A SEPARATION THEOREM FOR TOTALLY-SEWN 4-POLYTOPES

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Abstract. The Separation Problem, originally posed by K. Bezdek in [1], asks for the minimum number $s(O, K)$ of hyperplanes needed to strictly separate an interior point O in a convex body K from all faces of K. It is conjectured that $s(O, K) \leq 2^d$ in d-dimensional Euclidean space. We prove this conjecture for the class of all totally-sewn neighbourly 4-dimensional polytopes.

1. Introduction

The Gohberg-Markus-Hadwiger Covering Problem is a well-known unsolved problem in convex geometry. It seeks the minimum number $h(K)$ of smaller homothetic copies of a convex body K (compact convex set in $\mathbb{R}^d$ with non-empty interior) whose union covers K. It is conjectured that $h(K) \leq 2^d$, and that equality holds only for affine d-cubes. The Gohberg-Markus-Hadwiger Covering Problem is solved completely only in two dimensions, cf. [7]. In higher dimensional spaces there are only partial results. For a detailed overview of this topic we refer to [7] and [10].

In this article we consider the Separation Problem which was raised by K. Bezdek [1]. Let K be a convex body in $\mathbb{R}^d$ and $O \in \operatorname{int} K$ an interior point. The separation number $s(O, K)$ of O in K is defined as the minimum number of hyperplanes that strictly separate O from all faces of K. Bezdek proved in [1] that $s(O, K)$ is equal to the covering number $h(K^*)$ of the polar K^* of K, therefore, it is conjectured that $s(O, K) \leq 2^d$.

The evaluation of the separation number seems especially important for polytopes. There are only a few special classes of polytopes for which this has been accomplished. In particular, we mention here that in [2] and [3] it was shown that $s(O, P) < 2^d$ in the case that P is a cyclic d-polytope.

The celebrated Upper Bound Theorem of McMullen [11] states that among all d-polytopes with a fixed number of vertices, the neighbourly d-polytopes have the maximum number of facets. Thus, it is natural to investigate $s(O, P)$ for neighbourly d-polytopes P. Since interesting neighbourly polytopes exist only in $\mathbb{R}^d$ for

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$d \geq 4$, it is also natural to first restrict our attention to neighbourly 4-polytopes. Although a lot of information is known about neighbourly polytopes in general, only a few constructions yield infinite families of such polytopes. The most well-known such construction is the "sewing" operation introduced by Shemer [12]. Starting from a cyclic d-polytope, the sewing procedure of Shemer produces an infinite family of neighbourly d-polytopes each of which is obtained from the previous one by adding one new vertex in a suitable way. Neighbourly d-polytopes obtained from a cyclic d-polytope by a sequence of sewings are called called *totally-sewn*. Totally-sewn neighbourly 4-polytopes constitute a positive percentage of all neighbourly 4-polytopes with n vertices although this percentage decreases as n increases, cf. [12]. Moreover, each neighbourly 4-polytope has totally-sewn subpolytopes and this may yield a method of extending the present result to all neighbourly 4-polytopes.

The conjecture that $s(O, P) \leq 9$ for neighbourly 4-polytopes was formulated in [4]. This (stronger) conjecture was verified in [6] for those P that have at most ten vertices or that have the property that their vertices form a special configuration that resembles a "pentagram". It was demonstrated in [8] that semi-cyclic 4-polytopes possess this special pentagram property (for the definition of semi-cyclic 4-polytopes see, for example, page 125 in [5]). It was also shown in [6] that there exist totally-sewn neighbourly 4-polytopes that do not have the pentagram property.

In [5], it was proved that $s(O, P) \leq 16$ for a special class of totally-sewn neighbourly 4-polytopes that have the so-called *decreasing universal edge property*, cf. Section 3 of [5]. In this paper we build on the ideas developed in [5] and extend them to the whole class of totally-sewn neighbourly 4-polytopes. Our main result is the Separation Theorem (Theorem 5.1), which asserts that if P is a totally-sewn neighbourly 4-polytope, then $s(O, P) \leq 16$ for any point O in the interior of P. However, it still remains open whether the stronger conjecture [4] that $s(O, P) \leq 9$ holds for all totally-sewn neighbourly 4-polytopes.

The rest of the paper is organized as follows. In Section 2, we introduce vertex sewing and vertex types. In Section 3, we examine the universal edge types and the vertex types of P. We consider the separation of an interior point O of P from facets of P based upon the location (specific in Section 4, and generic in Section 5) of O in P.

2. Definitions

In this paper, we will work in $\mathbb{R}^4$. The convex hull and the affine hull of a set $X \subset \mathbb{R}^4$ will be denoted by $[X]$ and $\langle X \rangle$, respectively. We will use the following notations for the vertices, edges, and facets of a polytope P: $\mathcal{V}(P)$, $\mathcal{E}(P)$, and $\mathcal{F}(P)$, respectively. For $x \in \mathcal{V}(P)$, the vertex figure of P at x will be denoted by P/x. If $E = [x, y] \in \mathcal{E}(P)$, then the quotient polytope P/E is a vertex figure of P/x at the vertex that corresponds to E in P/x.

A 4-dimensional polytope P is *neighbourly* if for any $x, y \in \mathcal{V}(P)$, $x \neq y$, the segment $[x, y]$ is an edge of P. It is known that 4-dimensional neighbourly poytopes are simplicial. From now on, the symbol P will always denote a convex neighbourly 4-polytope. For basic geometric and combinatorial properties of neighbourly polytopes we refer to [9] and [12].

An edge $E = [x, y] \in \mathcal{E}(P)$ is *universal* if for any $z \in \mathcal{V}(P) \setminus \{x, y\}$, the triangle $[E, z]$ is a 2-face of P. The set of universal edges of P will be denoted by $\mathcal{U}(P)$.

Lemma 2.1 (cf. [12] and [13])**.** *If $|\mathcal{V}(P)| \geq 7$, then the following are equivalent:*

- $E = [x, y] \in \mathcal{U}(P)$.
- $[E, z]$ *is a 2-face of P for any $z \in \mathcal{V}(P) \setminus \{x, y\}$.*
- *x and y lie on the same side of every hyperplane determined by vertices of P.*
- *The quotient polytope P/E is a convex polygon with $|\mathcal{V}(P)| - 2$ vertices.*

Since P/E is convex polygon, there is a readily understandable description of all the facets of P that contain a universal edge E and how these facets are related to one another. We use this property and the fact that the vertices of any universal edge are not separable to determine the location of the vertices of P relative to the hyperplanes that contain E.

Next, we describe the sewing procedure of Shemer [12] in $\mathbb{R}^4$. Assume that P has n vertices and $E = [x, y]$ is a universal edge. Let $\mathcal{F}(E, P)$ denote the set of facets that contain E. By Lemma 2.1, $\mathcal{F}(E, P)$ has $n - 2$ elements. We label the vertices $\mathcal{V}(P) \setminus \{x, y\} = \{z_1, z_2, \ldots, z_{n-2}\}$ in such a way that

$$\mathcal{F}(E, P) = \{[E, z_i, z_{i+1}] \mid i = 1, \ldots, n-2 \text{ and } z_{n-1} = z_1\}.$$

To keep the notation simple, we denote the edge determined by $[E, z_i, z_{i+1}]$ in P/E by $[z_i, z_{i+1}]$.

Let $F = [E, z_i, z_{i+1}] \in \mathcal{F}(E, P)$, and let $\mathcal{F}(E, F, P) = \mathcal{F}(E, P) \setminus \{F\}$ be the set of facets of P that contain E and are different from F. Then there exists a point $\overline{x} \in \mathbb{R}^4$ (cf. [12]) which is beyond each facet in $\mathcal{F}(E, F, P)$ and beneath all other facets of P. The polytope $\overline{P} = [P, \overline{x}]$ is neighbourly (cf. [12]), and it is clear from the location of $\overline{x}$ that $\mathcal{V}(\overline{P}) = \mathcal{V}(P) \cup \{\overline{x}\}$. We say that $\overline{P}$ is obtained from P by *sewing $\overline{x}$ through* $\mathcal{F}(E, F, P)$. The universal edges of $\overline{P}$ were characterized in [12] as follows:

$$\mathcal{U}(\overline{P}) = \mathcal{U}^0(P) \cup \{[x, \overline{x}], [\overline{x}, y]\},$$

where $\mathcal{U}^0(P) = \{E^0 \in \mathcal{U}(P) \mid E^0 \cap F = \emptyset \text{ or } |E^0 \cap \{z_i, z_{i+1}\}| = 1\}$.

A polytope P with $n \geq 8$ vertices is *totally-sewn* if there exist subpolytopes $P_7, \ldots, P_n$ of P with the property that $|\mathcal{V}(P_m)| = m$ and P_{m+1} is obtained from P_m by sewing. Since P_7 is cyclic (cf. [12]), we may label its vertices such that $P_7 = [x_1, x_2, \ldots, x_7]$ and the vertices satisfy Gale's Evenness Condition in the order $x_1 < \cdots < x_7$. Then it is easy to check that

$$\mathcal{U}(P_7) = \{[x_{i-1}, x_i] \mid i = 1, \ldots, 7 \text{ and } x_0 = x_7\}.$$

Assume that $P_{m+1} = [P_m, x_{m+1}]$ and x_{m+1} is sewn through $\mathcal{F}(E_m, F_m, P_m)$ for $8 \leq m \leq n$. Then $P = [x_1, x_2, \ldots, x_n]$ and the sewing order determines an ordering on the vertices of P by $x_1 < x_2 < \cdots < x_n$. Note that there may be more than one sequence of sewings that produce the same polytope P from P_7. Once we fix a sequence of sewings, then we also fix the corresponding ordering of the vertices of P. If $1 \leq i < j \leq n$, then we may say that the vertex x_j *is after the vertex* x_i with respect to this ordering.

It is a great advantage of the sewing process that the universal edges and the facial structure of the new polytope can be completely characterized. The universal edges and facets of P_{m+1} constructed in the sewing process are described by the following statement.

Lemma 2.2 (cf. [6])**.** *Let $P_{m+1} = [P_m, x_{m+1}]$ and x_{m+1} be sewn through $\mathcal{F}(E_m, F_m, P_m)$ with $E_m = [x_a, x_b]$. Then*

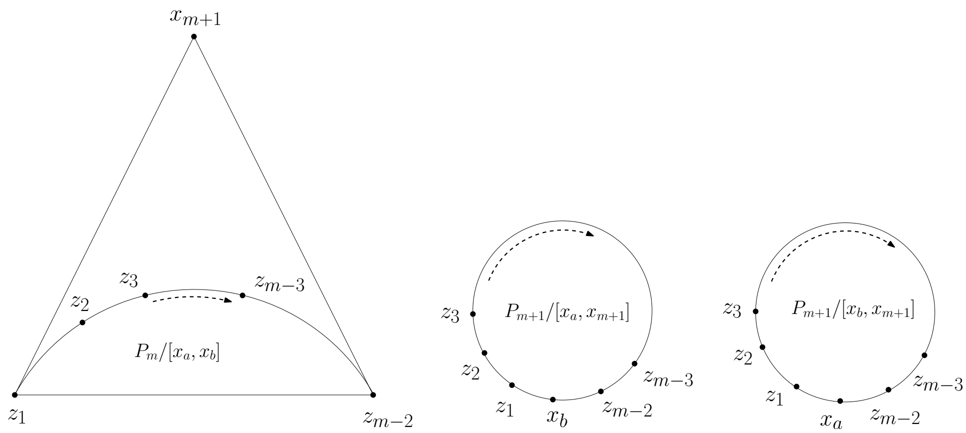


FIGURE 1.

- $\{[x_a, x_{m+1}], [x_{m+1}, x_b]\} \subset \mathcal{U}(P_{m+1})$,
- *any* $F \in \mathcal{F}(P_{m+1}) \setminus \mathcal{F}(P_m)$ *contains* $[x_a, x_{m+1}]$ *or* $[x_b, x_{m+1}]$, *and*
- *there is a labeling* $z_1, z_2, \ldots, z_{m-2}$ *of* $\mathcal{V}(P_m) \setminus \{x_a, x_b\}$ *so that*

$$\mathcal{F}(E_m, P_m) = \{[E_m, z_i, z_{i+1}] \mid i = 1, \ldots, m-2 \text{ and } z_{m-1} = z_1\},$$

and if $F_m = [x_a, x_b, z_1, z_{m-2}]$, *then*

$$\mathcal{F}(P_{m+1}) \setminus \mathcal{F}(P_m) = \{[x_a, x_b, x_{m+1}, z_1], [x_a, x_b, x_{m+1}, z_{m-2}], \\ [x_a, x_{m+1}, z_i, z_{i+1}], [x_b, x_{m+1}, z_i, z_{i+1}] \mid i = 1, \ldots, m-3\}.$$

We will frequently use in our arguments the representations of the quotient polytopes depicted in Figure 1. The first drawing describes the location of (the projection of) x_{m+1} in $P/[x_a, x_b]$, and the second and third figures indicate the locations of x_b and x_a in $P/[x_a, x_{m+1}]$ and $P/[x_b, x_{m+1}]$, respectively.

The most important ingredient in our proof is the concept of *type* of a vertex and a universal edge. This notion was introduced in [5]; below we recall the precise definition.

Let $6 \leq k < v \leq n$, and let x_v be sewn through $\mathcal{F}(E_{v-1}, F_{v-1}, P_{v-1})$ with $E_{v-1} \in \mathcal{U}(P_{v-1})$. Then either $E_{v-1} \in \mathcal{U}(P_k)$ or $E_{v-1} = [x_t, x_u]$ with $x_t < x_u$ and $x_u > x_k$. In the latter case x_u is sewn through $\mathcal{F}(E_{u-1}, F_{u-1}, P_{u-1})$ and either $E_{u-1} \in \mathcal{U}(P_k)$ or $E_{u-1} = [x_r, x_s]$ with $x_r < x_s$ and $x_s > x_k$. Iterating the above argument, we arrive at the conclusion that the vertex x_v originates from a unique universal edge $E \in \mathcal{U}(P_k)$ through a sequence of vertices that are sewn after x_k and before x_v. Let $\mathcal{U}(P_k) = \{E^1, \ldots, E^l\}$. Then $E = E^\alpha$ for some $1 \leq \alpha \leq l$ and we say that x_v *(E_{v-1}) is a vertex (universal edge) of type* α *with respect to* P_k. We note here that throughout the paper we use lower-case Greek letters to denote the vertex and universal edge types in P. We also recall from [5] the notations

$$V_k^\alpha = \{x_i \in \{x_{k+1}, \ldots, x_n\} \,|\, x_i \text{ is type } \alpha \text{ with respect to } P_k\},$$

and

$$V_k^\alpha(m) = V_k^\alpha \cap \{x_m, \ldots, x_n\} \text{ for } m > k.$$

In order to explain the importance of vertex (end edge) types, we used in [5] the similitude of sewing n coloured buttons on shirt. The buttons are the vertices and the colours are the vertex types of P. Once all the buttons are sewn in the order $x_7, \dots, x_n$, it turns out that the groupings of the colours are more important than the actual order of sewing.

Lemma 2.3 (Deletion process)**.** *Let $7 \le s \le n$ and $[x_p, x_q, x_r, x_s] \in \mathcal{F}(P_s)$ with $E_{s-1} = [x_r, x_t]$ and $x_t \notin \{x_p, x_q, x_r, x_s\}$. Then $[x_p, x_q, x_r, x_t] \in (\mathcal{F}(P_u) \cap \mathcal{F}(P_{s-1})) \setminus \mathcal{F}(P)$ with $u = \max\{p, q, r, t\}$.*

Proof. The assertion is a direct consequence of Lemmas 2.1 and 2.2. □

Note that the deletion process can be iterated if the facet $[x_p, x_q, x_r, x_t]$ does not contain the sewing edge E_{u-1}. In subsequent arguments we will iterate the deletion process in order to obtain facets at intermediate steps of the sewing process.

