A FRACTIONAL HELLY THEOREM FOR BOXES

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This paper is dedicated to Javier Bracho on occasion of his sixtieth birthday.

ABSTRACT. Let $\mathcal{F}$ be a family of n axis-parallel boxes in $\mathbb{R}^d$ and $\alpha \in (1-1/d, 1]$ a real number. There exists a real number $\beta(\alpha) > 0$ such that if there are $\alpha\binom{n}{2}$ intersecting pairs in $\mathcal{F}$, then $\mathcal{F}$ contains an intersecting subfamily of size βn. A simple example shows that the above statement is best possible in the sense that if $\alpha \leq 1 - 1/d$, then there may be no point in $\mathbb{R}^d$ that belongs to more than d elements of $\mathcal{F}$.

1. INTRODUCTION AND RESULTS

According to the classical theorem of Helly [1], if every $d + 1$-element subfamily of a finite family of convex sets in $\mathbb{R}^d$ has nonempty intersection, then the entire family has nonempty intersection. Although the number $d + 1$ in Helly's theorem cannot be lowered in general, it can be reduced for some special families of convex sets. For example, if any two elements in a finite family of axis-parallel boxes in $\mathbb{R}^d$ intersect, then all members of the family intersect, cf. [2].

Katchalski and Liu [7] proved the following generalization of Helly's theorem for the case when not all but only a fraction of $d + 1$-element subfamilies have a nonempty intersection in a family of convex sets.

Fractional Helly Theorem. (Katchalski and Liu [7]) *Assume that $\alpha \in (0, 1]$ is a real number and $\mathcal{F}$ is a family of n convex sets in $\mathbb{R}^d$. If at least $\alpha\binom{n}{d+1}$ of the $(d+1)$-tuples of $\mathcal{F}$ intersect, then $\mathcal{F}$ contains an intersecting subfamily of size $\frac{\alpha}{d+1}n$.*

The bound on the size of the intersecting subfamily was later improved by Kalai [6] from $\frac{\alpha}{d+1}n$ to $(1 - (1 - \alpha)^{1/(d+1)})n$, and this bound is best possible.

In this paper, we study the fractional behaviour of finite families of axis-parallel boxes, or boxes for short. We note that the boxes can be either open or closed, our statements hold for both cases. Our aim is to prove a statement similar to the Fractional Helly Theorem.

The intersection graph $\mathcal{G}_\mathcal{F}$ of a finite family $\mathcal{F}$ of boxes is a graph whose vertex set is the set of elements of $\mathcal{F}$, and two vertices are connected by an edge in $\mathcal{G}_\mathcal{F}$ precisely when the corresponding boxes in $\mathcal{F}$ have nonempty intersection.

Recall that for two integers $n \geq m \geq 1$, the Turán-graph $\mathcal{T}(n, m)$ is a complete m-partite graph on n vertices in which the cardinalities of the m vertex classes are

as close to each other as possible. Let $t(n, m)$ denote the number of edges of the Turán graph $\mathcal{T}(n, m)$. It is known that $t(n, m) \leq (1 - \frac{1}{m})\frac{n^2}{2}$, and equality holds if m divides n. Furthermore,

$$\lim_{n \to \infty} \frac{t(n, m)}{\frac{n^2}{2}} = 1 - \frac{1}{m}. \tag{1}$$

For more information on the properties of Turán graphs see, for example, the book of Diestel [3].

The following example shows that we cannot hope for a statement for boxes that is completely analogous to the Fractional Helly Theorem.

Example 1. Let $n \geq d + 1$ and $m, k \geq 0$ be integers such that $n = md + k$ and $0 \leq k \leq d - 1$. Let $n_1, \ldots, n_d$ be positive integers with $n = n_1 + \cdots + n_d$ and $n_i = \lceil \frac{n}{d} \rceil$ for $1 \leq i \leq k$ and $n_i = \lfloor \frac{n}{d} \rfloor$ for $k + 1 \leq i \leq d$. For $1 \leq i \leq d$, consider $n_i - 1$ hyperplanes orthogonal to the ith coordinate direction. These hyperplanes cut $\mathbb{R}^d$ into n_i pairwise disjoint open slabs $B'_{ij}, j = 1, \ldots, n_i$. Let C be a large open axis-parallel box that intersects each slab and let $\mathcal{F}_i$ consist of the open boxes $B_{ij} = C \cap B'_{ij}$. Define $\mathcal{F}$ as the union of the $\mathcal{F}_i$.

