Strengthened volume inequalities for L_p zonoids of even isotropic measures *

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Abstract

We strengthen the volume inequalities for L_p zonoids of even isotropic measures and for their duals, which are originally due to Ball, Barthe and Lutwak, Yang, Zhang. The special case $p = \infty$ yields a stability version of the reverse isoperimetric inequality for centrally symmetric convex bodies. Adding to known inequalities and stability results for the reverse isoperimetric inequality of arbitrary convex bodies, we state a conjecture on volume inequalities for L_p zonoids of general centred (non-symmetric) isotropic measures.

We achieve our main results by strengthening Barthe's measure transportation proofs of the rank one case of the geometric Brascamp-Lieb and reverse Brascamp-Lieb inequalities by estimating the derivatives of the transportation maps for a special class of probability density functions. Based on our argument, we phrase a conjecture about a possible stability version of the Brascamp-Lieb and reverse Brascamp-Lieb inequalities.

We also establish some geometric properties of the distribution of general isotropic measures that are essential to our argument. In particular, we prove a measure theoretic analog of the Dvoretsky-Rogers lemma.

1 Introduction

According to the classical isoperimetric inequality, Euclidean balls minimize the surface area among convex bodies of given volume in Euclidean space $\mathbb{R}^n$. We call a subset of $\mathbb{R}^n$ a convex body if it is compact, convex and has non-empty interior. Let B^n be the Euclidean unit ball centred at the origin, and let $S(\cdot)$ and $V(\cdot)$ denote the surface area and the volume functional in $\mathbb{R}^n$, respectively. The isoperimetric inequality can be stated in the form

$$\frac{S(B^n)^n}{V(B^n)^{n-1}} \le \frac{S(K)^n}{V(K)^{n-1}},$$

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where equality holds if and only if K is a Euclidean ball. Recently, N. Fusco, F. Maggi, A. Pratelli [25] proved an essentially optimal stability version of the isoperimetric inequality. It states that if K is a convex body with $V(K) = V(B^n)$ and if $S(B^n) \geq (1-\varepsilon)S(K)$ holds for some small $\varepsilon > 0$, then K is close to some translate $B^n + x$, $x \in \mathbb{R}^n$, of the unit ball; namely,

$$V(K\Delta(B^n + x)) \leq \gamma\varepsilon^{1/2},$$

where $\gamma > 0$ depends only on n, and Δ denotes the symmetric difference of sets.

Stability estimates for the planar isoperimetric inequality go back to the works of Minkowski and Bonnesen. However, a systematic exploration is much more recent. We refer to the survey articles of H. Groemer [27, 28] for an introduction to geometric stability results. The recent monograph [46] by R. Schneider provides an up-to-date treatment of the topic including applications. Here we only note that the stability estimate related to the isoperimetric inequality obtained in [25] was extended to a stability version of the Brunn-Minkowski inequality by A. Figalli, F. Maggi, A. Pratelli [23, 24].

Aiming at a reverse isoperimetric inequality, F. Behrend [10] suggested to consider equivalence classes of convex bodies with respect to non-singular linear transformations. C.M. Petty [45] proved (see also A. Giannopoulos, M. Papadimitrakis [26]) that there is an essentially unique representative minimizing the isoperimetric ratio in each equivalence class. The unique minimizer in an equivalence class is characterized by the property that its suitably normalized area measure is isotropic. We give a precise definition of isotropic measures later. This characterization yields that cubes minimize the isoperimetric ratio within the class of parallelotopes, and regular simplices within the class of simplices.

The functional that assigns to each equivalence class the minimum of the isoperimetric ratio within that class is affine invariant and upper semi-continuous, therefore it attains its maximum on the affine equivalence classes of convex bodies. In the Euclidean plane, the method of F. Behrend [10] yields that the maximum is attained by the affine equivalence class of triangles, and by the affine equivalence class of parallelograms if the convex body is assumed to be centrally symmetric. The extension of these results to higher dimensions proved to be quite difficult. Decades after Behrend's paper, K.M. Ball in [1, 3] managed to establish reverse forms of the isoperimetric inequality in arbitrary dimensions. More precisely, the largest isoperimetric ratio is attained by simplices according to [3], and by parallelotopes among centrally symmetric convex bodies according to [1]. Since the reverse isoperimetric inequality and a related inequality for general centred (not necessarily even) isotropic measures are discussed in K.J. Böröczky, D. Hug [13], in this paper we concentrate on centrally symmetric convex bodies.

In order to state the result of K.M. Ball [1] about centrally symmetric convex bodies, we set $W^n = [-1, 1]^n$, and note that $S(W^n) = n2^n = nV(W^n)$.

Theorem A (K.M. Ball) *For any centrally symmetric convex body K in $\mathbb{R}^n$, there exists some $\Phi \in \mathrm{GL}(n)$ such that*

$$\frac{S(\Phi K)^n}{V(\Phi K)^{n-1}} \leq \frac{S(W^n)^n}{V(W^n)^{n-1}}. \tag{1}$$

The case of equality in Theorem A was settled by F. Barthe [6]. He proved that if the left side of (1) is minimized over all $\Phi \in \mathrm{GL}(n)$, then equality holds precisely when K is a parallelotope.

Our first objective is to prove a stability version of the reverse isoperimetric inequality for centrally symmetric convex bodies. Following [23–25], we define an affine invariant distance of origin symmetric convex bodies K and M based on the volume difference. Let $\alpha = V(K)^{-1/n}$, $\beta = V(M)^{-1/n}$, and define

$$\delta_{\rm vol}(K,M) = \min\left\{V\left(\Phi(\alpha K)\Delta(\beta M)\right) :\, \Phi \in {\rm SL}(n)\right\},$$

where ${\rm SL}(n)$ is the group of linear transformations of $\mathbb{R}^n$ of determinant one. In fact, $\delta_{\rm vol}(\cdot\,,\cdot)$ induces a metric on the linear equivalence classes of origin symmetric convex bodies.

The John ellipsoid of a convex body K in $\mathbb{R}^n$ is the unique maximum volume ellipsoid contained in K. If K is origin symmetric, then its John ellipsoid is also origin symmetric. Note that each convex body has an affine image whose John ellipsoid is B^n. The John ellipsoid is a frequently used tool in geometric analysis, and, in particular, it was used by K.M. Ball in the proof of the reverse isoperimetric inequality. Since we will use the John ellipsoid in our arguments, below we review its basic properties (see (2)). For a more detailed treatment of the topic, we refer to K.M. Ball [4], P.M. Gruber [30] and R. Schneider [46].

Theorem 1.1 *Let K be an origin symmetric convex body in $\mathbb{R}^n$, $n \geq 3$, whose John ellipsoid is a Euclidean ball, and let $\varepsilon \in [0,1)$. If $\delta_{\rm vol}(K,W^n) \geq \varepsilon$, then*

$$\frac{S(K)^n}{V(K)^{n-1}} \leq (1-\gamma\,\varepsilon^3)\frac{S(W^n)^n}{V(W^n)^{n-1}},$$

where $\gamma = n^{-cn^3}$ for some absolute constant $c > 0$.

Although the stability order (the exponent 3 of ε) in Theorem 1.1 is probably not optimal, it is close to the optimum. Considering a convex body K which is obtained from W^n by cutting off simplices of height ε at the vertices of W^n, one can see that the exponent of ε must be at least 1 in Theorem 1.1.

Another common affine invariant distance between convex bodies is the Banach-Mazur metric $\delta_{\rm BM}(K,M)$, which we define here only for origin symmetric convex bodies K and M. Let

$$\delta_{\rm BM}(K,M) = \log\min\{\lambda \geq 1 :\, K \subseteq \Phi(M) \subseteq \lambda\, K \text{ for some } \Phi \in {\rm GL}(n)\}.$$

We note that $\delta_{\rm vol} \leq 2n^2\delta_{\rm BM}$ (see, say, [13]). Furthermore, $\delta_{\rm BM} \leq \gamma\, \delta_{\rm vol}^{\frac{1}{n}}$, where γ depends only on the dimension n (see [12, Section 5]). The example of a ball from which a cap is cut off shows that in the latter inequality the exponent $\frac{1}{n}$ cannot be replaced by anything larger than $\frac{2}{n+1}$.

Theorem 1.2 *Let K be an origin symmetric convex body in $\mathbb{R}^n$, $n \geq 3$, whose John ellipsoid is a Euclidean ball, and let $\varepsilon \in [0,1)$. If $\delta_{\rm BM}(K,W^n) \geq \varepsilon$, then*

$$\frac{S(K)^n}{V(K)^{n-1}} \leq (1-\gamma\,\varepsilon^n)\frac{S(W^n)^n}{V(W^n)^{n-1}},$$

where $\gamma = n^{-cn^3}$ for some absolute constant $c > 0$.

The stability order (the exponent n of ε) in Theorem 1.2 is again close to the optimum, but very likely it is not optimal. Considering a convex body K which is obtained from W^n by cutting off simplices of height ε at the vertices of W^n, one can see that the exponent of ε must be at least $n-1$ in Theorem 1.2.

In the planar case, a modification of the argument of F. Behrend [10] leads to stability results of optimal order.

Theorem 1.3 *Let K be an origin symmetric convex body in $\mathbb{R}^2$ which has a square as an inscribed parallelogram of maximum area. Let $\varepsilon \in [0,1)$. If $\delta_{\mathrm{vol}}(K,W^2) \geq \varepsilon$ or $\delta_{\mathrm{BM}}(K,W^2) \geq \varepsilon$, then*

$$\frac{S(K)^2}{V(K)} \leq \left(1 - \frac{\varepsilon}{54}\right) \frac{S(W^2)^2}{V(W^2)}.$$

Note that for an origin symmetric convex body K in $\mathbb{R}^2$ there always exists a linear transform $\Phi \in \mathrm{GL}(2)$ such that a square is an inscribed parallelogram of maximum area of ΦK. In particular, if we define $\mathrm{ir}(K) = \min\{S(\Phi K)^2/V(\Phi K) : \Phi \in \mathrm{GL}(2)\}$, for an origin symmetric convex body K in $\mathbb{R}^2$, and if $\varepsilon \in [0,1)$, then Theorem 1.3 implies that

$$\mathrm{ir}(K) \leq \left(1 - \frac{\varepsilon}{54}\right) \mathrm{ir}(W^2)$$

provided that $\delta_{\mathrm{vol}}(K,W^2) \geq \varepsilon$ or $\delta_{\mathrm{BM}}(K,W^2) \geq \varepsilon$.

As mentioned before, the proof of the reverse isoperimetric inequality by K.M. Ball [1, 3] is based on a volume estimate for convex bodies whose John ellipsoid is the unit ball B^n. Let S^{n-1} denote the Euclidean unit sphere. According to a classical theorem of F. John [33] (see also K.M. Ball [4]), B^n is the ellipsoid of maximal volume in an origin symmetric convex body K if and only if $B^n \subseteq K$ and there exist $\pm u_1, \ldots, \pm u_k \in S^{n-1} \cap \partial K$ and $c_1, \ldots, c_k > 0$ such that

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{Id}_n, \tag{2}$$

where $\otimes$ denotes the tensor product of vectors in $\mathbb{R}^n$, Id_n denotes the $n \times n$ identity matrix and ∂K is the boundary of K.

Following A. Giannopoulos, M. Papadimitrakis [26] and E. Lutwak, D. Yang, G. Zhang [42], we call an even Borel measure μ on the unit sphere S^{n-1} isotropic if

$$\int_{S^{n-1}} u \otimes u \, d\mu(u) = \mathrm{Id}_n.$$

In this case, equating traces of both sides we obtain that $\mu(S^{n-1}) = n$.

Using the standard notation $\langle \cdot\, , \cdot \rangle$ for the Euclidean scalar product and $\| \cdot \|$ for the induced norm in $\mathbb{R}^n$, the support function h_K of a convex compact set K in $\mathbb{R}^n$ at $v \in \mathbb{R}^n$ is defined as

$$h_K(v) = \max\{\langle v, x \rangle : \, x \in K\}.$$

For any $p \geq 1$ and an even measure μ on S^{n-1} not concentrated on any great subsphere, we define the L_p zonoid $Z_p(\mu)$ associated with μ by

$$h_{Z_p(\mu)}(v)^p = \int_{S^{n-1}} |\langle u, v\rangle|^p \, d\mu(u),$$

which is a zonoid in the classical sense if $p = 1$. In addition, let

$$Z_\infty(\mu) = \lim_{p\to\infty} Z_p(\mu) = \operatorname{conv}\operatorname{supp}\mu,$$

and for $1 \leq p \leq \infty$, let $Z_p^*(\mu)$ be the polar of $Z_p(\mu)$. In particular,

$$\begin{aligned} Z_p^*(\mu) &= \left\{x \in \mathbb{R}^n : \int_{S^{n-1}} |\langle x, u\rangle|^p \, d\mu(u) \leq 1\right\} \text{ for } p \in [1, \infty), \\ Z_\infty^*(\mu) &= \{x \in \mathbb{R}^n : \langle x, u\rangle \leq 1 \text{ for } u \in \operatorname{supp}\mu\}, \end{aligned}$$

and hence $Z_2(\mu) = B^n$ for any even isotropic measure μ.

It follows from D.R. Lewis [37] (see also E. Lutwak, D. Yang and G. Zhang [40,41]) that any n-dimensional subspace of L_p is isometric to $\|\cdot\|_{Z_p^*(\mu)}$ for some isotropic measure μ on S^{n-1}, where

$$\|x\|_{Z_p^*(\mu)} = \left(\int_{S^{n-1}} |\langle x, u\rangle|^p \, d\mu(u)\right)^{\frac{1}{p}}, \qquad x \in \mathbb{R}^n.$$

We call a measure ν on S^{n-1} a cross measure if there is an orthonormal basis $u_1, \ldots, u_n$ of $\mathbb{R}^n$ such that

$$\operatorname{supp}\nu = \{\pm u_1, \ldots, \pm u_n\}$$

and $\nu(\{u_i\}) = \nu(\{-u_i\}) = 1/2$ for $i = 1, \ldots, n$. In particular, any cross measure is even and isotropic. From now on, we fix a cross measure ν_n on S^{n-1}. We note that if $p \in [1, \infty]$, and $\Gamma(\cdot)$ is Euler's Gamma function, then

$$V(Z_p(\nu_n)) = \begin{cases} \frac{\Gamma(1+\frac{n}{2})\Gamma(1+\frac{p}{2})}{\Gamma(1+\frac{1}{2})\Gamma(1+\frac{n+p}{2})} & \text{if } p \geq 1, \\ \frac{2^n}{n!} & \text{if } p = \infty. \end{cases}$$

In addition,

$$V(Z_p^*(\nu_n)) = \begin{cases} 2^n \frac{\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} & \text{if } p \geq 1, \\ 2^n & \text{if } p = \infty. \end{cases}$$

The crucial statement leading to the reverse isoperimetric inequality is the case of $Z_\infty^*(\mu)$ in the following theorem.

Theorem B *If μ is an even isotropic measure on S^{n-1} and $p \in [1, \infty]$, then*

$$V(Z_p(\mu)) \;\geq\; V(Z_p(\nu_n)),$$

$$V(Z_p^*(\mu)) \quad \leq \quad V(Z_p^*(\nu_n)).$$

Assuming $p \neq 2$, equality holds if and only if μ is a cross measure.

Theorem B is the work of K.M. Ball [3] and F. Barthe [6] if μ is discrete. Their method has been extended to arbitrary even isotropic measures μ by E. Lutwak, D. Yang, and G. Zhang [40]. The measures on S^{n-1} which have an isotropic linear image are characterized by K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [14], building on the works of E.A. Carlen and D. Cordero-Erausquin [17], J. Bennett, A. Carbery, M. Christ and T. Tao [11] and B. Klartag [36]. We note that isotropic measures on $\mathbb{R}^n$ play a central role in the KLS conjecture by R. Kannan, L. Lovász and M. Simonovits [34]; see, for instance, F. Barthe and D. Cordero-Erausquin [8], O. Guedon and E. Milman [32] and B. Klartag [35].

For stating a stability version of Theorem B, a natural notion of distance between two isotropic measures μ and ν is the Wasserstein distance (also called the Kantorovich-Monge-Rubinstein distance) $\delta_W(\mu, \nu)$. To define it, we write $\angle(v, w)$ to denote the angle between two unit vectors v and w, which equals the geodesic distance of v and w on the unit sphere. Let $\mathrm{Lip}_1(S^{n-1})$ denote the family of Lipschitz functions with Lipschitz constant at most 1, that is to say, $f : S^{n-1} \to \mathbb{R}$ is in $\mathrm{Lip}_1(S^{n-1})$ if and only if $\|f(x) - f(y)\| \leq \angle(x, y)$ for $x, y \in S^{n-1}$. Then the Wasserstein distance of μ and ν is given by

$$\delta_W(\mu, \nu) = \max \left\{ \int_{S^{n-1}} f\, d\mu - \int_{S^{n-1}} f\, d\nu : \ f \in \mathrm{Lip}_1(S^{n-1}) \right\}.$$

What we actually need in this paper is the Wasserstein distance of an isotropic measure μ from the closest cross measure. Therefore, in the case of two isotropic measures μ and ν, we define

$$\delta_{\mathrm{WO}}(\mu, \nu) = \min \left\{ \delta_W(\mu, \Phi_* \nu) : \ \Phi \in \mathrm{O}(n) \right\},$$

where $\Phi_* \nu$ denotes the pushforward of ν by $\Phi : S^{n-1} \to S^{n-1}$.

Theorem 1.4 *Let μ be an even isotropic measure on S^{n-1}, $n \geq 2$, let $\varepsilon \in [0, 1)$, and let $p \in [1, \infty]$ with $p \neq 2$. If $\delta_{\mathrm{WO}}(\mu, \nu_n) \geq \varepsilon$, then*

$$\begin{aligned} V(Z_p(\mu)) \quad &\geq \quad (1 + \gamma \varepsilon^3) V(Z_p(\nu_n)), \\ V(Z_p^*(\mu)) \quad &\leq \quad (1 - \gamma \varepsilon^3) V(Z_p^*(\nu_n)), \end{aligned}$$

where $\gamma = n^{-cn^3} \min\{|p - 2|^2, 1\}$ for an absolute constant $c > 0$.

To state another stability version of Theorem B, in the case $p = \infty$, we use the "spherical" Hausdorff distance $\delta_H(X, Y)$ of compact sets $X, Y \subseteq S^{n-1}$ given by

$$\delta_H(X, Y) = \min \left\{ \max_{x \in X} \min_{y \in Y} \angle(x, y), \max_{y \in Y} \min_{x \in X} \angle(x, y) \right\}.$$

In addition, let

$$\delta_{HO}(X, Y) = \min \left\{ \delta_H(X, \Phi Y) : \ \Phi \in \mathrm{O}(n) \right\}.$$

We note that if $\delta_{HO}(\operatorname{supp}\mu, \operatorname{supp}\nu_n) \leq 1/(7n^2)$ for an even isotropic measure μ, then $\delta_{WO}(\mu, \nu_n) \leq 2n\delta_{HO}(\operatorname{supp}\mu, \operatorname{supp}\nu_n)$ according to Corollary 6.2. However, as we will see in Section 9, Theorem 1.4 implies the following seemingly stronger statement in the case $p = \infty$.

Corollary 1.5 *Let μ be an even isotropic measure on S^{n-1} and $\varepsilon \in [0, 1)$. If $\delta_{HO}(\operatorname{supp}\mu, \operatorname{supp}\nu_n) \geq \varepsilon$, then*

$$\begin{aligned} V(Z_\infty(\mu)) &\geq (1 + \gamma\varepsilon^3)V(Z_\infty(\nu_n)), \\ V(Z^*_\infty(\mu)) &\leq (1 - \gamma\varepsilon^3)V(Z^*_\infty(\nu_n)), \end{aligned}$$

where $\gamma = n^{-cn^3}$ for an absolute constant $c > 0$.