3. General properties of totally-sewn neighbourly 4-polytopes

Lemma 3.1. *Let $6 \le m < s < w \le n$, $x_s \in V_m^\alpha$, and let $x_w \in V_m^\beta$ with $E_{w-1} \in \mathcal{U}(P_s)$. Then $E_{w-1} \in \mathcal{U}(P_{s-1})$ and $E_{w-1} \cap E_{s-1} \subseteq E^\alpha \cap E^\beta$.*

Proof. Clearly, $E_{w-1} = E^\beta$ or E_{w-1} is a β type edge with respect to P_m. Since $x_s \in V_m^\alpha$, no new β type universal edge is constructed when x_s is sewn. Hence, $E_{w-1} \in \mathcal{U}(P_s)$ yields that $E_{w-1} \in \mathcal{U}(P_{s-1})$.

Let x_b be a common vertex of E_{s-1} and E_{w-1}. If $x_b \notin P_m$, then $E_{s-1} \neq E^\alpha$ which yields that $x_b \in V_m^\alpha$, and $E_{w-1} \neq E^\beta$ which yields that $x_b \in V_m^\beta$. This contradicts the fact that $V_m^\alpha \cap V_m^\beta = \emptyset$. Thus, $x_b \in P_m$ and $x_b \in E^\alpha \cap E^\beta$. □

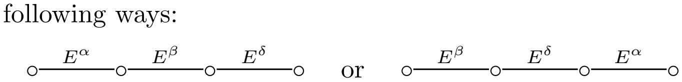
Lemma 3.2. *Let $6 \le k < m < u < w \le n$, $x_m \in V_k^\alpha$, $x_u \in V_k^\beta$, $x_w \in V_k^\delta$ and $\{x_m, x_u, x_w\} \subset F \in \mathcal{F}(P_w)$, $\alpha \neq \beta \neq \delta \neq \alpha$. Then in $\mathcal{U}(P_k)$, $E^\beta \cap E^\delta \neq \emptyset$ and $E^\alpha \cap (E^\beta \cup E^\delta) \neq \emptyset$.*

Proof. From $F \in \mathcal{F}(P_w)$ and Lemma 2.2 it follows that there exists a vertex $x_t \in F$ such that $[x_t, x_w] \in \mathcal{U}(P_w)$, $x_t \in E_{w-1} \in \mathcal{U}(P_{w-1})$. Let $E_{w-1} = [x_s, x_t]$. $E_{w-1} = E^\delta$ or E_{w-1} is a δ type edge with respect to P_k. $\{x_s, x_t\} \cap \{x_m, x_u\} = \emptyset$ implies that $F = [x_m, x_u, x_t, x_w]$.

Application of the deletion process (cf. Lemma 2.3) to the facet F yields that $\widetilde{F} = [x_m, x_u, x_s, x_t] \in \mathcal{F}(P_j)$ for $j = \max\{m, u, s, t\}$. If $x_j \in V_k^\delta$, then we iterate this deletion process. Hence, we may assume that, after a necessary number of iterations, $\widetilde{F} \in \mathcal{F}(P_u)$. Then $[\tilde{x}, x_u] \in \mathcal{U}(P_u)$ is a β type edge for some $\tilde{x} \in \{x_m, x_s, x_t\}$. Since $V_k^\alpha \cap V_k^\beta = \emptyset$, it follows that $\tilde{x} \neq x_m$, and we may assume that $\tilde{x} = x_s$. Now, by Lemma 3.1, $x_s \in E^\beta \cap E^\delta$.

Since $\widetilde{F} = [x_m, x_t, x_s, x_u] \in \mathcal{F}(P_u)$ and $[x_s, x_u] \in \mathcal{U}(P_u)$, there exists a vertex x_r such that $E_{u-1} = [x_r, x_s]$. Then $\widehat{F} = [x_m, x_t, x_s, x_r] \in \mathcal{F}(P_j)$ with $j = \max\{m, t, r\}$. Now, we may assume that, after a necessary number of iterations of the deletion process, $\widehat{F} \in \mathcal{F}(P_m)$. Then $[\hat{x}, x_m] \in \mathcal{U}(P_m)$ for some $\hat{x} \in \{x_t, x_s, x_r\}$. Since $\{x_s\} = E^\beta \cap E^\delta$, it follows that $\hat{x} \neq x_s$. If $\hat{x} = x_r$, then $\{x_r\} = E^\alpha \cap E^\beta$, and if $\hat{x} = x_t$, then $\{x_t\} = E^\alpha \cap E^\delta$ by Lemma 3.1 □

In summary, if $F \in \mathcal{F}(P)$ is a facet that satisfies the conditions of Lemma 3.2, then the universal edges E^α, E^β and E^δ form a path of length three in $\mathcal{U}(P_k)$ in one of the following ways:



Lemma 3.3. *Let $6 \le m < v < n$, $x_{m+1} \in V_m^\alpha$, $x_{v+1} \in V_m^\beta$ with $\{x_{v+1}\} = V_m^\beta \cap \{x_{m+1}, \dots, x_{v+1}\}$ and assume the $(P_v/E_v, z_i)$-configuration. Let $1 \le i < j \le v-2$, $\{z_i, z_j\} \subset V_m^\alpha$, $z_i = x_r$, $z_j = x_t$ and $[E_v, z_i, z_j] \in \mathcal{F}(P_t)$. Then (with suitable labeling) $\{z_{i+1}, \dots, z_{j-1}\} \subset V_m^\alpha$.*

Proof. We note that $E_m = E^\alpha$, $E_v = E^\beta$, $E_m \cap E_v \subset P_m$ and $E_v \in \mathcal{U}(P_w)$ for $m \le w \le v$. Clearly, we may assume that $i+1 < j$. Then

$$[E_v, z_i, z_j] \in \mathcal{F}(P_t) \setminus \mathcal{F}(P_v)$$

and there is a $t < u \le v$ such that

$$[E_v, z_i, z_j] \in \mathcal{F}(P_{u-1}) \setminus \mathcal{F}(P_u).$$

Then $x_u = z_k$ for some $i < k < j$, $E_{u-1} \subset [E_v, z_i, z_j] \neq F_{u-1}$ and either $E_{u-1} = E_v$ or $E_{u-1} = [z_i, z_j]$ or $|E_{u-1} \cap E_v| = 1 = |E_{u-1} \cap [z_i, z_j]|$. We note that $E_{u-1} = E_v$ yields that $x_u = x_{v+1}$; a contradiction. If $E_{u-1} = [z_i, z_j] = [x_r, x_t]$, then $x_t \in V_m^\alpha$ implies that E_{u-1} is an α type edge with respect to P_m and $z_k = x_u \in V_m^\alpha$.

If $E_{u-1} = [x_e, x_r]$ with $x_e \in E_v = E^\beta$, then $x_r \in V_m^\alpha$ implies that $\{x_e\} = E^\alpha \cap E^\beta$. Furthermore, E_{u-1} is α type and again $z_k = x_u \in V_m^\alpha$.

Since $[E_v, z_i, z_j] \in \mathcal{F}(P_{u-1}) \setminus \mathcal{F}(P_u)$ clearly implies that

$$\{[E_v, z_i, z_k], [E_v, z_k, z_j]\} \subset \mathcal{F}(P_u),$$

the assertion of the Lemma readily follows from iterations of the argument above for $\{z_i, z_k\}$ and $\{z_k, z_j\}$. □

In summary, Lemma 3.3 states that the α type vertices of P_{v+1} determine a connected arc in the polygon P_v/E_v.

4. Separation in totally-sewn neighbourly 4-polytopes

In this section we develop the basic tools that will be used in the proof of the main theorem. Some of the following statements are quoted from [5], and some are new. The arguments of the proofs are based on the lemmas of the previous section. The following statement is a direct consequence of Lemma 2.1.

Lemma 4.1. *Let $6 \le k < n$, $H \subset \mathbb{R}^4$ be a hyperplane spanned by the vertices of P_k and x be a point of $\mathbb{R}^4$.*

- 4.1.1 *If H strictly separates x and an endpoint of $E^\lambda \in \mathcal{U}(P_k)$, then H strictly separates x and V_k^λ.*
- 4.1.2 *If H strictly separates x and $x_u \in V_k^\lambda$, then H strictly separates x and $V_k^\lambda(u)$.*

Lemma 4.2 (cf. [5])**.** *Let Q be a 4-dimensional subpolytope of P and O be a point of $(\operatorname{int} P) \cap \partial Q$. Then O is strictly separated from any $F \in \mathcal{F}(P) \cap \mathcal{F}(Q)$ by one of at most three hyperplanes.*

Lemma 4.3. *Let $6 \le m < u, v \le n$, $x_{m+1} \in V_m^\alpha$, $\{x_u, x_v\} \subset V_m^\beta$ and $x_c \in \mathcal{V}(P_m) \setminus E^\alpha$ such that $\langle E^\alpha, x_c, x_{m+1}\rangle \cap [x_u, x_v] = \emptyset$. Then $\langle E^\alpha, x_c, x_g\rangle \cap [x_u, x_v] = \emptyset$ for all $x_g \in V_m^\alpha$.*

Proof. Let $x_g \in V_m^\alpha$ such that $[x_{m+1}, x_g] \in \mathcal{U}(P_g)$, and suppose, on the contrary, that $\langle E^\alpha, x_c, x_g\rangle \cap [x_u, x_v] \neq \emptyset$. Then $\langle E^\alpha, x_c, x_g\rangle$ strictly separates x_u and x_v, and we may assume that $\langle E^\alpha, x_c, x_v\rangle$ strictly separates x_{m+1} and x_g.

Let S denote the open region of $\mathbb{R}^4$ bounded by $\langle E^\alpha, x_c, x_{m+1}\rangle$ and $\langle E^\alpha, x_c, x_g\rangle$, that contains x_v. We note that if $x_i \in S$, then $\langle E^\alpha, x_c, x_i\rangle$ strictly separates x_{m+1} and x_g. In addition, $x_u \notin S$ and by Lemma 4.1 $S \cap \mathcal{V}(P) \subset \{x_{g+1}, \ldots, x_n\}$. Without loss of generality, we may assume that $S \cap V_m^\beta \subset \{x_v, \ldots, x_n\}$.

Let $E_{v-1} = [x_r, x_s]$. Then $\{x_r, x_s\} \subset V_m^\beta \cup \mathcal{V}(E^\beta)$, and both $[x_r, x_v]$ and $[x_s, x_v]$ are universal edges of P_v and thus neither $\langle E^\alpha, x_c, x_{m+1}\rangle$ nor $\langle E^\alpha, x_c, x_g\rangle$ strictly separates x_v from x_r or x_s. Since $\{x_r, x_s\} \subset \mathrm{cl}(S)$ and $\mathrm{cl}(S) \cap V_m^\beta = S \cap V_m^\beta$, it follows from $\{x_r, x_s\} \subset \{x_r, \ldots, x_{v-1}\}$ that $\{x_r, x_s\} \cap V_m^\beta = \emptyset$ and $E_{v-1} = [x_r, x_s] = E^\beta$ and thus x_v is the first β type vertex with respect to P_m. Therefore, $x_u > x_v$ and $x_v \in S$ yield that $x_u \in S$. This is a contradiction, and hence $\langle E^\alpha, x_c, x_g\rangle \cap [x_u, x_v] = \emptyset$.

The statement of the lemma follows from iterations of the above argument. □

The following lemma is the cornerstone of our argument.

Lemma 4.4. *Let $6 \le k \le m < n$, $O \in \mathrm{int}\, P_m$, $x_{m+1} \in V_k^\alpha$ and $\{x_{m+1}\} = V_k^\alpha \cap \{x_{k+1}, \ldots, x_{m+1}\}$. Let $\widetilde{F} \in \mathcal{F}(P)$ such that $\widetilde{F} \cap V_k^\alpha(m+1) \neq \emptyset$. Then O is separated from any $\widetilde{F}$ by one of at most five hyperplanes spanned by vertices of P.*

Proof. Note that $V_k^\alpha(m+1) = V_k^\alpha$. Let $E_m = [x_a, x_b] = E^\alpha$, $Q = [V_k^\alpha]$, and assume the $(P_m/E_m, y_i)$-configuration. By Lemma 4.1, O is (strictly) separated from Q by each hyperplane $\langle \hat{F}\rangle$ for $\hat{F} \in \mathcal{F}(E_m, P_m) \setminus \{F_m\}$, and so there are vertices x_g and x_h of Q such that O is separated from any $\hat{F}$ by one of the hyperplanes $H_1 = \langle E_m, y_l, x_g\rangle$, $H_2 = \langle E_m, y_l, y_{l+1}\rangle$, $H_3 = \langle E_m, y_{l+1}, x_h\rangle$ for some $1 \le l \le m-3$, see Figures 2 and 3. The location of O in $\mathrm{int}\, P_m$ determines the choice of l.

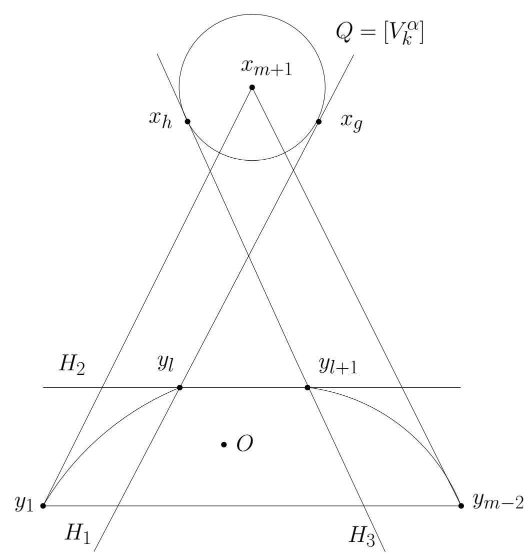


FIGURE 2. P_m/E_m

We examine first the facets $\widetilde{F}$ via the Deletion (process) and then consider the separation of $\widetilde{F}$ from O based upon the location of O.

We note that if $E^\lambda \in \mathcal{U}(P_k) \cap \mathcal{U}(P_m)$, then $V_k^\lambda = V_k^\lambda(m+1) = V_m^\lambda$, and if $E^\lambda \in \mathcal{U}(P_k) \setminus \mathcal{U}(P_m)$, and $E^\theta, \ldots, E^\psi$ are the λ types in $\mathcal{U}(P_m)$ with respect to

P_k, then $V_k^\lambda(m+1) = V_m^\theta \cup \dots \cup V_m^\psi$. Thus, $\widetilde{F} \cap V_m^\alpha \neq \emptyset$, and $\widetilde{F}$ contains at most two other types of vertices with respect to P_m by Lemmas 3.1 and 3.2. Let $X = \{x_{m+1}, \dots, x_n\}$ and $Y = \{y_1, \dots, y_{m-2}\}$.

i) Let $\widetilde{F} \cap X \subset V_m^\alpha$. Then $\widetilde{F} \cap Y \subseteq \{y_j, y_{j+1}\}$ for some $1 \leq j \leq m-3$.

Proof of i). Let $|\widetilde{F} \cap Y| \geq 2$, $x_u \in \widetilde{F} \cap V_m^\alpha \subseteq \{x_{m+1}, \dots, x_u\}$ and $E_{u-1} = [x_r, x_s]$. Since $x_u \in V_m^\alpha$ and $E^\alpha \cap Y = \emptyset$, it follows that $E_{u-1} \cap Y = \emptyset$, $E_{u-1} \not\subset \widetilde{F}$, $|\widetilde{F} \cap Y| = 2$ and x_r or x_s is in $\widetilde{F} \cap (V_m^\alpha \cup \{x_a, x_b\})$. Now, Deletion yields that there is an $F' \in \mathcal{F}(P_{m+1})$ such that $F' \cap Y = \widetilde{F} \cap Y$ and $x_{m+1} \in F'$. By Lemma 2.2, $F' \cap Y \subseteq \{y_j, y_{j+1}\}$ for some $1 \leq j \leq m-3$. □

ii) Let $\widetilde{F} \cap X \subset V_m^\alpha \cup V_m^\lambda$ and $\alpha \neq \lambda$. Then either $E^\lambda = [y_j, y_{j+1}]$ for some $1 \leq j \leq m-3$ and $\widetilde{F} \cap Y \subset E^\lambda$ or $E^\lambda = [E^\lambda \cap E^\alpha, y]$ for some $y \in \{y_1, y_{m-2}\}$ and $\widetilde{F} \cap Y$ is contained in $\{y_1, y_2\}$ or $\{y_{m-3}, y_{m-2}\}$.