This way we have obtained a family $\mathcal{F}$ of n boxes with the property that two elements of $\mathcal{F}$ intersect exactly if they belong to different $\mathcal{F}_i$. The intersection graph of $\mathcal{F}$ is $\mathcal{T}(n, d)$ and thus the number of intersecting pairs in $\mathcal{F}$ is $t(n, d)$. However, there is no point of $\mathbb{R}^d$ that belongs to any $d + 1$-element subfamily of $\mathcal{F}$. Thus, (1) shows that in a fractional Helly-type statement for boxes, the percentage α has to be greater than $1 - \frac{1}{d}$.

Let $n \geq k \geq d$ and let $T(n, k, d)$ denote the maximal number of intersecting pairs in a family $\mathcal{F}$ of n boxes in $\mathbb{R}^d$ with the property that no $k + 1$ boxes in $\mathcal{F}$ have a point in common.

Theorem 1. *With the above notation,*

$$T(n, k, d) < \frac{d - 1}{2d}n^2 + \frac{2k + d}{2d}n.$$

It is quite easy to precisely determine $T(n, k, d)$ when $d = 1$:

Proposition 1. $T(n, k, 1) = (k - 1)n - \binom{k}{2}$.

Theorem 1 directly implies the following corollary.

Corollary 1. *Assume that $\varepsilon > 0$ is a real number and $\mathcal{F}$ is a family of n boxes in $\mathbb{R}^d$. If at least $\left(\frac{d-1}{2d} + \varepsilon\right)n^2$ pairs of $\mathcal{F}$ intersect, then $\mathcal{F}$ contains an intersecting subfamily of size $dn\varepsilon - \frac{d}{2} + 1$.*

The proof of Corollary 1 is given in Subsection 2.2. Corollary 1 yields the next theorem, which is our main result.

Fractional Helly Theorem for boxes. *For every $\alpha \in (1 - \frac{1}{d}, 1]$ there exists a real number $\beta(\alpha) > 0$ such that, for every family $\mathcal{F}$ of n boxes in $\mathbb{R}^d$, if an α fraction of pairs are intersecting in $\mathcal{F}$, then $\mathcal{F}$ has an intersecting subfamily of cardinality at least βn.*

Kalai's lower bound $\beta(\alpha) = 1 - (1 - \alpha)^{1/(d+1)}$ for the size of the intersecting subfamily in the fractional Helly theorem yields that if $\alpha \to 1$, then $\beta(\alpha) \to 1$ as well. The same holds for families of parallel boxes as stated in the following theorem.

Theorem 2. *Let $\mathcal{F}$ be a family of n boxes in $\mathbb{R}^d$, and let $\alpha \in (1 - \frac{1}{d^2}, 1]$ be a real number. If at least $\alpha\binom{n}{2}$ pairs of boxes in $\mathcal{F}$ intersect, then there exists a point that belongs to at least $(1 - d\sqrt{1-\alpha})n$ elements of $\mathcal{F}$.*

Simple calculations show that Corollary 1 does not imply Theorem 2 so we provide a separate proof for it in Section 2.

2. Proofs

2.1. **Proof of Theorem 1.** It is enough to prove that if no $k+1$ elements of $\mathcal{F}$ have a point in common, then there are at least $\frac{n^2 - 2(k+d)n}{2d}$ non-intersecting pairs. We may assume by standard arguments that the boxes in $\mathcal{F}$ are all open, so $B \in \mathcal{F}$ is of the form $B = (a_1(B), b_1(B)) \times \cdots \times (a_d(B), b_d(B))$. We assume without loss of generality that all numbers $a_i(B), b_i(B)$ $(B \in \mathcal{F})$ are distinct. For $B \in \mathcal{F}$ we define $\deg B$ to be the number of boxes in $\mathcal{F}$ that intersect B.