We note that the order ε^3 of the error term in Corollary 1.5 can be improved to ε if $n = 2$ according to Theorem 11.1.

The proof of Theorem B is based on the rank one case of the geometric Brascamp-Lieb inequality. An essential tool in our approach is the proof provided by F. Barthe [5,6], which is based on mass transportation. Therefore, we review the argument from [5] in Section 2. At the end of that section, we outline the arguments leading to Theorem 1.1, Theorem 1.2 and Theorem 1.4 and we describe the structure of the paper. We also indicate in Section 2 the type of stability result that can be expected concerning the Brascamp-Lieb inequality (see Conjecture 2.2). Along the way of proving our main statements, we also establish some properties of arbitrary (not only even) isotropic measures in Section 5 that might be useful in other applications as well. From our results on the properties of general isotropic measures, we emphasise Lemma 5.4, which can be regarded as a measure theoretic version of the Dvoretzky-Rogers lemma.

Let us point out that the corresponding question in the non-symmetric setting is wide open. We call an isotropic measure μ on S^{n-1} centred if

$$\int_{S^{n-1}} u \, d\mu(u) = o.$$

Here and in the following, we write o for the origin (the zero vector). For a centred isotropic measure μ on S^{n-1}, and for $p \in [1, \infty)$, we define the non-symmetric L_p zonoid $Z_p(\mu)$ by

$$h_{Z_p(\mu)}(v)^p = 2 \int_{S^{n-1}} \max\{0, \langle v, u\rangle\}^p \, d\mu(u),$$

and hence its polar body is

$$Z^*_p(\mu) = \left\{ x \in \mathbb{R}^n : \int_{S^{n-1}} \max\{0, \langle x, u\rangle\}^p \, d\mu(u) \leq \frac{1}{2} \right\}.$$

This notion (for any discrete measure on S^{n-1}, not only isotropic ones), occurs in M. Weberndorfer [47] in connection with reverse versions of the Blaschke-Santaló inequality. The factor 2 is included to match the earlier definition for even isotropic measures. The difference to the case of even isotropic measures is that if $p = 2$ and μ is a non-even centered isotropic measure, then $Z_2(\mu)$ is typically not a Euclidean ball but has constant squared width; namely, $h_{Z_p(\mu)}(v)^2 + h_{Z_p(\mu)}(-v)^2$ is constant for $v \in S^{n-1}$.

Conjecture 1.6 *Let μ be a centered isotropic measure on S^{n-1} and $p \in [1, \infty)$. If ν is an isotropic measure on S^{n-1} such that* $\operatorname{supp} \nu$ *consists of the vertices of a regular simplex, then*

$$V(Z_p(\mu)) \ \geq \ V(Z_p(\nu)), \tag{3}$$

$$V(Z_p^*(\mu)) \ \leq \ V(Z_p^*(\nu)). \tag{4}$$

If μ is a centered isotropic measure on S^{n-1}, then $Z_\infty(\mu) = \operatorname{conv} \operatorname{supp} \mu$. In particular, if $p = \infty$, then (4) was proved by K.M. Ball in [3] for discrete μ, (3) was proved by F. Barthe in [6] again for discrete μ, and the case of general centered isotropic μ was handled by E. Lutwak, D. Yang and G. Zhang [42]. Again for $p = \infty$, a stability improvement of (4) was established in [13].

An inequality related to the case $p = 2$ of Conjecture 1.6 is proved by E. Lutwak, D. Yang, G. Zhang [43].

2 A brief review of the Brascamp-Lieb and the reverse Brascamp-Lieb inequality

The rank one geometric Brascamp-Lieb inequality (5), identified by K.M. Ball [1] as an essential case of the rank one Brascamp-Lieb inequality, due to H.J. Brascamp, E.H. Lieb [15], and the reverse form (6), due to F. Barthe [5, 6], read as follows. If $u_1, \ldots, u_k \in S^{n-1}$ are distinct unit vectors and $c_1, \ldots, c_k > 0$ satisfy

$$\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{Id}_n,$$

and $f_1, \ldots, f_k$ are non-negative measurable functions on $\mathbb{R}$, then

$$\int_{\mathbb{R}^n} \prod_{i=1}^k f_i(\langle x, u_i \rangle)^{c_i}\, dx \ \leq \ \prod_{i=1}^k \left(\int_{\mathbb{R}} f_i \right)^{c_i}, \tag{5}$$

$$\int_{\mathbb{R}^n}^* \sup_{x = \sum_{i=1}^k c_i \theta_i u_i} \prod_{i=1}^k f_i(\theta_i)^{c_i}\, dx \ \geq \ \prod_{i=1}^k \left(\int_{\mathbb{R}} f_i \right)^{c_i}. \tag{6}$$

Here we write $\int f_i$ to denote the integral of f_i over $\mathbb{R}$ with respect to the Lebesgue measure. In (6), the supremum extends over all $\theta_1, \ldots, \theta_k \in \mathbb{R}$. Since the integrand need not be a measurable function, we have to consider the outer integral. If $k = n$, then $u_1, \ldots, u_n$ form an orthonormal basis and therefore $\theta_1, \ldots, \theta_k$ are uniquely determined for a given $x \in \mathbb{R}^n$.

According to F. Barthe [6], if equality holds in (5) or in (6) and none of the functions f_i is identically zero or a scaled version of a Gaussian, then there is an origin symmetric regular crosspolytope in $\mathbb{R}^n$ such that $u_1, \ldots, u_k$ are among its vertices. Conversely, equality holds in (5) and (6) if each f_i is a scaled version of the same centered Gaussian, or if $k = n$ and $u_1, \ldots, u_n$ form an orthonormal basis.

A thorough discussion of the rank one Brascamp-Lieb inequality can be found in E. Carlen, D. Cordero-Erausquin [17]. The higher rank case, due to E.H. Lieb [38], is reproved and further explored by F. Barthe [6] (including a discussion of the equality case), and is again carefully analysed by J. Bennett, T. Carbery, M. Christ, T. Tao [11]. In particular, see F. Barthe, D. Cordero-Erausquin, M. Ledoux, B. Maurey [9] for an enlightening review of the relevant literature and an approach via Markov semigroups in a quite general framework.

F. Barthe [5, 6] provided concise proofs of (5) and (6) based on mass transportation (see also K.M. Ball [4] for (5)). We sketch the main ideas of his approach, since it will be the starting point of subsequent refinements.

We assume that each f_i is a positive continuous probability density both for (5) and (6), and let $g(t) = e^{-\pi t^2}$ be the Gaussian density. For $i = 1, \ldots, k$, we consider the transportation map $T_i : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_{-\infty}^{t} f_i(s)\, ds = \int_{-\infty}^{T_i(t)} g(s)\, ds.$$

It is easy to see that T_i is bijective, differentiable and

$$f_i(t) = g(T_i(t)) \cdot T_i'(t), \qquad t \in \mathbb{R}. \tag{7}$$

To these transportation maps, we associate the smooth transformation $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Theta(x) = \sum_{i=1}^{k} c_i T_i(\langle u_i, x\rangle)\, u_i, \qquad x \in \mathbb{R}^n,$$

which satisfies

$$d\Theta(x) = \sum_{i=1}^{k} c_i T_i'(\langle u_i, x\rangle)\, u_i \otimes u_i.$$

In this case, $d\Theta(x)$ is positive definite and $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ is injective (see [5, 6]). We will need the following two estimates due to K.M. Ball [1] (see also [6] for a simpler proof of (i)).

Lemma 2.1 (i) *For any $t_1, \ldots, t_k > 0$, we have*

$$\det\left(\sum_{i=1}^{k} t_i c_i u_i \otimes u_i\right) \geq \prod_{i=1}^{k} t_i^{c_i}.$$

(ii) *If $z = \sum_{i=1}^{k} c_i \theta_i u_i$ for $\theta_1, \ldots, \theta_k \in \mathbb{R}$, then*

$$\|z\|^2 \leq \sum_{i=1}^{k} c_i \theta_i^2.$$

Therefore, using first (7), then Lemma 2.1 (i) with $t_i = T_i'(\langle u_i, x\rangle)$, the definition of Θ and Lemma 2.1 (ii), and finally the transformation formula, the following argument leads to the Brascamp-Lieb inequality (5)

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle u_i, x\rangle)^{c_i}\, dx = \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} g(T_i(\langle u_i, x\rangle))^{c_i}\right) \left(\prod_{i=1}^{k} T_i'(\langle u_i, x\rangle)^{c_i}\right) dx \tag{8}$$

$$\leq \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} e^{-\pi c_i T_i(\langle u_i, x\rangle)^2} \right) \det \left(\sum_{i=1}^{k} c_i T_i'(\langle u_i, x\rangle)\, u_i \otimes u_i \right) dx \tag{9}$$

$$\leq \int_{\mathbb{R}^n} e^{-\pi \|\Theta(x)\|^2} \det\left(d\Theta(x)\right)\, dx$$

$$\leq \int_{\mathbb{R}^n} e^{-\pi\|y\|^2}\, dy = 1.$$

The Brascamp-Lieb inequality (5) for arbitrary non-negative integrable functions f_i follows by scaling and approximation.

For the reverse Brascamp-Lieb inequality (6), we consider the inverse S_i of T_i, and hence

$$\int_{-\infty}^{t} g(s)\, ds = \int_{-\infty}^{S_i(t)} f_i(s)\, ds,$$

$$g(t) = f_i(S_i(t)) \cdot S_i'(t), \qquad t \in \mathbb{R}. \tag{10}$$

In addition,

$$d\Psi(x) = \sum_{i=1}^{k} c_i S_i'(\langle u_i, x\rangle)\, u_i \otimes u_i$$

holds for the smooth transformation $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Psi(x) = \sum_{i=1}^{k} c_i S_i(\langle u_i, x\rangle)\, u_i, \qquad x \in \mathbb{R}^n.$$

In particular, $d\Psi(x)$ is positive definite and $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is injective (see [5,6]). Therefore, the transformation formula, Lemma 2.1 (i), and (10) imply that

$$\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^{k} c_i\theta_i u_i} \prod_{i=1}^{k} f_i(\theta_i)^{c_i}\, dx$$

$$\geq \int_{\mathbb{R}^n}^{*} \left(\sup_{\Psi(y)=\sum_{i=1}^{k} c_i\theta_i u_i} \prod_{i=1}^{k} f_i(\theta_i)^{c_i} \right) \det\left(d\Psi(y)\right)\, dy$$

$$\geq \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} f_i(S_i(\langle u_i, y\rangle))^{c_i} \right) \det \left(\sum_{i=1}^{k} c_i S_i'(\langle u_i, y\rangle)\, u_i \otimes u_i \right) dy \tag{11}$$

$$\geq \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} f_i(S_i(\langle u_i, y\rangle))^{c_i} \right) \left(\prod_{i=1}^{k} S_i'(\langle u_i, y\rangle)^{c_i} \right) dy \tag{12}$$

$$= \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} g(\langle u_i, y\rangle)^{c_i} \right) dy = \int_{\mathbb{R}^n} e^{-\pi\|y\|^2}\, dy = 1.$$

Again, the reverse Brascamp-Lieb inequality (6) for arbitrary non-negative integrable functions f_i follows by scaling and approximation.

We observe that Lemma 2.1 (i) shows that the optimal constant in the geometric Brascamp-Lieb inequality is 1. The stability version of Lemma 2.1 (i) (with $v_i = \sqrt{c_i}u_i$), Lemma 3.1, is an essential tool in proving a stability version of the Brascamp-Lieb inequality leading to Theorem 1.4.

Even if we do not use it in this paper, we point out that F. Barthe [7] proved "continuous" versions of the Brascamp-Lieb and the reverse Brascamp-Lieb inequalities that work for any isotropic measure μ on S^{n-1} (see (13) and (14) below). Here we only consider the case in which all non-negative real functions involved coincide with a "nice" probability density function, which is the common case in geometric applications. So let $f : \mathbb{R} \to [0,\infty)$ be such that $\int_{\mathbb{R}} f = 1$ and $\mathrm{supp}(f) = [a,b]$ for some $a, b \in [-\infty,\infty]$. Further, we assume that f is positive and continuous on $[a,b]$. According to [7], we have

$$\int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log f(\langle x,u\rangle)\,d\mu(u)\right)\,dx \le 1. \tag{13}$$

For the reverse inequality, let $h : \mathbb{R}^n \to [0,\infty)$ be a measurable function which satisfies

$$h\left(\int_{S^{n-1}} \theta(u)\,u\,d\mu(u)\right) \ge \exp\left(\int_{S^{n-1}} \log f(\theta(u))\,d\mu(u)\right)$$

for any continuous function $\theta : \mathrm{supp}\,\mu \to \mathbb{R}$. Then we have

$$\int_{\mathbb{R}^n} h(x)\,dx \ge 1. \tag{14}$$

Let us briefly discuss how K.M. Ball [1] and F. Barthe [6] used the Brascamp-Lieb inequality and its reverse form to prove the discrete version of Theorem B. In this section, we write μ to denote the isotropic measure on S^{n-1} whose support is $\{u_1,\ldots,u_k\}$ with $\mu(\{u_i\}) = c_i$, and we assume that μ is an even measure. For $i = 1,\ldots,k$, we consider the probability densities on $\mathbb{R}$ (see (20)) given by

$$f_i(t) = \frac{1}{2\Gamma(1+\frac{1}{p})}\,e^{-|t|^p}, \qquad t \in \mathbb{R},$$

if $p \in [1,\infty)$, and $f_i = \frac{1}{2}\mathbf{1}_{[-1,1]}$ if $p = \infty$, where $\mathbf{1}_{[-1,1]}(t) = 1$ if $t \in [-1,1]$, and zero otherwise. We will frequently use the following observation due to K. Ball [3]. If K is an orgin symmetric convex body in $\mathbb{R}^n$ with associated norm $\|\cdot\|_K$ and if $p \in [1,\infty)$, then

$$V(K) = \frac{1}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n} e^{-\|x\|_K^p}\,dx,$$

where

$$\|x\|_K = \min\{\lambda \ge 0 : x \in \lambda K\}, \qquad x \in \mathbb{R}^n.$$

In particular, if $p \in [1, \infty)$, then

$$\begin{aligned} V(Z_p^*(\mu)) &= \frac{1}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \exp\left(-\sum_{i=1}^k c_i |\langle x, u_i\rangle|^p\right) dx \\ &= \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \prod_{i=1}^k f_i(\langle x, u_i\rangle)^{c_i}\, dx \qquad (15) \\ &\leq \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma(1+\frac{n}{p})} \prod_{i=1}^k \left(\int_{\mathbb{R}} f_i\right)^{c_i} = \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma(1+\frac{n}{p})}. \qquad (16) \end{aligned}$$

On the other hand, if $p = \infty$, then using $f_i = \frac{1}{2}\mathbf{1}_{[-1,1]}$, we have

$$V(Z_\infty^*(\mu)) = 2^n \int_{\mathbb{R}^n} \prod_{i=1}^k f_i(\langle x, u_i\rangle)^{c_i}\, dx \leq 2^n \prod_{i=1}^k \left(\int_{\mathbb{R}} f_i\right)^{c_i} = 2^n.$$

Equality in (16) leads to equality in the Brascamp-Lieb inequality, and hence $k = 2n$ and $u_1, \ldots, u_k$ form the vertices of a regular crosspolytope in $\mathbb{R}^n$.

For the lower bound on the volume of the L_p zonotopes and $p \in [1, \infty]$, let us choose $p^* \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{p^*} = 1$. If $p \in [1, \infty)$, then an (auxiliary) origin symmetric convex body is defined by

$$M_p(\mu) = \left\{\sum_{i=1}^k c_i \theta_i u_i : \sum_{i=1}^k c_i |\theta_i|^p \leq 1\right\}.$$

We drop the reference to μ, if it does not cause any misunderstanding. In particular,

$$\|x\|_{M_p} = \left(\inf_{x=\sum_{i=1}^k c_i\theta_i u_i} \sum_{i=1}^k c_i |\theta_i|^p\right)^{\frac{1}{p}}, \qquad x \in \mathbb{R}^n.$$

In addition, we define

$$M_\infty(\mu) = \left\{\sum_{i=1}^k c_i \theta_i u_i : |\theta_i| \leq 1 \text{ for } i = 1, \ldots, k\right\}.$$

We claim that if $p \in [1, \infty]$, then

$$M_p(\mu) \subseteq Z_{p^*}(\mu). \qquad (17)$$

Let $x \in M_p(\mu)$, and hence $x = \sum_{i=1}^k c_i \theta_i u_i$ with $\sum_{i=1}^k c_i |\theta_i|^p \leq 1$ if $p \in [1, \infty)$, and $|\theta_i| \leq 1$ for $i = 1, \ldots, k$ if $p = \infty$. If $p \in (1, \infty)$, then it follows from Hölder's inequality that, for any $v \in \mathbb{R}^n$, we have

$$\langle x, v\rangle = \sum_{i=1}^k c_i \theta_i \langle u_i, v\rangle \leq \left(\sum_{i=1}^k c_i |\theta_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^k c_i |\langle u_i, v\rangle|^{p^*}\right)^{\frac{1}{p^*}} \leq h_{Z_{p^*}}(v).$$

If $p = 1$, then

$$\langle x, v\rangle = \sum_{i=1}^{k} c_i\theta_i\langle u_i, v\rangle \leq \max_{i=1,\dots,k} |\langle u_i, v\rangle| = h_{Z_\infty}(v).$$

In addition, if $p = \infty$, then

$$\langle x, v\rangle = \sum_{i=1}^{k} c_i\theta_i\langle u_i, v\rangle \leq \sum_{i=1}^{k} c_i|\langle u_i, v\rangle| = h_{Z_1}(v).$$

Now if $p \in [1, \infty)$, then we deduce from (17) and the reverse Brascamp-Lieb inequality (6) that

$$\begin{aligned} V(Z_{p^*}(\mu)) &\geq V(M_p(\mu)) = \frac{1}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n} \exp\left(-\|x\|_{M_p}^p\right)\,dx \\ &= \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^{k} c_i\theta_i u_i} \prod_{i=1}^{k} f_i(\theta_i)^{c_i}\,dx \qquad (18) \\ &\geq \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\prod_{i=1}^{k}\left(\int_{\mathbb{R}} f_i\right)^{c_i} = \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}. \qquad (19) \end{aligned}$$

Finally, if $p = \infty$, then $f_i = \frac{1}{2}\mathbf{1}_{[-1,1]}$ and

$$V(Z_1(\mu)) \geq V(M_\infty(\mu)) = 2^n\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^{k} c_i\theta_i u_i} \prod_{i=1}^{k} f_i(\theta_i)^{c_i}\,dx \geq 2^n\prod_{i=1}^{k}\left(\int_{\mathbb{R}} f_i\right)^{c_i} = 2^n.$$

Equality in (19) leads to equality in the reverse Brascamp-Lieb inequality, and hence $k = 2n$ and $u_1, \dots, u_k$ form the vertices of a regular crosspolytope in $\mathbb{R}^n$.

