The general nilpotent operator system

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Abstract

In this paper we show that a consistent logical system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz-logic, which means that nilpotent logical systems are wider than we have earlier thought. Using more than one generator functions we examine three naturally derived negations in these systems. It is shown that the coincidence of the three negations leads back to a system which is isomorphic to Łukasiewicz-logic. Consistent nilpotent logical structures with three different negations are also provided.

1 Introduction

One of the most significant problems of fuzzy set theory is the proper choice of set-theoretic operations [23, 27]. Triangular norms and conorms have thoroughly been examined in the literature [16, 11, 10, 13]. The most well-characterized class of t-norms are the so-called representable t-norms. They are derived from the solution of the associative functional equation [1]. The two main types of representable t-norms are the strict and non-strict or nilpotent t-norms.

t-norms and t-conorms are often used as conjunctions and disjunctions in logical structures [12], [19]. Łukasiewicz fuzzy logic [14, 18, 20, 21] is the logic where the conjunction is the Łukasiewicz t-norm. It has been introduced for philosophical reasons by Łukasiewicz in [18] and it is among the most significant and widely examined non-classical logics.

The class of non-strict t-norms has preferable properties which make them more usable in building up logical structures. Among these properties are the fulfillment of the law of contradiction and the excluded middle, the continuity of the implication or the coincidence of the residual and the S-implication [8, 26]. Due to the fact that all continuous Archimedean (i.e. representable) nilpotent t-norms are isomorphic to the Łukasiewicz t-norm [13], the previously studied nilpotent systems were all isomorphic to the well-known Łukasiewicz-logic.

In this paper we show that a logical system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz-logic. Of course, this lack of isomorphy is not the result of introducing a new operator family, it simply means that the system itself is built up in a significantly different way using more than one generator functions.

The paper is organized as follows. After some preliminaries in Section 2 we give a characterization of negation operators in Section 3, as negations will have an important role to play in Section 4. After considering the class of connective systems generated by nilpotent operators, we examine their structural properties in Section 4. We show examples for bounded systems, i.e. consistent nilpotent systems which are not isomorphic to Łukasiewicz-logic. Necessary and sufficient conditions are given for these systems to satisfy the De Morgan law, classification property and consistency. A wide range of examples for consistent and non-consistent bounded systems can be found in Section 5.

2 Preliminaries – basic fuzzy connectives

2.1 t-norms and t-conorms

First, we recall some basic notations and results regarding t-norms, t-conorms and negation operators that will be useful in the sequel.

A triangular norm (*t-norm* for short) T is a binary operation on the closed unit interval $[0,1]$ such that $([0,1],T)$ is an abelian semigroup with neutral element 1 which is totally ordered, i.e., for all x_1, x_2, y_1, $y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$ we have $T(x_1,y_1) \leq T(x_2,y_2)$, where $\leq$ is the natural order on $[0,1]$.

A triangular conorm (*t-conorm* for short) S is a binary operation on the closed unit interval $[0,1]$ such that $([0,1],S)$ is an abelian semigroup with neutral element 0 which is totally ordered.

Standard examples [5, 16] of t-norms are the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$, the Łukasiewicz t-norm $T_{\mathbf{L}}$ given by $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$, and the drastic product $T_{\mathbf{D}}$ with $T_{\mathbf{D}}(1,x) = T_{\mathbf{D}}(x,1) = x$, and $T_{\mathbf{D}}(x,y) = 0$ otherwise.

Standard examples of t-conorms are the maximum $S_{\mathbf{M}}$, the probabilistic sum $S_{\mathbf{P}}$, the Łukasiewicz t-conorm $S_{\mathbf{L}}$ given by $S_{\mathbf{L}}(x,y) = \min(x+y,1)$, and the drastic sum $S_{\mathbf{D}}$ with $S_{\mathbf{D}}(0,x) = S_{\mathbf{D}}(x,0) = x$, and $S_{\mathbf{D}}(x,y) = 1$ otherwise.

A continuous t-norm T is said to be *Archimedean* if $T(x,x) < x$ holds for all $x \in (0,1)$, *strict* if T is strictly monotone i.e. $T(x,y) < T(x,z)$ whenever $x \in (0,1]$ and $y < z$, and *nilpotent* if there exist $x, y \in (0,1)$ such that $T(x,y) = 0$.

From the duality between t-norms and t-conorms we can easily get the following properties as well. A continuous t-conorm S is said to be *Archimedean* if $S(x,x) > x$ holds for every $x, y \in (0,1)$, *strict* if S is strictly monotone i.e. $S(x,y) < S(x,z)$ whenever $x \in [0,1)$ and $y < z$, and *nilpotent* if there exist $x, y \in (0,1)$ such that $S(x,y) = 1$.