Proof of ii). Since $E^\lambda \in \mathcal{U}(P_m) \cap \mathcal{U}(P_{m+1})$, we have that either $E^\lambda \cap E^\alpha = \emptyset$, $[E^\lambda, E^\alpha] \in \mathcal{F}(P_m)$ (cf. Theorem 3.4, [12]) and $E^\lambda = [y_j, y_{j+1}]$ for some $1 \leq j \leq m-3$ or $E^\lambda \cap E^\alpha \neq \emptyset$ and $E^\lambda = [x, y]$ for some $x \in \{x_a, x_b\}$ and $y \in \{y_1, y_{m-2}\}$.

Let $x_p \in \widetilde{F} \cap V_m^\alpha \subseteq \{x_{m+1}, \dots, x_p\}$ and $x_v \in \widetilde{F} \cap V_m^\lambda \subseteq \{x_{m+1}, \dots, x_v\}$. Since $E_{p-1} \cap Y \subseteq E^\alpha \cap Y = \emptyset$ and $E_{v-1} \cap Y \subseteq E^\lambda \cap Y$, we may assume that

$$[E_{p-1}, x_p, x_v] \neq \widetilde{F} \neq [E_{v-1}, x_p, x_v].$$

Then $E_{p-1} \not\subset \widetilde{F}$ and $E_{v-1} \not\subset \widetilde{F}$, and it follows from Deletion that $\widetilde{F} \cap Y \subseteq (E_{p-1} \cup E_{v-1}) \cap Y$ in case $E^\alpha \cap E^\lambda = \emptyset$.

Let $E^\alpha \cap E^\lambda \neq \emptyset$ and $\widetilde{F} = [x_p, x_v, \widetilde{x}, y_i]$ with $y_i \notin E^\lambda$. By Deletion, we obtain that $[x_{m+1}, E^\alpha \cap E^\lambda, x_{w+1}, y_i] \in \mathcal{F}(P_{w+1})$ with $E_w = E^\lambda$ and $[x_{m+1}, E^\lambda, y_i] \in \mathcal{F}(P_{m+1}) \cap \mathcal{F}(P_w)$. Hence Lemma 2.2 yields that $y_i = y_2$ in case $E^\lambda = [E^\lambda \cap E^\alpha, y_1]$ and $y_i = y_{m-3}$ in case that $E^\lambda = [E^\alpha \cap E^\lambda, y_{m-2}]$. □

iii) Let $\widetilde{F} \cap V_m^\delta \neq \emptyset \neq \widetilde{F} \cap V_m^\eta$ and $\alpha \neq \delta \neq \eta \neq \alpha$. Then $E^\delta \cap E^\eta \neq \emptyset$ and $E^\alpha \cap (E^\delta \cup E^\eta) \neq \emptyset$.

Proof of iii). Let $\widetilde{F} = [x_p, x_q, x_r, \widetilde{x}]$ with $x_p \in V_m^\alpha$, $x_q \in V_m^\delta$ and $x_r \in V_m^\eta$. It is clear that none of E_{p-1}, E_{q-1} and E_{r-1} is contained in $\widetilde{F}$, so by Lemma 3.2,

$$\{\widetilde{x}\} \in \{E^\alpha \cap E^\delta, E^\alpha \cap E^\eta, E^\delta \cap E^\eta\}$$

and we need only to verify that $E^\delta \cap E^\eta \neq \emptyset$. Let $x_{u+1} \in V_m^\delta$ with $E_u = E^\delta$, and $x_{t+1} \in V_m^\eta$ with $E_t = E^\eta$.

If $\{\widetilde{x}\} = E^\alpha \cap E^\eta = E_m \cap E_t$ then $q = \min\{p, q, r\}$ by Lemma 3.2. Now $\widetilde{F} = [x_p, \widetilde{x}, x_r, x_q]$ and Deletion yields that $[x_{m+1}, \widetilde{x}, x_{t+1}, x_q] \in \mathcal{F}(P_{t+1})$, $[x_{m+1}, E_t, x_q] \in \mathcal{F}(P_t) \cap \mathcal{F}(P_q)$ and $E_{q-1} \cap E_t \neq \emptyset$. By Lemma 3.1, $E_{q-1} \cap E_t = E^\delta \cap E^\eta$.

If $\{\widetilde{x}\} = E^\alpha \cap E^\delta = E_m \cap E_u$ then $r = \min\{p, q, r\}$, $[x_{m+1}, \widetilde{x}, x_{u+1}, x_r] \in \mathcal{F}(P_{u+1})$, $[x_{m+1}, E_u, x_r] \in \mathcal{F}(P_r) \cap \mathcal{F}(P_u)$ and $\emptyset \neq E_{r-1} \cap E_u = E^\eta \cap E^\delta$. □

iv) Let $\widetilde{F} = [x_p, x_q, x_r, \widetilde{x}]$ with $x_p \in V_m^\alpha$, $x_q \in V_m^\delta$, $x_r \in V_m^\eta$, $x_{u+1} \in V_m^\delta$ with $E_u = E^\delta = [x_a, y_1]$, and $x_{t+1} \in V_m^\eta$ with $E_t = E^\eta = [y_1, y_2]$. Then

- *$\widetilde{x} = x_a$, $[x_{m+1}, E_u, x_r] \in \mathcal{F}(P_r) \cap \mathcal{F}(P_u)$ and $t < u$, or*
- *$\widetilde{x} = y_1$, and either $[x_{t+1}, E_u, x_p] \in \mathcal{F}(P_u) \cap \mathcal{F}(P_{t+1})$ and $t < u$, or $[x_{u+1}, E_t, x_p] \in \mathcal{F}(P_t) \cap \mathcal{F}(P_{u+1})$ and $u < t$.*

Proof of iv). From the Proof of iii), we have that $\widetilde{x} \in \{x_a, y_1\}$ and that we need only to verify the consequences of $\widetilde{x} = y_1$.

Let $\widetilde{F} = [x_q, y_1, x_r, x_p]$. Then $p = \min\{p, q, r\}$ and it follows by Deletion that $F' = [x_{u+1}, y_1, x_{t+1}, x_p] \in \mathcal{F}(P_{u+1}) \cup \mathcal{F}(P_{t+1})$. The assertions are now immediate. □

We now consider the separation of O from $\widetilde{F}$. We note that the location of O is as indicated on Figure 2 or (due to symmetry) Figure 3. From i), ii), iii) and iv), we can determine what types of vertices $\widetilde{F}$ may possess.

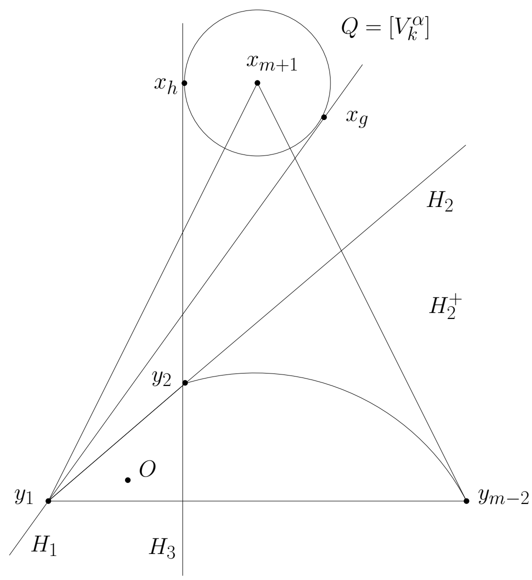


FIGURE 3. P_m/E_m

Case 1. *For some $2 \leq l < m/2$, O is separated from any $\widehat{F}$ as indicated in Figure 2.*

If $[y_l, y_{l+1}] \in \mathcal{U}(P_m)$, then we denote it by E^β, and if $V_m^\beta \neq \emptyset$, then let $x_q \in V_m^\beta$ with $E_{q-1} = E^\beta$. From i), ii), iii), and Lemma 4.1, we obtain that H_1, H_2 or H_3 separate O from any $\widetilde{F}$ such that $\widetilde{F} \cap V_m^\beta = \emptyset$.

Let $\widetilde{F} \cap V_m^\beta \neq \emptyset$. Then $E^\beta \cap \{y_1, y_{m-2}\} = \emptyset$ and iii) yield that $\widetilde{F} \cap X \subset V_m^\alpha \cup V_m^\beta$, and ii) yields that $\widetilde{F} \cap Y \subseteq \{y_l, y_{l+1}\}$. Since Lemma 4.1 implies that H_2 separates O and $\widetilde{F}$ in the case H_2 separates O and x_q, we may assume that H_2 does not separate O and x_q. Then with a $(P_{q-1}/E_{q-1}, z_i)$-configuration (cf. Figure 4), we have that $H_2 = \langle E_m, y_l, y_{l+1} \rangle = \langle x_a, x_b, E_{q-1} \rangle$ is not a supporting hyperplane of P_{q-1}, and

$$z_1 \leq z_r = x_a < x_b = z_s \leq z_{q-3}$$

for some $1 \leq r < r+1 < s \leq q-3$. Now, Lemma 3.3 yields

$$V_m^\alpha \cap \{z_1, \dots, z_{q-3}\} \subset \{z_1, \dots, z_r\} \cup \{z_s, \dots, z_{q-3}\},$$

and it follows from Lemma 4.1 and $\widetilde{F} \cap Y \subset E_{q-1}$ that O is separated from any such $\widetilde{F}$ by $H_4 = \langle E_{q-1}, x_a, x_{g'} \rangle$ or $H_5 = \langle E_{q-1}, x_b, x_{h'} \rangle$ for suitably chosen $x_{g'}$, and $x_{h'}$ in V_m^β.

In summary, O is separated from any $\widetilde{F}$ by one of H_1, H_2, H_3, H_4 and H_5.

Case 2. *O is separated from any $\widehat{F}$ as depicted in Figure 3.*

If $[x_a, y_1] \in \mathcal{U}(P_m)$ then let $E^\delta = [x_a, y_1]$, and if $V_m^\delta \neq \emptyset$ then let $x_{u+1} \in V_m^\delta$ with $E_u = E^\delta$. If $[y_1, y_2] \in \mathcal{U}(P_m)$ then let $E^\eta = [y_1, y_2]$, and if $V_m^\eta \neq \emptyset$ then let $x_{t+1} \in V_m^\eta$ with $E_t = E^\eta$. Let H_2^+ denote the open half-space determined by

$$H_2 = \langle E_m, y_1, y_2 \rangle = \langle x_a, x_b, y_1, y_2 \rangle$$

that contains O. As in Case 1, we obtain from i), ii), iii) and Lemma 4.1 that

- O is separated from any $\widetilde{F}$ by H_2 or H_3 in the case that $H_2^+ \cap \{x_{u+1}, x_{t+1}\} = \emptyset$, or
- O is separated from any $\widetilde{F}$ by one of H_2, H_3 and suitable determined H_4 and H_5 in the case that $|H_2^+ \cap \{x_{u+1}, x_{t+1}\}| = 1$ and $V_m^\delta = \emptyset$ or $V_m^\eta = \emptyset$.

Let $|H_2^+ \cap \{x_{u+1}, x_{t+1}\}| \geq 1$ and $V_m^\delta \neq \emptyset \neq V_m^\eta$. Then $H_2 = \langle E_u, x_b, y_2 \rangle = \langle E_t, x_a, x_b \rangle$ and either $m < u < t$ or $m < t < u$.

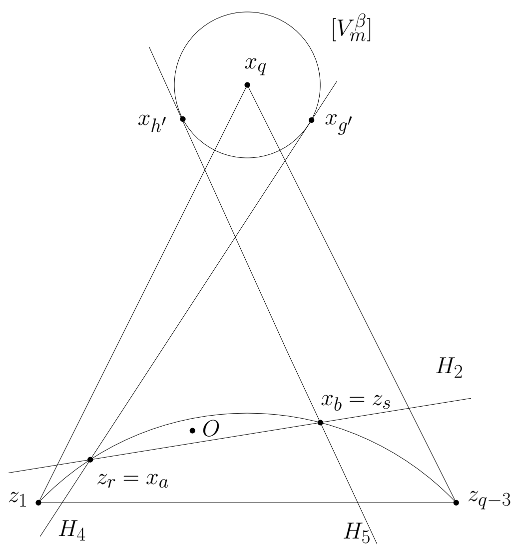


FIGURE 4. P_{q-1}/E_{q-1}

Let $m < u < t$ and refer to Figures 5 and 6 for configurations $(P_u/E_u, z_i)$ and $(P_t/E_t, w_j)$. Since $E_t = [y_1, y_2] \in \mathcal{U}(P_u) \cap \mathcal{U}(P_{u+1})$ and $E_t \cap E_u = \{y_1\}$, it follows that $y_2 \in \{z_1, z_{u-2}\}$ and we may assume that $y_2 = z_1$. Thus, there is no $F \in \mathcal{F}(P)$ such that $F \cap V_m^\delta \neq \emptyset$ and $\{y_2, z_j\} \subset F$ for some $i \leq j \leq u-2$. For Figure 6, we note that $\{x_a, x_b\} = \{w_i, w_j\}$ for some $1 \leq i < i+1 < j \leq t-2$.

Finally, iv) and $u < t$ yield that if $\widetilde{F} \cap V_m^\delta \neq \emptyset \neq \widetilde{F} \cap V_m^\eta$ for some $\widetilde{F}$ then $[x_{u+1}, y_1, y_2, \widetilde{F} \cap V_m^\alpha] \in \mathcal{F}(P_t)$ by Deletion. This means that H_2 does not separate x_{u+1} and $\widetilde{F} \cap V_m^\alpha$. Thus, by Lemma 4.1, $x_{m+1} \notin H_2^+$ implies that $x_{u+1} \notin H_2^+$.

We note that by i), ii) and Lemma 4.1, O is separated from any $\widetilde{F}$ such that $\widetilde{F} \cap (V_m^\delta \cup V_m^\eta) = \emptyset$ by H_2 or H_3. We now consider $\widetilde{F}$ such that $\widetilde{F} \cap (V_m^\delta \cup V_m^\eta) \neq \emptyset$.

Let $x_{u+1} \notin H_2^+$. Then $x_{t+1} \in H_2^+$, $V_m^\alpha \cap \{z_1, \dots, z_{u-2}\} \subset \{z_1, \dots, z_i\}$ and

$$(V_m^\alpha \cup V_m^\delta) \cap \{w_1, \dots, w_{t-2}\} \subset \{w_1, \dots, w_i\} \cup \{w_j, \dots, w_{t-2}\}.$$

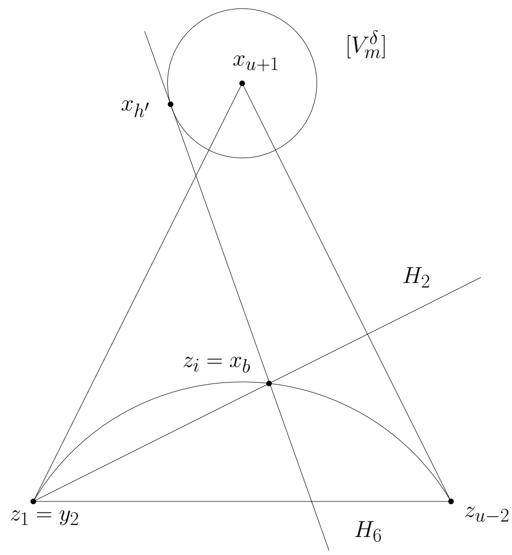


FIGURE 5. P_u/E_u

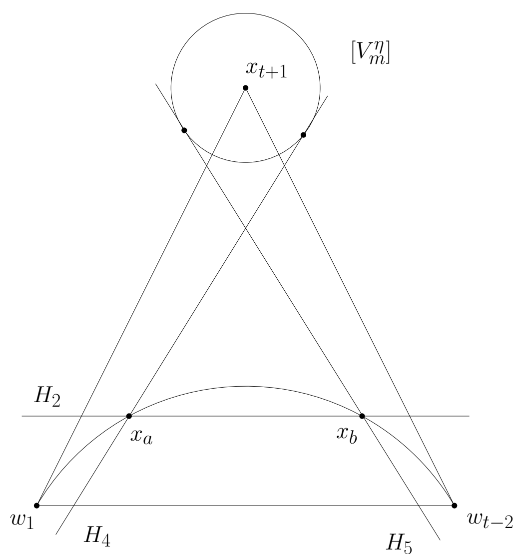


FIGURE 6. P_t/E_t

Thus, O is separated from any $\widetilde{F}$ such that $\widetilde{F} \cap X \subset V_m^\alpha \cup V_m^\delta$ by H_2, and O is separated from any $\widetilde{F}$ such that $\widetilde{F} \cap X \subset V_m^\alpha \cup V_m^\delta \cup V_m^\eta$ and $\widetilde{F} \cap V_m^\eta \neq \emptyset$ by H_4 or H_5.

