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VARIANCE ESTIMATES FOR RANDOM DISC-POLYGONS IN SMOOTH CONVEX DISCS

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Abstract

In this paper we prove asymptotic upper bounds on the variance of the number of vertices and the missed area of inscribed random disc-polygons in smooth convex discs whose boundary is C^2_+. We also consider a circumscribed variant of this probability model in which the convex disc is approximated by the intersection of random circles.

Keywords: Disc-polygon; random approximation; variance

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1. Introduction and results

Let K be a convex disc (compact convex set with nonempty interior) in the Euclidean plane $\mathbb{R}^2$. We use the notation B^2 for the origin-centred unit-radius closed circular disc, and S^1 for its boundary, the unit circle. The area of Lebesgue measurable subsets of $\mathbb{R}^2$ is denoted by $A(\cdot)$. Assume that the boundary ∂K is of class C^2_+, that is, two times continuously differentiable and the curvature at every point of ∂K is strictly positive. Let $\kappa(x)$ denote the curvature at $x \in \partial K$, and let κ_{m} (κ_{M}) be the minimum (maximum) of $\kappa(x)$ over ∂K. It is known (see [29, Section 3.2]) that in this case a closed circular disc of radius $r_{\mathrm{m}} = 1/\kappa_{\mathrm{M}}$ rolls freely in K, that is, for each $x \in \partial K$, there exists a $p \in \mathbb{R}^2$ with $x \in r_{\mathrm{m}}B^2 + p \subset K$. Moreover, K slides freely in a circle of radius $r_{\mathrm{M}} = 1/\kappa_{\mathrm{m}}$, which means that for each $x \in \partial K$ there is a vector $p \in \mathbb{R}^2$ such that $x \in r_{\mathrm{M}}\partial B^2 + p$ and $K \subset r_{\mathrm{M}}B^2 + p$. The latter yields that for any two points $x, y \in K$, the intersection of all closed circular discs of radius $r \geq r_{\mathrm{M}}$ containing x and y, denoted by $[x, y]_r$ and called the r-spindle of x and y, is also contained in K. Furthermore, for any $X \subset K$, the intersection of all radius $r \geq r_{\mathrm{M}}$ circles containing X, called the closed r-hyperconvex hull (or r-hull for short) and denoted by $\mathrm{conv}_r(X)$, is contained in K. The concept of hyperconvexity, also called spindle convexity or r-convexity, can be traced back to Mayer [21]. For a systematic treatment of geometric properties of hyperconvex sets and further references, see, for example, [10] and [19], and in a more general setting [20]. The notion of convexity arises naturally in many questions where a convex set can be represented as the intersection of equal radius closed balls. As recent examples of such problems, we mention the Kneser–Poulsen conjecture; see, for example, [7]–[9], and inequalities for intrinsic volumes in [22]. A more complete list can be found in [10], for short overviews, see also [15], [16], and [18].

Let K be a convex disc with C^2_+ boundary, and let $x_1, x_2, \ldots$ be independent random points chosen from K according to the uniform probability distribution, and write $X_n = \{x_1, \ldots, x_n\}$.

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The classical convex hull $\mathrm{conv}(X_n)$ is a random convex polygon in K. The geometric properties of $\mathrm{conv}(X_n)$ have been investigated extensively in the literature. For more information on this topic and further references we refer the reader to the surveys [1], [28], [30], [37], and the book [31].

Here we examine the following random model. Let $r \geq r_{\mathrm{M}}$, and let $K_n^r = \mathrm{conv}_r(X_n)$ be the r-hull of X_n, which is a (uniform) random disc-polygon in K. Let $f_0(K_n^r)$ denote the number of vertices (and also the number of edges) of K_n^r, and let $A(K_n^r)$ denote the area of K_n^r. The asymptotic behaviour of the expectation of the random variables $A(K_n^r)$ and $f_0(K_n^r)$ was investigated by Fodor *et al.* [18], where (among others) the following two theorems were proved.

Theorem 1. (Fodor *et al.* [18, Theorem 1.1, p. 901].) *Let K be a convex disc whose boundary is of class C_+^2. For any $r > r_{\mathrm{M}}$, it holds that*

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n^r))n^{-1/3} = \sqrt[3]{\frac{2}{3A(K)}}\,\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\kappa(x)-\frac{1}{r}\right)^{1/3}\mathrm{d}x,$$

and

$$\lim_{n\to\infty} \mathbb{E}(A(K\setminus K_n^r))n^{2/3} = \sqrt[3]{\frac{2A(K)^2}{3}}\,\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\kappa(x)-\frac{1}{r}\right)^{1/3}\mathrm{d}x.$$

Theorem 2. (Fodor *et al.* [18, Theorem 1.2, Equation (1.7), p. 901].) *For $r > 0$, let $K = rB^2$ be the closed circular disc of radius r. Then*

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n^r)) = \tfrac{1}{2}\pi^2, \tag{1}$$

and

$$\lim_{n\to\infty} \mathbb{E}(A(K\setminus K_n^r))n = \tfrac{1}{3}r^2\pi^3.$$

We denote by $\Gamma(\cdot)$ Euler's gamma function, and integration on ∂K is with respect to arc-length.

Observe that in Theorem 2 the expectation $\mathbb{E}(f_0(K_n^r))$ of the number of vertices tends to a constant as $n \to \infty$. This is a surprising fact that has no clear analogue in the classical convex case. A similar phenomenon was recently established in [6] concerning the expectation of the number of facets of certain spherical random polytopes in halfspheres; see [6, Theorem 3.1].

We note that Theorem 1 can also be considered as a generalization of the classical asymptotic results of Rényi and Sulanke about the expectation of the vertex number and missed area of classical random convex polygons in smooth convex discs (see [25], [26]) in the sense that it reproduces the formulas of Rényi and Sulanke in the limit as $r \to \infty$; see [18, Section 3].

Obtaining information on the second-order properties of random variables associated with random polytopes is much more difficult than on first-order properties. It is only recently that variance estimates, laws of large numbers, and central limit theorems have been proved in various models; see, for example, [2]–[5], [13], [17], [23], [24], and [32]–[36]. For an overview, see [1] and [30].

In this paper we prove the following asymptotic estimates for the variance of $f_0(K_n^r)$ and $A(K_n^r)$ in the spirit of Reitzner [23].