Proposition 1. *[17, 4] A function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm iff it has a continuous additive generator, i.e. there exists a continuous strictly decreasing function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x,y) = t^{-1}(min(t(x)+t(y),t(0)), \quad x,y \in [0,1]. \tag{1}$$

Proposition 2. *[17, 4] A function $S : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-conorm iff it has a continuous additive generator, i.e. there exists a continuous strictly increasing function $s : [0,1] \to [0,\infty]$ with $s(0) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$S(x,y) = s^{-1}(min(s(x)+s(y),s(1)), \quad x,y \in [0,1]. \tag{2}$$

Proposition 3. *[13]*
A t-norm T is strict if and only if $t(0) = \infty$ holds for each continuous additive generator t of T.
A t-norm T is nilpotent if and only if $t(0) < \infty$ holds for each continuous additive generator t of T.
A t-conorm S is strict if and only if $s(1) = \infty$ holds for each continuous additive generator s of S.
A t-conorm S is nilpotent if and only if $s(1) < \infty$ holds for each continuous additive generator s of S.

In both of the above mentioned Propositions 1 and 2 we can allow the generator functions to be strictly increasing *or* strictly decreasing, which will result in the fact that they will be determined up to a (not necessarily positive) multiplicative constant. For an increasing generator function t of a t-conorm and similarly for a decreasing generator function s of a t-conorm, *min* in (1) and (2) has to be replaced by *max*. In this case we will have $t(0) = \pm\infty$ and $s(1) = \pm\infty$ for strict norms and similarly, $t(0) < \infty$ *or* $t(0) > -\infty$ and $s(1) < \infty$ *or* $s(1) > -\infty$ for the nilpotent ones.

Proposition 4. *[13] Let T be a continuous Archimedean t-norm.*
If T is strict, then it is isomorphic to the product t-norm $T_{\mathbf{P}}$, i.e., there exists an automorphism of the unit interval ϕ such that $T_\phi = \phi^{-1}\left(T(\phi(x), \phi(y))\right) = T_{\mathbf{P}}$.
If T is nilpotent, then it is isomorphic to the Łukasiewicz t-norm $T_{\mathbf{L}}$, i.e., there exists an automorphism of the unit interval ϕ such that $T_\phi = \phi^{-1}\left(T(\phi(x), \phi(y))\right) = T_{\mathbf{L}}$.

From the definitions of t-norms and t-conorms it follows immediately that t-norms are conjunctive, while t-conorms are disjunctive aggregation functions. Therefore, they are widely used as conjunctions and disjunctions in multivalued logical structures.

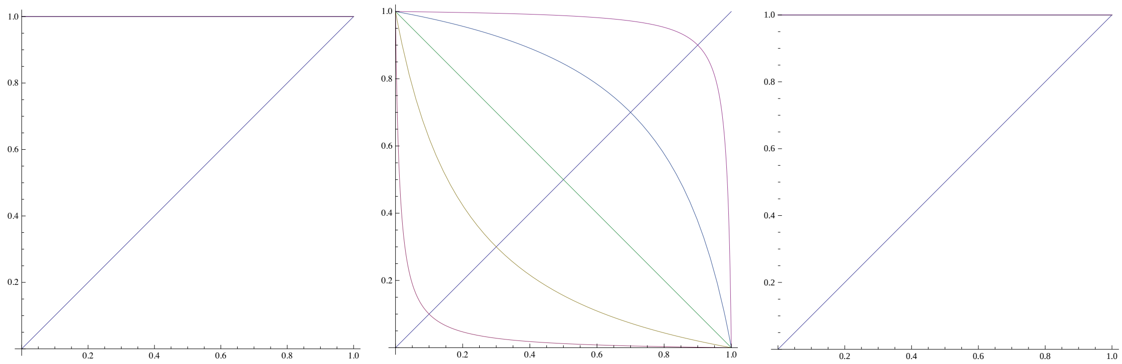
The logical system based on the nilpotent Łukasiewicz t-norm as conjunction is called *Łukasiewicz-logic* [14, 18, 20].

Henceforth we refer to t-norms as conjunctions ($c(x, y)$) and t-conorms as disjunctions ($d(x, y)$).

2.2 Negations

Definition 1. *A unary operation $n : [0, 1] \to [0, 1]$ is called a negation if it is non-increasing and compatible with classical logic, i.e. $n(0) = 1$ and $n(1) = 0$.*
A negation is strict if it is also strictly decreasing and continuous.
A negation is strong, if it is also involutive, i.e. $n(n(x)) = x$.

Due to the continuity and strict monotonicity of n, for continuous negations there always exists some ν_*, for which $n(\nu_*) = \nu_*$ holds. ν_* is called the *neutral value* of the negation and the notation n_{ν_*} stands for a negation operator with neutral value ν_*. In the literature ν_* is often denoted by e. In Figure 1 we can see some negations with different ν_* values.



(a) $n_0(x)$ (b) $\nu_* = 0.1, 0.25, 0.75, 0.9$ (c) $n_1(x)$

Figure 1: Continuous negations with different ν_* values and the drastic negations as limit cases

Drastic negations [28] (see Figure 1) are the so-called intuitionistic and dual intuitionistic negations (denoted by n_0 and n_1 respectively):

$$n_0(x) = \begin{cases} 1 & if \quad x = 0 \\ 0 & if \quad x > 0 \end{cases} \quad \text{and} \quad n_1(x) = \begin{cases} 1 & if \quad x < 1 \\ 0 & if \quad x = 1 \end{cases}$$

These drastic negations are neither continuous nor strictly decreasing, therefore they are not strict negations, but we can get them as limits of negations.

