\gamma}{2}] \subseteq [0, \frac{\pi}{2}]$ and $\mu \in [0, \pi)$.

To verify (19), we first prove that its right-hand side is positive, and then we only have to show that

$$A^2(\xi) - \Big(\sin\mu\cos\xi - \sin^2\frac{\gamma}{2}\cot\mu\Big)^2 \le 0. \tag{20}$$

Observe that the right-hand side of (19) is positive if and and only if

$$\sin^2\mu\cos\xi - \sin^2\frac{\gamma}{2}\cos\mu > 0.$$

If $\cos\mu < 0$, this is obvious, because $\xi \in [0, \frac{\gamma}{2}] \subseteq [0, \frac{\pi}{2}]$. If $\cos\mu \ge 0$, then using $\sin^2\frac{\gamma}{2} - 4\sin^2\frac{\mu}{2}\cos^2\frac{\xi}{2} = s(c-s) < 0$ we obtain that

$$\begin{aligned}\sin^2\mu\cos\xi - \sin^2\frac{\gamma}{2}\cos\mu &> \sin^2\mu\cos\xi - 4\sin^2\frac{\mu}{2}\cos^2\frac{\xi}{2}\cos\mu = 4\sin^2\frac{\mu}{2}\Big(\cos^2\frac{\mu}{2}\cos\xi - \cos^2\frac{\xi}{2}\cos\mu\Big)\\ &= 4\sin^2\frac{\mu}{2}\Big(\sin^2\frac{\mu}{2}\cos^2\frac{\xi}{2} - \sin^2\frac{\xi}{2}\cos^2\frac{\mu}{2}\Big) = 4\sin^2\frac{\mu}{2}\sin\frac{\mu+\xi}{2}\sin\frac{\mu-\xi}{2} > 0.\end{aligned}$$

Thus the right-hand side of (19) is indeed positive.

To prove (20), we first compute from (18) that

$$\begin{aligned}A^2(\xi) &= \Big(4\sin^2\frac{\mu}{2}\cos^2\frac{\xi}{2} - \sin^2\frac{\gamma}{2}\Big)\Big(\sin^2\frac{\gamma}{2} - 4\sin^2\frac{\xi}{2}\cos^2\frac{\mu}{2}\Big)\\ &= \Big(\Big(4\sin^2\frac{\mu}{2} - \sin^2\frac{\gamma}{2}\Big) - 4\sin^2\frac{\mu}{2}\sin^2\frac{\xi}{2}\Big)\Big(\sin^2\frac{\gamma}{2} - 4\sin^2\frac{\xi}{2}\cos^2\frac{\mu}{2}\Big)\\ &= 4\sin^2\frac{\mu}{2}\Big(\sin^2\frac{\gamma}{2} - 4\sin^2\frac{\xi}{2}\cos^2\frac{\mu}{2}\Big) - \sin^2\frac{\gamma}{2}\Big(\sin^2\frac{\gamma}{2} - 4\sin^2\frac{\xi}{2}\cos^2\frac{\mu}{2}\Big)\\ &\quad - 4\sin^2\frac{\mu}{2}\sin^2\frac{\xi}{2}\sin^2\frac{\gamma}{2} + 4\sin^2\frac{\mu}{2}\sin^2\frac{\xi}{2}4\sin^2\frac{\xi}{2}\cos^2\frac{\mu}{2}\\ &= 4\sin^2\frac{\mu}{2}\sin^2\frac{\gamma}{2} - 4\sin^2\mu\sin^2\frac{\xi}{2} - \sin^4\frac{\gamma}{2} + 4\sin^2\frac{\gamma}{2}\sin^2\frac{\xi}{2}\cos^2\frac{\mu}{2} - 4\sin^2\frac{\mu}{2}\sin^2\frac{\xi}{2}\sin^2\frac{\gamma}{2}\\ &\quad + 4\sin^2\mu\sin^4\frac{\xi}{2}\\ &= 4\sin^2\mu\sin^4\frac{\xi}{2} - 4\sin^2\frac{\xi}{2}\Big(\sin^2\mu - \sin^2\frac{\gamma}{2}\cos^2\frac{\mu}{2} + \sin^2\frac{\mu}{2}\sin^2\frac{\gamma}{2}\Big) + 4\sin^2\frac{\mu}{2}\sin^2\frac{\gamma}{2} - \sin^4\frac{\gamma}{2}\\ &= 4\sin^2\mu\sin^4\frac{\xi}{2} - 4\sin^2\frac{\xi}{2}\Big(\sin^2\mu - \sin^2\frac{\gamma}{2}\cos\mu\Big) + 4\sin^2\frac{\mu}{2}\sin^2\frac{\gamma}{2} - \sin^4\frac{\gamma}{2}.\end{aligned}$$

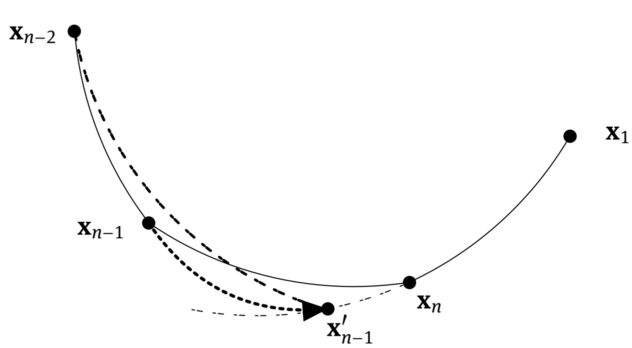
Substituting this into the left-hand side of (20), we obtain