For the order of magnitude, we use the following common symbols: if for two functions $f, g\colon I \to \mathbb{R}$, $I \subset \mathbb{R}$, there is a constant $\gamma > 0$ such that $|f| \leq \gamma g$ on I, then we write $f \ll g$ or $f = O(g)$. If $f \ll g$ and $g \ll f$, then this fact is indicated by the notation $f \approx g$.

Theorem 3. *With the same hypotheses as in Theorem 1, it holds that*

$$\mathrm{var}(f_0(K_n^r)) \ll n^{1/3}, \tag{2}$$

and

$$\mathrm{var}(A(K_n^r)) \ll n^{-5/3}, \tag{3}$$

where the implied constants depend only on K and r.

In the special case when K is the closed circular disc of radius r, we prove the following theorem.

Theorem 4. *With the same hypotheses as in Theorem 2, it holds that*

$$\mathrm{var}(f_0(K_n^r)) \approx constant, \tag{4}$$

and

$$\mathrm{var}(A(K_n^r))) \ll n^{-2}, \tag{5}$$

where the implied constants depend only on r.

From Theorem 3 we can conclude the following strong laws of large numbers. Since the proof follows a standard argument based on Chebysev's inequality and the Borel–Cantelli lemma (see, for example, [13, p. 2294] or [23, Section 5], and [3, p. 174]), we omit the details.

Theorem 5. *With the same hypotheses as in Theorem 1, it holds with probability* 1 *that*

$$\lim_{n\to\infty} f_0(K_n^r)n^{-1/3} = \sqrt[3]{\frac{2}{3A(K)}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\kappa(x)-\frac{1}{r}\right)^{1/3}\mathrm{d}x,$$

and

$$\lim_{n\to\infty} A(K\setminus K_n^r)n^{2/3} = \sqrt[3]{\frac{2A(K)^2}{3}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\kappa(x)-\frac{1}{r}\right)^{1/3}\mathrm{d}x.$$

In the theory of random polytopes there is more information on models in which the polytopes are generated as the convex hull of random points from a convex body K than on polyhedral sets produced by random closed half-spaces containing K. For some recent results and references in this direction, see, for example, [11], [12], [17], and the survey [30].

In Section 5 we consider a model of random disc-polygons that contain a given convex disc with C^2_+ boundary. In this circumscribed probability model, we give asymptotic formulas for the expectation of the number of vertices of the random disc-polygon, and the area difference and the perimeter difference of the random disc-polygon and K; see Theorem 6. Furthermore, Corollary 1 provides an asymptotic upper bound on the variance of the number of vertices of the circumscribed random polygons.

The outline of the paper is as follows. In Section 2 we collect some geometric facts that are needed for the arguments. Theorem 3 is proved in Section 3, and Theorem 4 is verified in Section 4. In Section 5 we discuss a different probability model in which K is approximated by the intersection of random closed circular discs containing K. This model is a kind of dual to the inscribed one.

2. Preparations

We note that it is enough to prove Theorem 3 for the case when $r_{\mathrm{M}} < 1$ and $r = 1$, and Theorem 4 for $r = 1$. The general statements then follow by a simple scaling argument. Therefore, from now on we assume that $r = 1$ and to simplify notation we write K_n for K_n^1.

Let $\overline{B}^2$ denote the open unit ball of radius 1 centred at the origin o. A *disc-cap* (of radius 1) of K is a set of the form $K \setminus (\overline{B}^2 + p)$ for some $p \in \mathbb{R}^2$.

We start with recalling the following notation from [18]. Let x and y be two points from K. The two unit circles passing through x and y determine two disc-caps of K, which we denote by $D_-(x, y)$ and $D_+(x, y)$, respectively, such that $A(D_-(x, y)) \le A(D_+(x, y))$. For brevity of notation, we write $A_-(x, y) = A(D_-(x, y))$ and $A_+(x, y) = A(D_+(x, y))$. In [18, see Lemma 3] it was shown that if the boundary of K is of class C_+^2 ($r_{\mathrm{M}} < 1$) then there exists a $\delta > 0$ (depending only on K) with the property that for any $x, y \in \operatorname{int} K$, it holds that $A_+(x, y) > \delta$.

We need some further technical lemmas about general disc-caps. Let $u_x \in S^1$ denote the (unique) outer unit normal to K at the boundary point x, and $x_u \in \partial K$ the unique boundary point with outer unit normal $u \in S^1$.

Lemma 1. (Fodor *et al.* [18, Lemma 4.1, p. 905].) *Let K be a convex disc with C_+^2 smooth boundary and assume that $\kappa_{\mathrm{m}} > 1$. Let $D = K \setminus (\overline{B}^2 + p)$ be a nonempty disc-cap of K (as above). Then there exists a unique point $x_0 \in \partial K \cap \partial D$ such that there exists a $t \ge 0$ with $B^2 + p = B^2 + x_0 - (1 + t)u_{x_0}$. We refer to x_0 as the* vertex *of D and to t as the* height of D.

Let $D(u, t)$ denote the disc-cap with vertex $x_u \in \partial K$ and height t. Note that for each $u \in S^1$, there exists a maximal positive constant $t^*(u)$ such that $(B + x_u - (1 + t)u) \cap K \ne \emptyset$ for all $t \in [0, t^*(u)]$. For simplicity, we let $A(u, t) = A(D(u, t))$ and let $\ell(u, t)$ denote the arc-length of $\partial D(u, t) \cap (\partial B + x_u - (1 + t)u)$.

We need the following limit relations about the behaviour of $A(u, t)$ and $\ell(u, t)$ which we recall from [18, Lemma 4.2, p. 905]:

$$\lim_{t\to 0^+} \ell(u_x, t)t^{-1/2} = 2\sqrt{\frac{2}{\kappa(x) - 1}}, \qquad \lim_{t\to 0^+} A(u_x, t)t^{-3/2} = \frac{4}{3}\sqrt{\frac{2}{\kappa(x) - 1}}. \tag{6}$$

It is clear that (6) implies that $A(u, t)$ and $\ell(u, t)$ satisfy the following relations uniformly in u:

$$\ell(u_x, t) \approx t^{1/2}, \qquad A(u_x, t) \approx t^{3/2}, \tag{7}$$

where the implied constants depend only on K.