Let $x_{u+1} \in H_2^+$. Then $\widetilde{F} \cap V_m^\delta = \emptyset$ or $\widetilde{F} \cap V_m^\eta = \emptyset$ for any $\widetilde{F}$ by the preceding, and $V_m^\alpha \cap \{z_1, \ldots, z_{u-2}\} \subset \{z_i, \ldots, z_{u-2}\}$. Let $\widetilde{F} \cap V_m^\delta \neq \emptyset$. Then as already noted, we have that $y_2 \notin \widetilde{F}$ and $\widetilde{F} \cap X \subset V_m^\alpha \cup V_m^\delta$. Thus, it follows from ii) and Lemma 4.1 that O is separated from any such $\widetilde{F}$ by H_6. Let $\widetilde{F} \cap V_m^\eta \neq \emptyset$. Then $\widetilde{F} \cap X \subset V_m^\alpha \cup V_m^\eta$

by iii), and either $x_{t+1} \notin H_2^+$ and H_2 separates O from any such $\widetilde{F}$ or $x_{t+1} \in H_2^+$ and H_4 or H_5 separate O from any such $\widetilde{F}$.

In summary, O is separated from any $\widetilde{F}$ by one of H_2, H_3, H_4, H_5 and H_6.

Finally, let $m < t < u$. Then we may consider Figure 5 to represent P_t/E_t, and Figure 6 to represent P_u/E_u. From iv), we obtain that if $\widetilde{F} \cap V_m^\delta \neq \emptyset \neq \widetilde{F} \cap V_m^\eta$ for some $\widetilde{F}$ then $x_{t+1} \notin H_2^+$. We now argue as above and obtain that O is separated from any $\widetilde{F}$ by one of analogously labeled H_2, H_3, H_4, H_5 and H_6. □

In the following Corollaries: we assume the hypotheses and the notation of Lemma 4.4, and determine locations of O that do not require five separating hyperplanes determined by vertices of P.

Corollary 4.5. *Let $1 \leq i < j < p \leq m-3$, and O be contained in either the open region bounded by $\langle E_m, y_i, y_j\rangle$, $\langle E_m, y_i, y_p\rangle$ and $\langle E_m, y_j, y_p\rangle$ or the relatively open region in $\langle E_m, y_i, y_p\rangle$ bounded by $\langle E_m, y_i\rangle$ and $\langle E_m, y_p\rangle$, and $x \in \{x_a, x_b\}$. If $\{[y_i, x], [y_i, y_j]\} \not\subset \mathcal{U}(P_m)$ then O is separated from any $\widetilde{F}$ by one of at most four hyperplanes. Moreover, if $1 < i$, $[y_i, y_j] \notin \mathcal{U}(P_m)$ and $[y_j, y_p] \notin \mathcal{U}(P_m)$ then three hyperplanes suffice.*

Proof. Let $i < 1$, $[y_i, y_j] \notin \mathcal{U}(P_m)$ and $[y_j, y_p] \notin \mathcal{U}(P_m)$. Then O is separated from any $\widetilde{F}$ by one of H_6, H_6' and $\langle E_m, y_i, y_j\rangle$ or one of H_7, H_7' and $\langle E_m, y_j, y_p\rangle$; cf. Figure 7. In case $[y_i, y_j]$ or $[y_j, y_p]$ is in $\mathcal{U}(P_m)$, we replace $\langle E_m, y_i, y_j\rangle$ or $\langle E_m, y_j, y_p\rangle$ by two hyperplanes.

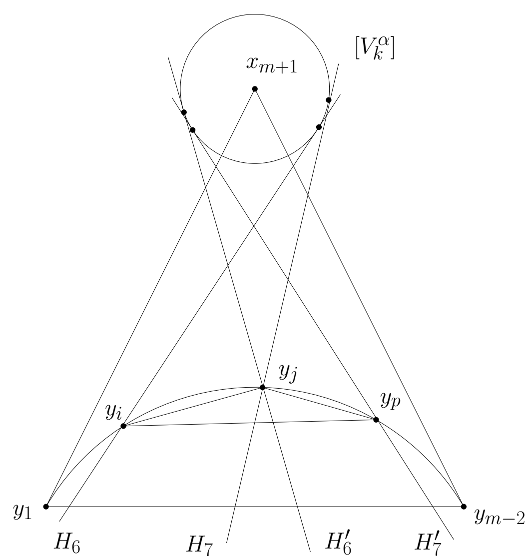


FIGURE 7. $P_m/E_m = [x_a, x_b]$

If $i = 1$, and $E^\delta = [E^\delta \cap E^\alpha, y_1] \in \mathcal{U}(P_m)$ with $V_m^\delta \neq \emptyset$ then we replace H_6 by two hyperplanes. □

A similar argument yields

Corollary 4.6. *Let $1 \leq i < i+2 < p \leq m-3$, O be contained in the relatively open region in $\langle E_m, y_i, y_p \rangle$ bounded by $\langle E_m, y_i \rangle$ and $\langle E_m, y_p \rangle$, and $x \in \{x_a, x_b\}$. Then O is separated from any $\widetilde{F}$ by one of at most four hyperplanes. Moreover, if either $1 < i$ or $i = 1$ and $[y_1, x] \notin \mathcal{U}(P_m)$ then three hyperplanes suffice.*

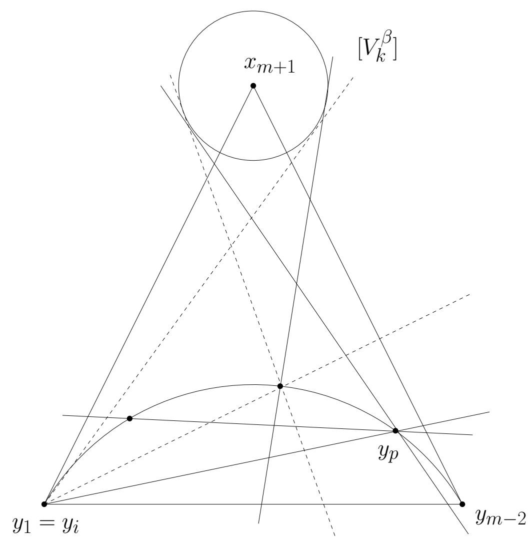


FIGURE 8. P_m/E_m

Let $\mathcal{U}'(P_6) = \{[x_6, x_1]\} \cup \{[x_i, x_{i+1}] | i = 1, \ldots, 5\}$. We note that $\mathcal{U}'(P_6) \subset \mathcal{U}(P_6)$ and that any $x_w \in \{x_7, \ldots, x_n\}$ is an η type with respect to P_6 for some $E^\eta \in \mathcal{U}'(P_6)$, and specifically, we start the sewing process with $E_6 = [x_6, x_1]$. Let $V^\eta = V_6^\eta$ and denote the elements of $\mathcal{U}'(P_6)$ as indicated below.

$$\mathcal{U}'(P_6) : \underset{x_6}{\circ} \overset{\alpha}{\text{———}} \underset{x_1}{\circ} \overset{\beta}{\text{———}} \underset{x_2}{\circ} \overset{\delta}{\text{———}} \underset{x_3}{\circ} \overset{\theta}{\text{———}} \underset{x_4}{\circ} \overset{\mu}{\text{———}} \underset{x_5}{\circ} \overset{\lambda}{\text{———}} \underset{x_6}{\circ}$$

We let $E^\beta = E_{r-1}$ in case $V^\beta \neq \emptyset$, and introduce similarly $E_{s-1} = E^\delta$, $E_{t-1} = E^\theta$, $E_{u-1} = E^\mu$, and $E_{v-1} = E^\lambda$.

Finally, let $H_1' = \langle x_1, x_6, x_2, x_3 \rangle$, $H_2' = \langle x_1, x_6, x_3, x_4 \rangle$, $H_3' = \langle x_1, x_6, x_4, x_5 \rangle$, $H_4' = \langle x_1, x_2, x_3, x_4 \rangle$, $H_5' = \langle x_1, x_2, x_4, x_5 \rangle$, $H_6' = \langle x_1, x_2, x_5, x_6 \rangle$, $H_7' = \langle x_2, x_3, x_4, x_5 \rangle$, $H_8' = \langle x_2, x_3, x_5, x_6 \rangle$, $H_9' = \langle x_3, x_4, x_5, x_6 \rangle$, and note that $\mathcal{F}(P_6) = \{H_i' \cap P | i = 1, \ldots, 9\}$.

Since $E^\alpha = [x_1, x_6]$ and $F_6 = [x_1, x_6, x_2, x_5]$, it follows that $H_i' \cap P_6 \notin \mathcal{F}(P)$ for $i = 1, 2, 3$. It is clear that if $V^\eta \neq \emptyset$ for some $\eta \neq \alpha$, then $|\mathcal{F}(P_6) \cap \mathcal{F}(P)| \leq 4$. It is straightforward but tedious to check the following:

Remark 1. *If there are $k \geq 2$ types of vertices with respect to P_6, then*

$$|\mathcal{F}(P_6) \cap \mathcal{F}(P)| \leq 6 - k.$$

We consider $\{x_7, \ldots, x_n\} = V^\alpha \cup V^\beta \cup V^\delta \cup V^\theta \cup V^\mu \cup V^\lambda$, the vertex type that is sewn last and facets of P that contain a last sewn type vertex. Then we consider the vertex type that is sewn second last and the facets of P that contain a vertex of second last sewn type but not a vertex of last sewn type. We reiterate this process as often as necessary. Lemma 4.7 states that if $O \in \operatorname{int} P_6$, then O is separated from any such class of facets by one of at most three hyperplanes.

Lemma 4.7. *Let $O \in \operatorname{int} P_6$, $V^\eta \neq \emptyset$ and $E_{w-1} = E^\eta \in \mathcal{U}'(P_6)$. Let $\widetilde{F} \in \mathcal{F}(P)$ such that $\widetilde{F} \cap V^\eta \neq \emptyset$ and $\widetilde{F} \cap V^\tau = \emptyset$ for all $E^\tau \in \mathcal{U}'(P_6)$ such that $E^\tau = E_{z-1}$ and $x_w < x_z$. Then O is separated from any such $\widetilde{F}$ by one of three hyperplanes spanned by the vertices of P; moreover, at least one of the three separating hyperplanes may be chosen from $\{H'_i \mid i = 1, \dots, 9\} = \mathcal{H}'$.*

Proof. Let $E_{w-1} = [x_a, x_b]$ and assume the $(P_{w-1}/E_{w-1}, y_i)$-configuration. Then $P_6 = [x_a, x_b, y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}]$ with $1 \leq i_1 < i_2 < i_3 < i_4 \leq w-3$; cf. Figure 9.

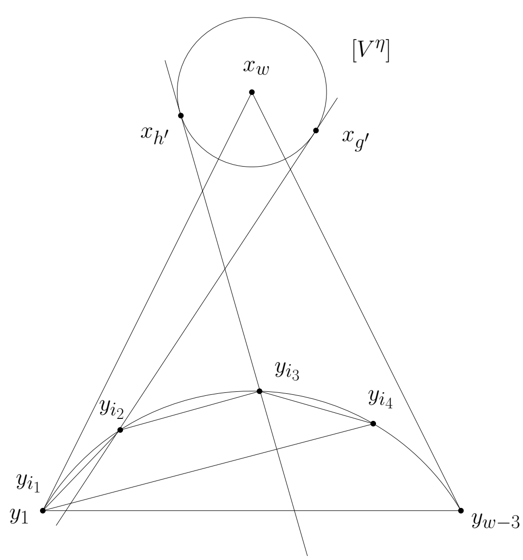


FIGURE 9. P_{w-1}/E_{w-1}

We use the same notation as in the proof of Lemma 4.4 for $k = 6$ and $m = w-1$. We recall that three hyperplanes do not suffice to separate $O \in \operatorname{int} P_6 \subseteq \operatorname{int} P_{w-1}$ from any such $\widetilde{F}$ only if $\widetilde{F} \cap V^\psi_{w-1} \neq \emptyset$ with $E^\psi \in \mathcal{U}(P_{w-1})$ and either $E^\psi = [y_l, y_{l+1}]$ or $E^\psi = [x_a, y_1]$ (for example); cf. Figures 2 and 3.

If $1 < i_1 < i_1 + 1 < i_2 < i_2 + 1 < i_3 < i_3 + 1 < i_4 < w - 3$, then any such E^ψ is separated from O by one of H_1, H_2 and H_3. In this case $1 < i_j \leq l < l + 1 \leq i_{j+1}$ for some $1 \leq j \leq 3$. Then there exist $x_{g'}, x_{h'} \in V^\eta$ such that H_1, H_2 and H_3 can be replaced by the hyperplanes $\langle E_{w-1}, y_{i_j}, x_{g'} \rangle$, $\langle E_{w-1}, y_{i_j}, y_{i_{j+1}} \rangle$ and $\langle E_{w-1}, y_{i_{j+1}}, x_{h'} \rangle$, respectively. We note that the hyperplane $\langle E_{w-1}, y_{i_j}, y_{i_{j+1}} \rangle$ is listed in $\mathcal{H}'$. To simplify notation we rename $H_1 = \langle E_{w-1}, y_{i_j}, x_{g'} \rangle$ and $H_3 = \langle E_{w-1}, y_{i_{j+1}}, x_{h'} \rangle$.

If $E^\psi = [y_l, y_{l+1}] = [y_{i_j}, y_{i_{j+1}}]$ and $y_{l+1} = y_{i_4}$ or if $E^\psi = [x_a, y_1]$ and $y_1 = y_{i_1}$, then $E^\psi \in \mathcal{U}'(P_6)$, $E^\psi = E_{z-1}$ for some $x_z > x_w$, and hence, $\widetilde{F} \cap V^\psi = \emptyset$. □

Remark 2. *We note that in the case when $i_1 = 1$ and $i_4 = w - 3$, the hyperplanes $\langle E_{w-1}, y_{i_1}, y_{i_2} \rangle$, $\langle E_{w-1}, y_{i_2}, y_{i_3} \rangle$ and $\langle E_{w-1}, y_{i_3}, y_{i_4} \rangle$ have the same separation properties as H_1, $\langle E_{w-1}, y_{i_j}, y_{i_{j+1}} \rangle$ and H_3 regardless of the position of O in P_6. Thus,* *they can be used in place of H_1, $\langle E_{w-1}, y_{i_j}, y_{i_{j+1}}\rangle$ and H_3. Note also that they are all listed in $\mathcal{H}'$.*

5. Proof of the main theorem

Theorem 5.1 (Separation Theorem)**.** *Let $O \in \operatorname{int} P_6$. Then O is separated from any facet of P by one of $s(O)$ hyperplanes determined by the vertices of P, and $s(O) \leq 16$.*

Proof. We assume the notation introduced in Section 4 and used in Lemma 4.7. Regarding Lemma 4.7, we simplify by referring to the $\widetilde{F}$ as *η-facets* and to the three separating hyperplanes as *η-separators*. Let $F \in \mathcal{F}(P)$.

We note that $x_7 \in V^\alpha$ and that from P_6/E_6: the α-separators are H_1', H_2' and H_3'. It is noteworthy that H_2' separates O from precisely those α-facets that are disjoint from $\{x_2, x_5\}$.

We may now assume that there are $k \geq 2$ types of vertices. Then Remark 1 and Lemma 4.7 yield that $s(O) \leq (6-k) + 3k$ and we may assume also that $k = 6$. We show that among the eighteen separating hyperplanes yielded by Lemma 4.7, there are at least two distinct ones in $\mathcal{H}'$ such that each is either redundant (separates O from facets that are separated from O by some other hyperplane) or used twice. We do this considering the order in which the six vertex types are sewn.