The main idea in deriving a stability version of (16) and (19) is to establish a stronger version of (9) and (12), respectively, based on the stronger version of Lemma 2.1 (i) which is stated in Lemma 3.1. In order to apply the estimate of Lemma 3.1, we need some basic bounds on the derivatives of the transportation maps involved. These bounds are proved in Section 4. The technical Sections 5 and 6 also serve as a preparation for the proof of the core statement Proposition 7.2 providing the stabiliy version of (9). The argument for the estimate strenghtening (12) is similar, and is reviewed in Section 8. This finally completes the proof of Theorem 1.4. The stability versions of the reverse isoperimetric inequality in the origin symmetric case (Theorem 1.1 and Theorem 1.2) and the strengthening of Theorem 1.4 for $p = \infty$ stated in Corollary 1.5 are proved in Section 9.

The methods of this paper are very specific for our particular choice of the functions f_i, and no method is known to the authors that could lead to a stability version of the Brascamp-Lieb inequality (5) or of its reverse form (6) in general. However, the proof of Theorem 1.4 suggests the following conjecture.

Conjecture 2.2 *If f is an even probability density function on $\mathbb{R}$ with variance 1, $t \mapsto g(t) = \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}$, $t \in \mathbb{R}$, is the density of the standard normal distribution, and μ is an even isotropic measure on S^{n-1} supported at $u_1, \dots, u_k \in S^{n-1}$ with $\mu(\{u_i\}) = c_i$, then*

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f(\langle x, u_i\rangle)^{c_i}\, dx \quad \leq \quad \exp\left(-\gamma \min\{1, \|f-g\|_1\}^{\alpha} \cdot \delta_{\mathrm{WO}}(\mu, \nu_n)^{\alpha}\right),$$

$$\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^{k} c_i\theta_i u_i} \prod_{i=1}^{k} f(\theta_i)^{c_i}\, dx \quad \geq \quad \exp\left(\gamma \min\{1, \|f-g\|_1\}^{\alpha} \cdot \delta_{\mathrm{WO}}(\mu, \nu_n)^{\alpha}\right),$$

where $\gamma > 0$ depends on n and $\alpha > 0$ is an absolute constant.

3 An auxiliary analytic stability result

To obtain a stability version of Theorem B, we need a stability version of the Brascamp-Lieb inequality and its reverse form in the special cases we use. For this we need some analytic inequalities such as estimates of the derivatives of the corresponding transportation maps, which will be provided in Section 4. Moreover, we will use the following strengthened form of Lemma 2.1 (i) and a basic algebraic inequality, which were both established in [13, Section 4].

Lemma 3.1 *Let $k \geq n+1$, $t_1, \dots, t_k > 0$, and let $v_1, \dots, v_k \in \mathbb{R}^n$ satisfy $\sum_{i=1}^{k} v_i \otimes v_i = \mathrm{Id}_n$. Then*

$$\det\left(\sum_{i=1}^{k} t_i v_i \otimes v_i\right) \geq \theta^* \prod_{i=1}^{k} t_i^{\langle v_i, v_i\rangle},$$

where

$$\theta^* = 1 + \frac{1}{2} \sum_{1 \leq i_1 < \dots < i_n \leq k} \det[v_{i_1}, \dots, v_{i_n}]^2 \left(\frac{\sqrt{t_{i_1} \cdots t_{i_n}}}{t_0} - 1\right)^2,$$

$$t_0 = \sqrt{\sum_{1 \leq i_1 < \dots < i_n \leq k} t_{i_1} \cdots t_{i_n} \det[v_{i_1}, \dots, v_{i_n}]^2}.$$

In order to estimate θ^* from below, we use the following observation from [13].

Lemma 3.2 *If $a, b, x > 0$, then*

$$(xa-1)^2 + (xb-1)^2 \geq \frac{(a^2-b^2)^2}{2(a^2+b^2)^2}.$$

4 The transportation maps

We note that for $p \geq 1$, we have

$$\int_{\mathbb{R}} e^{-|t|^p} \, dt = \frac{2}{p} \int_0^\infty e^{-s} s^{\frac{1}{p}-1} \, ds = 2\Gamma(1 + \tfrac{1}{p}). \tag{20}$$

Thus for $p \in [1, \infty]$, we consider the density functions on $\mathbb{R}$ given by

$$\varrho_p(s) = \begin{cases} \frac{1}{2\Gamma(1+\frac{1}{p})} \, e^{-|s|^p} & \text{if } p \in [1, \infty), \\ \frac{1}{2}\mathbf{1}_{[-1,1]}(s) & \text{if } p = \infty. \end{cases}$$

In particular, $s \mapsto \varrho_2(s) = \pi^{-1/2} e^{-s^2}$, $s \in \mathbb{R}$, is the Gaussian density function. In addition, we define the transportation maps $\varphi_p, \psi_p : \mathbb{R} \to \mathbb{R}$ for $p \in [1, \infty)$, $\varphi_\infty : (-1, 1) \to \mathbb{R}$ and $\psi_\infty : \mathbb{R} \to (-1, 1)$ by

$$\int_{-\infty}^{t} \varrho_p(s) \, ds = \int_{-\infty}^{\varphi_p(t)} \varrho_2(s) \, ds, \tag{21}$$

$$\int_{-\infty}^{\psi_p(t)} \varrho_p(s) \, ds = \int_{-\infty}^{t} \varrho_2(s) \, ds. \tag{22}$$

Here φ_p and ψ_p are odd and inverses of each other.

In the following, we use that

$$s - s^2 \leq \log(1 + s) \leq s \ \text{ if } s \geq -\tfrac{1}{2},$$

and the following properties of the Γ function:

(i) $\log \Gamma(t)$ is strictly convex for $t > 0$;

(ii) $\Gamma(1) = \Gamma(2) = 1$;

(iii) $\Gamma(1 + \frac{1}{2.3}) < \Gamma(1 + \frac{1}{2}) = \sqrt{\pi}/2$;

(iv) Γ has a unique minimum on $(0, \infty)$ at $x_{\min} = 1.4616\ldots$ with $\Gamma(x_{\min}) = 0.885603\ldots$ In particular, $\Gamma(t) > 0.8856$ for $t > 0$, Γ is strictly decreasing on $[0, x_{\min}]$ and strictly increasing on $[1.5, \infty)$.

We deduce from (i)–(iv) that the density functions involved satisfy

$$\frac{1}{2e} \leq \varrho_p(s) < \frac{1}{2 \cdot 0.8856} \qquad \text{for } p \in [1, \infty] \text{ and } s \in [0, 1]. \tag{23}$$

We note that $e/0.8856 < 3.1$, and hence

$$\varphi_p(s) \in [0, 1) \qquad \text{for } s \in [0, \tfrac{1}{3.1}]. \tag{24}$$

In fact, assuming that $\varphi_p(\frac{1}{3.1}) \geq 1 = \varphi_p(t)$, $t \in (0, \frac{1}{3.1}]$, we have

$$\frac{3.1^{-1}}{2 \cdot 0.8856} > \int_0^t \varrho_p(s)\,ds = \int_0^1 \varrho_2(s)\,ds \geq \frac{1}{2e},$$

a contradiction. Then, (23) and (7) yield that

$$\frac{1}{3.1} < \varphi_p'(s), \psi_p'(s) < 3.1 \quad \text{for } p \in [1, \infty] \text{ and } s \in [0, \tfrac{1}{3.1}]. \tag{25}$$

The following simple estimate will play a crucial role in the proofs of Lemma 4.2 and Lemma 4.3.

Lemma 4.1 *For* $p \in (1, 3) \setminus \{2\}$ *and* $\nu > 0$*, let* $f(t) = \nu t - pt^{p-1}$ *for* $t \in [0, 1]$.

(a) *If* $p \in (1, 2)$*,* $f(\tau) \leq 0$ *for some* $\tau \in (0, 1]$ *and* $t \in (0, \tau/2]$*, then*

$$f(t) < -\frac{p(p-1)(2-p)}{2^{4-p}} \cdot t^{p-1}.$$

(b) *If* $p \in (2, 3)$*,* $f(\tau) \geq 0$ *for some* $\tau \in (0, 1]$ *and* $t \in (0, \tau/2]$*, then*

$$f(t) > \frac{p(p-1)(p-2)}{2^{4-p}} \cdot t^{p-1}.$$

Remark Naturally, the bound could be linear in t with a factor depending on ν, but this way the only influence of ν is on the value of τ. We only use Lemma 4.1 when $1.5 \leq p \leq 2.3$ and $t > c$ for a positive absolute constant c anyway.

Proof: Let $p \in (1, 2)$. Since f is convex on $[0, \tau]$, $\tau \leq 1$, $f(0) \leq 0$ and $f(\tau) \leq 0$, we have $f(2t) \leq 0$ for $t \in [0, \tau/2]$. Taylor's formula yields that if $t \in (0, \tau/2]$, then there exist $\tau_1 \in (0, t)$ and $\tau_2 \in (t, 2t)$ such that

$$\begin{aligned} 0 &\geq \frac{1}{2}\left(f(0) + f(2t)\right) = \frac{1}{2}\left(f(t) - f'(t)t + \frac{1}{2}f''(\tau_1)t^2 + f(t) + f'(t)t + \frac{1}{2}f''(\tau_2)t^2\right) \\ &= f(t) + \frac{1}{2}\,\frac{f''(\tau_1) + f''(\tau_2)}{2}\,t^2, \end{aligned}$$

where $0 < \tau_i < 2t \leq \tau$. From $f''(\tau_i) = -p(p-1)(p-2)\tau_i^{p-3} > p(p-1)(2-p)(2t)^{p-3}$, $i = 1, 2$, we deduce the estimate

$$f(t) < -\frac{1}{2}p(p-1)(2-p)(2t)^{p-3} \cdot t^2 = -\frac{p(p-1)(2-p)}{2^{4-p}} \cdot t^{p-1}.$$

If $p \in (2, 3)$, then $f(t) = \nu t - pt^{p-1}$ is concave on $[0, \tau]$, and a similar argument yields (b). □

Lemma 4.2 *Let $p \in [1, \infty] \setminus \{2\}$ and $t \in (0, \frac{1}{8})$. Then*

$$\varphi_p''(t) < -\frac{2-p}{48} \cdot t \quad \textit{if} \quad p \in [1, 2), \tag{26}$$

$$\varphi_p''(t) > \frac{p-2}{5} \cdot t^{1.3} \quad \textit{if} \quad p \in (2, 3], \tag{27}$$

$$\varphi_p''(t) > 0.2 \cdot t^{1.3} \qquad \textit{if} \quad p \in (3, \infty]. \tag{28}$$

Proof: For brevity of notation, let $\varphi = \varphi_p$. We have $\varphi(0) = 0$ as φ is odd. Since φ is strictly increasing, $\varphi(t) > 0$ if $t > 0$.

Let $p \in [1, \infty) \setminus \{2\}$. For $t > 0$, differentiating (21) yields the formula

$$\frac{e^{-t^p}}{2\Gamma(1+\frac{1}{p})} = \frac{e^{-\varphi(t)^2}\varphi'(t)}{2\Gamma(1+\frac{1}{2})},$$

and by differentiating again, we obtain

$$\frac{-p\Gamma(1+\frac{1}{2})}{\Gamma(1+\frac{1}{p})} \cdot e^{-t^p} t^{p-1} = -2e^{-\varphi(t)^2}\varphi(t)\varphi'(t)^2 + e^{-\varphi(t)^2}\varphi''(t).$$

In particular, we get

$$\varphi'(t) \;=\; \frac{\Gamma(1+\frac{1}{2})}{\Gamma(1+\frac{1}{p})}\, e^{\varphi(t)^2 - t^p}, \tag{29}$$

$$\varphi''(t) \;=\; (2\varphi(t)\varphi'(t) - pt^{p-1})\varphi'(t). \tag{30}$$

In the following argument, we use the value

$$t_p = (2/p)^{\frac{1}{p-2}} \quad \text{for } p \in [1, \infty) \setminus \{2\}.$$

The function $p \mapsto t_p$ is continuously extended to $p = 2$ by $t_2 = e^{-1/2}$, and then this function is increasing on $[1, \infty)$. In particular, $t_p \geq 1/2$ for $p \in [1, \infty)$.

Moreover, we apply the fact that

$$\text{for given } t \in (0, 1/e),\ p \mapsto pt^{p-1} \text{ is a decreasing function of } p \geq 1. \tag{31}$$

First, we show that for $1 \leq p < 2$ and $t \in (0, 1/4)$, we have $\varphi''(t) < -\frac{2-p}{48} \cdot t$, which proves (26).

In this case, $\varphi'(0) < 1$ by (29), (i), (ii) and (iv). Since φ' is continuous, there exists a largest $s_p \in (0, \infty]$ such that $\varphi'(t) < 1$ if $0 < t < s_p$. Thus, if $t \in (0, s_p)$, then $\varphi(t) < t$, and in turn (30) yields that

$$\varphi''(t) = (2\varphi(t)\varphi'(t) - pt^{p-1})\varphi'(t) < (2t - pt^{p-1})\varphi'(t).$$

For $1 \leq p < 2$ and $t \in [0, t_p]$, we have $2t - pt^{p-1} \leq 0$. In particular, $\varphi'(t)$ is monotone decreasing on $(0, \min\{s_p, t_p\})$, which in turn implies that $s_p \geq t_p$. We deduce from (25) that

$$\varphi''(t) < \frac{2t - pt^{p-1}}{3.1} \qquad \text{for } t \in (0, \tfrac{1}{3.1}). \tag{32}$$

Now we distinguish two cases. If $1.5 \leq p < 2$, then we deduce from (32) and Lemma 4.1 (a) that

$$\varphi''(t) < -\frac{p(p-1)(2-p)}{3.1 \cdot 2^{4-p}} \cdot t^{p-1} < -\frac{\frac{3}{4}(2-p)}{3.1 \cdot 2^{2.5}} \cdot t < -\frac{2-p}{24} \cdot t \qquad \text{for } t \in (0, \tfrac{1}{4}). \tag{33}$$

If $1 \leq p \leq 1.5$, then when estimating the right-hand side of (32) for a given $t \in (0, \frac{1}{4})$, we may assume that $p = 1.5$ according to (31). In other words, using Lemma 4.1 (a), inequality (33) yields that if $1 \leq p \leq 1.5$ and $t \in (0, \frac{1}{4})$, then

$$\varphi''(t) < \frac{2t - pt^{p-1}}{3.1} \leq \frac{2t - 1.5t^{0.5}}{3.1} \leq -\frac{2-1.5}{24} \cdot t \leq -\frac{2-p}{48} \cdot t.$$

Second, if $2 < p \leq 2.3$ and $t \in (0, \frac{1}{4})$, then we show that $\varphi''(t) > \frac{p-2}{2} \cdot t^{1.3}$.

In this case, $\varphi'(0) > 1$ by (29), (i), (iii) and (iv). Since φ' is continuous, there exists a largest $s_p \in (0, \infty]$ such that $\varphi'(t) > 1$ if $0 < t < s_p$. Thus if $t \in (0, s_p)$, then $\varphi(t) > t$, and in turn (30) yields that

$$\varphi''(t) = (2\varphi(t)\varphi'(t) - pt^{p-1})\varphi'(t) > (2t - pt^{p-1})\varphi'(t).$$

For $p > 2$ and $t \in [0, t_p]$, we have $2t - pt^{p-1} \geq 0$. In particular, $\varphi'(t)$ is monotone increasing on $(0, \min\{s_p, t_p\})$, which, in turn, implies that $s_p \geq t_p$. We deduce that

$$\varphi''(t) > 2t - pt^{p-1} \qquad \text{if } t \in (0, \tfrac{1}{2}). \tag{34}$$

We deduce from (34) and Lemma 4.1 (b) that

$$\varphi''(t) > \frac{p(p-1)(p-2)}{2^{4-p}} \cdot t^{p-1} > \frac{2(p-2)}{2^2} \cdot t^{1.3} = \frac{p-2}{2} \cdot t^{1.3} \qquad \text{if } t \in (0, \tfrac{1}{4}).$$

If $p \geq 2.3$ and $t \in (0, \frac{1}{8})$, then $\varphi''(t) > 0.2 \cdot t^{1.3}$, which completes the proof of (27).

In this case, $\varphi'(0) > \sqrt{\pi}/2$ by (29), (i)–(iv). Since φ' is continuous, there exists largest $s_p \in (0, \frac{1}{4}]$ such that $\varphi'(t) > \sqrt{\pi}/2$ if $0 < t < s_p$. Thus if $t \in (0, s_p]$, then $\varphi(t) > (\sqrt{\pi}/2) \cdot t$. From (31) we see that

$$2\varphi(t)\varphi'(t) - pt^{p-1} \geq \frac{\pi}{2}\, t - pt^{p-1} \geq \frac{\pi}{2}\, t - 2.3t^{1.3} \geq 0$$

for $0 < t \leq s_p \leq 1/4$. Hence (30) yields that

$$\varphi''(t) = (2\varphi(t)\varphi'(t) - pt^{p-1})\varphi'(t) > \left(\frac{\pi}{2}\, t - 2.3t^{1.3}\right) \cdot \frac{\sqrt{\pi}}{2}$$

for $t \in (0, s_p]$. In particular, we conclude that $s_p = \frac{1}{4}$, and hence Lemma 4.1 (b) yields that

$$\varphi''(t) > \frac{(\sqrt{\pi}/2) \cdot 2.3 \cdot 1.3 \cdot 0.3}{2^{1.7}} \cdot t^{1.3} > 0.2 \cdot t^{1.3} \qquad \text{for } t \in (0, \tfrac{1}{8}).$$

If $p = \infty$ and $t > 0$, then $\varphi''(t) > t$, which completes the proof of (28). Differentiating (21) we deduce for $t \in (-1, 1)$ that

$$\varphi'(t) = \Gamma\left(1 + \frac{1}{2}\right) e^{\varphi(t)^2} = \frac{\sqrt{\pi}}{2} e^{\varphi(t)^2}, \tag{35}$$

$$\varphi''(t) = 2\varphi(t)\varphi'(t)^2. \tag{36}$$

As $\varphi(t) > 0$ for $t > 0$, we have $\varphi''(t) \geq 0$ by (36), and hence $\varphi'(t)$ is monotone increasing for $t \geq 0$. Therefore $\varphi'(t) \geq \varphi'(0) = \sqrt{\pi}/2$ by (35), which, in turn, again by (36) yields that

$$\varphi''(t) \geq 2\left(\frac{\sqrt{\pi}}{2}\right)^3 t > t \qquad \text{for } t \in (0, 1).$$

Thus we have proved all estimates of Lemma 4.2 for φ''. $\square$

Lemma 4.3 *Let $p \in [1, \infty] \setminus \{2\}$. For $t \in (0, \frac{1}{10})$, we have*

$$\psi_p''(t) > \frac{2-p}{16} \cdot t \qquad \textit{if} \quad p \in [1, 2), \tag{37}$$

$$\psi_p''(t) < -\frac{p-2}{11} \cdot t^{1.3} \quad \textit{if} \quad p \in (2, 3], \tag{38}$$

$$\psi_p''(t) < -\frac{1}{11} \cdot t^{1.3} \qquad \textit{if} \quad p \in (3, \infty]. \tag{39}$$

Proof: To simplify notation, let $\psi = \psi_p$. We have $\psi(0) = 0$ as ψ is odd. Therefore $\psi(t) > 0$ if $t > 0$. Turning to ψ'', we only sketch the main steps. In this case, differentiating (22) yields the formulas

$$\psi'(t) = \frac{\Gamma(1 + \frac{1}{p})}{\Gamma(1 + \frac{1}{2})} e^{\psi(t)^p - t^2},$$

$$\psi''(t) = (p\psi(t)^{p-1}\psi'(t) - 2t)\psi'(t). \tag{40}$$

First, for $1 \leq p < 2$ and $t \in (0, \frac{1}{8})$ we show that $\psi''(t) > \frac{2-p}{16} \cdot t$, which proves (37).