Let D be a disc-cap of K with vertex x. For a line $e \subset \mathbb{R}^2$ with $e \perp u_x$, let e_+ denote the closed half-plane containing x. Then there exist a maximal cap $C_-(D) = K \cap e_+ \subset D$, and a minimal cap $C_+(D) = e'_+ \cap K \supset D$.

Claim 1. *There exists a constant $\hat{c}$ depending only K such that if the height of the disc-cap D is sufficiently small, then*

$$\hat{c}(C_-(D) - x) \supset (C_+(D) - x).$$

Proof. Denote by h_- (h_+) the height of $C_-(D)$ ($C_+(D)$, respectively), which is the distance of x and e (e', respectively). By convexity, it is enough to find a constant $\hat{c} > 0$ such that for all disc-caps of K with sufficiently small height $h_+/h_- < \hat{c}$ holds.

Choose an arbitrary $R \in (1/\kappa_{\mathrm{m}}, 1)$, and consider $\hat{B} = RB^2 + x - Ru_x$, the disc of radius R that supports K in x. Clearly, $\hat{B} \supset K$ implies that $D = K \cap (\overline{B}^2 + p) \subset (\hat{B} \cap (\overline{B}^2 + p) = \hat{D}$.

Also, for the respective heights $\hat{h}_-$ and $\hat{h}_+$ of $C_-(\hat{D})$ and $C_+(\hat{D})$, we have $\hat{h}_- = h_-$ and $\hat{h}_+ > h_+$. Thus, it is enough to find $\hat{c}$ such that $\hat{h}_+/\hat{h}_- < \hat{c}$. The existence of such $\hat{c}$ is clear from elementary geometry. □

Let $x_i, x_j (i \neq j)$ be two points from X_n, and let $B(x_i, x_j)$ be one of the unit discs that contain x_i and x_j on its boundary. The shorter arc of $\partial B(x_i, x_j)$ forms an edge of K_n if the entire set X_n is contained in $B(x_i, x_j)$. Note that it may happen that the pair x_i, x_j determines two edges of K_n if the above condition holds for both unit discs that contain x_i and x_j on its boundary.

We recall that the Hausdorff distance $d_{\mathrm{H}}(A, B)$ of two nonempty compact sets $A, B \subset \mathbb{R}^2$ is

$$d_{\mathrm{H}}(A, B) := \max\Big\{\max_{a\in A}\min_{b\in B} d(a, b), \max_{b\in B}\min_{a\in A} d(a, b)\Big\},$$

where $d(a, b)$ is the Euclidean distance of a and b.

First, we note that for the proof of Theorem 3, similar to [23], we may assume that the Hausdorff distance $d_{\mathrm{H}}(K, K_n)$ of K and K_n is at most ε_K, where $\varepsilon_K > 0$ is a suitably chosen constant. This can be seen in the following way. Assume that $d_{\mathrm{H}}(K, K_n) \geq \varepsilon_K$. Then there exists a point x on the boundary of K_n such that $\varepsilon_K B^2 + x \subset K$. There exists a supporting circle of K_n through x that determines a disc-cap of height at least ε_K. By the above remark, the probability content of this disc-cap is at least $c_K > 0$, where c_K is a suitable constant depending on K and ε_K. Then

$$\mathbb{P}(d_{\mathrm{H}}(K, K_n) \geq \varepsilon_K) \leq (1 - c_K)^n. \tag{8}$$

Our main tool in the variance estimates is the Efron–Stein inequality [14], which has previously been used to provide upper estimates on the variance of various geometric quantities associated with random polytopes in convex bodies; see [23], and for further references in this topic we recommend the recent surveys [1] and [30].

3. Proof of Theorem 3

We present the proof of the asymptotic upper bound on the variance of the vertex number in detail. Since the argument for the variance of the missed area is very similar, we only indicate the key steps in the last few paragraphs of this section. Our argument is similar to the one in [23, Sections 4 and 6]. The basic idea of the argument rests on the Efron–Stein inequality, which bounds the variance of a random variable (in our case the vertex number or the missed area) in terms of expectations. To calculate the involved expectations, we use some basic geometric properties of disc-caps and the integral transformation [18, pp. 907–909], see also [27]. Finally, the asymptotic estimate (11) in [13, p. 2290] for the order of magnitude of beta integrals yields the desired asymptotic upper bound.

For the number of vertices of K_n, the Efron–Stein inequality [14] states the following:

$$\operatorname{var} f_0(K_n) \leq (n + 1)\mathbb{E}(f_0(K_{n+1}) - f_0(K_n))^2.$$

Let x be an arbitrary point of K and let $x_i x_j$ be an edge of K_n. Following [23], we say that the edge $x_i x_j$ is visible from x if x is not contained in K_n and it is not contained in the unit disc of the edge $x_i x_j$. For a point $x \in K \setminus K_n$, let $\mathcal{F}_n(x)$ denote the set of edges of K_n that can be seen from x, and for $x \in K_n$, set $\mathcal{F}_n(x) = \varnothing$. Let $F_n(x) = |\mathcal{F}_n(x)|$.

Let x_{n+1} be a uniform random point in K chosen independently from X_n. If $x_{n+1} \in K_n$ then $f_0(K_{n+1}) = f_0(K_n)$. If, on the other hand, $x_{n+1} \notin K_n$ then

$$f_0(K_{n+1}) = f_0(K_n) + 1 - (F_n(x_{n+1}) - 1) = f_0(K_n) - F_n(x_{n+1}) + 2.$$

Therefore,

$$|f_0(K_{n+1}) - f_0(K_n)| \leq 2F_n(x_{n+1}),$$

and, by the Efron–Stein jackknife inequality,

$$\operatorname{var}(f_0(K_n)) \leq (n+1)\mathbb{E}(f_0(K_{n+1}) - f_0(K_n))^2 \leq 4(n+1)\mathbb{E}(F_n^2(x_{n+1})). \tag{9}$$

Similar to [23], we introduce the following notation; see [23, p. 2147]. Let $I = (i_1, i_2)$, $i_1 \neq i_2$, $i_1, i_2 \in \{1, 2, \dots\}$ be an ordered pair of indices. Denote by F_I the shorter arc of the unique unit circle incident with x_{i_1} and x_{i_2} on which x_{i_1} follows x_{i_2} in the positive cyclic ordering of the circle. Let $\mathbf{1}(A)$ denote the indicator function of the event A. For the sake of brevity, we use the notation $x_1, x_2, \dots$ for the integration variables as well.