Definition 2. *A continuous, strictly increasing function* $\varphi : [a,b] \to [a,b]$ *with boundary conditions* $\varphi(a) = a, \varphi(b) = b$ *is called an automorphism of* $[a,b]$.

The well-known representation theorem was obtained by Trillas.

Theorem 1. *[25] n is a strong negation if and only if*

$$n(x) = f_n(x)^{-1}(1 - f_n(x)),$$

where $f_n : [0,1] \to [0,1]$ *is an automorphism of* $[0,1]$.

An extension of this result for strict negations was given by Fodor as follows.

Proposition 5. *(Lemma 6.2. in [10]) n is a strict negation iff*

$$n(x) = f_n^{-1}\left(n'\left(f_n(x)\right)\right) \tag{3}$$

where f_n*, called the generator function of* n*,* $f_n : [0;1] \to [0;\infty]$ *is a strictly monotone, continuous function with* $f_n(0) = 0$ *and* $f_n(1) = 1$ *and* n' *is a strong negation.*

Example 1. *For* $f_n(x) = x^2$ *and* $n'(x) = \frac{1-x}{1+x}$ *we get* $n(x) = \sqrt{\frac{1-x^2}{1+x^2}}$.

3 Characterization of strict negation operators

Our main goal in this section is to present a representation of strict negations with a wide range of examples, since negations will have a very important role to play in the next section.

First let us see some further examples for negation operators. Hamacher proved [15] that the only negation having polynomial form is $1-x$, the so-called standard negation, introduced by Zadeh [29]. He also proved that if an involutive negation belongs to the class of rational polynomials, then it has the following form:

$$n_\lambda(x) = \frac{1-x}{1+\lambda x}, \quad \text{where} \quad \lambda > -1. \tag{4}$$

Sugeno had the same result from the concept of fuzzy measures and integrals [24].

In the literature, generally the standard negation $1-x$ or infrequently $\frac{1-x}{1+x}$ ((4) for λ =1) are used. Here we make suggestions about using different types of negations as well. The negation operators can be characterized by their neutral values. In [6] (see also [7]) Dombi introduced the following negation formula by expressing $n_\lambda(x)$ with the help of its neutral element ν_*:

$$n_{\nu_*}(x) = \frac{1}{1 + (\frac{1-\nu_*}{\nu_*})^2 \frac{x}{1-x}}. \tag{5}$$

Note that if $\nu_* \to 0$, then $\lim n_{\nu_*}(x) = n_0(x)$, if $\nu_* \to 1$, then $\lim n_{\nu_*}(x) = n_1(x)$ and for $\nu_* = \frac{1}{2}$ we get the standard negation.

Yager introduced (see [28])

$$n(x) = (1 - x^\alpha)^{1/\alpha}, \quad \alpha > 0. \tag{6}$$

Both this type of negation operator and the above-mentioned n_λ reduce to the standard negation when $\alpha = 1$ and $\lambda = 0$ respectively.

It is easy to see that the neutral value of the negation operator in (6) is $2^{-\frac{1}{\alpha}}$. If we write this negation operator by using its neutral value as a parameter, we get $n(x) = \left(1 - x^{-\frac{1}{\log_2 \nu_*}}\right)^{-\log_2 \nu_*}$.

Note that the representation in Proposition 5 is not unique. It is not always easy to find a generator function. The following propositions state that there can be infinitely many generator functions for a negation operator.

Proposition 6. *Let $\nu_* \in (0,1)$, $f : [0,1] \to [0,1]$*

$$f(x) = \begin{cases} \frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\cdot\frac{1-x}{x}\right)^{\alpha}} & \textit{if } x \neq 0 \\ 0 & \textit{if } x = 0. \end{cases}$$

is a generator function of the negation n_{ν_*} (see (5)) for any $\alpha \neq 0$.

Proof. It can easily be seen that $f^{-1}(x) = \frac{1}{1+\frac{\nu_*}{1-\nu_*}\left(\frac{1-x}{x}\right)^{\frac{1}{\alpha}}}$, and $1 - f(x) = \frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\cdot\frac{1-x}{x}\right)^{-\alpha}}$, hence $f^{-1}(1 - f(x)) = \frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\frac{x}{1-x}} = n_{\nu_*}(x)$.

□

Remark 1. *Note that in Proposition 6, if f is a generator function of n, then f^{-1} also generates n.*

Proposition 7. *In Theorem 1 (Trillas) the generator function can also be decreasing.*

Proof. We shall prove that if f_n is a generator function of n, then $g_n(x) = 1 - f_n(x)$ is also a generator function of n. If f_n is the generator function of n, then $n(x) = f^{-1}(1 - f_n(x))$. If $g_n(x) = 1 - f_n(x)$ then $g_n^{-1}(x) = f_n^{-1}(1-x)$. With this generator function the negation has the following form: $g^{-1}(1 - g(x)) = $ $= g^{-1}(1 - (1 - f_n(x))) = g^{-1}(f_n(x)) = f_n^{-1}(1 - f_n(x))$. Since f_n is increasing, g_n is decreasing. □

For the neutral element ν_*, using the representation theorem, we get $\nu_* = f^{-1}(1 - f(\nu_*))$, so $\nu_* = f^{-1}\left(\frac{1}{2}\right)$.