If $p \in [1, 2)$, then $\psi'(0) > 1$ by (i), (ii) and (iv). Arguments similar to those in the proof of Lemma 4.2 yield

$$\psi''(t) = (p\psi(t)^{p-1}\psi'(t) - 2t)\psi'(t) > pt^{p-1} - 2t \qquad \text{for } t \in (0, \tfrac{1}{2}). \tag{41}$$

If $1.5 \le p < 2$, then we deduce from (41) and Lemma 4.1 (a) that

$$\psi''(t) > \frac{p(p-1)(2-p)}{2^{4-p}} \cdot t^{p-1} > \frac{\frac{3}{4}(2-p)}{2^{2.5}} \cdot t > \frac{2-p}{8} \cdot t \qquad \text{for } t \in (0, \tfrac{1}{8}).$$

If $1 \le p \le 1.5$, then when estimating the right-hand side of (41) for a given $t \in (0, \frac{1}{e})$, we may assume that $p = 1.5$ according to (31). In other words, (41) yields that if $1 \le p \le 1.5$ and $t \in (0, \frac{1}{e})$, then

$$\psi''(t) > pt^{p-1} - 2t \ge 1.5t^{0.5} - 2t \ge \frac{2-1.5}{8} \cdot t \ge \frac{2-p}{16} \cdot t. \tag{42}$$

Next, for $2 < p \le 2.3$ and $t \in (0, \frac{1}{4})$, we prove that $\psi''(t) < -\frac{p-2}{3} \cdot t^{1.3}$.

If $p \in (2, 2.3]$, then $\psi'(0) < 1$ by (i)–(iv), and arguments similar to the ones used in the proof of Lemma 4.2 yield

$$\psi''(t) = (p\psi(t)^{p-1}\psi'(t) - 2t)\psi'(t) < -(2t - pt^{p-1})\psi'(t) < -\frac{2t - pt^{p-1}}{3.1} < 0 \qquad \text{for } t \in (0, \tfrac{1}{3.1}).$$

We deduce from Lemma 4.1 (b) that

$$\psi''(t) < -\frac{p(p-1)(p-2)}{3.1 \cdot 2^{4-p}} \cdot t^{p-1} < -\frac{2(p-2)}{3.1 \cdot 2^2} \cdot t^{1.3} < -\frac{p-2}{7} \cdot t^{1.3} \qquad \text{for } t \in (0, \tfrac{1}{8}).$$

Let $p \ge 2.3$ and $t \in (0, \frac{1}{10})$. We now show that $\psi''(t) < -t^{1.3}/11$, which completes the proof of (38).

In this case, $\psi'(0) < 2/\sqrt{\pi}$ by (i)–(iv). There exists a maximal $s_p \in (0, \frac{1}{5}]$ such that if $t \in (0, s_p)$, then $\psi'(t) < 2/\sqrt{\pi}$. Thus if $t \in (0, s_p]$, then $\psi(t) < (2/\sqrt{\pi}) \cdot t$, and, in turn, (40) yields that

$$\psi''(t) = (p\psi(t)^{p-1}\psi'(t) - 2t)\psi'(t) < \left(\left(\frac{2}{\sqrt{\pi}}\right)^p pt^{p-1} - 2t\right)\psi'(t). \tag{43}$$

Given $t \in (0, \frac{1}{2}]$,

$$\frac{d}{dp} \log\left[\left(\frac{2}{\sqrt{\pi}}\right)^p pt^{p-1}\right] = \frac{1}{p} + \log\frac{2t}{\sqrt{\pi}} < 0 \quad \text{for } p \in (2, \infty),$$

and hence (43) yields that if $t \in (0, s_p]$, then

$$\begin{aligned} \psi''(t) &= (p\psi(t)^{p-1}\psi'(t) - 2t)\psi'(t) \\ &< \left(\left(\frac{2}{\sqrt{\pi}}\right)^{2.3} 2.3t^{1.3} - 2t\right)\psi'(t) = f(t)\left(\frac{2}{\sqrt{\pi}}\right)^{2.3}\psi'(t), \end{aligned} \tag{44}$$

where

$$f(t) = 2.3t^{1.3} - 2\left(\frac{\sqrt{\pi}}{2}\right)^{-2.3} t.$$

Here $f(\frac{1}{5}) < 0$, thus with $\tau = \frac{1}{5}$, Lemma 4.1 (b) yields that

$$f(t) < -\frac{2.3 \cdot 1.3 \cdot 0.3}{2^{1.7}} \cdot t^{1.3} < -0.27 \cdot t^{1.3} \qquad \text{for } t \in (0, \tfrac{1}{10}).$$

We conclude from (25) and (44) that

$$\psi''(t) < -\frac{(\frac{2}{\sqrt{\pi}})^{2.3} \cdot 0.27 \cdot t^{1.3}}{3.1} < -\frac{t^{1.3}}{11} \qquad \text{for } t \in (0, \tfrac{1}{10}).$$

Finally, for $p = \infty$ and $t \in (0, \frac{1}{3.1})$, we show $\psi''(t) < -\frac{2}{3.1} \cdot t$, which completes the proof of (39).

Differentiating (22) we deduce that if $t > 0$, then

$$\begin{aligned} \psi'(t) &= \frac{1}{\Gamma\left(1 + \frac{1}{2}\right)} e^{-t^2} = \frac{2}{\sqrt{\pi}} e^{-t^2}, \\ \psi''(t) &= -2t\psi'(t). \end{aligned}$$

We conclude from (25) that $\psi''(t) < -\frac{2t}{3.1}$ for $t \in (0, \frac{1}{3.1})$.

In summary, we have established all estimates of Lemma 4.3 for ψ''. $\square$

5 Basic estimates on isotropic measures and a measure theoretic version of the Dvoretzky-Rogers lemma

The main result of this section is Lemma 5.4. It states that for any isotropic measure μ on S^{n-1}, there exist spherical caps $X_1, \ldots, X_n \subseteq S^{n-1}$ whose μ-measure is bounded from below and which have the additional property that for any vectors $w_i \in X_i$, $i \in \{1, \ldots, n\}$, also the determinant $|\det[w_1, \ldots, w_n]|$ is bounded from below.

For $\alpha \in (0, \frac{\pi}{2}]$ and $v \in S^{n-1}$, we consider the (closed and open, respectively) spherical caps

$$\begin{aligned} \Omega(v, \alpha) &= \{u \in S^{n-1} : \langle u, v \rangle \geq \cos\alpha\}, \\ \widetilde{\Omega}(v, \alpha) &= \{u \in S^{n-1} : \langle u, v \rangle > \cos\alpha\}. \end{aligned}$$

Claim 5.1 *If μ is an isotropic measure on S^{n-1}, $v \in S^{n-1}$, and $\alpha \in (0, \frac{\pi}{2})$, then*

$$\mu\left(\widetilde{\Omega}(v, \alpha)\right) + \mu\left(\widetilde{\Omega}(-v, \alpha)\right) \geq 1 - n\cos^2\alpha.$$

Proof: For given $v \in S^{n-1}$ and $\alpha \in (0, \frac{\pi}{2})$, let $X = \{u \in S^{n-1} : |\langle u, v \rangle| \leq \cos\alpha\}$. Since μ is isotropic, we have $\mu(X) \leq n$, and

$$1 = \langle v, v \rangle = \int_{S^{n-1}} \langle u, v \rangle^2 \, d\mu(u) = \int_{\widetilde{\Omega}(v,\alpha) \cup \widetilde{\Omega}(-v,\alpha)} \langle u, v \rangle^2 \, d\mu(u) + \int_X \langle u, v \rangle^2 \, d\mu(u)$$

$$\leq \mu\left(\widetilde{\Omega}(v,\alpha) \cup \widetilde{\Omega}(-v,\alpha)\right) + n\cos^2\alpha. \quad \square$$

Observe that if $\cos\alpha \geq 1/\sqrt{n}$ in the preceding claim, then the conclusion holds trivially. The next claim follows from a standard argument but we are not aware of any reference.

Claim 5.2 *If μ is a Borel measure on S^{n-1}, $p \in S^{n-1}$, and $0 < \beta < \alpha < \frac{\pi}{2}$, then there exists a point $v \in \Omega(p,\alpha)$ such that*

$$\mu\left(\Omega(p,\alpha) \cap \Omega(v,\beta)\right) \geq \mu(\Omega(p,\alpha)) \cdot \frac{\sin^{n-1}\beta}{\sqrt{2\pi n}};$$

if $\mu(\Omega(p,\alpha)) > 0$, then $v \in \Omega(p,\alpha)$ can be chosen such that the inequality is strict.

Proof: We define the Borel measure $\bar{\mu}$ on S^{n-1} by $\bar{\mu}(X) = \mu(X \cap \Omega(p,\alpha))$ for Borel sets $X \subseteq S^{n-1}$. Let ν be the Haar probability measure on $\mathrm{SO}(n)$. Hence, if $X \subseteq S^{n-1}$ is a Borel set and $u \in S^{n-1}$, then

$$\nu(\{g \in \mathrm{SO}(n) :\ gu \in X\}) = \frac{\mathcal{H}^{n-1}(X)}{\mathcal{H}^{n-1}(S^{n-1})},$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure (its restriction to Borel subsets of S^{n-1} equals spherical Lebesgue measure). We deduce that

$$\begin{aligned}
\mu\left(\Omega(p,\alpha)\right) \cdot \frac{\mathcal{H}^{n-1}(\Omega(p,\beta))}{\mathcal{H}^{n-1}(S^{n-1})} &= \bar{\mu}\left(S^{n-1}\right) \cdot \frac{\mathcal{H}_{n-1}(\Omega(p,\beta))}{\mathcal{H}_{n-1}(S^{n-1})} \\
&= \int_{S^{n-1}} \int_{\mathrm{SO}(n)} \mathbf{1}_{\Omega(p,\beta)}(gu)\, d\nu(g)\, d\bar{\mu}(u) \\
&= \int_{\mathrm{SO}(n)} \int_{S^{n-1}} \mathbf{1}_{\Omega(p,\beta)}(gu)\, d\bar{\mu}(u)\, d\nu(g) \\
&= \int_{\mathrm{SO}(n)} \bar{\mu}(\Omega(g^{-1}p,\beta))\, d\nu(g) \\
&= \int_{\mathrm{SO}(n)} \mu(\Omega(p,\alpha) \cap \Omega(g^{-1}p,\beta))\, d\nu(g).
\end{aligned}$$

Hence there exists some $v_0 \in S^{n-1}$ such that

$$\mu\left(\Omega(p,\alpha) \cap \Omega(v_0,\beta)\right) \geq \mu\left(\Omega(p,\alpha)\right) \cdot \frac{\mathcal{H}^{n-1}(\Omega(p,\beta))}{\mathcal{H}^{n-1}(S^{n-1})}.$$

To finish the proof, we can assume that $\mu(\Omega(p,\alpha)) > 0$. Finally, if $v \in \Omega(p,\alpha)$ is the closest point to v_0, then

$$\Omega(p,\alpha) \cap \Omega(v_0,\beta) \subseteq \Omega(p,\alpha) \cap \Omega(v,\beta).$$

To conclude the proof, we use that $\mathcal{H}^{n-1}(\Omega(p,\beta)) > \kappa_{n-1}\sin^{n-1}\beta$, $\mathcal{H}^{n-1}(S^{n-1}) = n\kappa_n$, where κ_i denotes the volume of the i-dimensional unit ball, and the basic inequality $\frac{\kappa_{n-1}}{n\kappa_n} > \frac{1}{\sqrt{2\pi n}}$, which follows from (i); see [48, p. 564, l. 2]. $\Box$

Claim 5.3 *If $b_1,\dots,b_n \in S^{n-1}$, and $s_1,\dots,s_n \in \mathbb{R}^n$ satisfy $\|s_i\| \le |\det[b_1,\dots,b_n]|/4n$, then*

$$|\det[b_1+s_1,\dots,b_n+s_n]| \ge |\det[b_1,\dots,b_n]|/2.$$

Proof: Let $D = |\det[b_1,\dots,b_n]|/4n$. Since for any $r_1,\dots,r_n \in \mathbb{R}^n$ we have

$$|\det[r_1,\dots,r_n]| \le \|r_1\|\cdots\|r_n\|,$$

we deduce from the linearity of the determinant and $e^t \le 1+2t$ for $t\in[0,1]$ that

$$\begin{aligned}
|\det[b_1+s_1,\dots,b_n+s_n]| &\ge |\det[b_1,\dots,b_n]| - \sum_{i=1}^{n}\binom{n}{i}D^i \\
&= 4nD - (1+D)^n + 1 \\
&\ge 4nD - e^{nD} + 1 \\
&\ge 4nD - 2nD \ge 2nD = |\det[b_1,\dots,b_n]|/2. \quad \Box
\end{aligned}$$

Lemma 5.4 can be considered as a measure theoretic version of the Dvoretzky-Rogers lemma (see A. Dvoretzky, C. A. Rogers [21], S. Brazitikos, A. Giannopoulos, P. Valettas, B.-H. Vritsiou [16], and for a non-symmetric version, M. Naszodi [44]).

Lemma 5.4 *Let $\beta = 2^{-(n+1)}n^{-(n+1)/2}$. If μ is an isotropic measure on S^{n-1}, then there exist $v_1,\dots,v_n \in S^{n-1}$ such that $\mu(\Omega(v_i,\beta)) \ge \beta^n$, for $i=1,\dots,n$, and such that if $w_i \in \Omega(v_i,\beta)$, for $i\in\{1,\dots,n\}$, then $|\det[w_1,\dots,w_n]| \ge 2n\beta$.*

Proof: Let $\alpha_n \in (0,\frac{\pi}{2})$ satisfy $\cos\alpha_n = \frac{1}{2\sqrt{n}}$. First, we will construct $v_i, p_i \in S^{n-1}$ by induction on $i\in\{1,\dots,n\}$ in such a way that

$$\begin{aligned}
\mu(\Omega(v_i,\beta)) &\ge \beta^n, && (45)\\
\mu(\Omega(p_i,\alpha_n)) &\ge 3/8, && (46)\\
v_i &\in \Omega(p_i,\alpha_n), && (47)\\
\langle p_i, v_j\rangle &= 0 \ \text{ for } 1\le j<i\le n. && (48)
\end{aligned}$$

For this, let $p \in S^{n-1}$. According to Claim 5.1, we can choose $p_1 \in \{p,-p\}$ such that

$$\mu(\Omega(p_1,\alpha_n)) \ge \frac{1-n\cos^2\alpha_n}{2} = \frac{3}{8}.$$

Thus, since $\beta < 1 < \alpha_n$, Claim 5.2 yields the existence of a point $v_1 \in \Omega(p_1, \alpha_n)$ satisfying (45).

If $i \geq 2$, and v_j, p_j are known for $j = 1, \ldots, i-1$, then we choose $p_i' \in S^{n-1}$ satisfying (48). Again, Claim 5.1 provides $p_i \in \{p_i', -p_i'\}$ satisfying (46). In addition, a point $v_i \in \Omega(p_i, \alpha_n)$ satisfying (45) is provided by Claim 5.2.

We deduce from (47) that if $i \in \{1, \ldots, n\}$, then $\langle p_i, v_i \rangle \geq \frac{1}{2\sqrt{n}}$. Combined with (48), for $i \in \{2, \ldots, n\}$ this yields that

$$\operatorname{dist}\left(v_i, \operatorname{aff}\{v_1, \ldots, v_{i-1}\}\right) \geq \frac{1}{2\sqrt{n}}.$$

In particular,

$$|\det[v_1, \ldots, v_n]| \geq 2^{-(n-1)} n^{-(n-1)/2} = 4n\beta.$$

Next let $w_i \in \Omega(v_i, \beta)$ for $i = 1, \ldots, n$, and hence $\|s_i\| < \beta$ for $s_i = w_i - v_i$ and $i = 1, \ldots, n$. Therefore Claim 5.3 implies the lemma. $\Box$

The following Lemma 5.5 uses the notation of Lemma 5.4.

Lemma 5.5 *For an isotropic measure μ on S^{n-1}, let $v_1, \ldots, v_n \in S^{n-1}$ and β be as in Lemma 5.4. For every $i \in \{1, \ldots, n\}$ and $\eta \in (0, \beta)$,*

(i) *there exists $q_i \in \Omega(v_i, \beta)$ such that*

$$\mu(\Omega(v_i, \beta) \cap \Omega(q_i, \eta)) \geq \frac{\beta^n}{4n},$$

(ii) *or there exist $\Psi_1, \Psi_2 \subseteq \Omega(v_i, \beta)$ such that*

$$\mu(\Psi_j) \geq \frac{\beta^n}{4n} \qquad \text{for } j = 1, 2,$$
$$\|a_1 - a_2\| \geq \frac{\eta}{\sqrt{n}} \qquad \text{for } a_1 \in \Psi_1 \text{ and } a_2 \in \Psi_2.$$

The points q_1, q_2 and the sets Ψ_1, Ψ_2 can be chosen independently of $\eta \in (0, \beta)$.

Proof: Let $i \in \{1, \ldots, n\}$ and $\eta \in (0, \beta)$ be fixed.

If there exists $q_i \in \Omega(v_i, \beta)$ such that $\mu(\{q_i\}) \geq \frac{\beta^n}{4n}$, then (i) is satisfied. Therefore we assume that

$$\mu(\{q\}) < \frac{\beta^n}{4n} \qquad \text{for all } q \in \Omega(v_i, \beta). \tag{49}$$

We choose an orthonormal basis $w_1, \ldots, w_{n-1}$ for $v_i^\perp$. It follows from (49) that there exist $-1 < s_j \leq t_j < 1$ for $j = 1, \ldots, n-1$ such that

$$\mu\left(\{x \in \Omega(v_i, \beta) : \langle w_j, x \rangle < s_j\}\right) \leq \frac{\beta^n}{4n} \leq \mu\left(\{x \in \Omega(v_i, \beta) : \langle w_j, x \rangle \leq s_j\}\right),$$

$$\mu\left(\{x\in\Omega(v_i,\beta):\,\langle w_j,x\rangle>t_j\}\right)\le\frac{\beta^n}{4n}\le\mu\left(\{x\in\Omega(v_i,\beta):\,\langle w_j,x\rangle\ge t_j\}\right).$$

We may assume that $t_1-s_1\ge\ldots\ge t_{n-1}-s_{n-1}$, and we define $\Psi_1=\{x\in\Omega(v_i,\beta):\langle w_1,x\rangle\le s_1\}$ and $\Psi_2=\{x\in\Omega(v_i,\beta):\,\langle w_1,x\rangle\ge t_1\}$. In addition, let $q_i\in\Omega(v_i,\beta)$ be such that $\langle q_i,w_j\rangle=(s_j+t_j)/2$ for $j=1,\ldots,n-1$, and let

$$\Psi=\{x\in\Omega(v_i,\beta):s_j\le\langle w_j,x\rangle\le t_j,\ j=1,\ldots,n-1\}.$$

If $t_1-s_1\ge\eta/\sqrt{n}$, then Ψ_1 and Ψ_2 satisfy (ii). Finally, we assume that $t_1-s_1<\eta/\sqrt{n}$, and hence $t_j-s_j<\eta/\sqrt{n}$ for $j=1,\ldots,n-1$. On the one hand,

$$\mu(\Psi)\ge\mu(\Omega(v_i,\beta))-2n\cdot\frac{\beta^n}{4n}\ge\frac{\beta^n}{2}.$$

On the other hand, $\|z-(q_i|v_i^\perp)\|\le\eta/2$ for $z\in\Psi|v_i^\perp$. Since $\langle u,v_i\rangle>1/2$ for $u\in\Omega(v_i,\beta)$, we deduce that $\Psi\subseteq\Omega(q_i,\eta)$. In turn, we conclude (i). □

6 Even isotropic measures and the cross measure

As a consequence of Claim 5.1, we estimate the Wasserstein distance.