We wish to estimate the expectation $\mathbb{E}(F_n^2(x_{n+1}))$ under the condition that $d_{\mathrm{H}}(K, K_n) < \varepsilon_K$. To compensate for the cases in which $d_{\mathrm{H}}(K, K_n) \geq \varepsilon_k$, using (8), we add an error term $O((1 - c_K)^n)$. Thus,

$$\begin{aligned}
&\mathbb{E}(F_n(x_{n+1})^2)\\
&\quad= \frac{1}{A(K)^{n+1}} \int_K \int_{K^n} \bigg(\sum_I \mathbf{1}(F_I \in \mathcal{F}_n(x_{n+1}))\bigg)^2 \,\mathrm{d}X_n \,\mathrm{d}x_{n+1}\\
&\quad= \frac{1}{A(K)^{n+1}} \int_K \int_{K^n} \bigg(\sum_I \mathbf{1}(F_I \in \mathcal{F}_n(x_{n+1}))\bigg)\bigg(\sum_J \mathbf{1}(F_J \in \mathcal{F}_n(x_{n+1}))\bigg) \,\mathrm{d}X_n \,\mathrm{d}x_{n+1}\\
&\quad\leq \frac{1}{A(K)^{n+1}} \sum_I \sum_J \int_K \int_{K^n} \mathbf{1}(F_I \in \mathcal{F}_n(x_{n+1}))\mathbf{1}(F_J \in \mathcal{F}_n(x_{n+1}))\\
&\qquad\qquad \times \mathbf{1}(d_{\mathrm{H}}(K, K_n) \leq \varepsilon_K) \,\mathrm{d}X_n \,\mathrm{d}x_{n+1} + O((1 - c_K)^n).
\end{aligned} \tag{10}$$

Choose ε_K so small that $A(K \setminus K_n) < \delta$. Note that with this choice of ε_K only one of the two shorter arcs determined by x_{i_1} and x_{i_2} can determine an edge of K_n.

Now we fix the number k of common elements of I and J, that is, $|I \cap J| = k$. Let F_1 denote one of the shorter arcs spanned by x_1 and x_2, and let F_2 be one of the shorter arcs determined by x_{3-k} and x_{4-k}. Since the random points are independent, we have

$$\begin{aligned}
(10) &\ll \frac{1}{A(K)^{n+1}} \sum_{k=0}^{2} \binom{n}{2}\binom{2}{k}\binom{n-2}{2-k}\\
&\quad \times \int_K \int_{K^n} \mathbf{1}(F_1 \in \mathcal{F}_n(x_{n+1}))\mathbf{1}(F_2 \in \mathcal{F}_n(x_{n+1}))\\
&\qquad \times \mathbf{1}(d_{\mathrm{H}}(K, K_n) \leq \varepsilon_K) \,\mathrm{d}X_n \,\mathrm{d}x_{n+1} + O((1 - c_K)^n)\\
&\ll \frac{1}{A(K)^{n+1}} \sum_{k=0}^{2} n^{4-k} \int_K \cdots \int_K \mathbf{1}(F_1 \in \mathcal{F}_n(x_{n+1}))\mathbf{1}(F_2 \in \mathcal{F}_n(x_{n+1}))\\
&\qquad \times \mathbf{1}(d_{\mathrm{H}}(K, K_n) \leq \varepsilon_K) \,\mathrm{d}X_n \,\mathrm{d}x_{n+1} + O((1 - c_K)^n).
\end{aligned} \tag{11}$$

Since the roles of F_1 and F_2 are symmetric, we may assume that $\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)$, where $D_1 = D_-(x_1, x_2)$ and $D_2 = D_-(x_{3-k}, x_{4-k})$ are the corresponding disc-caps, and $\operatorname{diam}(\cdot)$ denotes the diameter of a set. Thus,

$$
\begin{aligned}
(11) \ll \frac{1}{A(K)^{n+1}} \sum_{k=0}^{2} n^{4-k} \int_K \int_{K^n} & \mathbf{1}(F_1 \in \mathcal{F}_n(x_{n+1})) \\
& \times \mathbf{1}(F_2 \in \mathcal{F}_n(x_{n+1}))\mathbf{1}(\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)) \\
& \times \mathbf{1}(d_{\mathrm{H}}(K, K_n) \leq \varepsilon_K)\, \mathrm{d}X_n\, \mathrm{d}x_{n+1} + O((1-c_K)^n).
\end{aligned} \tag{12}
$$

Clearly, x_{n+1} is a common point of the disc-caps D_1 and D_2, so we may write

$$
\begin{aligned}
(12) \leq \frac{1}{A(K)^{n+1}} \sum_{k=0}^{2} n^{4-k} \int_K \int_{K^n} & \mathbf{1}(F_1 \in \mathcal{F}_n(x_{n+1})) \\
& \times \mathbf{1}(D_1 \cap D_2 \neq \varnothing)\mathbf{1}(\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)) \\
& \times \mathbf{1}(d_{\mathrm{H}}(K, K_n) \leq \varepsilon_K)\, \mathrm{d}X_n\, \mathrm{d}x_{n+1} + O((1-c_K)^n).
\end{aligned} \tag{13}
$$

In order for F_1 to be an edge of K_n, it is necessary that $x_{5-k}, \ldots x_n \in K \setminus D_1$, and for $F_1 \in \mathcal{F}_n(x_{n+1})$ x_{n+1} must be in D_1. Therefore,

$$
\begin{aligned}
(13) &\ll \frac{1}{A(K)^{n+1}} \sum_{k=0}^{2} n^{4-k} \int_K \cdots \int_K (A(K) - A(D_1))^{n-4+k} A(D_1) \\
&\qquad\qquad \times \mathbf{1}(D_1 \cap D_2 \neq \varnothing)\mathbf{1}(\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)) \\
&\qquad\qquad \times \mathbf{1}(d_{\mathrm{H}}(K, K_n) \leq \varepsilon_K)\, \mathrm{d}x_1 \cdots \mathrm{d}x_{4-k} + O((1-c_K)^n) \\
&\ll \sum_{k=0}^{2} n^{4-k} \int_K \cdots \int_K \left(1 - \frac{A(D_1)}{A(K)}\right)^{n-4+k} \frac{A(D_1)}{A(K)} \\
&\qquad\qquad \times \mathbf{1}(D_1 \cap D_2 \neq \varnothing)\mathbf{1}(\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)) \\
&\qquad\qquad \times \mathbf{1}(d_{\mathrm{H}}(K, K_n) \leq \varepsilon_K)\, \mathrm{d}x_1 \cdots \mathrm{d}x_{4-k} + O((1-c_K)^n).
\end{aligned} \tag{14}
$$