For the generator function $g(x) = \frac{a^x - 1}{a-1}$, where $a > 0, a \neq 1$, we get

$$n(x) = \log_a (a + 1 - a^x). \tag{7}$$

If we choose the inverse function $g^{-1}(x) = \log_a (x(a-1) + 1)$ for the generator function, we obtain

$$n(x) = \frac{1-x}{1 + x(a-1)}, \tag{8}$$

which was mentioned above.

In this section we considered three basic families of strict negations generated by rational, power and exponential functions. (See also Tables 1, 3 and 4.)

4 Nilpotent connective systems

Next, instead of operators in themselves, we consider connective systems.

Definition 3. *The triple (c, d, n), where c is a t-norm, d is a t-conorm and n is a strong negation, is called a connective system.*

Definition 4. *A connective system is nilpotent, if the conjunction c is a nilpotent t-norm, and the disjunction d is a nilpotent t-conorm.*

Definition 5. *Two connective systems (c_1, d_1, n_1) and (c_2, d_2, n_2) are isomorphic, if there exists a bijection $\phi : [0,1] \to [0,1]$ such that*

$$\phi^{-1}\left(c_1\left(\phi(x), \phi(y)\right)\right) = c_2(x, y)$$

$$\phi^{-1}\left(d_1\left(\phi(x), \phi(y)\right)\right) = d_2(x, y)$$

$$\phi^{-1}\left(n_1\left(\phi(x)\right)\right) = n_2(x).$$

In the nilpotent case, the generator functions of the disjunction and the conjunction being determined up to a multiplicative constant can be normalized the following way:

$$f_c(x) := \frac{t(x)}{t(0)}, \qquad f_d(x) := \frac{s(x)}{s(1)}.$$

Remark 2. *Thus, the normalized generator functions are uniquely defined.*

We will use normalized generator functions for conjunctions and disjunctions well. This means that the normalized generator functions of conjunctions, disjunctions and negations are

$$f_c, f_d, f_n : [0,1] \to [0,1].$$

We will suppose that f_c is continuous and strictly decreasing, f_d is continuous and strictly increasing and f_n is continuous and strictly monotone.

Note that by using Proposition 7, there are two special negations generated by the normalized additive generators of the conjunction and the disjunction.

Definition 6. *The negations n_c and n_d generated by f_c and f_d respectively,*

$$n_c(x) = f_c^{-1}(1 - f_c(x))$$

and

$$n_d(x) = f_d^{-1}(1 - f_d(x))$$

are called natural negations.

This means that for a connective system with normalized generator functions f_c, f_d and f_n we can associate three negations by (3), n_c, n_d and n.

Definition 7. *Let us define the cutting operation $[\ ]$ by*

$$[x] = \begin{cases} 0 & if \quad x < 0 \\ x & if \quad 0 \le x \le 1 \\ 1 & if \quad 1 < x \end{cases}$$

and let the notation $[\ ]$ also act as 'brackets' when writing the argument of an operator, so that we can write $f[x]$ instead of $f([x])$.

Remark 3. *The cutting operator has also been defined in [22].*

Proposition 8. *With the help of the cutting operator, we can write the conjunction and disjunction in the following form, where f_c and f_d are decreasing and increasing normalized generator functions respectively.*

$$c(x,y) = f_c^{-1}[f_c(x) + f_c(y)], \tag{9}$$

$$d(x,y) = f_d^{-1}[f_d(x) + f_d(y)]. \tag{10}$$

Proof. From (1) we know that

$$c(x,y) = f_c^{-1}\left(\min(f_c(x) + f_c(y), f_c(0)\right) = f_c^{-1}\left(\min(f_c(x) + f_c(y), 1\right) = f_c^{-1}[f_c(x) + f_c(y)],$$

and similarly, from (2)

$$d(x,y) = f_d^{-1}\left(\min(f_d(x) + f_d(y), f_d(0)\right) = f_d^{-1}\left(\min(f_d(x) + f_d(y), 1\right) = f_d^{-1}[f_d(x) + f_d(y)].$$

□

Remark 4. *Note that in Proposition 8 it is necessary to use normalized generator functions as the following example shows. This fact supports the use of normalized functions.*

Example 2. *Let $f_c(x) = 2 - 2x$.*

$$c\left(\frac{1}{2}, \frac{1}{2}\right) = f_c^{-1}\left(min\left(f_c(x) + f_c(y), f_c(0)\right)\right) = f_c^{-1}(2) = 0,$$

while

$$f_c^{-1}\left[f_c\left(\frac{1}{2}\right) + f_c\left(\frac{1}{2}\right)\right] = f_c^{-1}\left[2 - 1 + 2 - 1\right] = f_c^{-1}[2] = f_c^{-1}(1) = \frac{1}{2}.$$