We prove Theorem 1 by induction on n. The starting case $n = k$ is simple since then $\frac{n^2 - 2(k+d)n}{2d} < 0$. In the induction step $n - 1 \to n$ we consider two cases.

Case 1. *When there is a box B with $\deg B \leq (1 - \frac{1}{d})n + \frac{2k+1}{2d}$.*

By induction, we have at least $\frac{(n-1)^2 - 2(k+d)(n-1)}{2d}$ non-intersecting pairs after removing B from $\mathcal{F}$. Since B is involved in at least $(n-1) - \left(1 - \frac{1}{d}\right)n - \frac{2k+1}{2d}$ non-intersecting pairs, there are at least

$$\frac{(n-1)^2 - 2(k+d)(n-1)}{2d} - 1 + \frac{n}{d} - \frac{2k+1}{2d} = \frac{n^2 - 2(k+d)n}{2d}$$

non-intersecting pairs in $\mathcal{F}$, indeed.

Case 2. *For every $B \in \mathcal{F}$ $\deg B \geq (1 - \frac{1}{d})n + \frac{2k+1}{2d}$.*

We show by contradiction that this cannot happen which finishes the proof.

We define d distinct boxes $B_1, \ldots, B_d \in \mathcal{F}$ the following way. Set

$$c_1 = \min\{b_1(B) : B \in \mathcal{F}\}$$

and define B_1 via $c_1 = b_1(B_1)$. The box B_1 is uniquely determined as all $b_1(B)$ are distinct numbers. Assume now that $i < d$ and that the numbers $c_1, \ldots, c_{i-1}$, and boxes $B_1, \ldots, B_{i-1}$ have been defined. Set

$$c_i = \min\{b_i(B) : B \in \mathcal{F} \setminus \{B_1, \ldots, B_{i-1}\}\}$$

and define B_i via $c_i = b_i(B_i)$ which is unique, again.

Let $\mathcal{F}' = \mathcal{F} \setminus \{B_1, \ldots, B_d\}$. We partition $\mathcal{F}'$ into $d+2$ parts. Let $\mathcal{F}_0$ be the set of all boxes of $\mathcal{F}'$ that intersect every B_i. For $i = 1, \ldots, d$ let $\mathcal{F}_i$ be the set of all boxes in $\mathcal{F}'$ that intersect every B_j for $j \neq i$ but do not intersect B_i. Let $\mathcal{F}^*$ be the set of all boxes of $\mathcal{F}'$ that intersect at most $d-2$ of the B_i boxes. As this is a partition of $\mathcal{F}'$ we have

$$|\mathcal{F}_0| + \sum_{i=1}^{d} |\mathcal{F}_i| + |\mathcal{F}^*| = |\mathcal{F}'| = n - d.$$

Note that $|\mathcal{F}_0| \leq k$ since every box in $\mathcal{F}_0$ contains the point $(c_1, \ldots, c_d)$.

Let N be the number of intersecting pairs between $\{B_1, \ldots, B_d\}$ and $\mathcal{F}'$. Each B_i intersects at least $\deg B_i - (d-1)$ boxes from $\mathcal{F}'$ as B_i may intersect B_j for all

$j \in [d], j \neq i$. Since every $\deg B_i \geq (1 - \frac{1}{d})n + \frac{2k+1}{2d}$ we have

$$d\left((1 - \frac{1}{d})n + \frac{2k+1}{2d} - (d-1)\right) \leq N$$

Every box in $\mathcal{F}_0$ intersects every B_i, $i \in [d]$, every box in $\mathcal{F}_i$ intersects every B_j except for B_i and every box in $\mathcal{F}^*$ intersects at most $(d-2)$ of the B_i. Consequently

$$N \leq d|\mathcal{F}_0| + (d-1)\sum_{i=1}^{d} |\mathcal{F}_i| + (d-2)|\mathcal{F}^*|.$$