Reitzner (see [23, pp. 2149–2150]) proved that if $D_1 \cap D_2 \neq \varnothing$, $d_{\mathrm{H}}(K, K_n) \leq \varepsilon_K$, and $\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)$ then there exists a constant $\bar{c}$ (depending only on K) such that $C_+(D_2) \subset \bar{c}(C_+(D_1) - x_{D_1}) + x_{D_1}$, where x_{D_1} is the vertex of D_1. Combining this with Claim 1 we find that there is a constant c_1 depending only on K, such that $D_2 \subset c_1(D_1 - x_{D_1}) + x_{D_1}$. Hence, $A(D_2) \leq c_1^2 A(D_1)$ and, therefore,

$$
\begin{aligned}
\int_K \cdots \int_K & \mathbf{1}(D_1 \cap D_2 \neq \varnothing)\mathbf{1}(\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)) \\
& \times \mathbf{1}(d_{\mathrm{H}}(K, K_n) \leq \varepsilon_K)\, \mathrm{d}x_3 \cdots \mathrm{d}x_{4-k} \\
& \ll A(D_1)^{2-k}.
\end{aligned}
$$

We continue by estimating (14) term by term (omitting the $O((1-c_K)^n)$ term).

$$n^{4-k}\int_K\cdots\int_K\left(1-\frac{A(D_1)}{A(K)}\right)^{n-4+k}\frac{A(D_1)}{A(K)}\mathbf{1}(D_1\cap D_2\neq\varnothing)$$
$$\times\,\mathbf{1}(\operatorname{diam}C_+(D_1)\geq\operatorname{diam}C_+(D_2))\mathbf{1}(d_{\mathrm{H}}(K,K_n)\leq\varepsilon_K)\,\mathrm{d}x_1\cdots\mathrm{d}x_{4-k}$$
$$\ll n^{4-k}\int_K\int_K\left(1-\frac{A(D_1)}{A(K)}\right)^{n-4+k}\left(\frac{A(D_1)}{A(K)}\right)^{3-k}\mathbf{1}(d_{\mathrm{H}}(K,K_n)\leq\varepsilon_K)\,\mathrm{d}x_1\,\mathrm{d}x_2. \quad (15)$$

Now, we use the following parametrization of (x_1, x_2) the same way as in [18] to transform the integral. Let

$$(x_1,x_2)=\Phi(u,t,u_1,u_2),$$

where $u, u_1, u_2 \in S^1$, and $0\leq t\leq t_0(u)$ are chosen such that

$$D(u,t)=D_1=D_-(x_1,x_2)\quad\text{and}\quad(x_1,x_2)=(x_u-(1+t)u+u_1,x_u-(1+t)u+u_2).$$

More information on this transformation can be found in [18, pp. 907–909]. Here we just recall that the Jacobian of Φ is

$$|J\Phi|=\left(1+t-\frac{1}{\kappa(x_u)}\right)|u_1\times u_2|,$$

where $u_1\times u_2$ denotes the cross product of u_1 and u_2.

Let $L(u,t)=\partial D_1\cap\operatorname{int}K$ then we obtain

$$(15)\ll n^{4-k}\int_{S^1}\int_0^{t^*(u)}\int_{L(u,t)}\int_{L(u,t)}\left(1-\frac{A(u,t)}{A(K)}\right)^{n-4+k}\left(\frac{A(u,t)}{A(K)}\right)^{3-k}$$
$$\times\left(1+t-\frac{1}{\kappa(x_u)}\right)|u_1\times u_2|\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}t\,\mathrm{d}u$$
$$=n^{4-k}\int_{S^1}\int_0^{t^*(u)}\left(1-\frac{A(u,t)}{A(K)}\right)^{n-4+k}\left(\frac{A(u,t)}{A(K)}\right)^{3-k}$$
$$\times\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t)-\sin\ell(u,t))\,\mathrm{d}t\,\mathrm{d}u. \quad (16)$$

From now on the evaluation follows in a standard way. First, we split the domain of integration with respect to t into two parts. Let $h(n)=(c\ln n/n)^{2/3}$, where $c>0$ is a sufficiently large absolute constant. Using (7), it follows that $A(u,t)\geq\gamma t^{3/2}$ uniformly in $u\in S^1$; hence,

$$n^{4-k}\int_{S^1}\int_{h(n)}^{t^*(u)}\left(1-\frac{A(u,t)}{A(K)}\right)^{n-4+k}\left(\frac{A(u,t)}{A(K)}\right)^{3-k}$$
$$\times\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t)-\sin\ell(u,t))\,\mathrm{d}t\,\mathrm{d}u$$
$$\ll n^{4-k}\int_{S^1}\int_{h(n)}^{t^*(u)}\left(1-\frac{A(u,t)}{A(K)}\right)^{n-4+k}\mathrm{d}t\,\mathrm{d}u$$
$$\ll n^{4-k}\int_{S^1}\int_{h(n)}^{t^*(u)}\left(1-\frac{\gamma t^{3/2}}{A(K)}\right)^{n-4+k}\mathrm{d}t\,\mathrm{d}u$$
$$\ll n^{4-k}\left(1-\frac{\gamma h(n)^{3/2}}{A(K)}\right)^{n-4+k}$$

$$= n^{4-k}\left(1-\frac{\gamma(c\ln n)}{nA(K)}\right)^{n-4+k}$$
$$\ll n^{-2/3}$$

if $\gamma c/A(K)$ is sufficiently large.