Remark 5. *Note that using the cutting function defined above we can omit applying the min and max operators. In the literature, the use of the pseudo-inverse was replaced by the forms (1) and (2), which is now replaced by (9) and (10).*

Definition 8. *A connective system is called Łukasiewicz system, if it is isomorphic to $([x+y-1], [x+y], 1-x)$, i.e. it has the form $(\phi^{-1}[\phi(x)+\phi(y)-1], \phi^{-1}[\phi(x)+\phi(y)], \phi^{-1}[1-\phi(x)])\quad for\ \forall x, y \in [0,1]$.*

Proposition 9. *For nilpotent t-norms and t-conorms Definition 6 is equivalent to the following definition (also denoted by N_T and N_S, see [16] (p. 232.) and [2] Definition 2.3.1.):*

$$n_c(x) = N_T(x) = \ sup\ \{y \in [0,1] \mid c(x,y) = 0\}, \quad x \in [0,1],$$

$$n_d(x) = N_S(x) = \ inf\ \{y \in [0,1] \mid d(x,y) = 1\}, \quad x \in [0,1].$$

Proof. For the conjunction, $c(x,y) = f_c^{-1}[f_c(x) + f_c(y)] = 0$ iff $f_c(x) + f_c(y) \geq 1$, from which $y \leq f_c^{-1}\left(1 - f_c(x)\right) = n_c(x)$. For $y = n_c(x)$, $c(x, n_c(x)) = 0$ is trivial. The proof is similar for the disjunction as well. □

4.1 Structural properties of connective systems

Definition 9. ***Classification property*** *means that the law of contradiction holds, i.e.*

$$c(x, n(x)) = 0, \qquad \forall x, y \in [0,1], \tag{11}$$

and the excluded third principle holds as well, i.e.

$$d(x, n(x)) = 1, \qquad \forall x, y \in [0,1]. \tag{12}$$

Definition 10. *The* ***De Morgan identity*** *means that*

$$c(n(x), n(y)) = n(d(x, y)) \tag{13}$$

or

$$d(n(x), n(y)) = n(c(x, y)). \tag{14}$$

Remark 6. *These two forms of the De Morgan law are equivalent, if the negation is involutive. The first De Morgan law holds with a strict negation n if and only if the second holds with n^{-1} (see page 18 in [11])*

Definition 11. *A connective system is said to be* ***consistent****, if the classification property (Definition 9) and the De Morgan identity (Definition 10) hold.*

4.1.1 Classification Property

Now we will examine the conditions that the connectives and their normalized generator functions in a connective system must satisfy, if we want the classification property to hold.

Proposition 10. *(See also [11] 1.5.4. and 1.5.5., and [2] 2.3.2.) In a connective system (c, d, n) the classification property holds iff*

$$n_d(x) \leq n(x) \leq n_c(x), \quad for \quad \forall x \in [0, 1]$$

where n_c and n_d are the natural negations of c and d, respectively.

Proof. From the excluded third principle, we have $d(x, n(x)) = 1$. Using the normalized generator function, $f_d^{-1}[f_d(x)+f_d(n(x))] = 1$. It means that $f_d(x)+f_d(n(x)) \geq 1$, from which $f_d(n(x)) \geq 1-f_d(x)$. f_d and its inverse f_d^{-1} are strictly increasing, thus we get the left hand side of the inequality:

$$n(x) \geq f_d^{-1}(1 - f_d(x)) = n_d(x).$$

Similarly, we get the right hand side from the law of contradiction $c(x, n(x)) = 0$. Using the normalized generator function we get $f_c^{-1}[f_c(x) + f_c(n(x))] = 0$. From the definition of the cutting function $f_c(x) + f_c(n(x)) \geq 1$, which means that $f_c(n(x)) \geq 1 - f_c(x)$. Since f_c and f_c^{-1} are strictly decreasing,

$$n(x) \leq f_c^{-1}(1 - f_c(x)) = n_c(x),$$

$$n_d(x) \leq n(x) \leq n_c(x).$$

□

Remark 7. *Generally, in a consistent system only one negation is used in the literature. The logical connectives are usually generated by a single generator function.*

$$c(x, y) = f^{-1}\left[f(x) + f(y) - 1\right],$$

$$d(x, y) = f^{-1}\left[f(x) + f(y)\right],$$

$$n(x) = f^{-1}\left(1 - f(x)\right),$$

where $f : [0, 1] \to [0, 1]$ is a continuous, strictly increasing function.

The question arises immediately, whether the use of more than one negation is possible. We will consider this possibility later in detail (see 4.2.1).

Next we give examples for connective systems in which the classification property holds, but which does not fulfil the De Morgan law.

In Section 5, we present an overview of all the examples included in the following part of our paper. The examples from the rational family will be considered in detail in 4.2.1.