So we have

$$\begin{aligned}
d\left((1 - \frac{1}{d})n + \frac{2k+1}{2d} - (d-1)\right) &\leq d|\mathcal{F}_0| + (d-1)\sum_{i=1}^{d} |\mathcal{F}_i| + (d-2)|\mathcal{F}^*| \\
&= |\mathcal{F}_0| + (d-1)\left(|\mathcal{F}_0| + \sum_{1}^{d} |\mathcal{F}_i| + |\mathcal{F}^*|\right) - |\mathcal{F}^*| \\
&= |\mathcal{F}_0| + (d-1)(n-d) - |\mathcal{F}^*|.
\end{aligned}$$

Simplifying the inequality and using $|\mathcal{F}_0| \leq k$ give

$$k + \frac{1}{2} \leq |\mathcal{F}_0| - |\mathcal{F}^*| \leq k - |\mathcal{F}^*|$$

implying $|\mathcal{F}^*| \leq -\frac{1}{2}$, which is a contradiction.

2.2. **Proof of Corollary 1.** If no point of $\mathbb{R}^d$ belongs to $dn\varepsilon - \frac{d}{2} + 1$ elements of $\mathcal{F}$, then by Theorem 1 the number of intersecting pairs of $\mathcal{F}$ is smaller than

$$\frac{d-1}{2d}n^2 + \frac{2(dn\varepsilon - \frac{d}{2}) + d}{2d}n = \left(\frac{d-1}{2d} + \varepsilon\right)n^2,$$

which yields a contradiction.

2.3. **Proof of Theorem 2.** Let π_i denote the orthogonal projection to the ith dimension in $\mathbb{R}^d$, that is, $\pi_i(B) = (a_i(B), b_i(B))$ for $B \in \mathcal{F}$. Set $\varepsilon = 1 - \alpha$. Define $T_i = \{\pi_i(B) : B \in \mathcal{F}\}$; this is a family of n intervals, and all but at most $\varepsilon\binom{n}{2}$ of the pairs in T_i intersect. According to the sharp version of the fractional Helly theorem (cf. [6]), T_i contains an intersecting subfamily T_i' of size $(1 - \sqrt{\varepsilon})n$, let c_i be a common point of all the intervals in T_i'. Define $D_i = \{B \in \mathcal{F} : c_i \notin \pi_i(B)\}$. Then $\mathcal{F} \setminus \bigcup_1^d D_i$ consists of at least $(1 - d\sqrt{\varepsilon})n = (1 - d\sqrt{1-\alpha})n$ boxes and all of them contain the point $(c_1, \ldots, c_d)$.

2.4. **Proof of Proposition 1.** Let $k \in \{1, \ldots, n\}$ be an integer, and let $\mathcal{F}$ be the family of open intervals $(i, i+k)$ for $i = 1, 2, \ldots, n$. Thus $\mathcal{F}$ consists of n intervals, no $k+1$ of them have a point in common, and there are $(k-1)n - \binom{k}{2}$ intersecting pairs in $\mathcal{F}$. Consequently $T(n, k, 1) \geq (k-1)n - \binom{k}{2}$.

Next we show, by induction on n that $T(n, k, 1) \leq (k-1)n - \binom{k}{2}$. Let $\mathcal{F}$ be a family of n intervals such that no $k + 1$ of them have a common point. We assume that these intervals are closed which is no loss of generality. The statement is clearly true when $n = k$. Let $[a, b] \in F$ be the interval where b is minimal. Since any interval intersecting $[a, b]$ contains b, there are at most $k - 1$ intervals intersecting $[a, b]$. Removing $[a, b]$ from $\mathcal{F}$ and applying induction, we find there

are at most $(k-1)(n-1) - \binom{k}{2}$ intersecting pairs in $\mathcal{F} \setminus \{[a,b]\}$. That is, there are at most $k - 1 + (k-1)(n-1) - \binom{k}{2} = (k-1)n - \binom{k}{2}$ intersecting pairs in $\mathcal{F}$.

3. Acknowledgements

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