Therefore, it is enough to estimate the following part of (16):

$$n^{4-k}\int_{S^1}\int_0^{h(n)}\left(1-\frac{A(u,t)}{A(K)}\right)^{n-4+k}\left(\frac{A(u,t)}{A(K)}\right)^{3-k} \times\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t)-\sin\ell(u,t))\,\mathrm{d}t\,\mathrm{d}u. \tag{17}$$

Using (7) and the Taylor series of the sine function, we obtain $\ell(u,t)-\sin\ell(u,t)\ll t^{3/2}$. Since $\kappa(x)>1$ for all $x\in\partial K$, it follows that $0<1+t-\kappa(x_u)^{-1}\ll 1$. We also use (7) to estimate $A(u,t)$, similarly as before. Assuming that n is large enough, we obtain

$$(17)\ll n^{4-k}\int_{S^1}\int_0^{h(n)}\left(1-\frac{\gamma t^{3/2}}{A(K)}\right)^{n-4+k}(t^{3/2})^{3-k}\cdot 1\cdot t^{3/2}\,\mathrm{d}t\,\mathrm{d}u$$
$$\ll n^{4-k}\int_0^{h(n)}\left(1-\frac{\gamma t^{3/2}}{A(K)}\right)^{n-4+k}t^{(12-3k)/2}\,\mathrm{d}t$$
$$\ll n^{-2/3},$$

where the last inequality follows directly from [13, Equation (11), p. 2290]. Together with (9), this yields the desired upper estimate for var $f_0(K_n)$.

As the argument for the case of the missing area is very similar, we only highlight the major steps.

Again, we use the Efron–Stein inequality [14], which states the following for the missed area:

$$\operatorname{var} A(K\setminus K_n)\le(n+1)\mathbb{E}(A(K_{n+1})-A(K_n))^2.$$

Therefore, we need to estimate $\mathbb{E}(A(K_{n+1})-A(K_n))^2$. Following the ideas of Reitzner [23], we see that

$$\mathbb{E}(A(K_{n+1})-A(K_n))^2\ll\sum_I\sum_J\int_K\int_{K^n}\mathbf{1}(F_1\in\mathcal{F}_n(x_{n+1}))A(D_1)\mathbf{1}(F_2\in\mathcal{F}_n(x_{n+1}))A(D_2) \times\mathbf{1}(d_{\mathrm{H}}(K,K_n)\le\varepsilon_K)\,\mathrm{d}X_n\,\mathrm{d}x_{n+1}. \tag{18}$$

From here, we closely follow the proof of (2), the only major difference being the extra $A(D_1)A(D_2)\le A^2(D_1)$ factor in the integrand. After similar calculations as for the vertex number, we obtain

$$(18)\ll n^{4-k}\int_{S^1}\int_0^{h(n)}\left(1-\frac{A(u,t)}{A(K)}\right)^{n-4+k}\left(\frac{A(u,t)}{A(K)}\right)^{5-k} \times\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t)-\sin\ell(u,t))\,\mathrm{d}t\,\mathrm{d}u.$$
$$\ll n^{4-k}\int_0^{h(n)}(1-c_Kt^{3/2})^{n-4+k}t^{(20-3k)/2}\,\mathrm{d}t$$
$$\ll n^{-8/3},$$

which proves (3) (the missing factor n comes from the Efron–Stein inequality).

4. The case of the circle

In this section we prove Theorem 4. In particular, we give a detailed proof of the estimate (4) for the variance of the number of vertices of the random disc-polygon. The case of the missed area (5) is very similar.

Without loss of generality, we may assume that $K = B^2$, and that $r = 1$.

We begin by recalling from [18] that for any $u \in S^1$ and $0 \le t \le 2$, it holds that

$$\ell(u,t) = 2\arcsin\sqrt{1-\tfrac{1}{4}t^2}, \quad \text{and} \quad A(u,t) = A(t) = t\sqrt{1-\tfrac{1}{4}t^2} + 2\arcsin\tfrac{1}{2}t.$$

Proof of Theorem 4 (Equation (4)). From (1) and Chebyshev's inequality, it follows that

$$1 = \mathbb{P}\left(\left| f_0(K_n^1) - \frac{\pi^2}{2}\right| > 0.05\right) \le \frac{\operatorname{var}(f_0(K_n^1))}{0.05^2};$$

thus,

$$\operatorname{var}(f_0(K_n^1)) \ge 0.05^2.$$

This proves that $\operatorname{var}(f_0(K_n^1)) \gg$ constant.

In order to prove the asymptotic upper bound in (4), we use a modified version of the argument of the previous section. With the same notation as in Section 3, the Efron–Stein inequality for the vertex number yields that

$$\operatorname{var}(f_0(K_n^1)) \ll n\mathbb{E}(F_n(x_{n+1}))^2.$$

Following a similar line of argument as above, we obtain

$$\begin{aligned} &n\mathbb{E}(F_n(x_{n+1}))^2 \\ &\quad= \frac{n}{\pi^{n+1}} \int_{(B^2)^{n+1}} \left(\sum_I \mathbf{1}(F_I \in \mathcal{F}_n(x_{n+1}))\right) \\ &\qquad\qquad \times \left(\sum_J \mathbf{1}(F_J \in \mathcal{F}_n(x_{n+1}))\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \mathrm{d}x_{n+1} \\ &\quad\le \frac{n}{\pi^{n+1}} \sum_I \sum_J \int_{(B^2)^{n+1}} \mathbf{1}(F_I \in \mathcal{F}_n(x_{n+1}))\mathbf{1}(F_J \in \mathcal{F}_n(x_{n+1}))\, \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \mathrm{d}x_{n+1}. \end{aligned} \tag{19}$$

Now, let $|I \cap J| = k$, where $k = 0, 1, 2$, and let $F_1 = x_1x_2$ and $F_2 = x_{3-k}x_{4-k}$. By the independence of the random points (and by also taking into account their order), we have

$$\begin{aligned} (19) &\ll \frac{n}{\pi^{n+1}} \sum_{k=0}^{2} \binom{n}{2}\binom{2}{k}\binom{n-2}{2-k} \int_{(B^2)^{n+1}} \mathbf{1}(F_1 \in \mathcal{F}_n(x_{n+1})) \\ &\qquad\qquad \times \mathbf{1}(F_2 \in \mathcal{F}_n(x_{n+1}))\, \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \mathrm{d}x_{n+1}. \\ &\ll \frac{1}{\pi^{n+1}} \sum_{k=0}^{2} n^{5-k} \int_{(B^2)^{n+1}} \mathbf{1}(F_1 \in \mathcal{F}_n(x_{n+1}))\mathbf{1}(F_2 \in \mathcal{F}_n(x_{n+1}))\, \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \mathrm{d}x_{n+1}. \end{aligned} \tag{20}$$