Case 1. $x_s < x_t, x_u, x_v, x_r$.

We assume the $(P_{s-1}/E_{s-1}, y_i)$-configuration with

$$F_s = [E^\delta, y_1, y_{s-3}] = [x_2, x_3, y_1, y_{s-3}]$$

and note that $\{E^\beta = [x_1, x_2], E^\theta = [x_3, x_4]\} \subset \mathcal{U}(P_{s-1}) \cap \mathcal{U}(P_s)$ yield that $\{y_1, y_{s-3}\} = \{x_1, x_4\}$ and that the δ-separators are H_1' (a second use), H_7' and H_8'.

If $x_t < x_u$, then we assume the $(P_{t-1}/E_{t-1}, z_i)$-configuration with

$$F_{t-1} = [E^\theta, z_1, z_{t-3}] = [x_3, x_4, z_1, z_{t-3}],$$

and note that $E_{u-1} = E^\mu = [x_4, x_5] \in \mathcal{U}(P_{t-1}) \cap \mathcal{U}(P_t)$ yields that $x_5 \in \{z_1, z_{t-3}\}$. Let $x_5 = z_1$, say. Then $(x_6, x_1, x_2) = (z_i, z_j, z_l)$ with $1 < i < j < l < t-3$ or $1 < l < j < i < t-3$.

With $1 < i < j < l < t-3$ and depending upon the location (cf. Figure 10), we obtain that the θ-separators are either H_9', H_2' (a second use) and H_2^θ, or H_1^θ, H_4' and H_3^θ. In the latter case, H_1^θ separates O from α-facets that are disjoint from $\{x_2, x_5\}$; that is, H_2' is redundant.

If $1 < l < j < i < t-3$ then we obtain by a similar argument that H_2' is again either redundant or used twice.

If $x_u < x_t$ then we argue as above with the $(P_{u-1}/E_{u-1}, w_i)$-configuration, $F_{u-1} = [E^\mu, w_1, w_{u-3}] = [x_4, x_5, w_1, w_{u-3}]$, $x_3 = w_1$, and obtain that H_3' is either a μ-separator (a second use) or it is rendered redundant by a μ-separator.

With respect to $\mathcal{U}'(P_6)$; it follows that with respect to $E^\alpha = [x_6, x_1]$, there is a symmetry between $E^\delta = E_{s-1}$ and $E^\mu = E_{u-1}$. Thus, with relabeling, the argument above also handles the case $x_u < x_s, x_t, x_v, x_r$.

Case 2. $x_t < x_r, x_s, x_u, x_v$.

From the $(P_{t-1}/E_{t-1}, z_i)$-configuration, we obtain that (cf. Figure 10) $\{z_1, z_{t-3}\} = \{x_2, x_5\}$ and the θ-separators are H_9', H_2' (a second use) and H_4'.

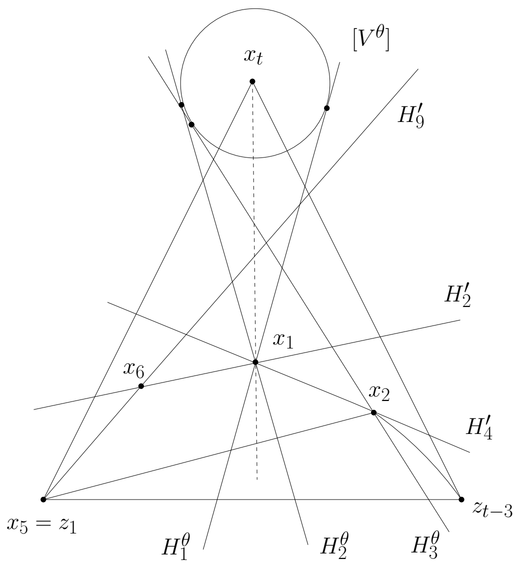


FIGURE 10. $P_{t-1}/[x_3, x_4]$

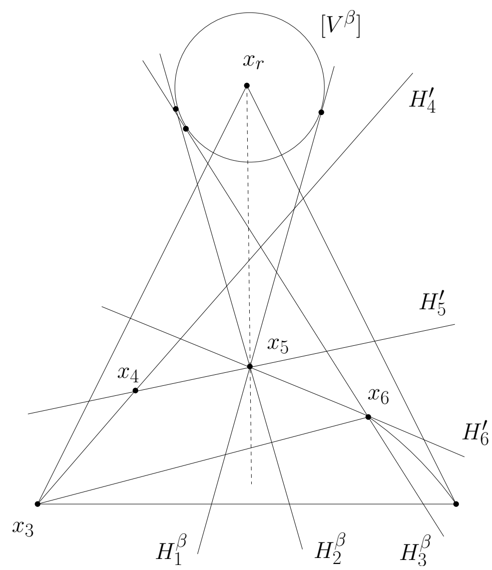


FIGURE 11. $P_{r-1}/[x_1, x_2]$

With the method of argument now evident, it is easy to check that $x_r < x_s$ yields that either H_1' is a β-separator or it is rendered redundant by a β-separator, or H_4' is a β-separator or it is rendered redundant by a β-separator. If $x_s < x_r$, then we obtain also that either H_1' or H_4' is redundant or used twice.

Case 3. $x_r < x_s, x_t, x_u, x_v$.

From a $(P_{r-1}/[x_1,x_2],w_i)$-configuration, we obtain (cf. Figure 11) that the β-separators are either H'_4, H'_5 and H^β_2 or H^β_1, H'_6 and H^β_3.

If $x_t < x_s, x_u$, then the θ-separators are H'_2 (a second use), H'_4 and H'_9. Thus, H'_4 is either used a second time or rendered redundant by H^β_1.

If $x_u < x_t, x_v$, then the μ-separators are H'_3 (a second use), H'_5 and H'_7, and H'_5 is either used a second time or rendered redundant by H^β_1.

We may now assume that neither x_t nor x_u is the third type of vertex sewn. In $\mathcal{U}'(P_6)$, we have that E_{s-1} and E_{v-1} are symmetric with respect to $E^\alpha \cup E^\beta$ and thus, we may assume without loss of generality that $x_v < x_s, x_t, x_u$. Then from above, we have that either $x_u < x_s, x_t$ or $x_s < x_u, x_t$.

Let $x_u < x_s, x_t$. Then the μ-separators are either H'_7, H'_5 and H^μ_2 or H^μ_1, H'_3 and H^μ_3. Hence, H'_3 is either used a second time or rendered redundant by H^μ_2. Finally, similar arguments yield that if $x_t < x_s$ then H'_2 is either redundant or used a second time, and if $x_s < x_t$ then H'_1 is either redundant or used a second time.

Let $x_s < x_u, x_t$. Then H'_1 is either redundant or used twice. From $x_t < x_u$ ($x_u < x_t$), we obtain that H'_2 (H'_3) is either redundant or used twice.

Lastly, we remark that with relabeling, the argument above also handles the case $x_v < x_r, x_s, x_t, x_u$. □

Theorem 5.2. *Let* $7 \le u \le n$ *and* $O \in (\operatorname{int} P_u) \cap (\operatorname{bd} P_{u-1})$. *Then* $s(O) \le 16$.

Proof. Let $F \in \mathcal{F}(P)$, $x_u \in V^\beta_{u-1}$, $E_{u-1} = [x_a, x_b]$ and consider the $(P_{u-1}/E_{u-1}, y_i)$-configuration.

Case 1. $O \in \operatorname{relint}[x_a, x_b, y_k, y_{k+1}]$ *for some* $2 \le k \le u-5$.

Let $H_0 = \langle x_a, x_b, y_k, y_{k+1}\rangle$ and H^+_0 and H^-_0 denote the two closed half-spaces determined by H_0. We assume that $x_u \in H^+_0$. If $[y_k, y_{k+1}] \in \mathcal{U}(P_{u-1})$, then let $E^\lambda = [y_k, y_{k+1}]$.

We note that by Lemma 4.2, O is separated from any F such that $F \subset H^-_0$ by one of at most three hyperplanes, and hence, O is separated from any F such that either $F \subset H^-_0$ or $F \subset H^+_0$ by one of at most six hyperplanes. By Lemma 4.4, O is separated from any F such that $F \cap V^\lambda_{u-1} \neq \emptyset$ by one of at most five hyperplanes.

We now assume that $F \cap V^\lambda_{u-1} = \emptyset$, $F \not\subset H^-_0$ and $F \not\subset H^+_0$. Note that any facet containing $[y_k, y_{k+1}]$ and a β type vertex is contained in H^+_0. Thus $|F \cap \{y_k, y_{k+1}\}| \le 1$ and O is separated from any such F by one of $H_1 = \langle x_a, x_b, y_k, x_g\rangle$ and $H_2 = \langle x_a, x_b, y_{k+1}, x_h\rangle$ with $\{x_g, x_h\} \subset V^\beta_{u-1}$ and H_1 and H_2 are supporting hyperplanes of $[V^\beta_{u-1}]$.

In summary, $s(O) \le 13$.

Case 2. $O \in \operatorname{relint}[x_a, x_b, y_1, y_2]$.

Let $H_0 = \langle x_a, x_b, y_1, y_2\rangle$ and $E^\lambda = [y_1, y_2]$ in case $[y_1, y_2] \in \mathcal{U}(P_{u-1})$. We note that if y_1 is a vertex of a second universal edge of P_{u-1}, then it is either $[y_1, x_a]$ or $[y_1, x_b]$. We may assume that if such a universal edge exists, then it is $E^\delta = [y_1, x_b]$.

Remark 3. *Let* $y_1 \in F$, $F \cap V^\beta_{u-1} \neq \emptyset$ *and* $F \cap (V^\lambda_{u-1} \cup V^\delta_{u-1}) = \emptyset$. *Then* $\mathcal{V}(F) \subset \{x_a, x_b, y_1, y_2\} \cup V^\beta_{u-1}$.

Proof of Remark 3. If $F = [y_1, x_w, x_z, \tilde{x}]$ with $x_w \in V^\beta_{u-1}$ and $x_z \in V^\theta_{u-1}$, $\theta \neq \beta$, and either $F \in \mathcal{F}(P_w)$ or $F \in \mathcal{F}(P_z)$, then the deletion process, $\{y_1\} = E^\delta \cap E^\lambda$

and $\{\theta, \beta\} \cap \{\lambda, \delta\} = \emptyset$ yield that $E^\beta \cap E^\theta \neq \emptyset$, that is, $E^\theta = [x_a, y_{u-3}]$ and $[y_1, x_u, x_a, y_{u-3}] \in \mathcal{F}(P_u)$. By Lemma 2.1, we have a contradiction. □

Let $F \cap (V^\lambda_{u-1} \cup V^\delta_{u-1}) = \emptyset$. Then O is separated from any such F by one of at most seven hyperplanes for the following reasons. If $y_1 \notin F$, then H_2 suffices (see Figure 12 with $k = 1$). If $y_1 \in F$ and $F \cap V^\beta_{u-1} = \emptyset$, then $F \subset H_0^-$. If $y_1 \in F$ and $F \cap V^\beta_{u-1} \neq \emptyset$, then $F \subset H_0^+$ by Remark 3 above.

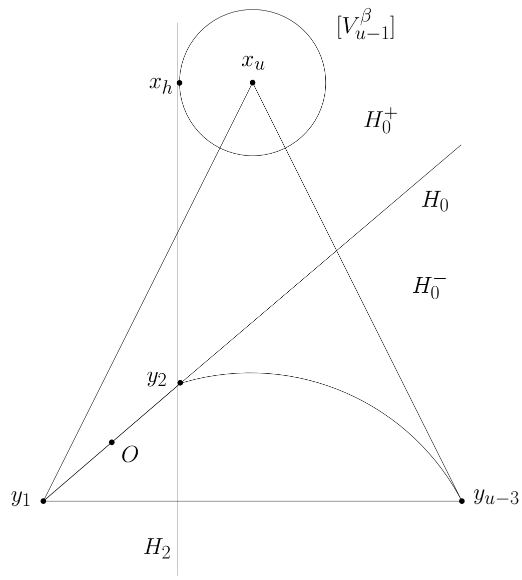


FIGURE 12. $P_{u-1}/[x_a, x_b]$

Let $F \cap (V^\lambda_{u-1} \cup V^\delta_{u-1}) \neq \emptyset$. If V^λ_{u-1} or V^δ_{u-1} is empty, then five hyperplanes suffice to separate O from any such F by Lemma 4.4 and $s(O) \leq 7 + 5$. Let $V^\lambda_{u-1} \neq \emptyset \neq V^\delta_{u-1}$ with $E_{r-1} = E^\delta$ and $E_{q-1} = E^\lambda$.

Let $r < q$ and consider the $(P_{r-1}/E_{r-1}, z_i)$-configuration. Since $[y_1, y_2] \in \mathcal{U}(P_{r-1}) \cap \mathcal{U}(P_r)$, we have that $y_2 \in \{z_1, z_{r-3}\}$. Let $y_2 = z_1$. Since $r > u \geq 7$ and $[x_a, x_b, y_1, y_2] = [E_{r-1}, z_1, x_a] \notin \mathcal{F}(P_{r-1})$, it follows that $x_a \notin \{z_2, z_{r-3}\}$.

We recall that H_0 strictly separates x_u from $u-5$ vertices of P_{u-1}. Hence, if H_0 separates x_u and x_r, then $u \geq 7$ yields that $x_a \neq z_3$ as well. Now, O is separated from any F such that $F \cap V^\delta_{u-1} \neq \emptyset$ and $F \cap V^\lambda_{u-1} \neq \emptyset$ by one of at most three hyperplanes (cf. Figure 13), and hence $s(O) \leq 7 + 3 + 5$ by Lemma 4.4.

Let x_u and x_r be on the same side of H_0. We consider the $(P_{q-1}/E_{q-1}, w_i)$-configuration with $F_{q-1} = [E_{q-1}, w_1, w_{q-3}] = [y_1, y_2, w_1, w_{q-3}]$. We note on the one hand that x_q and at least two elements from $\{w_2, \ldots, w_{q-4}\}$ are contained in the same open half-space determined by H_0. On the other hand, none of $[x_a, y_1]$,$[x_a, y_2]$,$[x_b, y_1]$ and $[x_b, y_2]$ is in $\mathcal{U}(P_{q-1})$. Hence O is separated from any F such that $F \cap V^\lambda_{u-1} \neq \emptyset$ by one of at most four hyperplanes (cf. Figure 14), and so $s(O) \leq 7 + 4 + 5$.

We argue similarly if $q < r$.

Case 3. *$O \in [x_a, x_b, y_k]$ for some $2 \leq k \leq u - 4$.*

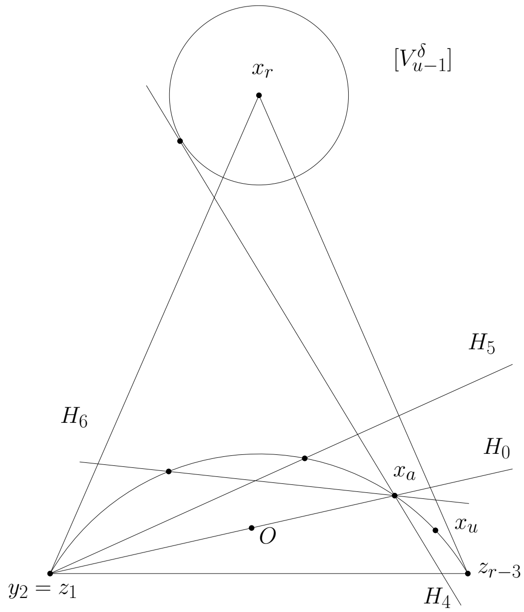


FIGURE 13. $P_{r-1}/[x_b, y_1]$

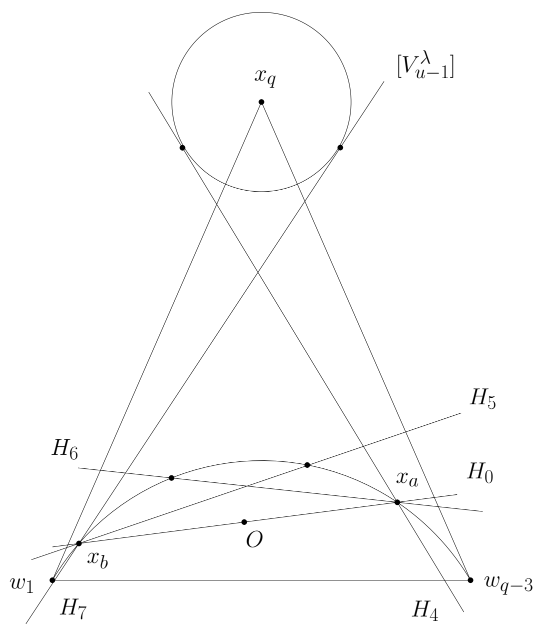


FIGURE 14. $P_{q-1}/[y_1, y_2]$

By symmetry, we may assume that $2 \leq k \leq u-5$. If $[y_k, y_{k+1}] \in \mathcal{U}(P_{u-1})$, then let $E^{\lambda} = [y_k, y_{k+1}]$ and $H_0 = \langle x_a, x_b, y_k, y_{k+1} \rangle$.