Lemma 6.1 *Let μ be an even isotropic measure, and let ν be a cross measure on S^{n-1} with* $\operatorname{supp}\nu=\{\pm w_1,\ldots,\pm w_n\}$. *If $\delta\in[0,\frac{\pi}{4})$ and $\omega\in[0,1)$ are such that*

$$\mu\left(S^{n-1}\backslash\bigcup_{i=1}^{n}(\Omega(w_i,\delta)\cup\Omega(-w_i,\delta))\right)\le\omega,$$

then

$$\delta_W(\mu,\nu)\le 2n\delta+2\pi n^2\omega.$$

Proof: We write $w_{i+n}=-w_i$ for $i=1,\ldots,n$. Since $\widetilde{\Omega}\left(w_i,\frac{\pi}{2}-\delta\right)$ is disjoint from $\Omega(w_j,\delta)$ for $i\ne j$, it follows from Claim 5.1 that for each $i=1,\ldots,n$, we have

$$\begin{aligned}\mu\left(\Omega(w_i,\delta)\cup\Omega(-w_i,\delta)\right) &\ge \mu\left(\widetilde{\Omega}\left(w_i,\frac{\pi}{2}-\delta\right)\cup\widetilde{\Omega}\left(-w_i,\frac{\pi}{2}-\delta\right)\right)-\omega\\ &> 1-n\sin^2\delta-\omega>1-n\delta^2-\omega.\end{aligned}$$

Since μ is even, we get

$$\mu\left(\Omega(w_i,\delta)\right)-\frac{1}{2}\ge-\frac{n\delta^2+\omega}{2}.$$

Since $\mu(S^{n-1})=n$, μ is even, and $\delta<\pi/4$ we deduce for $i=1,\ldots,n$ that

$$n\ \ge\ 2\mu\left(\Omega(w_i,\delta)\right)+\sum_{j:j\notin\{i,i+n\}}\mu\left(\Omega(w_j,\delta)\cup\Omega(-w_j,\delta)\right)+0$$

$$\geq \ 2\mu\left(\Omega(w_i,\delta)\right) + (n-1)(1-n\delta^2-\omega),$$

and hence

$$\mu(\Omega(w_i,\delta)) \leq \frac{1}{2}\left(n-(n-1)(1-n\delta^2-\omega)\right) \leq \frac{1+n^2\delta^2+n\omega}{2},$$

for $i=1,\ldots,2n$. Thus, for $i=1,\ldots,2n$ we get

$$\left|\mu(\Omega(w_i,\delta))-\frac{1}{2}\right| \leq \frac{n^2\delta^2+n\omega}{2}.$$

For $f\in \mathrm{Lip}_1(S^{n-1})$, we may assume that $f(w_1)=0$, since $\mu(S^{n-1})=\nu(S^{n-1})=n$, and hence $|f(u)|\leq\pi$ for $u\in S^{n-1}$. Therefore

$$\begin{aligned}
&\int_{S^{n-1}} f\,d\mu - \int_{S^{n-1}} f\,d\nu \\
&= \sum_{i=1}^{2n}\left(\int_{\Omega(w_i,\delta)}(f(u)-f(w_i))\,d\mu(u) + \int_{\Omega(w_i,\delta)} f(w_i)\,d\mu(u) - \frac{f(w_i)}{2}\right) \\
&\qquad + \int_{S^{n-1}\setminus(\cup_{i=1}^{2n}\Omega(w_i,\delta))} f(u)\,d\mu(u) \\
&\leq 2n\left(\delta\cdot\frac{1+n^2\delta^2+n\omega}{2} + \pi\cdot\frac{n^2\delta^2+n\omega}{2}\right) + \pi\omega \\
&\leq 2n\delta + 2\pi n^2\omega,
\end{aligned}$$

which yields the assertion. □

We deduce the following estimate for the Wasserstein distance.

Corollary 6.2 *Let μ be an even isotropic measure, and let ν be a cross measure on S^{n-1}. If $\delta_H(\mathrm{supp}\,\mu,\mathrm{supp}\,\nu)<\pi/4$, then*

$$\delta_W(\mu,\nu)\leq 2n\delta_H(\mathrm{supp}\,\mu,\mathrm{supp}\,\nu).$$

Finally, we consider the stability of optimal symmetric coverings of S^{n-1} by $2n$ congruent spherical caps, where a symmetric covering is an arrangement invariant under the antipodal map. It is a well-known conjecture that in an optimal covering of S^{n-1} by $2n$ congruent spherical caps, the spherical centers of the caps are vertices of a regular crosspolytope (see, say, L. Fejes Tóth [22]). This conjecture has been verified by L. Fejes Tóth [22] for $n\leq 3$, and by L. Dalla, D. G Larman, P. Mani-Levitska, C. Zong [18] for $n=4$. The case when the $2n$ congruent spherical caps are symmetric (see Lemma 6.3 (i)) should be known, but we could not find any reference for the cases $n\geq 5$.

Lemma 6.3 *Let $n\geq 2$, let $t\in(0,(2\cdot 4^{n-2}\sqrt{(n-1)!})^{-1})$, and let $u_1,\ldots,u_n\in S^{n-1}$.*

(i) *If there exist $i < j$ such that $|\langle u_i, u_j\rangle| \geq \sin t$, then there exists $u \in S^{n-1}$ such that*

$$|\langle u_i, u\rangle| \leq \frac{1}{\sqrt{n}} - \frac{t}{4n^{3/2}} \text{ for } i = 1, \dots, n.$$

(ii) *If $|\langle u_i, u_j\rangle| \leq \sin t$ for all $i < j$, then there exists a cross measure ν such that*

$$\delta_H(\operatorname{supp} \nu, \{\pm u_1, \dots, \pm u_n\}) \leq 4^{n-2}\sqrt{(n-1)!} \cdot t.$$

Proof: For the proof of (i) we may assume that $|\langle u_1, u_2\rangle| \geq \sin t$. We construct sequences $a_2, \dots, a_n > 0$ and $w_1, \dots, w_n \in S^{n-1}$ such that $w_i \in \operatorname{lin}\{u_1, \dots, u_i\}$, and possibly after exchanging some of the vectors u_i by $-u_i$, we have

$$\langle w_i, u_j\rangle = a_i \text{ for } i = 1, \dots, n \text{ and } j = 1, \dots, i.$$

More precisely, let $w_1 = u_1$, and if $i \in \{2, \dots, n\}$ and $w_1, \dots, w_{i-1}$ have already been determined, then we choose the direction of u_i in such a way that $\langle u_i, w_{i-1}\rangle \leq 0$. This algorithm determines $a_2, \dots, a_n > 0$ and $w_1, \dots, w_n \in S^{n-1}$, and subsequently we prove that

$$\langle w_i, u_j\rangle = a_i \leq \frac{1}{\sqrt{i}} - \frac{t}{4i^{3/2}} \text{ for } i = 2, \dots, n \text{ and } j = 1, \dots, i. \tag{50}$$

To verify (50), we use the elementary fact that if o is a vertex of a triangle and if the two sides meeting at o are of length a and b and enclose an angle γ, then the distance of o from the line of the third side is

$$h = \frac{ab\sin\gamma}{\sqrt{a^2 + b^2 - 2ab\cos\gamma}}. \tag{51}$$

In addition, we use that if $f(a) = \frac{a}{\sqrt{1+a^2}}$ for $a \in (0, s)$ and $s > 0$, then

$$f'(a) = \frac{1}{(1+a^2)^{3/2}} > \frac{1}{(1+s^2)^{3/2}}. \tag{52}$$

We start with the case $i = 2$. Since $\langle u_1, u_2\rangle \leq 0$, we have $\angle(u_1, u_2) \geq \frac{\pi}{2} + t$ and $w_2 = (u_1 + u_2)/\|u_1 + u_2\|$. Therefore (51) yields that

$$\langle w_2, u_1\rangle = \langle w_2, u_2\rangle \leq \frac{\cos t}{\sqrt{2 + 2\sin t}} < \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{1 + \sin t}} < \frac{1}{\sqrt{2}} \cdot \left(1 - \frac{\sin t}{4}\right) < \frac{1}{\sqrt{2}} - \frac{t}{8\sqrt{2}}.$$

Next assume that $2 \leq i < n$ and (50) holds. We observe that $a_i w_i \in \operatorname{aff}\{u_1, \dots, u_i\}$ and a_{i+1} is the distance of o from $\operatorname{aff}\{u_1, \dots, u_{i+1}\}$, which is then at most the distance of o from $\operatorname{aff}\{a_i w_i, u_{i+1}\}$, that is in turn the height of the triangle $[o, a_i w_i, u_{i+1}]$ corresponding to o. Since $\langle u_{i+1}, w_i\rangle \leq 0$, we deduce first from (51), then from (52) with $a_i < s = \frac{1}{\sqrt{i}}$ that

$$a_{i+1} \leq \frac{a_i}{\sqrt{1 + a_i^2}} = f(a_i) < f(s) - \frac{t}{4i^{3/2}(1+s^2)^{3/2}} = \frac{1}{\sqrt{i+1}} - \frac{t}{4(i+1)^{3/2}}.$$

Finally, (50) yields (i) with $u = w_n$.

For (ii), let $v_1, \ldots, v_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $v_i \in \mathrm{lin}\{u_1, \ldots, u_i\}$ and $\langle v_j, u_j \rangle \geq 0$ for $j = 1, \ldots, n$, and hence $v_1 = u_1$. We verify that

$$\angle(v_i, u_i) \leq 4^{i-2}\sqrt{(i-1)!} \cdot t \ \text{ for } i = 2, \ldots, n \tag{53}$$

by induction on $i = 2, \ldots, n$.

If $i = 2$, then readily $\angle(v_2, u_2) \leq t$. If (53) holds for all $j \leq i$ for some $i \in \{2, \ldots, n-1\}$, then

$$\left|\angle(u_{i+1}, v_j) - \frac{\pi}{2}\right| \leq \left|\angle(u_{i+1}, u_j) - \frac{\pi}{2}\right| + \angle(u_j, v_j) < 2 \cdot 4^{i-2}\sqrt{(i-1)!} \cdot t$$

for $j = 1, \ldots, i$. In other words, $\langle u_{i+1}, v_j \rangle < 2 \cdot 4^{i-2}\sqrt{(i-1)!} \cdot t$ for $j = 1, \ldots, i$, which in turn yields that

$$\sin \angle(u_{i+1}, v_{i+1}) = \sqrt{\sum_{j=1}^{i} \langle u_{i+1}, v_j \rangle^2} \leq 2 \cdot 4^{i-2}\sqrt{(i-1)!}\sqrt{i} \cdot t = 2 \cdot 4^{i-2}\sqrt{i!} \cdot t.$$

Thus we conclude $\angle(u_{i+1}, v_{i+1}) < 4^{i-1}\sqrt{i!} \cdot t$. $\Box$

Lemma 6.3 yields the following statement with factor $4n^{3/2} \cdot 4^{n-2}\sqrt{(n-1)!} < 4^n n!$.

Corollary 6.4 *Let $n \geq 2$, let $t \in (0, \frac{1}{4^n n!})$, and let $u_1, \ldots, u_n \in S^{n-1}$. If*

$$\Omega\left(u, \arccos\left(\frac{1}{\sqrt{n}} - t\right)\right) \cap \{\pm u_1, \ldots, \pm u_n\} \neq \emptyset$$

for any $u \in S^{n-1}$, then there exists a cross measure ν such that

$$\delta_H(\mathrm{supp}\, \nu, \{\pm u_1, \ldots, \pm u_n\}) \leq 4^n n! \cdot t.$$

Remark The condition in Corollary 6.4 is equivalent to saying that the spherical caps $\Omega\left(\pm u_i, \arccos\left(\frac{1}{\sqrt{n}} - t\right)\right)$, $i = 1, \ldots, n$, cover S^{n-1}.

7 The volume of Z_p^*

In this section, we prove the stability result for the volume of Z_p^*, which is stated in Theorem 1.4. The remaining part of this theorem is established in Section 8.

The main ingredient for the proof in this section is stated as Proposition 7.2. We start with a preparatory claim.

Claim 7.1 *For $u, u_0 \in S^{n-1}$ with $\langle u, u_0 \rangle \geq 0$, we have $V(\Xi_{u,u_0}) \geq \kappa_n / 240^n$, where*

$$\Xi_{u,u_0} = \left\{ y \in 0.1\, B^n : \langle y, u \rangle \geq \frac{1}{30}, \langle y, u_0 \rangle \geq \frac{1}{30}, \langle y, u - u_0 \rangle \geq \frac{\|u - u_0\|}{120} \right\}. \tag{54}$$

Proof: Let γ be half of the angle of u and u_0, and hence $\gamma \in [0, \frac{\pi}{4}]$. The set

$$\Xi_0 = \left\{ y \in 0.1\, B^n : \langle y, u \rangle \geq \frac{1}{30},\ \langle y, u_0 \rangle \geq \frac{1}{30} \right\}$$

contains a ball of radius r with center $\frac{0.1-r}{\|u+u_0\|}\,(u+u_0)$ provided that

$$(0.1 - r)\cos\gamma \geq \tfrac{1}{30} + r.$$

Since $\cos\gamma \geq 1/\sqrt{2}$, we may choose

$$r = \frac{0.1 - (\sqrt{2}/30)}{\sqrt{2}+1} > \frac{1}{60}.$$

Therefore Ξ_{u,u_0} contains a ball of radius $r/4 > 1/240$. $\Box$

Proposition 7.2 *If $p \in [1, \infty) \setminus \{2\}$, μ is an even discrete isotropic measure on S^{n-1}, and*

$$V(Z_p^*(\mu)) \geq (1-\varepsilon) V(Z_p^*(\nu_n))$$

for some $\varepsilon \in (0,1)$, then there exists a cross measure ν on S^{n-1} such that

$$\delta_W(\mu, \nu) \leq n^{cn^3} \max\{|p-2|^{-\frac{2}{3}}, 1\} \cdot \varepsilon^{\frac{1}{3}}$$

for some absolute constant $c > 0$.

Proof: What we actually prove is that for any $0 < \eta < \beta^n/(2n)$, we have

$$V(Z_p^*(\mu)) < (1 - n^{-cn^3} \min\{(p-2)^2, 1\} \cdot \eta^3) V(Z_p^*(\nu_n)) \tag{55}$$

or there exists a cross measure ν satisfying

$$\delta_W(\mu, \nu) \leq n^{cn}\eta \tag{56}$$

for some absolute constant $c > 0$.

Let $\operatorname{supp}\mu = \{\bar{u}_1, \ldots, \bar{u}_{\bar{k}}\}$, and let $\bar{c}_i = \mu(\{\bar{u}_i\})$. For $c_0 = \min\{\bar{c}_i : i = 1, \ldots, \bar{k}\}$ and $i = 1, \ldots, \bar{k}$, we define $\bar{m}_i = \min\{m \in \mathbb{Z} : m \geq 1 \text{ and } \bar{c}_i/m \leq c_0\}$, and let $k = \sum_{i=1}^{\bar{k}} \bar{m}_i$. We consider $\xi : \{1, \ldots, k\} \to \{1, \ldots, \bar{k}\}$ such that $\#\xi^{-1}(\{i\}) = \bar{m}_i$ for $i = 1, \ldots, \bar{k}$, and define

$$u_i = \bar{u}_{\xi(i)} \text{ and } c_i = \bar{c}_{\xi(i)}/\bar{m}_{\xi(i)}$$

for $i = 1, \ldots, k$. The system $(u_1, \ldots, u_k, c_1, \ldots, c_k)$ is even (i.e. origin symmetric) in the following sense: Any $u \in S^{n-1}$ occurs as u_i exactly as many times as $-u$, and if $u_i = -u_j$, then $c_i = c_j$.

In particular, $\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{Id}_n$ and $\sum_{i=1}^k c_i = n$, and for any Borel $X \subseteq S^{n-1}$, we have

$$\mu(X) = \sum_{u_i \in X} c_i.$$

The reason for the renormalization is that

$$c_0/2 < c_i \leq c_0 \ \text{ for } i = 1, \ldots, k. \tag{57}$$

In addition, let $\varphi = \varphi_p$ be defined as in (21), let $g(t) = e^{-\pi t^2}$, and let $f_i = \varrho_p$, for $i = 1, \ldots, k$.

We define the map $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Theta(y) = \sum_{i=1}^k c_i \varphi(\langle y, u_i \rangle)\, u_i,$$

and hence the differential of Θ is

$$d\Theta(y) = \sum_{i=1}^k c_i \varphi'(\langle y, u_i \rangle)\, u_i \otimes u_i,$$

where $d\Theta(y)$ is positive definite, and $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ is injective. Applying first (15) and then (8), we get

$$\begin{aligned} V(Z_p^*(\mu)) &\leq \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \left(\prod_{i=1}^k g(\varphi(\langle u_i, x \rangle))^{c_i} \right) \left(\prod_{i=1}^k \varphi'(\langle u_i, x \rangle)^{c_i} \right) dx \\ &= \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \exp\left(-\pi \sum_{i=1}^k c_i \varphi(\langle u_i, x \rangle)^2 \right) \left(\prod_{i=1}^k \varphi'(\langle u_i, x \rangle)^{c_i} \right) dx. \end{aligned} \tag{58}$$

For each fixed $y \in \mathbb{R}^n$, we estimate the product of the two terms in (58) after the integral sign. To estimate the first term in (58), we apply Lemma 2.1 (ii) with $\theta_i = \varphi(\langle y, u_i \rangle)$, $i = 1, \ldots, k$, and hence the definition of Θ yields

$$\exp\left(-\pi \sum_{i=1}^k c_i \varphi(\langle y, u_i \rangle)^2 \right) \leq e^{-\pi \|\Theta(y)\|^2}. \tag{59}$$

To estimate the second term, we apply Lemma 3.1 with $v_i = \sqrt{c_i} \cdot u_i$ and $t_i = \varphi'(\langle y, u_i \rangle)$, at each $y \in \mathbb{R}^n$, and write $\theta^*(y)$ and $t_0(y)$ to denote the corresponding $\theta^* \geq 1$ and $t_0 > 0$. In particular, if $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, k\}$ and $y \in \mathbb{R}^n$, then we put

$$\mathsf{N}(i_1, \ldots, i_n; y) = c_{i_1} \cdots c_{i_n} \det[u_{i_1}, \ldots, u_{i_n}]^2 \left(\frac{\sqrt{\varphi'(\langle y, u_{i_1} \rangle) \cdots \varphi'(\langle y, u_{i_n} \rangle)}}{t_0(y)} - 1 \right)^2. \tag{60}$$

Therefore, for

$$\theta^*(y) = 1 + \frac{1}{2}\sum_{1\le i_1<\ldots<i_n\le k} \mathsf{N}(i_1,\ldots,i_n;y) \tag{61}$$

Lemma 3.1 yields that

$$\prod_{i=1}^{k} \varphi'(\langle y,u_i\rangle)^{c_i} \le \theta^*(y)^{-1}\det\left(d\Theta(y)\right). \tag{62}$$

From (59) and (62), we conclude that

$$V(Z_p^*(\mu)) \le \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n} \theta^*(y)^{-1}e^{-\pi\|\Theta(y)\|^2}\det\left(d\Theta(y)\right)\,dy. \tag{63}$$

To provide a lower bound for $\theta^*(y)$, we use (25) and (24), hence

$$\frac{1}{3.1} < \varphi'(s) < 3.1 \text{ and } \varphi(s) < 1 \;\text{ for } p\in[1,\infty] \text{ and } s\in[0,\tfrac{1}{3.1}]. \tag{64}$$

We consider the vectors $v_1,\ldots,v_n\in S^{n-1}$ provided by Lemma 5.4 such that

$$\begin{aligned} \mu(\Omega(v_i,\beta)) &> \beta^n \quad \text{for } i=1,\ldots,n;\\ |\det[w_1,\ldots,w_n]| &\ge 2n\beta \quad \text{for } w_i\in\Omega(v_i,\beta) \text{ and } i\in\{1,\ldots,n\};\\ \beta &= 2^{-(n+1)}n^{-(n+1)/2}. \end{aligned} \tag{65}$$

The remaining discussion is split into three cases, where the first two correspond to the two cases in Lemma 5.5.