$$\begin{aligned}
&A^2(\xi) - \Big(\sin\mu\cos\xi - \sin^2\frac{\gamma}{2}\cot\mu\Big)^2\\
&\quad= 4\sin^2\mu\sin^4\frac{\xi}{2} - 4\sin^2\frac{\xi}{2}\Big(\sin^2\mu - \sin^2\frac{\gamma}{2}\cos\mu\Big) + 4\sin^2\frac{\mu}{2}\sin^2\frac{\gamma}{2} - \sin^4\frac{\gamma}{2}\\
&\qquad - \sin^2\mu\cos^2\xi - \sin^4\frac{\gamma}{2}\cot^2\mu + 2\sin\mu\cos\xi\sin^2\frac{\gamma}{2}\cot\mu\\
&\quad= 4\sin^2\mu\sin^4\frac{\xi}{2} - 4\sin^2\frac{\xi}{2}\Big(\sin^2\mu - \sin^2\frac{\gamma}{2}\cos\mu\Big) + 4\sin^2\frac{\mu}{2}\sin^2\frac{\gamma}{2} - \sin^4\frac{\gamma}{2}\\
&\qquad - \sin^2\mu\Big(1 - 2\sin^2\frac{\xi}{2}\Big)^2 - \sin^4\frac{\gamma}{2}\cot^2\mu + 2\sin\mu\Big(1 - 2\sin^2\frac{\xi}{2}\Big)\sin^2\frac{\gamma}{2}\cot\mu\\
&\quad= -4\sin^2\frac{\xi}{2}\Big(\sin^2\mu - \sin^2\frac{\gamma}{2}\cos\mu\Big) + 4\sin^2\frac{\mu}{2}\sin^2\frac{\gamma}{2} - \sin^4\frac{\gamma}{2} - \sin^2\mu + 4\sin^2\mu\sin^2\frac{\xi}{2}\\
&\qquad - \sin^4\frac{\gamma}{2}\cot^2\mu + 2\sin\mu\sin^2\frac{\gamma}{2}\cot\mu - 4\sin^2\frac{\xi}{2}\sin^2\frac{\gamma}{2}\cos\mu\\
&\quad= 4\sin^2\frac{\mu}{2}\sin^2\frac{\gamma}{2} - \sin^4\frac{\gamma}{2} - \sin^2\mu - \sin^4\frac{\gamma}{2}\cot^2\mu + 2\sin^2\frac{\gamma}{2}\cos\mu\\
&\quad= \sin^2\frac{\gamma}{2}\Big(4\sin^2\frac{\mu}{2} + 2\cos\mu\Big) - \sin^4\frac{\gamma}{2}\Big(1 + \cot^2\mu\Big) - \sin^2\mu\\
&\quad= \sin^2\frac{\gamma}{2}\Big(4\sin^2\frac{\mu}{2} + 2\Big(1 - 2\sin^2\frac{\mu}{2}\Big)\Big) - \sin^4\frac{\gamma}{2}\frac{1}{\sin^2\mu} - \sin^2\mu\\
&\quad= -(\sin^2\mu - \sin^2\frac{\gamma}{2})^2/\sin^2\mu.
\end{aligned}$$

This is clearly non-positive, hence (20) is proved.

If $\frac{d\hat{A}(\xi)}{d\xi} = 0$, then by the first formula of this proof, either $\xi = 0$ or $\mu = 0$ or $\sin^2\mu = \sin^2(\gamma/2)$ by our last formula. As $2\mu = \alpha + \beta$, we can exclude the second case. If $\sin\mu = \sin(\gamma/2)$ then $\mu = \gamma/2$, hence $\alpha + \beta = \gamma$, which means that $\mathbf{z}$ is on the reflection of the arc $\mathbf{xy}$ (side of $\mathbf{xyz}$) to the straight line $\mathbf{xy}$, i.e. $\mathbf{xyz}$ is not a proper disc-triangle but a spindle. This finishes the proof of Lemma 4.1. □

Now we proceed from disc-triangles to general disc-polygons with an arbitrary number of sides. Let D be a disc-polygon with vertices $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ ($n \geq 4$). Assume that the vertices are labeled in a cyclic order on the boundary of D such that the side $\mathbf{x}_{n-2}\mathbf{x}_{n-1}$ is not shorter than the side $\mathbf{x}_{n-1}\mathbf{x}_n$. We apply Lemma 4.1 to the disc-triangle $\mathbf{x}_{n-2}\mathbf{x}_{n-1}\mathbf{x}_n$ in such a way that $\mathbf{x}_{n-1}$ plays the role of the vertex $\mathbf{z}$. We continuously move $\mathbf{x}_{n-1}$ as described in Lemma 4.1 while all other vertices of D remain fixed and the perimeter of D also remains fixed.



The extreme position of $\mathbf{x}_{n-1}$ is when it is incident with the extension of the arc of the side $\mathbf{x}_n\mathbf{x}_1$. Denote this new point by $\mathbf{x}'_{n-1}$. By Lemma 4.1, area($\mathbf{x}_{n-2}\mathbf{x}'_{n-1}\mathbf{x}_n$) < area($\mathbf{x}_{n-2}\mathbf{x}_{n-1}\mathbf{x}_n$). The points $\mathbf{x}_1, \dots, \mathbf{x}'_{n-1}, \mathbf{x}_n$ determine a new disc-polygon D' with fewer vertices than D ($\mathbf{x}_n$ is no longer a vertex), the same perimeter, and with area(D') < area(D).

Since a general hyperconvex disc may be approximated by disc n-gons arbitrarily well with respect to Hausdorff distance, a simple continuity argument finishes the proof of Theorem 1.3 for general hyperconvex discs.

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