If $F \cap (V_{u-1}^{\beta} \cup V_{u-1}^{\lambda}) = \emptyset$, then $F \subset H_0^-$, $O \in \operatorname{bd} H_0^-$ and we apply Lemma 4.2. If $F \cap V_{u-1}^{\lambda} \neq \emptyset$, then we apply Lemma 4.4. If $F \cap V_{u-1}^{\lambda} = \emptyset$ and $F \cap V_{u-1}^{\beta} \neq \emptyset$, then F is contained in a closed half-space that is bounded by H_4 or H_5 (cf. Figure 15) with O on its boundary, and we apply Lemma 4.2 twice. Thus, $s(O) \leq 3 + 5 + 6$.

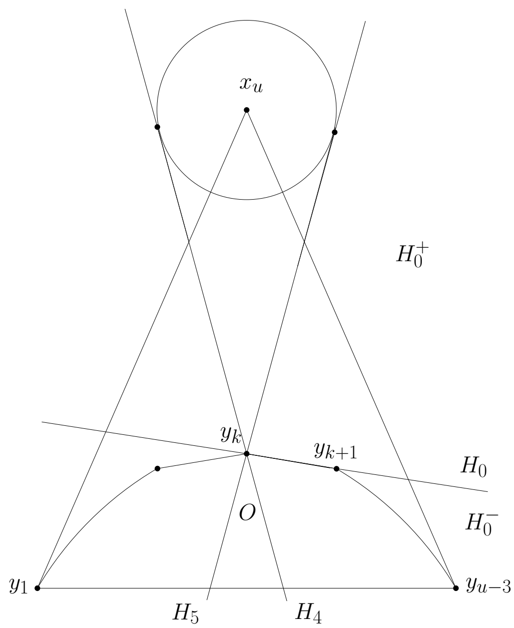


FIGURE 15. $P_{u-1}/[x_a, x_b]$

□

Theorem 5.3. *Let $7 \leq u \leq n$ and $O \in (\operatorname{int} P_u) \setminus P_{u-1}$. Then $s(O) \leq 16$.*

Proof. Let $x_u \in V_{u-1}^{\beta}$, $E_{u-1} = [x_a, x_b]$, $\widetilde{H}_i = \langle x_a, x_b, x_u, y_i \rangle$ for $i = 1, \ldots, u-3$, and $E^{\alpha} = [x_u, x_a]$ and $E^{\eta} = [x_u, x_b]$ in $\mathcal{U}(P_u)$.

We note that $V_{u-1}^{\beta} = \{x_u\} \cup V_u^{\alpha} \cup V_u^{\eta}$, and set $E_s = E^{\alpha}$ in the case $V_u^{\alpha} \neq \emptyset$, and $E_t = E^{\eta}$ in the case $V_u^{\eta} \neq \emptyset$. Finally, let $F \in \mathcal{F}(P)$.

Our subsequent arguments will depend initially upon the location of O and then upon various properties of F.

Case 1. *Let $O \in \operatorname{int}[x_a, x_b, x_u, y_k, y_{k+1}]$ for some $2 \leq k \leq u-5$.*

If $[y_k, y_{k+1}] \in \mathcal{U}(P_u)$, then let $E^{\lambda} = [y_k, y_{k+1}]$; furthermore, let $E_r = E^{\lambda}$ in the case $V_u^{\lambda} \neq \emptyset$.

Remark 4.

(4.1) If $V_u^{\alpha} \neq \emptyset \neq V_u^{\lambda}$, then $F_s \neq [x_u, x_a, y_k, y_{k+1}] \neq F_r$ and hence, $[x_u, x_a, y_k, y_{k+1}] \notin \mathcal{F}(P_{s+1}) \cup \mathcal{F}(P_{r+1}) \cup \mathcal{F}(P)$.

(4.2) If $V_u^{\eta} \neq \emptyset \neq V_u^{\lambda}$, then $F_t \neq [x_u, x_b, y_k, y_{k+1}] \neq F_r$ and hence, $[x_u, x_b, y_k, y_{k+1}] \notin \mathcal{F}(P_{t+1}) \cup \mathcal{F}(P_{r+1}) \cup \mathcal{F}(P)$.

(4.3) No $F' \in \mathcal{F}(P_u)$ contains $[x_u, y_1, y_{u-3}]$, and hence, no $F'' \in \mathcal{F}(P)$ contains $[x_u, y_1, y_{u-3}]$.

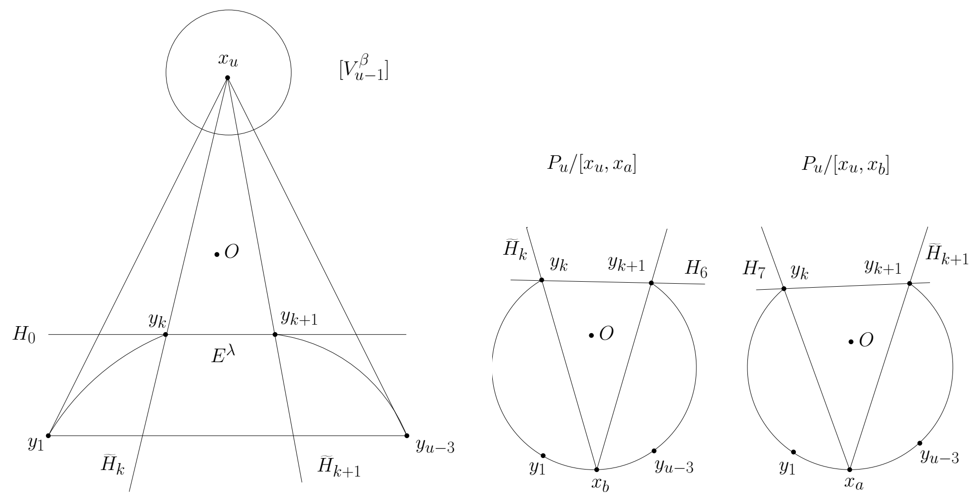


FIGURE 16. $P_{u-1}/[x_a, x_b]$, and $P_u/[x_u, x_a]$ and $P_u/[x_u, x_b]$

(4.4) If $F \in \mathcal{F}(P) \setminus \mathcal{F}(P_u)$, $F \cap V_{u-1}^{\beta} = \{x_u\}$ and $F \cap V_u^{\lambda} = \emptyset$, then $[y_k, y_{k+1}] \not\subset F$.

Proof of (4.4). Let $\hat{F} = [x_u, y_k, y_{k+1}, x_w] \in \mathcal{F}(P_w)$ and $x_w \in V_u^{\theta}$ with $\beta \neq \theta \neq \lambda$. We may assume that $[y_{k+1}, x_w] \in \mathcal{U}(P_w)$. Then $y_{k+1} \in E^{\theta} \in \mathcal{U}(P_u)$ and E^{θ} is disjoint from E^{α} or E^{η}. We may assume that $E^{\theta} \cap E^{\alpha} = \emptyset$. Then

$$\{[x_u, x_a, E^{\theta}], [x_u, x_a, y_k, y_{k+1}], [x_u, x_a, y_{k+1}, y_{k+2}]\} \subset \mathcal{F}(P_u)$$

yields that $E^{\theta} = [y_{k+1}, y_{k+2}]$. From the deletion process applied to $\hat{F}$, it follows that $[x_u, y_k, y_{k+1}, y_{k+2}] \in \mathcal{F}(P_u)$; a contradiction. □

Let $V_u^{\lambda} = \emptyset$. Then O is separated from any F by

- H_0 in the case $F \cap V_{u-1}^{\beta} = \emptyset$, and
- one of ten hyperplanes spanned by the vertices of P in the case $F \cap (V_u^{\alpha} \cup V_u^{\eta}) \neq \emptyset$ (cf. Lemma 4.4).

Let $F \cap V_{u-1}^{\beta} = \{x_u\}$. If $[y_k, y_{k+1}] \not\subset F$, then O is separated from any such F by one of $\widetilde{H}_k$ and $\widetilde{H}_{k+1}$ by Lemma 4.1 and Lemma 3.3. If $[y_k, y_{k+1}] \subset F$, then $F \in \mathcal{F}(P_u)$ by (4.4) and O is separated from any such F by one of H_6 and H_7.

In summary, $s(O) \leq 1 + 10 + 2 + 2$.

Let $V_u^{\lambda} \neq \emptyset$. Then O is separated from any F such that $F \cap V_u^{\lambda} \neq \emptyset$ by one of at most five hyperplanes by Lemma 4.4.

Henceforth, let $F \cap V_u^{\lambda} = \emptyset$.

If $F \cap V_{u-1}^{\beta} = \emptyset$, then H_0 suffices to separate O from any such F.

Let $F \cap V_{u-1}^{\beta} \neq \emptyset$ and we consider two cases.

If one of V_u^{α} or V_u^{η} is empty, we may assume that $V_u^{\eta} = \emptyset$. Then O is separated from any F such that $F \cap V_u^{\alpha} \neq \emptyset$ by one of at most five hyperplanes by Lemma 4.4. Finally, let $F \cap V_{u-1}^{\beta} = \{x_u\}$. Then O is separated from any such F by one of $\widetilde{H}_k$, $\widetilde{H}_{k+1}$, and H_7 (applying (4.1)) except if $[x_b, y_1] = E^{\theta} \in \mathcal{U}(P_{u-1})$, $[x_b, y_{u-3}] = E^{\delta} \in \mathcal{U}(P_{u-1})$ and $F \cap V_u^{\theta} \neq \emptyset \neq F \cap V_u^{\delta}$. Let $F = [x_u, x_v, \widetilde{x}, x_z]$ with $x_v \in V_u^{\theta}$, $x_z \in V_u^{\delta}$, and, say, $v < z$. Then $u < v < z$ and Lemma 3.2 yield that $E^\theta \cap E^\delta \neq \emptyset$; a contradiction. In summary, $s(O) \leq 5+1+5+3$.

Let $V_u^\alpha \neq \emptyset \neq V_u^\eta$ with $s < t$, say. Then O is separated from any F such that $F \cap V_u^\eta \neq \emptyset$ by one of five hyperplanes, and we consider P_s/E_s with the $(P_s/E_s, z_i)$-configuration. Then $E_t = E^\eta \in \mathcal{U}(P_s) \cap \mathcal{U}(P_{s+1})$ yields that, say, $(x_b, y_k, y_{k+1}) = (z_1, z_i, z_j)$ with $i < j$. Then O is separated from any F such that $F \cap V_u^\alpha \neq \emptyset$ by one of $\widetilde{H}_k$, H_6 and a hyperplane through $[x_a, x_u, y_{k+1}]$ that supports $[V_u^\eta]$.

Finally, let $F \cap V_{u-1}^\beta = \{x_u\}$. Then (4.1) and (4.2) yield that O is separated from any such F by one of $\widetilde{H}_k$ or $\widetilde{H}_{k+1}$. In summary, $s(O) \leq 5+1+5+3+2$.

Case 2. *Let* $O \in \operatorname{int}[x_a, x_b, x_u, y_1, y_2]$.

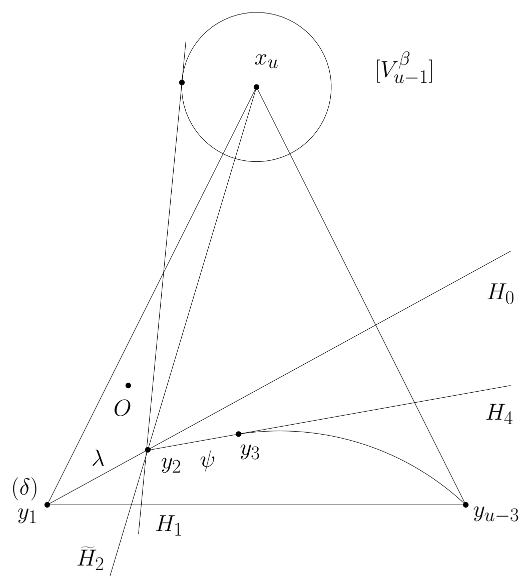
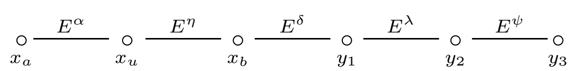


FIGURE 17. $P_{u-1}/[x_a, x_b]$

Let us consider the $(P_{u-1}/E_{u-1}, y_i)$-configuration with $E^\alpha = E_s$, $E^\eta = E_t$, $E\delta = E_r$, $E^\lambda = E_q$ and $E^\psi = E_w$ in case these various edge and vertex types exist.



Remark 5. *Let* $F' \in \mathcal{F}(P)$ *such that* $y_1 \in F'$, $F' \cap V_{u-1}^\beta \neq \emptyset$ *and* $F' \cap (V_u^\delta \cup V_u^\lambda) = \emptyset$.

(5.1) $\mathcal{V}(F') \subset V_{u-1}^\beta \cup \{x_a, x_b, y_1, y_2\}$.

Proof of (5.1). If $F' \cap V_{u-1}^\theta \neq \emptyset$ for some $\theta \neq \beta$, then $\{\beta, \theta\} \cap \{\delta, \lambda\} = \emptyset$, $\{y_1\} = E^\delta \cap E^\lambda$ and the deletion process yield that $E^\beta \cap E^\theta \neq \emptyset$. Then $E^\beta = [x_a, x_b]$, $E^\delta = [x_b, y_1]$ and Lemma 2.1 yield that $x_a \in E^\theta$ and $[x_u, x_b, E^\theta] \in \mathcal{F}(P_u)$. Hence, $E^\theta = [x_a, y_{u-3}]$, $[E^\delta, E^\theta] \in \mathcal{F}(P_u)$ and we have a contradiction by Remark (4.3). $\square$

(5.2) *Let* $F' \in \mathcal{F}(P_u)$. *Then*
$F' = [x_u, x_a, y_1, y_2]$ *and* V_u^α *or* V_u^λ *is empty, or*
$F' = [x_u, x_b, y_1, y_2]$ *and* V_u^λ *or* V_u^η *is empty, or*
$F' = [x_u, x_a, x_b, y_1]$ *and* V_u^α *or* V_u^δ *is empty.*

(5.3) *O is separated from any F' such that $F' \cap (V_u^\alpha \cup \{x_u\}) \neq \emptyset$ ($F' \cap (V_u^\eta \cup \{x_u\}) \neq \emptyset$) by one of at most two hyperplanes spanned by the vertices of P and containing E^α (respectively E^η).*

Proof of (5.3). We note that $O \in \operatorname{int} P_s$, apply (5.1) and assume the $(P_s/E_s, z_i)$-configuration. Then either $y_1 = z_j$ for some $2 \leq j \leq u-3$ and $\langle E^\alpha, z_{j-1}, z_j \rangle$ and $\langle E^\alpha, z_j, z_{j+1} \rangle$ are the separating hyperplanes, or if $j = 1$ then $\langle E^\alpha, z_1, z_2 \rangle$ and if $j = s-2$ then $\langle E^\alpha, z_{s-3}, z_{s-2} \rangle$ suffices. We argue similarly for $O \in \operatorname{int} P_t$. □

(5.4) *Let* $F'' \in \mathcal{F}(P)$ *such that* $F'' \cap V_{u-1}^\beta = \{x_u\}$, $F'' \cap (V_u^\delta \cup V_u^\lambda) = \emptyset$ *and* $F'' \not\subset P_u$. *Then* $y_1 \notin F''$.