Example 3. *Let $f_n(x) := x^2$, $f_c(x) := \sqrt{1-x}$ and $f_d(x) := \sqrt{x}$. This connective system fulfills the classification property but does not fulfill the De Morgan law. (See also Table 1.)*

We can get another example by using the rational family of normalized generators functions

$$f_n(x) = \frac{1}{1 + \frac{\nu}{1-\nu}\frac{1-x}{x}}, \quad f_n(0) = 0,$$

$$f_c(x) = \frac{1}{1 + \frac{\nu_c}{1-\nu_c}\frac{x}{1-x}}, \quad f_c(1) = 0,$$

$$f_d(x) = \frac{1}{1 + \frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}, \quad f_d(0) = 0,$$

choosing e.g. $\nu_d = 0.3$, $\nu_c = 0.7$ and $\nu = 0.5$. (See Table 4.)

The existence of such systems explains why we have to consider the De Morgan law in the following section.

4.1.2 The De Morgan Law

Now we will examine the conditions that the connectives and their normalized generator functions must satisfy if we want the connective system to fulfill the De Morgan law. Before stating Proposition 12 we need to solve the following functional equation.

Lemma 1. *Let $u : [0,1] \longrightarrow [0,1]$ be a continuous, strictly increasing function with $u(0) = 0$ and $u(1) = 1$. The functional equation*

$$[u(x) + u(y)] = u[x + y] \tag{15}$$

(where [ ] stands for the cutting operator defined in Definition 7) has a unique solution $u(x) = x$.

Proof.

- First we shall prove that $u[0] = 0$. Let us suppose that $u[0] = c$, where $0 \le c \le 1$. Then

  $$c = u[0 + 0] = [2u(0)],$$

  which means $c = [2c]$ i.e. $c = 1$, or $c = 0$, but $c = 1$ contradicts $u(0) = 0$.

- Second, we will show that $u[1] = 1$. Similarly, let us suppose that $u[1] = c$, where $0 \le c \le 1$. Then $c = u[1 + 1] = [2u(1)]$, which means $c = [2c]$ i.e. $c = 1$, or $c = 0$, but for $c = 0$ we get contradiction.

- Third, we will prove that $u\left(\frac{1}{2}\right) = \frac{1}{2}$.

  If $x < \frac{1}{2}$, then $2x < 1$. u is strictly increasing, therefore $u(2x) < 1$ as well. $u[2x] = u(2x) = 2u(x) = [2u(x)]$, because of the continuity of u, $\lim_{x\to\frac{1}{2}} u(2x) = u(1)$, $2\lim_{x\to\frac{1}{2}} u(x) = 1$, which implies $u\left(\frac{1}{2}\right) = \frac{1}{2}$.

- Similarly, we can prove that $u\left(\frac{1}{2^m}\right)=\frac{1}{2^m}$.
- Next, we will prove that $u\left(\frac{3}{4}\right)=\frac{3}{4}$.
  $u\left(\frac{3}{4}\right)=u\left(\frac{1}{2}+\frac{1}{4}\right)=u\left(\frac{1}{2}\right)+u\left(\frac{1}{4}\right)=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$.
- In a similar way, we obtain that for $u\left(\frac{k}{2^m}\right)=\frac{k}{2^m}$.
  Then, for any rational number from $[0,1]$, we have $u(x)=x$.
- Let r be any arbitrary irrational number from $[0,1]$. There exists a sequence of rational numbers q_n such that $\forall n: q_n \in [0,1]$ and $q_n \longrightarrow r$.

Because of the continuity of u we have $u(q_n) \longrightarrow u(r)$, which implies $u(r)=r$. □

We note that the solution of the following general form of the functional equation (15) can be found in the papers of M. Baczynski [3], [4] (Propositions 3.4. and 3.6.).

Proposition 11. *Fix real $a,b>0$. For a function $f:[0,a]\to[0,b]$, the following statements are equivalent.*

1. *f satisfies the functional equation*
   $f(min(x+y,a))=min(f(x)+f(y),b) \quad \forall x,y\in[0,a]$.
2. *Either $f=b$, or $f=0$, or*

$$f(x)=\begin{cases}0 \text{ if } & x=0\\ b \text{ if } & 0<x\le a\end{cases}$$

or there exists a unique constant $c\in[b/a,\infty)$ such that

$$f(x)=min(cx,b), \qquad x\in[0,a].$$

Remark 8. *Specially, for $a=b=1$ we get the statement of Lemma 1.*

Proposition 12. *If f_c is the normalized generator function of a conjunction in a connective system, f_d is a normalized generator function of the disjunction and n is a strong negation, then the following statements are equivalent:*

1. *The De Morgan law holds in the connective system. That is,*

$$c(n(x),n(y))=n(d(x,y)). \tag{16}$$

2. *The normalized generator functions of the conjunction, disjunction and negation operator obey the following equations (which are obviously equivalent to each other):*

$$n(x)=f_c^{-1}\left(f_d(x)\right)=f_d^{-1}\left(f_c(x)\right), \tag{17}$$

$$f_c(x)=f_d(n(x)) \qquad \text{or equivalently} \qquad f_d(x)=f_c(n(x)). \tag{18}$$

Proof. (18) ⇒ (16) is obvious.