Case 1 *There exist $l\in\{1,\ldots,n\}$ and $\Psi_1,\Psi_2\subseteq\Omega(v_l,\beta)$ such that*

$$\begin{aligned} \mu(\Psi_j) &\ge \frac{\beta^n}{4n} \quad \textit{for } j=1,2, \textit{ and}\\ \|a_1-a_2\| &\ge \frac{\eta}{\sqrt{n}} \quad \textit{for } a_1\in\Psi_1 \textit{ and } a_2\in\Psi_2. \end{aligned}$$

In this case, we prove

$$V(Z_p^*(\mu)) < \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}(1-n^{-cn^3}\min\{(p-2)^2,1\}\cdot\eta^2) \tag{66}$$

for some absolute constant $c>0$.

We may assume that $l=n$. For $j=1,2$, let

$$\Pi_j = \{i\in\{1,\ldots,k\}:\, u_i\in\Psi_j\}\neq\emptyset.$$

Possibly after interchanging the roles of Ψ_1 and Ψ_2, we may assume that $\#\Pi_1\le\#\Pi_2$. Let

$$\tau:\Pi_1\to\Pi_2 \;\text{ be an injective map.}$$

Given $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \ldots, n-1$ and $u_{i_n} \in \Psi_1$, we have have $u_{\tau(i_n)} \in \Psi_2$, and (57) and (65) yield

$$\left.\begin{array}{r} c_{i_1} \cdots c_{i_{n-1}} \cdot c_{i_n} \det[u_{i_1}, \ldots, u_{i_n}]^2 \\ c_{i_1} \cdots c_{i_{n-1}} \cdot c_{\tau(i_n)} \det[u_{i_1}, \ldots, u_{i_{n-1}}, u_{\tau(i_n)}]^2 \end{array}\right\} \geq 4n^2\beta^2 c_{i_1} \cdots c_{i_{n-1}} \cdot (c_{i_n}/2). \tag{67}$$

Since $\beta < \pi/4$, we have $\langle u_{i_n}, u_{\tau(i_n)} \rangle > 0$ if $u_{i_n} \in \Psi_1$. Claim 7.1 shows that $V(\Xi_{u,u_0}) \geq \kappa_n/240^n$ for $u, u_0 \in S^{n-1}$ with $\langle u, u_0 \rangle \geq 0$, where Ξ_{u,u_0} is defined in (54). In particular, if $y \in \Xi_{u_{i_n}, u_{\tau(i_n)}}$, then

$$\begin{aligned} \langle y, u_{i_n} \rangle, \langle y, u_{\tau(i_n)} \rangle \quad &< \quad \frac{1}{8}, \text{ and} \\ \langle y, u_{i_n} \rangle - \langle y, u_{\tau(i_n)} \rangle \quad &= \quad \langle y, u_{i_n} - u_{\tau(i_n)} \rangle \geq \frac{\eta}{120\sqrt{n}}. \end{aligned}$$

Next, φ'' is continuous, and Lemma 4.2 implies that if $t \in [\frac{1}{30}, 0.1]$, then

$$|\varphi''(t)| \geq \left\{ \begin{array}{lll} \frac{|p-2|}{48} \left(\frac{1}{30}\right)^{1.3} & > \frac{|p-2|}{2^{12}} & \text{if } p \in [1,3] \setminus \{2\}, \\ 0.2 \left(\frac{1}{30}\right)^{1.3} & > 2^{-9} & \text{if } p > 3. \end{array} \right. \tag{68}$$

Therefore,

$$|\varphi'(\langle y, u_{i_n} \rangle) - \varphi'(\langle y, u_{\tau(i_n)} \rangle)| \geq \left\{ \begin{array}{lll} \frac{|p-2|}{2^{12} 120\sqrt{n}}\, \eta & > \frac{|p-2|}{2^{19}\sqrt{n}}\, \eta & \text{if } p \in [1,3] \setminus \{2\}, \\ \frac{1}{2^9 120\sqrt{n}}\, \eta & > \frac{1}{2^{19}\sqrt{n}}\, \eta & \text{if } p > 3. \end{array} \right.$$

It follows from Lemma 3.2 and $0 < \varphi'(t) \leq 3.1$ for $p \in [1, \infty) \setminus \{2\}$ and $t \in (0, 0.1]$ (cf. (64)) that

$$\begin{aligned} &\left(\frac{\sqrt{\varphi'(\langle y, u_{i_1} \rangle) \cdots \varphi'(\langle y, u_{i_{n-1}} \rangle) \cdot \varphi'(\langle y, u_{i_n} \rangle)}}{t_0(y)} - 1 \right)^2 \\ &\quad + \left(\frac{\sqrt{\varphi'(\langle y, u_{i_1} \rangle) \cdots \varphi'(\langle y, u_{i_{n-1}} \rangle) \cdot \varphi'(\langle y, u_{\tau(i_n)} \rangle)}}{t_0(y)} - 1 \right)^2 \\ &\qquad \geq \frac{(\varphi'(\langle y, u_{i_n} \rangle) - \varphi'(\langle y, u_{\tau(i_n)} \rangle))^2}{2(\varphi'(\langle y, u_{i_n} \rangle) + \varphi'(\langle y, u_{\tau(i_n)} \rangle))^2} \geq \frac{\min\{1, (p-2)^2\}}{2^{45} n}\, \eta^2. \end{aligned}$$

Combining this estimate with (60) and (67) implies that if $p \in [1, \infty) \setminus \{2\}$ and $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \ldots, n-1$, $u_{i_n} \in \Psi_1$ and $y \in \Xi_{u_{i_n}, u_{\tau(i_n)}}$, then

$$\begin{aligned} &\mathsf{N}(i_1, \ldots, i_{n-1}, i_n; y) + \mathsf{N}(i_1, \ldots, i_{n-1}, \tau(i_n); y) \\ &\qquad \geq 4n^2\beta^2 c_{i_1} \cdots c_{i_{n-1}} \cdot (c_{i_n}/2) \frac{\min\{1, (p-2)^2\}}{2^{45} n}\, \eta^2. \end{aligned}$$

If $u_{i_n} \in \Psi_1$ and $y \in \mathbb{R}^n$, then we define

$$\varrho(i_n; y) = \begin{cases} 0 & \text{if } y \notin \Xi_{i_n,\tau(i_n)}; \\ \frac{\beta^2 n(p-2)^2}{2^{44}}\,\eta^2 & \text{if } y \in \Xi_{i_n,\tau(i_n)} \text{ and } p \in [1,3] \setminus \{2\}; \\ \frac{\beta^2 n}{2^{44}}\,\eta^2 & \text{if } y \in \Xi_{i_n,\tau(i_n)} \text{ and } p > 3. \end{cases}$$

In particular, if $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \ldots, n-1$, $u_{i_n} \in \Psi_1$ and $y \in \mathbb{R}^n$, then

$$\mathsf{N}(i_1, \ldots, i_{n-1}, i_n; y) + \mathsf{N}(i_1, \ldots, i_{n-1}, \tau(i_n), y) \geq c_{i_1} \cdots c_{i_n} \varrho(i_n; y). \tag{69}$$

Substituting (69) into (61), and then using (65), we see that if $y \in \mathbb{R}^n$, then

$$\begin{aligned} \theta^*(y) &\geq 1 + \frac{1}{2} \sum_{\substack{u_{i_j} \in \Omega(v_j,\beta),\ j=1,\ldots,n-1 \\ u_{i_n} \in \Psi_1}} c_{i_1} \cdots c_{i_{n-1}} \cdot c_{i_n} \varrho(i_n; y) \\ &= 1 + \frac{1}{2} \left(\prod_{j=1}^{n-1} \mu(\Omega(v_j, \beta)) \right) \sum_{u_{i_n} \in \Psi_1} c_{i_n} \varrho(i_n; y) \\ &\geq 1 + \frac{\beta^{n(n-1)}}{2} \sum_{u_{i_n} \in \Psi_1} c_{i_n} \varrho(i_n; y). \end{aligned}$$

Here

$$\frac{\beta^{n(n-1)}}{2} \sum_{u_{i_n} \in \Psi_1} c_{i_n} \varrho(i_n; y) \leq \frac{\beta^{n(n-1)}}{2}\, \mu(\Psi_1) \cdot \frac{\beta^2 n}{2^{44}}\, \eta^2 < 1,$$

and hence if $y \in \mathbb{R}^n$, then

$$\theta^*(y)^{-1} \leq 1 - \frac{\beta^{n(n-1)}}{4} \sum_{u_{i_n} \in \Psi_1} c_{i_n} \varrho(i_n; y). \tag{70}$$

We deduce from (63) and (70) that

$$\begin{aligned} V(Z_p^*(\mu)) &\leq \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \left(1 - \frac{\beta^{n(n-1)}}{4} \sum_{u_{i_n} \in \Psi_1} c_{i_n} \varrho(i_n; y) \right) e^{-\pi \|\Theta(y)\|^2} \det\left(d\Theta(y)\right)\, dy \\ &= \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} e^{-\pi \|\Theta(y)\|^2} \det\left(d\Theta(y)\right)\, dy \\ &\quad - \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \cdot \frac{\beta^{n(n-1)}}{4} \sum_{u_{i_n} \in \Psi_1} c_{i_n} \int_{\mathbb{R}^n} \varrho(i_n; y) e^{-\pi \|\Theta(y)\|^2} \det\left(d\Theta(y)\right)\, dy. \end{aligned}$$

Here, we use that

$$\int_{\mathbb{R}^n} e^{-\pi\|\Theta(y)\|^2} \det(d\Theta(y))\, dy \le \int_{\mathbb{R}^n} e^{-\pi\|z\|^2}\, dz = 1. \tag{71}$$

If $y \in \Xi_{i_n,\tau(i_n)}$, then (59), (62) and (64) yield that

$$e^{-\pi\|\Theta(y)\|^2} \ge \exp\left(-\pi\sum_{i=1}^k c_i\varphi(\langle y,u_i\rangle)^2\right) > \exp\left(-\pi\sum_{i=1}^k c_i\right) = e^{-\pi n}, \tag{72}$$

$$\det(d\Theta(y)) \ge \prod_{i=1}^k \varphi'(\langle y,u_i\rangle)^{c_i} \ge \prod_{i=1}^k 3.1^{-c_i} = 3.1^{-n}. \tag{73}$$

Therefore

$$V(Z_p^*(\mu)) \le \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\left(1 - \sum_{u_{i_n}\in\Psi_1} c_{i_n}\frac{\beta^{n(n-1)}}{4}\cdot\frac{V(\Xi_{i_n,\tau(i_n)})}{(3.1e^{\pi})^n}\cdot\frac{\beta^2 n\min\{(p-2)^2,1\}}{2^{44}}\cdot\eta^2\right).$$

Since $V(\Xi_{i_n,\tau(i_n)}) \ge \kappa_n/240^n$ if $u_{i_n} \in \Psi_1$, according to Claim 7.1, and

$$\sum_{u_{i_n}\in\Psi_1} c_{i_n} = \mu(\Psi_1) > \frac{\beta^n}{4n},$$

we conclude (66).

Case 2 *There exists $q_i \in \Omega(v_i,\beta)$, for $i = 1,\dots,n$, such that*

$$\mu(\Omega(q_i,\eta)) \ge \tfrac{\beta^n}{4n} \text{ for } i = 1,\dots,n, \text{ and} \tag{74}$$

$$\mu\left(\textstyle\bigcup_{i=1}^n(\Omega(q_i,2\eta)\cup\Omega(-q_i,2\eta))\right) \le n-\eta. \tag{75}$$

In this case, we prove

$$V(Z_p^*(\mu)) \le \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}(1 - n^{-cn^3}\min\{(p-2)^2,1\}\cdot\eta^3) \tag{76}$$

for some absolute constant $c > 0$. The argument is very similar to the one in Case 1.

Let

$$\widetilde{\Psi} = S^{n-1}\setminus\left(\bigcup_{i=1}^n(\Omega(q_i,2\eta)\cup\Omega(-q_i,2\eta))\right).$$

It follows from (65) that any $x \in \mathbb{R}^n$ can be written in the form

$$x = \sum_{i=1}^n \lambda_i(x) q_i.$$

Since $\mu(\widetilde{\Psi}) \geq \eta$ by (75), the triangle inequality ensures that there exists some $i \in \{1, \dots, n\}$ satisfying $|\lambda_i(x)| \geq 1/n$. Thus we may reindex $q_1, \dots, q_n$ in such a way that

$$\mu(\Psi) \geq \frac{\eta}{n} \qquad \text{for } \Psi = \{x \in \widetilde{\Psi} : |\lambda_n(x)| \geq 1/n\}. \tag{77}$$

We deduce from (65) that if $x \in \Psi$, then

$$|\det[q_1, \dots, q_{n-1}, x]| \geq |\det[q_1, \dots, q_{n-1}, q_n]|/n \geq 2\beta.$$

Next, for $u_{i_j} \in \Omega(q_j, \eta)$ for $j = 1, \dots, n-1$, we apply Claim 5.3 with $b_l = q_l$, $s_l = u_{i_l} - q_l$, for $l = 1, \dots, n-1$, $b_n = x \in \Psi$, and $s_n = 0$, where

$$|s_i| \leq \eta \leq \frac{\beta}{2n} = \frac{2\beta}{4n} \leq \frac{1}{4n}|\det[q_1, \dots, q_{n-1}, x]|, \quad i = 1, \dots, n.$$

Hence,

$$|\det[u_{i_1}, \dots, u_{i_{n-1}}, x]| \geq \frac{1}{2}|\det[q_1, \dots, q_{n-1}, x]| \geq \beta. \tag{78}$$

We observe that $\Psi = -\Psi$. Thus, for

$$\Pi_2 = \{i \in \{1, \dots, k\} : u_i \in \Psi\},$$

there exists $\Pi' \subseteq \Pi_2$ with $\#\Pi' = \frac{1}{2}\#\Pi_2$, and a bijection $\tilde{\tau} : \Pi' \to \Pi_2 \setminus \Pi'$ such that if $i \in \Pi'$ then $u_{\tilde{\tau}(i)} = -u_i$.

Since $\eta < \beta^n$, (74) implies that

$$\sum_{u_i \in \Omega(q_n, \eta)} c_i = \mu(\Omega(q_n, \eta)) \geq \frac{\beta^n}{4n} \geq \frac{\eta}{8n}.$$

Thus we can find a minimal (with respect to inclusion) set $\Pi_1 \subseteq \{1, \dots, k\}$ such that $u_i \in \Omega(q_n, \eta)$ for $i \in \Pi_1$ and

$$\sum_{i \in \Pi_1} c_i \geq \frac{\eta}{8n}, \tag{79}$$

By minimality and (57) it follows that

$$\frac{c_0}{2}\left(\#\Pi_1 - 1\right) \leq \frac{\eta}{8n}.$$

Moreover, by (77) and again by (57), we have

$$c_0 \#\Pi_2 \geq \sum_{j \in \Pi_2} c_j \geq \frac{\eta}{n},$$

and hence

$$\frac{c_0}{8}\#\Pi_2 \geq \frac{c_0}{2}\left(\#\Pi_1 - 1\right),$$

which yields $\#\Pi_2 \geq 4(\#\Pi_1 - 1)$ if $\#\Pi_1 \geq 2$. In any case, we deduce that $\#\Pi_2 \geq 2\#\Pi_1$.

We conclude that there exists an injective map $\tau : \Pi_1 \to \Pi_2$ such that if $i \in \Pi_1$, then

$$\langle u_i, u_{\tau(i)} \rangle \geq 0. \tag{80}$$

In addition, if $i \in \Pi_1$, then $u_i \in \Omega(q_n, \eta)$ and $u_{\tau(i)} \notin \Omega(q_n, 2\eta)$, and therefore

$$\|u_i - u_{\tau(i)}\| \geq \frac{\eta}{2}.$$

Given $u_{i_j} \in \Omega(q_j, \eta)$ for $j = 1, \dots, n-1$ and $i_n \in \Pi_1$, we have have $\tau(i_n) \in \Pi_2$, and (57), (65) and (78) yield

$$\left.\begin{array}{r} c_{i_1} \cdots c_{i_{n-1}} \cdot c_{i_n} \det[u_{i_1}, \dots, u_{i_n}]^2 \\ c_{i_1} \cdots c_{i_{n-1}} \cdot c_{\tau(i_n)} \det[u_{i_1}, \dots, u_{i_{n-1}}, u_{\tau(i_n)}]^2 \end{array}\right\} \geq \beta^2 c_{i_1} \cdots c_{i_{n-1}} \cdot (c_{i_n}/2). \tag{81}$$

We deduce from (80) that Claim 7.1 applies to $\Xi_{u_{i_n}, u_{\tau(i_n)}}$. In particular, we have $V(\Xi_{u_{i_n}, u_{\tau(i_n)}}) \geq \kappa_n/240^n$, and if $y \in \Xi_{u_{i_n}, u_{\tau(i_n)}}$, then

$$\begin{aligned} \langle y, u_{i_n} \rangle, \langle y, u_{\tau(i_n)} \rangle &< \frac{1}{8}; \\ \langle y, u_{i_n} \rangle - \langle y, u_{\tau(i_n)} \rangle &= \langle y, u_{i_n} - u_{\tau(i_n)} \rangle \geq \frac{\eta}{240} > \frac{\eta}{2^8}. \end{aligned}$$

It follows from (68) that

$$|\varphi'(\langle y, u_{i_n} \rangle) - \varphi'(\langle y, u_{\tau(i_n)} \rangle)| \geq \frac{\min\{|p-2|, 1\}}{2^{20}} \cdot \eta.$$

Since $0 < \varphi'(t) \leq 3.1$ for $t \in (0, 0.1]$, if $i_n \in \Pi_1$, then

$$\frac{(\varphi'(\langle y, u_{i_n} \rangle) - \varphi'(\langle y, u_{\tau(i_n)} \rangle))^2}{2(\varphi'(\langle y, u_{i_n} \rangle) + \varphi'(\langle y, u_{\tau(i_n)} \rangle))^2} \geq \frac{\min\{(p-2)^2, 1\}}{2^{47}} \cdot \eta^2.$$

Thus combining Lemma 3.2 and (81), we obtain that if $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \dots, n-1$, $i_n \in \Pi_1$ and $y \in \Xi_{u_{i_n}, u_{\tau(i_n)}}$, then

$$\mathsf{N}(i_1, \dots, i_{n-1}, i_n; y) + \mathsf{N}(i_1, \dots, i_{n-1}, \tau(i_n); y) \geq \frac{\beta^2 c_{i_1} \cdots c_{i_n}}{2} \cdot \frac{\min\{(p-2)^2, 1\}}{2^{47}} \cdot \eta^2.$$