Proof of (5.4). If $[x_u, y_1, \tilde{x}, x_z] \in \mathcal{F}(P_z)$, $x_z \in V_u^\theta$ and $\theta \notin \{\alpha, \eta, \delta, \lambda\}$, then $[x_z, \tilde{x}] \in \mathcal{U}(P_z)$ and $[x_u, y_1, E^\theta] \in \mathcal{F}(P_u)$ by the deletion process. Then $E^\theta \in \mathcal{U}(P_u)$ and (5.2) yield that $\theta = \alpha$, a contradiction. □

(5.5) *Let* $F'' \in \mathcal{F}(P)$ *such that* $F'' \cap V_{u-1}^\beta = \{x_u\}$ *and* $F'' \cap V_u^\delta \neq \emptyset \neq F'' \cap V_u^\lambda$. *Then* $\{x_r, x_t\} \subset \{x_{q+2}, \ldots, x_n\}$.

Proof of (5.5). Let $F'' = [x_u, \tilde{x}_r, \tilde{x}_q, \tilde{x}]$ with $\tilde{x}_r \in V_u^\delta$ and $\tilde{x}_q \in V_u^\lambda$. By the deletion process, we may assume that $\tilde{x} = y_1$. Let $x_r < x_q$. Then we may assume also that $[x_u, \tilde{x}_r, x_{q+1}, y_1] \in \mathcal{F}(P_{q+1})$, $[x_u, x_{r+1}, y_1, y_2] \in \mathcal{F}(P_{r+1})$ and $[x_u, x_b, y_1, y_2] \in \mathcal{F}(P_r) \cap \mathcal{F}(P_u)$. If $x_t < x_r$, then $[x_u, x_b, y_1, y_2] \in \mathcal{F}(P_t) \cap \mathcal{F}(P_{t+1})$, $F_t = [x_u, x_b, y_1, y_2]$ and $[y_1, y_2] \notin \mathcal{U}(P_{t+1})$; a contradiction. If $x_r < x_t$, then $\{E_t, E_q\} \subset \mathcal{U}(P_{r+1})$, $[E_t, E_q] = [x_u, x_b, y_1, y_2] \in \mathcal{F}(P_{r+1})$, $F_r = [x_u, x_b, y_1, y_2]$ and

$$\{[x_{r+1}, x_b, y_1, y_2], [x_{r+1}, x_b, y_1, x_u]\} \subset \mathcal{F}(P_{r+1}).$$

Now, $[x_{r+1}, x_u, y_1, y_2] \in \mathcal{F}(P_{r+1})$ and $[x_{r+1}, y_1] \in \mathcal{U}(P_{r+1})$ yield that $|\mathcal{V}(P_{r+1})| = 5$ by Lemma 2.1; a contradiction.

Since $x_q < x_r$, we may assume from F'' that $[x_u, x_{r+1}, \tilde{x}_q, y_1] \in \mathcal{F}(P_{r+1})$, $[x_u, x_{q+1}, x_b, y_1] \in \mathcal{F}(P_{q+1})$ and $[x_u, x_b, y_1, y_2] \in \mathcal{F}(P_q) \cap \mathcal{F}(P_u)$. If $x_t < x_q$, then $[x_u, x_b, y_1, y_2] \in \mathcal{F}(P_t) \cap \mathcal{F}(P_{t+1})$, $F_t = [x_u, x_b, y_1, y_2]$ and $[y_1, y_2] \notin \mathcal{U}(P_{t+1})$; a contradiction. □

Case 2.1. *O and x_u are separated by H_1 and $O \notin H_1$, cf. Figure 17.*

We note that O is separated from any F such that $F \cap (V_u^\delta \cup V_u^\lambda) \neq \emptyset$ by one of ten hyperplanes by Lemma 4.4.

Let $F \cap (V_u^\delta \cup V_u^\lambda) = \emptyset$. Then O is separated from any F by

- H_1 in the case $y_1 \notin F$,
- H_0 in the case $F \cap V_{u-1}^\beta = \emptyset$, and
- one of at most four hyperplanes in the case $y_1 \in F$ and $F \cap V_{u-1}^\beta \neq \emptyset$ from (5.1) and (5.3).

In summary, $s(O) \leq 10 + 1 + 1 + 4 = 16$.

Case 2.2. *O and x_u are on the same side of H_1.*

i) Let $F \cap V_{u-1}^{\beta} = \emptyset$. We note that O is separated from F by H_4 in the case that $F \cap V_u^{\psi} = \emptyset$ or O and x_{w+1} are on opposite sides of H_4. Let $F \cap V_u^{\psi} \neq \emptyset$ and assume that O and x_{w+1} are on the same side of H_4. Consider the $(P_w/E_w, z_i)$-configuration. Then $(x_a, x_b) = (z_k, z_l)$ and we may assume $1 \leq k < l \leq w-2$. Since O, x_{w+1} and x_u are on the same side of $H_4 = \langle E_w, z_k, z_l \rangle$, it follows that $x_u = z_i$ for some $k < i < l$. Now, by Lemma 3.3, we have that

$$\{z_{k+1}, \dots, z_{l-1}\} = V_{u-1}^{\beta} \cap \{z_1, \dots, z_{w-2}\}.$$

Thus, O is separated from any F above by one of H_4, H_5 and H_6; cf. Figure 18.

In summary, three hyperplanes suffice.

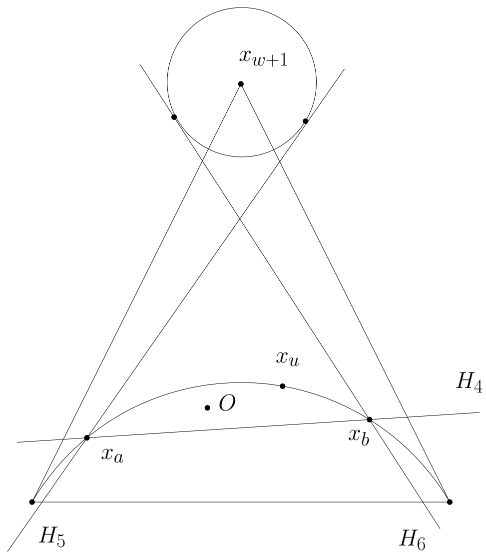


FIGURE 18. $P_w/[y_2, y_3]$

ii) Next, we consider F with the property that $F \cap V_{u-1}^{\beta} = \{x_u\}$. Then

- $\widetilde{H}_2$ separates O from F in the case $F \cap (V_u^{\delta} \cup V_u^{\lambda} \cup \{y_1\}) = \emptyset$,

and we argue below that

- one hyperplane separates O from F in the case that $F \cap V_u^{\delta} \neq \emptyset \neq F \cap V_u^{\lambda}$.

We let $\overline{H} = \langle x_u, x_b, y_1, y_2 \rangle$, and recall that $x_q < x_r, x_t$ by (5.5). Hence, $[x_u, x_b, y_1, y_2] \in \mathcal{F}(P_q) \setminus \{F_q\}$ and $\overline{H}$ strictly separates O and x_{q+1}. Since $\{x_u, y_1\} \subset \overline{H}$ and $\{y_1\} = E^{\lambda} \cap E^{\delta}$, it follows that $\overline{H}$ separates O from any such F in the case that $\overline{H}$ separates O and x_{r+1}. Let O and x_{r+1} be on the same side of $\overline{H}$. Then (cf. Figure 19) $x_u \in F$ yields that O is separated from any such $\widehat{F}$ by H, a hyperplane through $\langle x_b, y_1, x_u \rangle$ that supports $[V_u^{\delta}]$.

Now, we consider F such that $F \cap V_u^{\lambda} \neq \emptyset$ and $F \cap V_u^{\delta} = \emptyset$. We recall that $E^{\lambda} = E_q = [y_1, y_2]$ and consider the $(P_q/E_q, z_i)$-configuration. Then $x_u = z_k$ and we may assume that $1 \leq k \leq q-3$. From Lemma 3.2 and the hypotheses, we obtain that $\mathcal{V}(F) \subset \mathcal{V}(P_u) \cup V_u^{\lambda}$. It is then clear that $\langle E_q, x_u, z_{k+1} \rangle$ separates O from any F such that $|F \cap (V_u^{\lambda} \cup \{y_1, y_2\})| = 3$.

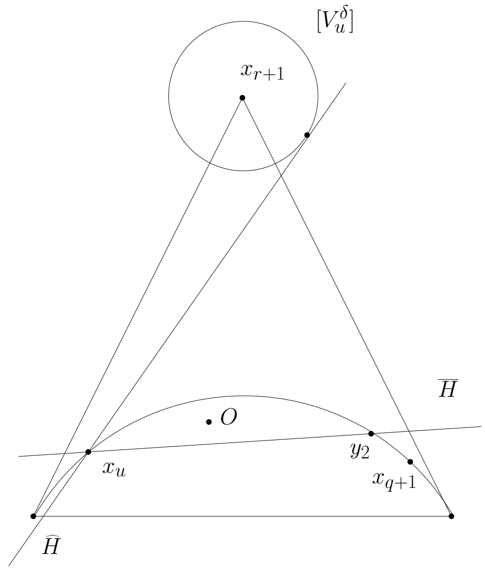


FIGURE 19. $P_r/[x_b, y_1]$

Let $F = [x_u, \widetilde{x}, x', x_v] \in \mathcal{F}(P_v)$, $x_v \in V_u^\lambda$, and $\widetilde{x} \in \mathcal{V}(P_u) \setminus \{y_1, y_2\}$. Then $[x', x_v] \in \mathcal{U}(P_v)$, $\widetilde{x} \notin E_{v-1}$ and the deletion process yield that $\widetilde{x} \in \{x_a, x_b\}$, $[x_u, \widetilde{x}, y_1, y_2] \in \mathcal{F}(P_u) \cap \mathcal{F}(P_q)$ and $\widetilde{x} \in \{z_{k-1}, z_{k+1}\}$. Thus, O is separated from F by $\langle E_q, x_u, \widetilde{x}\rangle$. We note that if $\widetilde{x} = x_a$ and $V_u^\alpha \neq \emptyset$, then $x_{s+1} > x_v$ and $F = F_s$. It is clear that if $x_{s+1} > x_v$ then $F \in \mathcal{F}(P_v)$ yields that $F \in \mathcal{F}(P_s) \cap \mathcal{F}(P_{s+1})$, and so, $F = F_s$. If $x_{s+1} < x_v$ then, as a simplification, we may assume $x' = x_p$ and $x_v > x_{s+1} > x_p$. Since $[x_u, x_a, E_{v-1}] \in \mathcal{F}(P_p) \cap \mathcal{F}(P_{v-1})$ by Deletion, it follows that $[x_u, x_a, E_{v-1}] \in \mathcal{F}(P_s) \cap \mathcal{F}(P_{s+1})$, $F_s = [x_u, x_a, E_{v-1}]$, $E_{v-1} \notin \mathcal{U}(P_{s+1})$ and $E_{v-1} \notin \mathcal{U}(P_{v-1})$; a contradiction. Similarly, if $\widetilde{x} = x_b$ then either $V_u^\eta = \emptyset$ or $x_{t+1} > x_v$ and $F = F_t$.

In summary, O is separated from any such F by $\langle E_q, x_u, z_{k-1}\rangle$ or $\langle E_q, x_u, z_{k+1}\rangle$ only if V_u^α or V_u^η is empty or $V_u^\alpha \neq \emptyset \neq V_u^\eta$ and $F \in \{F_s, F_t\}$.

If $V_u^\alpha \neq \emptyset \neq V_u^\eta$, then $F = F_s$ with the $(P_s/E_s, w_i)$-configuration implies that $x_b = w_j$ for some $1 < j < s-2$, and thus, $E^\eta = E_t = [x_u, x_b] \notin \mathcal{U}(P_{s+1})$ by Lemma 2.1 and $x_t < x_s$. On the other hand, $F = F_t$ yields that $x_s < x_t$ by a similar argument. This is a contradiction. Since F is now unique, we may assume that $\widetilde{x} = z_{k+1}$ and F is also separated from O by $\langle E_q, x_u, z_{k+1}\rangle$.

Next, we consider F such that $F \cap V_u^\delta \neq \emptyset$ and $V_u^\lambda = \emptyset$. We argue as above and obtain that, with the $(P_r/E_r, z_i)$-configuration and $x_u = z_k$ for some $1 \le k \le r-3$, O is separated from any F such that $|F \cap (V_u^\delta \cup \{x_b, y_1\})| = 3$ by $\langle E_r, x_u, z_{k+1}\rangle$. From $F \cap (V_u^\alpha \cup V_u^\eta \cup V_u^\lambda) = \emptyset$, it follows that if $F = [x_u, \widetilde{x}, x', x_v] \in \mathcal{F}(P_v)$, $\widetilde{x} \notin V_u^\delta \cup \{x_b, y_1\}$, and $\{x', x_v\} \cap V_u^\delta \neq \emptyset$, then $\widetilde{x} = x_a \in \{z_{k-1}, z_{k+1}\}$ (hence, we may assume that $\widetilde{x} = z_{k+1}$) then O is separated from F again by $\langle E_r, x_u, z_{k+1}\rangle$ and either $V_u^\alpha = \emptyset$ or $F = F_s$.

In summary, O is separated from any F such that F intersects exactly one of V_u^δ and V_u^λ by one of two hyperplanes in the case that $V_u^\alpha \neq \emptyset \neq V_u^\eta$, and by one of three hyperplanes otherwise.

Finally, let $F\cap(V_u^\delta\cup V_u^\lambda\cup\{y_1\})=\{y_1\}$. Then by (5.4) and (5.2), $F\in\mathcal{F}(P_u)$ and there are at most three such F and each is contained in $\{F_s,F_t,F_r,F_q\}$. From this it follows that if any of V_u^α, V_u^η, V_u^δ and V_u^λ is non-empty then there are at most two such F and at least one of them is empty. We observe that if only V_u^λ is empty and there are two such F, then they are $[x_u,x_a,y_1,y_2]=F_s$ (hence, $x_t<x_s$ from $E_t=[x_u,x_b]$) and $[x_u,x_b,y_1,y_2]=F_t=F_r$ (hence, $x_s<x_t$ from $E_s=[x_u,x_a]$); a contradiction.

From i) and the above, we know that O is separated from any F such that $F\cap(V_u^\alpha\cup V_u^\eta)=\emptyset$ by one of at most ten hyperplanes. Furthermore, ten are necessary only if V_u^α or V_u^η is empty. But in that case, O is separated from any F such that $F\cap(V_u^\alpha\cup V_u^\eta)\neq\emptyset$ by at most five hyperplanes by Lemma 4.4.

We assume that $V_u^\alpha\neq\emptyset\neq V_u^\eta$. Then i) and the above yield that O is separated from any F such that $F\cap(V_u^\alpha\cup V_u^\eta)=\emptyset$ by one of at most seven hyperplanes; furthermore, seven are necessary only if V_u^δ and V_u^λ are non-empty. Thus, by Lemma 4.4, we may assume that $V_u^\delta\neq\emptyset\neq V_u^\lambda$ and that there exists an $F''\in\mathcal{F}(P)$ such that $F''\cap V_{u-1}^\beta=\{x_u\}$ and $F''\cap V_u^\delta\neq\emptyset\neq F''\cap V_u^\lambda$. Thus, $x_r>x_q$ and $x_t>x_q$ by (5.5).

iii) Let $F\cap(V_u^\alpha\cup V_u^\eta)\neq\emptyset$ under the preceding assumptions. From the preceding, we need to verify that O is separated from any such F by one of at most nine hyperplanes.