(16) ⇒ (17): Let us write the De Morgan law using the normalized generator functions.

$$f_c^{-1}[f_c(n(x))+f_c(n(y))])=n(f_d^{-1}[f_d(x)+f_d(y)]).$$

Applying $f_c(x)$ to both sides of the equation we obtain

$$[f_c(n(x)) + f_c(n(y))] = f_c(n(f_d^{-1}[f_d(x) + f_d(y)])).$$

Let us substitute $x = f_d^{-1}(x)$. Then we have

$$[f_c(n(f_d^{-1}(x))) + f_c(n(f_d^{-1}(y)))] = f_c(n(f_d^{-1}[f_d(f_d^{-1}(x)) + f_d(f_d^{-1}(y))])).$$

From this, we get the following functional equation:

$$[f_c(n(f_d^{-1}(x))) + f_c(n(f_d^{-1}(y)))] = f_c(n(f_d^{-1}[x + y])).$$

If we use $u(x) := f_c(n(f_d^{-1}(x)))$, then we get the following form of the functional equation:

$$[u(x) + u(y)] = u[x + y].$$

We can readily see that function $u(x)$ satisfies the conditions of Lemma 1, i.e. it is a continuous, strictly monotone increasing function with $u(0) = 0$ and $u(1) = 1$. This means that by Lemma 1, $u(x) = x$. Hence, $f_c\left(n\left(f_d^{-1}(x)\right)\right) = x$. □

Remark 9. *Note that in Proposition 12 any two of n, f_c, f_d determine the third.*

However, note that this remark above does *not* mean that any two of n, f_c, f_d can be chosen arbitrary. If f_c and f_d are given and we want the De Morgan property to hold, we obtain n from (17). This means that for f_c and f_d the equation in (17) has to hold. Hence, in order to get an involutive negation, we must take notice of the appropriate relationship of the normalized generator functions as the following example shows.

Example 4. *Let $f_c(x) = 1 - x^{\alpha}$ and $f_d(x) = x^{\beta}$, where $\alpha \neq \beta$. Then*

$$f_c^{-1}\left(f_d(x)\right) = \sqrt[\alpha]{1 - x^{\beta}} \neq \sqrt[\beta]{1 - x^{\alpha}} = f_d^{-1}\left(f_c(x)\right).$$

Proposition 13. *If the De Morgan property holds in a connective system (c, d, n), then*

$$n_c\left(n(x)\right) = n\left(n_d(x)\right) \tag{19}$$

and similarly,

$$n_d\left(n(x)\right) = n\left(n_c(x)\right), \tag{20}$$

where n_c and n_d are the natural negations.

Proof. Because of the involutive property of n it is enough to prove (19).

$$n\left(f_c^{-1}\left(1 - f_c\left(n(x)\right)\right)\right) = f_d^{-1}\left(f_c\left(f_c^{-1}\left(1 - f_c\left(f_c^{-1}\left(f_d(x)\right)\right)\right)\right)\right) = n_d(x).$$

□

Corollary 1. *If the De Morgan law holds in a connective system (c, d, n), then*

$$n(x) = n_c(x) \text{ if and only if } n(x) = n_d(x), \tag{21}$$

where n_c and n_d are the natural negations.

Remark 10. *Note that we can readily see that if any two of* n, n_d, n_c *are equal, then the third is equal to them as well.*

Proposition 14. *Let* h *be the transformation for which* $h(f_c(x)) = f_d(x)$ *in a connective system in which the De Morgan property holds. Then* h *is a (strong) negation.*

Proof. By using the involutive property of n, we get

$$f_d^{-1}(f_c(x)) = f_c^{-1}(f_d(x)),$$
$$f_d(x) = f_c\left(f_d^{-1}(f_c(x))\right),$$
$$f_c(x) = f_d\left(f_c^{-1}(f_d(x))\right) = h(f_d(x)),$$
$$f_c^{-1}(x) = f_d^{-1}\left(h^{-1}(x)\right),$$
$$f_d\left(f_c^{-1}(x)\right) = h^{-1}(x) = h(x).$$

So h is also involutive. It is easy to see that $h(0) = 1, h(1) = 0$ and $h(x) = f_d\left(f_c^{-1}(x)\right)$ is strictly monotone decreasing. □

Now we give examples for consistent and non-consistent connective systems where the De Morgan property holds. For examples from the rational family of normalized generator functions see propositions 18 and 19.

Example 5. *If in a connective system the conjunction, the disjunction and the negation have the following forms*

$$f_n(x) = x, f_c(x) = (1-x)^\alpha, f_d(x) = x^\alpha,$$

then this connective system is consistent (i.e. the De Morgan law and the classification property hold), if and only if $0 < \alpha \leq 1$. *(See also Table 1.)*

Proof. It is easy to see, that from the Proposition 18 formula (17) is true for the mentioned normalized generator and negation functions:

$$x^\alpha = (1-(1-x))^\alpha,$$

which means that the De Morgan law holds.
It is easy to see that the classification property holds if and only if

$$x^\alpha + (1-x)^\alpha \geq 1,$$

which is only true if for $0 < \alpha \leq 1$. □

Remark 11. *Note that the example above shows that there exists a system in which the De Morgan property holds, whereas the classification property does not (for* $\alpha > 1$*). (See also Table 1.)*

For an example from the rational family of normalized generator functions (see propositions 18 and 19 and also Table 4)

$$f_n(x) = \frac{1}{1 + \frac{\nu}{1-\nu}\frac{1-x}{x}}, \quad f_n(0) = 0,$$
$$f_c(x) = \frac{1}{1 + \frac{\nu_c}{1-\nu_c}\frac{x}{1-x}}, \quad f_d(0) = 0,$$
$$f_d(x) = \frac{1}{1 + \frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}, \quad f_c(1) = 0,$$

we can choose e.g. $\nu = 0.6$, $\nu_c = 0.2$ *and* $\nu_d = 0.36$.