If $i_n \in \Pi_1$ and $y \in \mathbb{R}^n$, then we define

$$\varrho(i_n; y) = \begin{cases} 0 & \text{if } y \notin \Xi_{i_n, \tau(i_n)} \\ \frac{\beta^2 \min\{(p-2)^2, 1\}}{2^{48}} \cdot \eta^2 & \text{if } y \in \Xi_{i_n, \tau(i_n)}. \end{cases}$$

In particular, if $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \dots, n-1$, $i_n \in \Pi_1$ and $y \in \mathbb{R}^n$, then

$$\mathsf{N}(i_1, \dots, i_{n-1}, i_n; y) + \mathsf{N}(i_1, \dots, i_{n-1}, \tau(i_n), y) \geq c_{i_1} \cdots c_{i_n} \varrho(i_n; y). \tag{82}$$

Substituting (82) into (61) and then using (65), we obtain for $y \in \mathbb{R}^n$ that

$$\begin{aligned}\theta^*(y) &\geq 1 + \frac{1}{2} \sum_{\substack{u_{i_j} \in \Omega(v_j,\beta),\, j=1,\ldots,n-1 \\ i_n \in \Pi_1}} c_{i_1} \cdots c_{i_{n-1}} \cdot c_{i_n} \varrho(i_n; y) \\ &= 1 + \frac{1}{2} \left(\prod_{j=1}^{n-1} \mu(\Omega(v_j, \beta)) \right) \sum_{i_n \in \Pi_1} c_{i_n} \varrho(i_n; y) \\ &\geq 1 + \frac{\beta^{n(n-1)}}{2} \sum_{i_n \in \Pi_1} c_{i_n} \varrho(i_n; y).\end{aligned}$$

Similarly as before, we have

$$\frac{\beta^{n(n-1)}}{2} \sum_{i_n \in \Pi_1} c_{i_n} \varrho(i_n; y) \leq \frac{\beta^{n(n-1)}}{2} \mu(\Psi_1) \cdot \frac{\beta^2 n}{2^{48}} \cdot \eta^2 < 1,$$

and hence

$$\theta^*(y)^{-1} \leq 1 - \frac{\beta^{n(n-1)}}{4} \sum_{i_n \in \Pi_1} c_{i_n} \varrho(i_n; y). \tag{83}$$

We deduce from (63) and (83) that

$$\begin{aligned}V(Z_p^*(\mu)) \leq \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} & \int_{\mathbb{R}^n} e^{-\pi\|\Theta(y)\|^2} \det\left(d\Theta(y)\right)\, dy \\ & - \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \cdot \frac{\beta^{n(n-1)}}{4} \sum_{i_n \in \Pi_1} c_{i_n} \int_{\mathbb{R}^n} \varrho(i_n; y) e^{-\pi\|\Theta(y)\|^2} \det\left(d\Theta(y)\right)\, dy.\end{aligned}$$

Now we use again (71) as well as the estimates (72) and (73) if $y \in \Xi_{i_n,\tau(i_n)}$. Therefore

$$V(Z_p^*(\mu)) \leq \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \left(1 - \sum_{i_n \in \Pi_1} c_{i_n} \frac{\beta^{n(n-1)}}{4} \cdot \frac{V(\Xi_{i_n,\tau(i_n)})}{(3.1e^{\pi})^n} \cdot \frac{\beta^2 \min\{(p-2)^2, 1\}}{2^{48}} \cdot \eta^2 \right)$$

Since $V(\Xi_{i_n,\tau(i_n)}) \geq \kappa_n / 240^n$ if $i_n \in \Pi_1$ and by (79), we conclude (76).

Case 3 *There exists $q_i \in \Omega(v_i, \beta)$, for $i = 1, \ldots, n$, such that*

$$\mu\left(\bigcup_{i=1}^{n} (\Omega(q_i, 2\eta) \cup \Omega(-q_i, 2\eta)) \right) > n - \eta.$$

In this case, we prove that there exists a cross measure ν such that

$$\delta_W(\nu, \mu) \leq n^{cn} \eta \tag{84}$$

for some absolute constant $c > 0$.

We observe that $\frac{1}{2}(1 - n(\frac{1}{\sqrt{n}} - t)^2) > \eta$ for $t = 2\eta$, since $\eta < 1/(2n)$. Thus Claim 5.1 yields that $\Omega(u, \arccos(\frac{1}{\sqrt{n}} - 2\eta))$ intersects $\cup_{i=1}^n \Omega(\pm q_i, 2\eta)$ for any $u \in S^{n-1}$. In turn, we deduce that

$$\Omega\left(u, \arccos\left(\frac{1}{\sqrt{n}} - 4\eta\right)\right) \cap \{\pm q_1, \dots, \pm q_n\} \neq \emptyset$$

for any $u \in S^{n-1}$, since $4\eta < 1/(4^n n!)$. Therefore Corollary 6.4 implies that there exists a cross measure ν such that

$$\delta_H(\text{supp}\,\nu, \{\pm q_1, \dots, \pm q_n\}) \leq 4^n n! \cdot 4\eta.$$

In particular, (84) follows from Lemma 6.1.

According to Lemma 5.5, Cases 1, 2 and 3 cover all possible even isotropic measure μ. Thus, we have proved (55) in Cases 1 and 2, and (56) in Case 3. □

Proof of Theorem 1.4 in the case of $Z_p^*(\mu)$: Let $p \in [1, \infty) \setminus \{2\}$, and let μ be a discrete even isotropic measure on S^{n-1}. Assume that $\delta_{\mathrm{WO}}(\mu, \nu_n) \geq \varepsilon > 0$ for some $\varepsilon \in (0, 1)$. Then Proposition 7.2 yields that

$$V(Z_p^*(\mu)) \leq (1 - \gamma\varepsilon^3) V(Z_p^*(\nu_n)), \tag{85}$$

where $\gamma = n^{-cn^3} \min\{|p - 2|^2, 1\}$ for an absolute constant $c > 0$. Since any even isotropic measure can be weakly approximated by discrete even isotropic measures (see, for instance, F. Barthe [7]), we conclude (85), and in turn Theorem 1.4 in the case of $Z_p^*(\mu)$, for any even isotropic measure μ on S^{n-1} and $p \in [1, \infty) \setminus \{2\}$.

Since for any isotropic measure μ, we have

$$\lim_{p \to \infty} Z_p^*(\mu) = Z_\infty^*(\mu),$$

and the factor γ in (85) is independent of $p \in (2, \infty)$, we deduce the case $p = \infty$ as well. □

8 The case of the L_p zonoids in Theorem 1.4

The proof of Theorem 1.4 for $V(Z_p(\mu))$ is analogous to the argument for $V(Z_p^*(\mu))$. In particular, we may assume again that μ is a discrete even isotropic measure, and $p \in (1, \infty) \setminus \{2\}$. Let $p^* \in (1, \infty)$ be defined by $\frac{1}{p} + \frac{1}{p^*} = 1$. We prove that if $\eta \in (0, 1)$, then

$$V(Z_{p^*}(\mu)) > (1 - n^{-cn^3} \min\{(p - 2)^2, 1\} \cdot \eta^3) V(Z_{p^*}(\nu_n)) \tag{86}$$

or there exists a cross measure ν satisfying

$$\delta_W(\mu, \nu) \leq n^{cn} \eta \tag{87}$$

for some absolute constant $c > 0$. Since if $p \in [\frac{3}{2}, 3]$, then $p^* \in [\frac{3}{2}, 3]$ and $|p-2|/2 \leq |p^*-2| \leq 2|p-2|$, (86) and (87) yield Theorem 1.4 for $V(Z_p(\mu))$.

Again, let $\operatorname{supp}\mu = \{\bar{u}_1, \ldots, \bar{u}_{\bar{k}}\}$, and let $\bar{c}_i = \mu(\{\bar{u}_i\})$. For $c_0 = \min\{\bar{c}_i : i = 1, \ldots, \bar{k}\}$ and $i = 1, \ldots, \bar{k}$, we define $\bar{m}_i = \min\{m \in \mathbb{Z} : m \geq 1 \text{ and } \bar{c}_i/m \leq c_0\}$, and let $k = \sum_{i=1}^{\bar{k}} \bar{m}_i$. We consider $\xi : \{1, \ldots, k\} \to \{1, \ldots, \bar{k}\}$ such that $\#\xi^{-1}(\{i\}) = \bar{m}_i$ for $i = 1, \ldots, \bar{k}$, and define

$$u_i = \bar{u}_{\xi(i)} \text{ and } c_i = \bar{c}_{\xi(i)}/\bar{m}_{\xi(i)}$$

for $i = 1, \ldots, k$.

In particular, $\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{Id}_n$ and $\sum_{i=1}^k c_i = n$, and for any Borel $X \subseteq S^{n-1}$, we have

$$\mu(X) = \sum_{u_i \in X} c_i.$$

Again, we obtain

$$c_0/2 < c_i \leq c_0 \text{ for } i = 1, \ldots, k.$$

In addition, let $\psi = \psi_p$ be defined as in (22), let $g(t) = e^{-\pi t^2}$, and let $f_i = \varrho_p$, for $i = 1, \ldots, k$.

We define the map $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Psi(y) = \sum_{i=1}^k c_i \psi(\langle y, u_i \rangle)\, u_i.$$

Its differential

$$d\Psi(y) = \sum_{i=1}^k c_i \psi'(\langle y, u_i \rangle)\, u_i \otimes u_i$$

is positive definite, and $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is injective.

It follows by first applying (18), and then (11), that

$$\begin{aligned} V(Z_{p^*}(\mu)) \geq V(M_p(\mu)) &= \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n}^* \sup_{x = \sum_{i=1}^k c_i \theta_i u_i} \prod_{i=1}^k f_i(\theta_i)^{c_i}\, dx \\ &\geq \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \left(\prod_{i=1}^k f_i(\psi(\langle u_i, y \rangle))^{c_i} \right) \det \left(\sum_{i=1}^k c_i \psi'(\langle u_i, y \rangle)\, u_i \otimes u_i \right) dy. \end{aligned}$$

To estimate the second term, we apply Lemma 3.1 with $v_i = \sqrt{c_i} \cdot u_i$ and $t_i = \psi'(\langle y, u_i \rangle)$ at each $y \in \mathbb{R}^n$, and write $\theta^*(y)$ and $t_0(y)$ to denote the corresponding $\theta^* \geq 1$ and t_0. In particular, if $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, k\}$ and $y \in \mathbb{R}^n$, then we now set

$$\mathsf{N}(i_1, \ldots, i_n; y) = c_{i_1} \cdots c_{i_n} \det[u_{i_1}, \ldots, u_{i_n}]^2 \left(\frac{\sqrt{\psi'(\langle y, u_{i_1} \rangle) \cdots \psi'(\langle y, u_{i_n} \rangle)}}{t_0(y)} - 1 \right)^2.$$

Therefore, using again the notation

$$\theta^*(y) = 1 + \frac{1}{2} \sum_{1 \leq i_1 < \ldots < i_n \leq k} \mathsf{N}(i_1, \ldots, i_n; y),$$

Lemma 3.1 and (10) lead to

$$\begin{aligned}
V(Z_{p^*}(\mu)) &\geq \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n}\theta^*(y)\left(\prod_{i=1}^k f_i(\psi(\langle u_i,y\rangle))^{c_i}\right)\left(\prod_{i=1}^k \psi'(\langle u_i,y\rangle)^{c_i}\right)dy\\
&= \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n}\theta^*(y)\left(\prod_{i=1}^k g(\langle u_i,y\rangle)^{c_i}\right)dy\\
&= \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n}\theta^*(y)e^{-\pi\|y\|^2}\,dy.
\end{aligned}$$

Now (86) and (87), and hence Theorem 1.4 for $V(Z_p(\mu))$, can be proved in the same way as (55) and (56) in Proposition 7.2 were proved following (62).

9 Stability of the reverse isoperimetric inequality in the origin symmetric case

In this section, we turn to the proofs of Corollary 1.5 and of Theorems 1.1 and 1.2.

We may assume that the facets of the cube W^n touch B^n in the support of the reference cross measure ν_n, where $\operatorname{supp}\nu_n = \{\pm e_1,\ldots,\pm e_n\}$.

Lemma 9.1 *If μ is an even measure on S^{n-1} such that $\delta_H(\operatorname{supp}\mu,\operatorname{supp}\nu_n) < \alpha$ for some $\alpha\in(0,\frac{1}{3n})$, then $e^{-n\alpha}W^n \subseteq Z_\infty^*(\mu)\subseteq e^{2n\alpha}W^n$.*

Proof: First, we show that $Z_\infty^*(\mu)\subseteq e^{2n\alpha}W^n$. For this, let $x\in\mathbb{R}^n\setminus e^{2n\alpha}W^n$. Clearly, we may assume that $x_1=\max\{|x_1|,\ldots,|x_n|\}$. It follows that there is some $i\in\{1,\ldots,n\}$ such that

$$x_1 \geq |x_i| = |\langle x,e_i\rangle| > e^{2n\alpha} \geq \left(1-\frac{1}{2}\alpha^2-\sqrt{n-1}\,\alpha\right)^{-1} \geq \left(\cos\alpha-\sqrt{n-1}\sin\alpha\right)^{-1}, \quad (88)$$

where we used that $\alpha\in(0,\frac{1}{3n})$ for the third inequality. Since $\delta_H(\operatorname{supp}\mu,\operatorname{supp}\nu_n)<\alpha$, there is some $v\in\operatorname{supp}\mu$ such that $\angle(e_1,v)<\alpha$, hence

$$\langle e_1,v\rangle > \cos\alpha, \qquad \sum_{i=2}^n |\langle e_i,v\rangle| < \sqrt{n-1}\sin\alpha. \quad (89)$$

From (88) we deduce that

$$\langle x,v\rangle \geq x_1\langle e_1,v\rangle - x_1\sum_{i=2}^n|\langle e_i,v\rangle| > x_1\left(\cos\alpha-\sqrt{n-1}\sin\alpha\right) > 1,$$

and hence $x\notin Z_\infty^*(\mu)$.

In order to show that $e^{-n\alpha}W^n \subseteq Z^*_\infty(\mu)$, we put $\varrho = (1+\sqrt{n-1}\sin\alpha)^{-1}$. Since $\varrho \geq (1+n\alpha)^{-1} \geq e^{-n\alpha}$, we have $e^{-n\alpha}W^n \subseteq \varrho W^n$, and it is sufficient to show that $\varrho W^n \subseteq Z^*_\infty(\mu)$. For this, let $x \in \varrho W^n$, and let $v \in \operatorname{supp}\mu$ be arbitrary. Then there is some $i \in \{1,\dots,n\}$ such that $\angle(e_i, v) < \alpha$ or $\angle(-e_i, v) < \alpha$. We may assume that $i = 1$. Hence (89) is available again. Then $x = x_1e_1 + \ldots + x_ne_n$ with $|x_i| \leq \varrho$ satisfies

$$\langle x, v\rangle \leq \varrho \cdot 1 + \varrho\sqrt{n-1}\sin\alpha = 1,$$

which shows that $x \in Z^*_\infty(\mu)$. $\square$

For the proof of Theorem 1.2 (the case of the Banach-Mazur distance), we also need the following statement.

Lemma 9.2 *If $\tau \in (0, 1/4)$ and the o-symmetric convex bodies $K, Z \subset \mathbb{R}^n$ satisfy $K \subseteq Z$, $(1-\tau)W^n \subseteq Z$, $(1-2\tau)W^n \not\subseteq K$ and $V(Z) \leq V(W^n)$, then $V(K) \leq (1 - \frac{\tau^n}{2^n})V(W^n)$.*

Proof: Let $e_1, \dots, e_n$ be the orthonormal basis of $\mathbb{R}^n$ such that the facets of W_n touch S^{n-1} at $\{\pm e_1, \dots, \pm e_n\}$. Replacing e_i by $-e_i$ for $i \in \{1,\dots,n\}$ if necessary, we may assume, for some $t > 0$, that

$$\begin{aligned} t\sum_{i=1}^n e_i &\in \partial K, \text{ and} \\ t\sum_{i=1}^n \eta_i e_i &\in K \quad \text{for } \eta_i \in \{-1, 1\},\ i = 1,\dots,n. \end{aligned}$$

Since $(1-2\tau)W^n \not\subseteq K$, we have $t < 1 - 2\tau$. It follows that

$$\begin{aligned} (\operatorname{int} K) \cap \left(\tau[0,1]^n + t\sum_{i=1}^n e_i\right) &= \emptyset, \\ \tau[0,1]^n + t\sum_{i=1}^n e_i &\subseteq (1-\tau)W^n \subseteq Z. \end{aligned}$$

Therefore

$$V(K) \leq V(Z) - \tau^n \leq \left(1 - \frac{\tau^n}{2^n}\right)V(W^n). \quad \square$$

Proof of Corollary 1.5 We may assume that μ is not a cross measure. For an even isotropic measure μ and a sufficiently small $\varepsilon > 0$, we assume that

$$V(Z^*_\infty(\mu)) \geq (1-\varepsilon)V(Z^*_\infty(\nu_n)) \tag{90}$$

$$\text{or } V(Z_\infty(\mu)) \leq (1+\varepsilon)V(Z_\infty(\nu_n)), \tag{91}$$

and prove that

$$\delta_{HO}(\operatorname{supp}\mu, \operatorname{supp}\nu_n) < n^{cn^3}\varepsilon^{1/3}$$

for some absolute constant $c > 0$. How small ε should be is specified by (93).

According to Theorem 1.4, there exists an absolute constant $c_0 > 0$ such that if $n^{c_0 n^3}\varepsilon^{1/3} < 1$, then (90) implies that

$$\delta_{\mathrm{W}}(\mu, \nu_n) < n^{c_0 n^3}\varepsilon^{1/3}, \tag{92}$$

where $\operatorname{supp}\nu_n = \{\pm e_1, \ldots, \pm e_n\}$ for an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. In particular, $Z_\infty^*(\nu_n) = W^n$, and $Z_\infty(\nu_n)$ is the cross polytope $C^n = [\pm e_1, \ldots, \pm e_n]$, where $[z_1, \ldots, z_k]$ denotes the convex hull of points $z_1, \ldots, z_k \in \mathbb{R}^n$.

In the following argument, we require that

$$3n^2 6^n n! n^{c_0 n^3}\varepsilon^{1/3} < \pi/4. \tag{93}$$

We claim that for any $i \in \{1, \ldots, n\}$ there exists $u_i \in \operatorname{supp}\mu$ such that

$$\angle(u_i, e_i) \leq n^{c_0 n^3}\varepsilon^{1/3}. \tag{94}$$

We suppose that say for e_1, we have $\angle(e_1, u) > n^{c_0 n^3}\varepsilon^{1/3}$ for any $u \in \operatorname{supp}\mu$, and seek a contradiction. Naturally, also $\angle(-e_1, u) > n^{c_0 n^3}\varepsilon^{1/3}$ for any $u \in \operatorname{supp}\mu$. We consider the function $f \in \operatorname{Lip}_1(S^{n-1})$ defined by

$$f(u) = \max\left\{0, n^{c_0 n^3}\varepsilon^{1/3} - \angle(u, e_1), n^{c_0 n^3}\varepsilon^{1/3} - \angle(u, -e_1)\right\} \quad \text{for } u \in S^{n-1}.$$

Then we have

$$\int_{S^{n-1}} f\, d\nu_n = n^{c_0 n^3}\varepsilon^{1/3} \text{ and } \int_{S^{n-1}} f\, d\mu = 0,$$

contradicting (92), and proving (94). Writing μ_0 to denote any even measure on S^{n-1} with support $\{\pm u_1, \ldots, \pm u_n\}$, we deduce from (94) and Lemma 9.1 that

$$Z_\infty^*(\mu) \subseteq Z_\infty^*(\mu_0) \subseteq e^{2n\alpha}W^n \text{ for } \alpha = n^{c_0 n^3}\varepsilon^{1/3}. \tag{95}$$

Let $w = \sum_{i=1}^n e_i$, let $\varphi = \min\left\{\delta_{\mathrm{H}}(\mu, \nu_n), \frac{\pi}{4}\right\}$, and let $u \in \operatorname{supp}\mu$ be such that $\angle(u, e_i) \geq \varphi$ and $\angle(u, -e_i) \geq \varphi$ for $i = 1, \ldots, n$. In particular, $\varphi \in (0, \frac{\pi}{4}]$ as $\mu \neq \nu_n$. Possibly after changing the sign of some of the vectors $e_1, \ldots, e_n$, we may assume that $u \in \operatorname{pos}\{e_1, \ldots, e_n\}$. Let $u = (t_1, \ldots, t_n)$, where we may assume that

$$0 \leq t_1 \leq \ldots \leq t_n \leq \cos\varphi.$$

We prove that

$$\langle u, w\rangle \geq 1 + \frac{\varphi}{3}. \tag{96}$$

Our task is to minimize $\langle u, w\rangle = \sum_{i=1}^n t_i$ under the conditions that each $t_i \in [0, \cos\varphi]$ and $\sum_{i=1}^n t_i^2 = 1$. Solving this problem leads to

$$\langle u, w\rangle = \sum_{i=1}^n t_i \geq \cos\varphi + \sin\varphi = \sqrt{1 + \sin 2\varphi} > 1 + \frac{\sin 2\varphi}{3},$$

proving (96).