By symmetry, we may also assume that $A(D_1) \geq A(D_2)$; therefore,

$$(20) \ll \sum_{k=0}^{2} n^{5-k} \int_{(B^2)^{n+1}} \mathbf{1}(F_1 \in \mathcal{F}_n(x_{n+1}))\mathbf{1}(F_2 \in \mathcal{F}_n(x_{n+1})) \times \mathbf{1}(A(D_1) \geq A(D_2))\, \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \mathrm{d}x_{n+1}. \tag{21}$$

By integrating with respect to $x_{5-k}, \dots, x_n$ and x_{n+1}, we obtain

$$(21) \ll \sum_{k=0}^{2} n^{5-k} \int_{B^2} \cdots \int_{B^2} \left(1 - \frac{A(D_1)}{\pi}\right)^{n-4+k} \frac{A(D_1)}{\pi} \mathbf{1}(A(D_1) \geq A(D_2))\, \mathrm{d}x_1 \cdots \mathrm{d}x_{4-k} \tag{22}$$

If $A(D_1) \geq A(D_2)$ then D_2 is fully contained in the circular annulus whose width is equal to the height of the disc-cap D_1. The area of this annulus is not more than $4A(D_1)$. Therefore,

$$(22) \ll \sum_{k=0}^{2} n^{5-k} \int_{B^2} \int_{B^2} \left(1 - \frac{A(D_1)}{\pi}\right)^{n-4+k} A(D_1)^{3-k}\, \mathrm{d}x_1\, \mathrm{d}x_2.$$

As is common in these arguments, we may assume that $A(D_1)/\pi < c \log n/n$ for some suitable constant $c > 0$ that will be determined later. To see this, let $A(D_1)/\pi \geq c \log n/n$. Then

$$\begin{aligned}\left(1 - \frac{A(D_1)}{\pi}\right)^{n-4+k} A(D_1)^{3-k} &\leq \left(\frac{\pi c \log n}{n}\right)^{3-k} \exp\left(-\frac{c(n-4+k)\log n}{n}\right) \\ &\ll \left(\frac{\log n}{n}\right)^{3-k} n^{-c} \\ &\ll n^{-c}.\end{aligned}$$

If $c > 0$ is sufficiently large then the contribution of the $A(D_1)/\pi \geq c \log n/n$ case is $O(n^{-1})$. Thus,

$$n\mathbb{E}(F_n(x_{n+1})) \ll \sum_{k=0}^{2} n^{5-k} \int_{B^2} \int_{B^2} \left(1 - \frac{A(D_1)}{\pi}\right)^{n-4+k} A(D_1)^{3-k} \times \mathbf{1}\left(A(D_1) \leq \frac{c \log n}{n}\right) \mathrm{d}x_1\, \mathrm{d}x_2 + O(n^{-1}). \tag{23}$$

Now, we use the same type of reparametrization as in the previous section. Let $(x_1, x_2) = (-tu_1, -tu_2)$, $u \in S^1$, and $0 \leq t < c^* \log n/n$. Then

$$\begin{aligned}(23) &\ll \sum_{k=0}^{2} n^{5-k} \int_{S^1} \int_0^{c^* \log n/n} \int_{S^1} \int_{S^1} \left(1 - \frac{A(u,t)}{\pi}\right)^{n-4+k} A(u,t)^{3-k} \\ &\qquad \times t|u_1 \times u_2|\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}u\, \mathrm{d}t + O(n^{-1}) \\ &\ll \sum_{k=0}^{2} n^{5-k} \int_0^{c^* \log n/n} \left(1 - \frac{A(u,t)}{\pi}\right)^{n-4+k} A(u,t)^{3-k} \\ &\qquad \times t(l(t) - \sin l(t))\, \mathrm{d}t + O(n^{-1}).\end{aligned} \tag{24}$$

Using the fact that $l(t) \to \pi$ as $t \to 0^+$, and the Taylor series of $V(u,t)$ at $t=0$, we find that there exists a constant $\omega > 0$ such that

$$(24) \ll \sum_{k=0}^{2} n^{5-k} \int_0^{c^* \log n/n} (1-\omega t)^{n-4+k} t^{4-k}\, \mathrm{d}t + O(n^{-1}). \tag{25}$$

Now, using a formula for the asymptotic order of beta integrals (see [13, Equation (11), p. 2290]), we obtain

$$(25) \ll \sum_{k=0}^{2} n^{5-k} n^{-(5-k)} + O(n^{-1}) \ll \text{constant},$$

which completes the proof of the upper bound in (4). □

In order to prove the asymptotic upper bound (5), only slight modifications are needed in the above argument.

5. A circumscribed model

In this section we consider circumscribed random disc-polygons. Let $K \subset \mathbb{R}^2$ be a convex disc with C^2_+ smooth boundary, and $r \geq \kappa_{\mathrm{m}}^{-1}$. Consider the following set:

$$K^{*,r} = \{x \in \mathbb{R}^2 \mid K \subset rB^2 + x\},$$

which is also called the r-hyperconvex dual, or r-dual for short, of K. It is known that $K^{*,r}$ is a convex disc with C^2_+ boundary, and it also has the property that the curvature is at least $1/r$ at every boundary point. See [19] and the references therein for further details.

For $u \in S^1$, let $x(K,u) \in \partial K$ ($x(K^{*,r},u) \in \partial K^{*,r}$, respectively) be the unique point on ∂K ($\partial K^{*,r}$, respectively), where the outer unit normal to K (respectively, $K^{*,r}$) is u. For a convex disc $K \subset \mathbb{R}^2$ with $o \in \operatorname{int} K$, let $h_K(u) = \max_{x\in K}\langle x, u\rangle$ denote the support function of K. Let $\operatorname{per}(\cdot)$ denote the perimeter.