Let $x_s<x_t,x_r$ (simply $s<t,r$) and $F\cap V_u^\alpha\neq\emptyset$. Then $O\in\operatorname{int}P_s$, $x_b\in F_s$ and with reference to Figure 20, it is easy to check from arguments as in Lemma 4.4 that O is separated from any such F by

- one of at most three hyperplanes in the case that O is separated from $[V_u^\alpha]$ by $\widehat{H}_1$ and $\widehat{H}_2$ ($x_q<x_r$ yields that $x_b\in F_q$ and so, $\widehat{H}_0$ separates x_{q+1} from x_b and O), and
- one of at most two hyperplanes in the case that O is separated from $[V_u^\alpha]$ by $\widetilde{H}_1$ and $\widehat{H}_3$, and O is separated by $\widetilde{H}_1$ from x_{t+1} and x_{r+1}.

Hence we may now assume that O is separated from $[V_u^\alpha]$ by $\widetilde{H}_1$ and $\widehat{H}_3$, and O is not separated from x_{t+1} and x_{r+1} by $\widetilde{H}_1$.

Let $s<r<t$. If $\widetilde{H}_1$ separates O from x_{t+1}, then O and x_{r+1} are on the same side of $\widetilde{H}_1$ and we argue as in the proof of Lemma 4.4 with the $(P_r/E_r,w_i)$-configuration and obtain that O is separated from any F such that $F\cap V_u^\alpha\neq\emptyset$ by one of $\widetilde{H}_1$, $\widehat{H}_3$ and a hyperplane through $\langle E_r,x_a\rangle$ that supports $[V_u^\delta]$. In the case that $\widetilde{H}_1$ separates O from x_{r+1}, we argue with the $(P_t/E_t,w_i)$-configuration and obtain that O is separated from any F such that $F\cap V_u^\alpha\neq\emptyset$ by one of $\widetilde{H}_1$, $\widehat{H}_3$ and two hyperplanes that support $[V_u^\eta]$, one through $\langle E_t,x_a\rangle$ and one through $\langle E_t,y_1\rangle$.

Finally, we assume that O, x_{r+1} and x_{t+1} are all on the same side of $\widetilde{H}_1$. Then we argue with the $(P_r/E_r,w_i)$-configuration and the $(P_t/E_t,z_i)$- configuration and obtain from $x_r<x_t$ that O is separated from any F such that $F\cap V_u^\alpha\neq\emptyset$ by one of $\widehat{H}_3$, one hyperplane through $\langle E_r,x_a\rangle$ that supports $[V_u^\delta]$ and two hyperplanes that support $[V_u^\eta]$ (through $\langle E_t,x_a\rangle$ and $\langle E_t,y_1\rangle$).

In summary, if $s<r<t$, then O is separated from any F such that $F\cap V_u^\alpha\neq\emptyset$ by one of at most four hyperplanes. A similar argument yields the same result in the case that $s<t<r$. Hence, $s(O)\leq 7+4+5=16$ by Lemma 4.4 applied to F such that $F\cap V_u^\eta\neq\emptyset$.

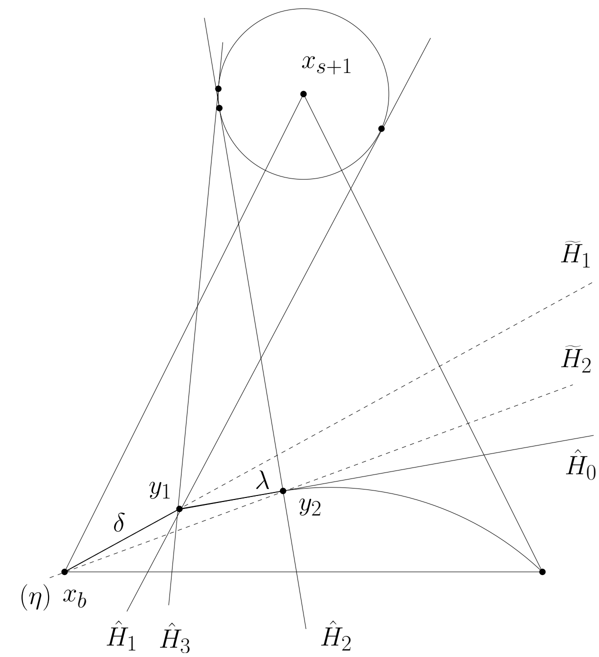


FIGURE 20. $P_s/E_s = [x_u, x_a]$

In the case that $x_r < x_s$, we obtain from $O \in \text{int}[x_a, x_b, x_u, y_1, y_2]$ that with the $(P_s/E_s, z_i)$-configuration: O is contained in a region bounded by $\langle E_s, z_i, z_j \rangle$, $\langle E_s, z_j, z_p \rangle$ and $\langle E_s, z_i, z_p \rangle$ with $1 \leq i < j < p \leq s-2$ and $\{z_i, z_j, z_p\} = \{x_b, y_1, y_2\}$. Then $[z_i, z_j] \notin \mathcal{U}(P_s)$ and $[z_j, z_p] \notin \mathcal{U}(P_s)$ from $q < r, s$, and we apply Corollary 4.5 to obtain $s(O) \leq 16$.

It remains to consider the case $t < s < r$ with the $(P_s/E_s, z_i)$-configuration and the $(P_r/E_r, w_i)$-configuration. The important fact is whether $\widetilde{H}_1$ separates O and x_{r+1}. If yes, then O is separated from any F such that $F \cap V_u^\alpha \neq \emptyset$ by one of at most three hyperplanes, each of which contains E_s. If not, then separation is by one of at most four hyperplanes, two containing E_s and the other two containing E_r.

Case 3. $O \in \text{relint}[x_a, x_b, x_u, y_k]$, $2 \leq k \leq u-4$.

Let $2 < k < u-4$ and $F \in \mathcal{F}(P)$. We show first that O is separated from any F by one of at most seven hyperplanes spanned by the vertices of P in the case that $F \cap (V_u^\alpha \cup V_u^\eta) \neq \emptyset$.

Clearly, we may assume that $V_u^\alpha \neq \emptyset \neq V_u^\eta$ by Lemma 4.4.

Recall that

$$\underset{x_a}{\circ} \overset{E_s}{\text{———}} \underset{x_u}{\circ} \overset{E_t}{\text{———}} \underset{x_b}{\circ} \overset{E_r}{\text{———}} \underset{y_1}{\circ} \overset{E_q}{\text{———}} \underset{y_2}{\circ}\ .$$

Let $s < t$. Then we note that $\mathcal{U}(P_t) \cap \{[x_a, x_u], [x_a, x_b]\} = \emptyset$. Since $2 < k < u-4$, we have also that $\mathcal{U}(P_t) \cap \{[y_k, x_u], [y_k, x_b]\} = \emptyset$, and the $(P_t/E_t, z_i)$-configuration with $\{x_a, y_k\} = \{z_i, z_p\}$ and $1 \leq i < i+2 < p \leq t-2$. Thus O is separated from any F such that $F \cap V_u^\eta \neq \emptyset$ by one of at most three hyperplanes by Corollary 4.6.

Since $2 < k < u-4$, a similar argument with the $(P_s/E_s, w_i)$-configuration yields that O is separated from any F such that $F \cap V_u^\alpha \neq \emptyset$ by one of at most four hyperplanes by Corollary 4.6. Clearly, the claim follows by a similar argument in the case that $t < s$.

Let $E^\theta = [y_{k-1}, y_k] = E_v$ and $E^\psi = [y_k, y_{k+1}] = E_w$ in case these edge and vertex types exist with respect to P_u.

Let $F \cap V_{u-1}^\beta = \emptyset$. Then O is separated from F by one of $H'_{k-1} = \langle x_a, x_b, y_{k-1}, y_k\rangle$ and $H'_{k+1} = \langle x_a, x_b, y_k, y_{k+1}\rangle$ in case $F \cap V_u^\theta = \emptyset$ or $F \cap V_u^\psi = \emptyset$ or H'_{k+1} separates O and x_{w+1} or H'_{k-1} separates O and x_{v+1}. In the remaining case, we are left to consider F such that $F \cap V_u^\theta \neq \emptyset \neq F \cap V_u^\psi$. We may assume that $v < w$ and that in the $(P_v/E_v, z_i)$-configuration, $(y_{k+1}, x_a, x_b) = (z_1, z_j, z_l)$ with $1 < j < l \leq v-2$. Then O is separated from any such F by a hyperplane through $\langle E_v, x_a\rangle$ that supports $[V_u^\theta]$.

Let $F \cap V_{u-1}^\beta = \{x_u\}$. Then $2 < k < u-4$ yields that $E^\beta \cap ([y_{k-1}, y_k], [y_k, y_{k+1}]) = \emptyset$, and either $F \cap V_u^\theta = \emptyset$ or $F \cap V_u^\psi = \emptyset$ by Lemma 3.2. Hence, F is contained in one of the closed half-spaces determined by $\widetilde{H}_k$ and O is separated from any such F by one of at most six hyperplanes by Lemma 4.2 (applied twice). In summary, $s(O) \leq 7 + 3 + 6$.

Let $k = 2$ and $F \in \mathcal{F}(P)$ and $V_u^\delta \neq \emptyset \neq V_u^\lambda$. We note again that O is separated from any F such that $F \cap V_{u-1}^\beta = \emptyset$ by one of at most three hyperplanes, and we claim that O is separated from any F such that $F \cap (V_u^\alpha \cup V_u^\eta) \neq \emptyset$ by one of at most seven hyperplanes.

Let $F \cap (V_u^\alpha \cup V_u^\eta) \neq \emptyset$ and $V_u^\alpha \neq \emptyset \neq V_u^\eta$. Let $s < t$. If $r < t$, then $\widetilde{H}_2$ separates y_1 and x_{r+1} from y_3 and y_{u-3}. Now we readily obtain from Corollary 4.6 that O is separated from any F such that $F \cap V_u^\eta \neq \emptyset$ by one of at most three hyperplanes. If $r > t$, then with the $(P_t/E_t, z_i)$-notation, we obtain that $y_1 \in \{z_1, z_{t-2}\}$. Since $\widetilde{H}_2 = \langle E_t, x_a, y_2\rangle$ separates y_1 and $\{y_3, \dots, y_{u-2}\}$, we obtain the result above again by Corollary 4.6.

It is clear from Corollary 4.6 that O is separated from any F such that $F \cap V_u^\alpha \neq \emptyset$ by one of at most four hyperplanes except possibly in the case that $s < q, r, t$ and the location of O in $\widetilde{H}_2$ as indicated in Figure 20.

From the above, we may assume that $F \cap V_u^\eta = \emptyset$. Then O is separated from any such F by one of at most four hyperplanes. Depending upon the location of O: either two for F such that $F \cap V_u^\lambda = \emptyset$ and two for F such that $F \cap V_u^\lambda \neq \emptyset$, or two for F such that $F \cap V_u^\delta = \emptyset$ and two for F such that $F \cap V_u^\delta \neq \emptyset$.

Let $t < s$. Then there is no $E \in \mathcal{U}(P_s)$ such that $|E \cap \{x_u, x_a\}| = 1 = |E \cap \{x_b, y_2\}|$. Thus, if $r < s$ or $q < s$, then we argue as in the preceding and obtain that O is separated from any F by

- one of at most three hyperplanes in the case that $F \cap V_u^\alpha \neq \emptyset$, and
- one of at most four hyperplanes in the case that $F \cap V_u^\alpha = \emptyset$ and $F \cap V_u^\eta \neq \emptyset$.

Finally, let $t < s < r, q$. Then with the $(P_s/E_s, w_i)$-configuration and $\{x_b, y_2\} = \{w_i, w_p\}$, we obtain that $|p - i| = 2$. We now apply Corollary 4.5 and obtain that O is separated from any F such that $F \cap V_u^\alpha \neq \emptyset$ by one of at most four hyperplanes.

From the $(P_t/E_t, z_i)$-configuration, it follows from $t < s < r$ that $\{x_a, y_i\} = \{z_1, z_{t-2}\}$. Let $x_a = z_1$ and $y_1 = z_{t-2}$. Then $y_2 = z_{t-3}$. Let $F \cap V_u^\alpha = \emptyset$. Then $F \cap V_u^\eta \neq \emptyset$. We now obtain, as above, that O is separated from any such F by one of at most three hyperplanes by Corollary 4.6.

Let $F \cap V_{u-1}^{\beta} = \{x_u\}$. Again, O is separated from any F such that $F \cap (V_u^{\delta} \cup V_u^{\lambda} \cup \{y_1\}) = \emptyset$ by one of at most three hyperplanes by Lemma 4.2.

Let $F \cap (V_u^{\delta} \cup V_u^{\lambda} \cup \{y_1\}) \neq \emptyset$. If $F \cap (V_u^{\delta} \cup V_u^{\lambda}) = \emptyset$, then $y_1 \in F$ and (5.1) imply $F \in \mathcal{F}(P_u)$, and (5.2) implies that $V_u^{\alpha}, V_u^{\eta}, V_u^{\delta}$ or V_u^{λ} is empty; a contradiction. Let $F \cap (V_u^{\delta} \cup V_u^{\lambda}) \neq \emptyset$. If $F \cap V_{u-1}^{\theta} \neq \emptyset$ for some $\theta \notin \{\beta, \delta, \lambda\}$ then Lemma 3.2 implies that $E^{\theta} \cap E^{\phi} \neq E^{\beta} \cap (E^{\theta} \cup E^{\psi})$ for some $\phi \in \{\delta, \lambda\}$. Since $E^{\theta} \cap E^{\phi} \neq \emptyset$ implies $\phi = \lambda$, and $E^{\beta} \cap (E^{\theta} \cup E^{\phi}) \neq \emptyset$ implies $\phi = \delta$, we have a contradiction. Thus, $F \subset Q = [V_u^{\delta} \cup V_u^{\lambda} \cup \{x_u, x_a, x_b, y_1, y_2\}]$, and O is separated from any such F by one of at most three hyperplanes by Lemma 4.2.

Let V_u^{δ} or V_u^{λ} be empty. We argue as above and again obtain that O is separated from F by

- one of at most three hyperplanes in each of the following cases: a) $F \cap V_{u-1}^{\beta} = \emptyset$, b) $F \cap V_{u-1}^{\beta} = \{x_u\}$ and $F \cap (V_u^{\delta} \cup V_u^{\lambda} \cup \{y_1\}) = \emptyset$, and c) $F \cap V_{u-1}^{\beta} = \{x_u\}$ and $F \cap (V_u^{\delta} \cup V_u^{\lambda} \cup \{y_1\}) \neq \emptyset$,
- one of at most seven hyperplanes in the case $F \cap (V_u^{\alpha} \cup V_u^{\eta}) \neq \emptyset$ and either $V_u^{\lambda} = \emptyset$ or $V_u^{\lambda} \neq \emptyset = V_u^{\delta}$ and $q < s$ or $q < t$, and
- one of at most eight hyperplanes in the case $F \cap (V_u^{\alpha} \cup V_u^{\eta}) \neq \emptyset$, $V_u^{\alpha} \neq \emptyset \neq V_u^{\eta}$, $V_u^{\lambda} \neq \emptyset = V_u^{\delta}$ and $q > s, t$.

Let $V_u^{\alpha} \neq \emptyset \neq V_u^{\eta}$, $V_u^{\lambda} \neq \emptyset = V_u^{\delta}$ and $q > s, t$. Let $F \in \mathcal{F}(P)$ such that $F \cap V_{u-1}^{\beta} = \{x_u\}$ and $F \cap (V_u^{\delta} \cup V_u^{\lambda} \cup \{y_1\}) \neq \emptyset$. We claim that O is separated from F by one hyperplane, and hence, $s(O) \leq 16$ in this case as well.

If there is an F such that $F \cap V_u^{\lambda} \neq \emptyset$ then (as noted above) $F \subset Q$ by Lemma 3.2, and $[x_u, \widetilde{x}, y_1, y_2] \in \mathcal{F}(P_u) \cap \mathcal{F}(P_q)$ for some $\widetilde{x} \in \{x_a, x_b\}$ by the Deletion process. Since $\widetilde{x} \in \{x_a, x_b\}$ implies that $q < s$ or $q < t$, it follows that $F \cap V_u^{\lambda} = \emptyset$ for all such F. Then $y_1 \in F$ and (5.1) imply that $F \in \mathcal{F}(P_u)$. Since none of V_u^{α}, V_u^{η} and V_u^{λ} is empty, it follows from (5.2) that $F = [x_u, x_a, x_b, y_1]$ is unique. $\square$

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