First, we assume that (90) holds. For the halfspace $H^+ = \{x \in \mathbb{R}^n : \langle x, u\rangle \geq 1\}$, we claim that

$$V(H^+ \cap W^n) \geq \frac{\varphi}{6^n n!} V(W^n). \tag{97}$$

For $i = 1, \dots, n$, let $s_i \in [0, 2]$ be maximal such that $w - s_i e_i \in H^+ \cap W^n$. Then we have $\langle w - s_i e_i, u\rangle = 1$ provided $s_i < 2$, thus (96) yields

$$s_i = \min\left\{2, \frac{\langle u, w\rangle - 1}{t_i}\right\} \geq \min\left\{2, \frac{\varphi}{3t_i}\right\},$$

where we use the convention $\frac{a}{0} = \infty$ for $a > 0$. We consider two cases. If $\varphi = \frac{\pi}{4}$, then $t_i < \varphi$, and hence $s_i \geq 1/3$ for $i = 1, \dots, n$. We deduce that

$$V(H^+ \cap W^n) \geq \frac{s_1 \cdots s_n}{n!} \geq \frac{1}{3^n n!} \geq \frac{\varphi}{6^n n!} V(W^n).$$

If $0 < \varphi < \frac{\pi}{4}$, then $t_n = \cos\varphi$, thus $t_i \leq \sin\varphi < \varphi$ for $i = 1, \dots, n-1$. In particular, $s_n > \frac{\varphi}{3}$, and $s_i > \frac{1}{3}$ for $i = 1, \dots, n-1$, and hence

$$V(H^+ \cap W^n) \geq \frac{s_1 \cdots s_n}{n!} \geq \frac{\varphi}{3^n n!} = \frac{\varphi}{6^n n!} V(W^n).$$

We deduce from $2n^2\alpha < 1$ (cf. (93)), (95) and (97) that

$$V(Z^*_\infty(\mu)) \leq e^{2n^2\alpha} V(W^n) - \frac{2\varphi}{6^n n!} V(W^n) \leq \left(1 + 4n^2 n^{c_0 n^3} \varepsilon^{1/3} - \frac{2\varphi}{6^n n!}\right) V(W^n).$$

Comparing to (90) yields that

$$\varphi < 3n^2 6^n n! n^{c_0 n^3} \varepsilon^{1/3},$$

where $\delta_{\mathrm{H}}(\mu, \nu_n) = \varphi$ by (93).

Finally we assume (91). We deduce from (95) and by duality that

$$e^{-2n\alpha} C^n \subseteq Z_\infty(\mu).$$

Let $T_o = [o, e^{-2n\alpha} e_1, \dots, e^{-2n\alpha} e_n]$ and $T_u = [u, e^{-2n\alpha} e_1, \dots, e^{-2n\alpha} e_n]$. Since the height of the simplex T_u corresponding to u is $n^{-1/2}(\langle u, w\rangle - e^{-2n\alpha})$, and the height of T_o corresponding to o is $n^{-1/2} e^{-2n\alpha}$, it follows from (96) that

$$V(T_u) \geq \frac{\varphi}{3} V(T_o) = \frac{\varphi}{3 \cdot 2^n} V(e^{-2n\alpha} C^n).$$

Since $u \in \operatorname{supp}\mu$, we have

$$V(Z_\infty(\mu)) \geq \left(1 + \frac{\varphi}{3 \cdot 2^n}\right) e^{-2n^2\alpha} V(C^n).$$

Comparing this to (91), we obtain

$$1 + \frac{\varphi}{3 \cdot 2^n} \leq e^{2n^2\alpha}(1 + \varepsilon) < e^{3n^2\alpha} < 1 + 6n^2 n^{c_0 n^3}.$$

We conclude $\varphi \leq 18 \cdot 2^n n^2 n^{c_0 n^3}$, where $\delta_{\rm H}(\mu, \nu_n) = \varphi$ by (93). $\Box$

Proofs of Theorems 1.1 and 1.2: Let K be an origin symmetric convex body such that B^n is the maximal volume ellipsoid contained in K, and suppose that

$$\frac{S(K)^n}{V(K)^{n-1}} \geq (1-\varepsilon)\frac{S(W^n)^n}{V(W^n)^{n-1}} \tag{98}$$

for a sufficiently small $\varepsilon > 0$. If C is a compact convex set with $B^n \subseteq C$, and S_C is the surface area measure of C, then

$$V(C) = \int_{S^{n-1}} \frac{h_C(u)}{n}\, dS_C(u) \geq \int_{S^{n-1}} \frac{1}{n}\, dS_C(u) = \frac{S(C)}{n},$$

with equality if $h_C(u) = 1$ for each $u \in \operatorname{supp} S_C$. Therefore $V(W^n) = S(W^n)/n$ and $V(K) \geq S(K)/n$, and hence (98) implies

$$V(K) \geq (1-\varepsilon)V(W^n). \tag{99}$$

Let μ be a discrete even isotropic measure satisfying $\operatorname{supp}\mu \subseteq S^{n-1} \cap \partial K$ provided by John's Theorem. In particular,

$$K \subseteq Z_\infty^*(\mu) \text{ and } V(Z_\infty^*(\mu)) \geq V(K) \geq (1-\varepsilon)V(W^n). \tag{100}$$

We deduce from Corollary 1.5 that, possibly after a suitable rotation, we may assume that

$$\delta_H(\operatorname{supp}\mu, \operatorname{supp}\nu_n) \leq n^{c_1 n^3} \varepsilon^{\frac{1}{3}}$$

for an absolute constant $c_1 > 0$. Applying now Lemma 9.1, we have

$$e^{-\omega\varepsilon^{\frac{1}{3}}} W^n \subseteq Z_\infty^*(\mu) \subseteq e^{\omega\varepsilon^{\frac{1}{3}}} W^n \tag{101}$$

for $\omega = n^{c_2 n^3}$ and an absolute constant $c_2 > 0$ (assuming that ε is sufficiently small).

To verify the estimate of Theorem 1.1 for $\delta_{\rm vol}$, let us write $\delta_{\rm sym}(C, M) = V(C \Delta M)$ to denote the distance of two compact convex sets according to the symmetric difference metric. For example, (101) yields

$$\delta_{\rm sym}(Z_\infty^*(\mu), W^n) \leq \left(e^{n\omega\varepsilon^{\frac{1}{3}}} - e^{-n\omega\varepsilon^{\frac{1}{3}}}\right) 2^n \leq n^{c_3 n^3} \varepsilon^{\frac{1}{3}} \cdot 2^n$$

for an absolute constant $c_3 > 0$. We note that $V(K) \leq V(Z_\infty^*(\mu)) \leq 2^n$ by K.M. Ball's Theorem B. Hence,

$$0 \leq \delta_{\rm sym}(Z_\infty^*(\mu), K) = V(Z_\infty^*(\mu)) - V(K) \leq V(Z_\infty^*(\mu)) - V(W^n) + 2^n\varepsilon \leq 2^n\varepsilon.$$

Let $\lambda \geq 1$ be such that $V(\lambda K) = 2^n$, and hence $V(\lambda K) - V(K) \leq \varepsilon \cdot 2^n$ according to (100). We conlude that

$$\delta_{\rm vol}(K, W^n) \;\leq\; 2^{-n}\delta_{\rm sym}(\lambda K, W^n)$$

$$
\begin{aligned}
&\leq 2^{-n}(\delta_{\text{sym}}(\lambda K, K) + \delta_{\text{sym}}(K, Z_\infty^*(\mu)) + \delta_{\text{sym}}(Z_\infty^*(\mu), W^n)) \\
&\leq n^{c_4 n^3} \varepsilon^{\frac{1}{3}},
\end{aligned}
$$

for an absolute constant $c_4 > 0$, and this completes the proof of Theorem 1.1.

Let us turn to the estimate of Theorem 1.2 for δ_{BM}. Let $\delta_{\text{BM}}(K, W^n) \geq \alpha$ for some $\alpha \in (0, 1)$. If

$$
e^{-\frac{\alpha}{5}} W^n \subseteq Z_\infty^*(\mu) \subseteq e^{\frac{\alpha}{5}} W^n, \tag{102}
$$

then $\delta_{\text{BM}}(K, W^n) \geq \alpha$ implies that $e^{-\frac{4\alpha}{5}} W^n \not\subseteq K$, and hence $(1 - \frac{2\alpha}{5}) W^n \not\subseteq K$. On the other hand, $(1 - \frac{\alpha}{5}) W^n \subseteq Z_\infty^*(\mu)$, thus Lemma 9.2 yields

$$
V(K) \leq \left(1 - \frac{\alpha^n}{10^n}\right) V(W^n). \tag{103}
$$

Finally, we assume that (102) does not hold. Since (99) leads to (101), we have $V(K) < (1 - \varepsilon) V(W^n)$ provided $\frac{\alpha}{5} = \omega \varepsilon^{\frac{1}{3}}$. In other words,

$$
V(K) \leq \left(1 - \frac{\alpha^3}{125\omega^3}\right) V(W^n) \tag{104}
$$

where $\frac{1}{125\omega^3} \geq n^{-c_5 n^3}$ for an absolute constant $c_5 > 0$. Combining (103) and (104) proves Theorem 1.2. $\square$

10 Proof of Theorem 1.3

In this section, we prove Theorem 1.3, which is the 2-dimensional (sharper) version of Theorems 1.1 and 1.2. The idea of our proof is essentially the one given by F. Behrend [10]. As before, let $[x_1, \ldots, x_k]$ denote the convex hull of the points $x_1, \ldots, x_k \in \mathbb{R}^2$. For the origin symmetric convex body $K \subseteq \mathbb{R}^2$ and $u \in \mathbb{R}^2 \setminus \{o\}$, we write $H(K, u)$ to denote the supporting line with exterior normal u, and $H(K, u)^-$ to denote the corresponding halfplane containing K.

Let $\varepsilon \in [0, \frac{1}{2})$. Let K be a planar origin symmetric convex body which has a square as an inscribed parallelogram of maximum area. Suppose that

$$
\frac{S(K)^2}{V(K)} \geq (1 - \varepsilon) \frac{S(W^2)^2}{V(W^2)}. \tag{105}
$$

Then we prove that

$$
\begin{aligned}
\delta_{\text{vol}}(K, W^2) &\leq 54\varepsilon \text{ and} & (106) \\
\delta_{\text{BM}}(K, W^2) &\leq 18\varepsilon. & (107)
\end{aligned}
$$

Let u_1, u_2 denote the standard basis of $\mathbb{R}^2$. We may assume that $W^2 = [-1,1]^2$ is a parallelogram of largest area contained in K, and hence $p_i \in \partial K \cap H(K, p_i)$ holds for the vertices $p_1 = u_2 + u_1$ and $p_2 = u_2 - u_1$ of W^2. It also follows that

$$K \subseteq \bigcap_{i=1}^{2} H(K, \pm p_i)^- = [\pm 2u_1, \pm 2u_2]. \tag{108}$$

Let $q_i \in \partial K \cap H(K, u_i)$ for $i = 1, 2$. In particular, (108) yields

$$\begin{aligned} q_1 &= (1 + t_1, s_1) \text{ where } t_1 \in [0,1] \text{ and } |s_1| \leq 1 - t_1, \\ q_2 &= (s_2, 1 + t_2) \text{ where } t_2 \in [0,1] \text{ and } |s_2| \leq 1 - t_2. \end{aligned}$$

Since K contains the parallelogram $P = [\pm q_1, \pm q_2]$, we have

$$\begin{aligned} V(W^2) &\geq V(P) = 2|\det[q_1, q_2]| = 2[(1+t_1)(1+t_2) - s_1 s_2] \\ &\geq 2[(1+t_1)(1+t_2) - (1-t_1)(1-t_2)] = 4(t_1 + t_2), \end{aligned}$$

and hence

$$t = \frac{t_1 + t_2}{2} \leq \frac{1}{2}.$$

We approximate K by suitable polygons to obtain

$$W^2 \subseteq Q \subseteq K \subseteq M \subseteq (1+t)W^2, \tag{109}$$

where

$$\begin{aligned} M &= \left(\bigcap_{i=1}^{2} H(K, \pm u_i)^- \right) \bigcap \left(\bigcap_{i=1}^{2} H(K, \pm p_i)^- \right) \text{ with } S(M) = (1 + (\sqrt{2} - 1)t) S(W^2), \\ Q &= [\pm p_1, \pm p_2, \pm q_1, \pm q_2] \text{ with } V(Q) = (1+t)V(W^2). \end{aligned}$$

We deduce from (105) and (109) that

$$(1 - \varepsilon)\frac{S(W^2)^2}{V(W^2)} \leq \frac{S(K)^2}{V(K)} \leq \frac{S(M)^2}{V(Q)} = \frac{(1 + (\sqrt{2}-1)t)^2 S(W^2)^2}{(1+t)V(W^2)}.$$

Since $\frac{1-t}{1+t} \geq \frac{1}{3}$ by $t \leq \frac{1}{2}$, we have

$$\varepsilon \geq 1 - \frac{(1 + (\sqrt{2}-1)t)^2}{1+t} = \frac{(3 - 2\sqrt{2})t(1-t)}{1+t} \geq \frac{(3 - 2\sqrt{2})t}{3} \geq \frac{t}{18}. \tag{110}$$

Therefore combining (109) and (110) leads to

$$\delta_{\rm BM}(K, W^2) \leq \log(1+t) \leq t \leq 18\varepsilon,$$

and combining (109) and (110) with an elementary argument leads to

$$\delta_{\rm vol}(K, W^2) \leq (1+t)^2 - 1 \leq 3t \leq 54\varepsilon.$$

We conclude (106) and (107), and in turn Theorem 1.3.

11 Even isotropic measures on S^1

The goal of this section is to prove the following improvement of Corollary 1.5 if $n = 2$.

Theorem 11.1 *Let μ be an even isotropic measure on S^1, and let $\varepsilon \in (0,1)$. If $\delta_{HO}(\operatorname{supp}\mu, \operatorname{supp}\nu_2) \geq \varepsilon$, then*

$$\begin{aligned} V(Z_\infty(\mu)) &\geq (1 + 0.25\,\varepsilon)V(Z_\infty(\nu_2)), \\ V(Z_\infty^*(\mu)) &\leq (1 - 0.1\,\varepsilon)V(Z_\infty^*(\nu_2)). \end{aligned}$$

We call a compact, symmetric set $X \subseteq S^1$ proper if for each $v \in S^1$ there is some $u \in X$ such that $\angle(u,v) \leq \pi/4$. A compact, symmetric set $X \subseteq S^1$ is proper if and only if the angle between consecutive points of X on S^1 is at most $\pi/2$. For a closed set $X \subseteq S^1$ we define

$$d_0(X) = \min\{\delta_H(X, \rho\{\pm e_1, \pm e_2\}) : \rho \in \mathrm{SO}(2)\},$$

where e_1, e_2 is an orthonormal basis of $\mathbb{R}^2$. If X is proper, then $d_0(X) \leq \pi/4$.

Note that if μ is an even isotropic measure on S^1, then Claim 5.1 shows that the support of μ is a proper set.

Lemma 11.2 *If $X \subseteq S^1$ is proper, $\eta \in (0, \pi/4)$ and $d_0(X) \geq \eta$, then there are $u, v \in X$ such that $\eta \leq \angle(u,v) \leq \frac{\pi}{2} - \eta$.*

Proof: Assume that for any pair $u, v \in X$ either $\angle(u,v) < \eta$ or $\angle(u,v) > \frac{\pi}{2} - \eta$. Let $u_1 \in X$ be arbitrary. Then there is no $v \in X$ such that $\angle(u,v) \in [\eta, \frac{\pi}{2} - \eta]$. The same is true for $-u_1 \in X$. Let $\bar{u}_1 \in S^1 \cap u_1^\perp$. Then there is some $u_2 \in X$ with $\angle(\bar{u}_1, u_2) < \eta$. Since X is closed and symmetric, we conclude that $d_0(X) < \eta$, a contradiction. $\square$

We turn to the proof of Theorem 11.1 and start with the second assertion. Let the assumptions be fulfilled. By an approximation argument (see Barthe [7]), we can assume that μ is discrete. In the following, we use property (P) which states that for $0 \leq \beta \leq \alpha < \pi/2$ the function

$$F(t) := \tan\left(\frac{\alpha + t}{2}\right) + \tan\left(\frac{\beta - t}{2}\right), \qquad t \in [0, \min\{\beta, \tfrac{\pi}{2} - \alpha\}],$$

is strictly increasing. Applying (P) repeatedly to angles between consecutive vectors of supp μ, Lemma 11.2 and symmetry, we obtain

$$V(Z_\infty^*(\mu)) \leq 2\left(\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) + \tan\left(\frac{\pi}{4}\right)\right)$$

for some $\alpha \in [\varepsilon, \frac{\pi}{2} - \varepsilon]$. Since

$$\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = 2(1 + \sin\alpha + \cos\alpha)^{-1}$$

and

$$\sin\alpha + \cos\alpha \geq 1 + 0.5\,\varepsilon \qquad \text{for } \alpha \in [\varepsilon, \tfrac{\pi}{2} - \varepsilon] \tag{111}$$

with $\varepsilon \in (0, \pi/4)$, we obtain

$$V(Z_\infty^*(\mu)) \leq 2\left(\frac{1}{1 + 0.25\,\varepsilon} + 1\right) < 4\,(1 - 0.1\,\varepsilon)\,,$$

which proves the second assertion.

For the first assertion, we argue similarly. Here we use the fact that for $0 \leq \beta \leq \alpha < \pi/2$ the function $G(t) = \sin(\alpha + t) + \sin(\beta - t)$, $t \in [0, \min\{\beta, \frac{\pi}{2} - \alpha\}]$, is strictly decreasing. Thus we obtain

$$V(Z_\infty(\mu)) \geq \sin(\alpha) + \sin\left(\frac{\pi}{2} - \alpha\right) + \sin\left(\frac{\pi}{2}\right) = \sin\alpha + \cos\alpha + 1$$

for some $\alpha \in [\varepsilon, \frac{\pi}{2} - \varepsilon]$. Now the first assertion follows from (111). $\square$

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