In the following lemma we collect some results from [19, Section 2].

Lemma 2. (Fodor *et al.* [19].) *With the notation above*

(i) $h_K(u) + h_{K^{*,r}}(-u) = r$ *for any* $u \in S^1$*;*

(ii) $\kappa_K^{-1}(x(u,K)) + \kappa_{K^{*,r}}^{-1}(x(-u,K^{*,r})) = r$ *for any* $u \in S^1$*;*

(iii) $\operatorname{per}(K) + \operatorname{per}(K^{*,r}) = 2r\pi$*;*

(iv) $A(K^{*,r}) = A(K) - r\operatorname{per}(K) + r^2\pi$.

Now we turn to the probability model. Let K be a convex disc with C^2_+ boundary, and let $r > \kappa_{\mathrm{m}}^{-1}$ as before. Let $X_n = \{x_1, \dots, x_n\}$ be a sample of n independent random points chosen from $K^{*,r}$ according to the uniform probability distribution, and define

$$K^{*,r}_{(n)} = \bigcap_{x\in X_n} rB^2 + x,$$

where $K^{*,r}_{(n)}$ is a random disc-polygon that contains K. Observe that, by definition $K^{*,r}_{(n)} = (\operatorname{conv}_r(X_n))^{*,r}$, and, consequently, $f_0(K^{*,r}_{(n)}) = f_0(\operatorname{conv}_r(X_n))$. We note that this is a very natural approach to define a random disc-polygon that is circumscribed about K that has no clear analogy in classical convexity. (If we take the limit as $r \to \infty$, the underlying probability measures do not converge.) The model is of special interest in the $K = K^{*,r}$ case, which happens exactly when K is of constant width r.

Theorem 6. *Assume that K has C^2_+ boundary, and let $r > \kappa_{\mathrm{m}}^{-1}$. With the notation above*

$$\lim_{n\to\infty} \mathbb{E}(f_0(K^{*,r}_{(n)}))n^{-1/3} = \sqrt[3]{\frac{2r}{3(A(K) - r\,\mathrm{per}(K) + r^2\pi)}}\,\Gamma\Big(\frac{5}{3}\Big)\int_{\partial K}\Big(\kappa(x) - \frac{1}{r}\Big)^{2/3}\,\mathrm{d}x. \tag{26}$$

Furthermore, if K has C^5_+ boundary then

$$\begin{aligned}\lim_{n\to\infty} n^{2/3}(\mathrm{per}\,K^{*,r}_{(n)} - \mathrm{per}\,K) &= \frac{(12(A(K) - r\,\mathrm{per}(K) + r^2\pi))^{2/3}}{36}\Gamma\Big(\frac{2}{3}\Big)\\ &\quad\times r^{-2/3}\int_{\partial K}\Big(\kappa(x) - \frac{1}{r}\Big)^{-1/3}\Big(4\kappa(x) - \frac{1}{r}\Big)\,\mathrm{d}x,\\ \lim_{n\to\infty} n^{2/3}A(K^{*,r}_{(n)} \setminus K) &= \frac{(12(A(K) - r\,\mathrm{per}(K) + r^2\pi))^{2/3}}{12}\\ &\quad\times\Gamma\Big(\frac{2}{3}\Big)r^{-2/3}\int_{\partial K}\Big(\kappa(x) - \frac{1}{r}\Big)^{-1/3}\,\mathrm{d}x.\end{aligned}$$

Proof. From Lemma 2, it follows that $K^{*,r}$ has also C^2_+ boundary. As $f_0(K^{*,r}_{(n)}) = f_0(\mathrm{conv}_r(X_n))$, from [18, Theorem 1.1], we immediately obtain

$$\lim_{n\to\infty} \mathbb{E}(f_0(K^{*,r}_{(n)}))n^{-1/3} = \sqrt[3]{\frac{2}{3A(K^{*,r})}}\,\Gamma\Big(\frac{5}{3}\Big)\int_{\partial K^{*,r}}\Big(\kappa(x) - \frac{1}{r}\Big)^{1/3}\,\mathrm{d}x.$$

Using Lemma 2, we proceed as follows:

$$\begin{aligned}&\int_{\partial K^{*,r}}\Big(\kappa(x) - \frac{1}{r}\Big)^{1/3}\,\mathrm{d}x\\ &\quad= \int_{S^1}\frac{(\kappa(x(K^{*,r}, u)) - 1/r)^{1/3}}{\kappa(x(K^{*,r}, u))}\,\mathrm{d}u\\ &\quad= \int_{S^1}\bigg[\bigg(\frac{\kappa(x(K, -u))}{r\kappa(x(K, -u)) - 1} - \frac{1}{r}\bigg)^{1/3}\bigg]\bigg[\frac{\kappa(x(K, -u))}{r\kappa(x(K, -u)) - 1}\bigg]^{-1}\,\mathrm{d}u\\ &\quad= \int_{S^1} r^{1/3}\frac{(\kappa(x(K, u)) - 1/r)^{2/3}}{\kappa(x(K, u))}\,\mathrm{d}u\\ &\quad= r^{1/3}\int_{\partial K}\Big(\kappa(x) - \frac{1}{r}\Big)^{2/3}\,\mathrm{d}x.\end{aligned}$$

Together with Lemma 2, this proves (26).

The rest of the theorem can be proved similarly, by using [18, Theorems 1.1 and 1.2], and Lemma 2. □

As an obvious consequence of Theorem 3, Lemma 2, and the definition of $K^{*,r}_{(n)}$, we obtain the following corollary.

Corollary 1. *Assume that K has C^2_+ boundary, and let $r > \kappa_{\mathrm{m}}^{-1}$. With the notation above*

$$\mathrm{var}(f_0(K^{*,r}_{(n)})) \ll n^{1/3}.$$

Remark 1. We note that if K is a convex disc of constant width r then $K^{*,r} = K$ (see, for example, [19]), and similar calculations to those in the proof of Theorem 6 provide some interesting integral formulas. For example, for a real p, we obtain

$$\int_{\partial K} \left(\kappa(x) - \frac{1}{r}\right)^p \mathrm{d}x = r^{1-2p} \int_{\partial K} \left(\kappa(x) - \frac{1}{r}\right)^{1-p} \mathrm{d}x.$$

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