Example 6. *If we express the normalized generator functions in Example 5 in terms of the neutral values of the related negations, we get*

$$f_n(x) = x, f_c(x) = (1-x)^{\frac{1}{log_{0.5}(1-\nu_c)}}, f_d(x) = x^{log_{\nu_d}(0.5)}.$$

This system fulfills the De Morgan identity iff $\nu_c + \nu_d = 1$, and is consistent iff $\nu_d \leq \frac{1}{2}$ also holds. (See also Table 1.)

4.2 Consistent Connective Systems

Now we consider the consistent connective systems (in which the De Morgan property and the classification property hold together).

Proposition 15. *1. If the connective system (c, d, n) is consistent, then $f_c(x) + f_d(x) \geq 1$ for any $x \in [0, 1]$, where f_c and f_d are the normalized generator functions of the conjunction c and the disjunction d respectively.*

2. If $f_c(x) + f_d(x) \geq 1$ for any $x \in [0, 1]$ and the De Morgan law holds, then the connective system (c, d, n) satisfies the classification property as well (which now means that the system is consistent).

Proof. By Proposition 10, the classification property holds if and only if

$$f_d^{-1}(1 - f_d(x)) = n_d(x) \leq n(x) \leq n_c(x) = f_c^{-1}(1 - f_c(x))$$

and by Proposition 12, the De Morgan identity holds if and only if

$$n(x) = f_d^{-1}(f_c(x)) = f_c^{-1}(f_d(x)).$$

From the right hand side of the inequality we get

$$f_c^{-1}(f_d(x)) \leq f_c^{-1}(1 - f_c(x)),$$

so

$$f_c(x) + f_d(x) \geq 1.$$

Similarly, we get the same from the left hand side of the inequality.

□

Remark 12. *Note that as Example 3 shows, $f_c(x) + f_d(x) \geq 1$ does not imply the De Morgan law, even if the classification property holds.*

Moreover, $f_c(x) + f_d(x) \geq 1$ without the De Morgan law does not imply the classification property either (for a counterexample we can chose $f_n = x^2$ and $\alpha = 0.7$ in Example 5).

Next, we show examples for consistent systems.

Example 7. *If in a connective system the generator function of the conjunction, the disjunction and the negation have the following forms*

$$f_c(x) = 1 - x^\alpha, f_d(x) = x^\alpha, f_n(x) = x^\alpha,$$

where $\alpha > 0$, then the De Morgan law and the classification property hold for every α. (See also Table 1.)

Example 8. *More generally, the connective system with generator functions*

$$f_c(x) = (1 - x^\alpha)^{\frac{\beta}{\alpha}}, f_d(x) = x^\beta, f_n(x) = x^\alpha,$$

where $\alpha, \beta > 0$ *is consistent if and only if* $\beta \leq \alpha$*. (See also Table 1.)*

Note that Example 8 reduces to Example 5 if $\alpha = 1$ and $0 < \beta \leq 1$ and to Example 7 if $\alpha = \beta$.

Proposition 16. *In a connective system the following equations are equivalent:*

$$f_c(x) + f_d(x) = 1 \tag{22}$$

$$n_c(x) = n_d(x), \tag{23}$$

where f_c, f_d *are the normalized generator functions of the conjunction and the disjunction and* n_c, n_d *are the natural negations.*

Proof. From $f_d(x) = 1 - f_c(x)$,

$$f_d^{-1}(x) = f_c^{-1}(1 - x)$$

and

$$n_d(x) = f_d^{-1}(1 - f_d(x)) = f_d^{-1}(1 - (1 - f_c(x))) = f_d^{-1}(f_c(x)) = n(x) = f_c^{-1}(1 - f_c(x)) = n_c(x).$$

□

Remark 13. *Let us suppose that in a connective system the De Morgan property holds. If condition (22) holds, then*

$$n_c(x) = n(x) = n_d(x),$$

and therefore the system is consistent.

Remark 14. *Note that if condition (22) holds, we get the the classical nilpotent (Łukasiewicz) logic.*

4.2.1 Bounded Systems

The question arises, whether we can use more than one generator functions in our connective system without losing consistency. In the literature only systems generated by only one generator function have been considered, see e.g. [2], Theorem 2.3.18. In these systems the natural negations of the conjunction and the disjunction coincide with the negation operator. Now we will examine the case when $n_c(x) \neq n_d(x) \neq n(x)$.

Definition 12. *A nilpotent connective system is called a bounded system, if*

$$f_c(x) + f_d(x) > 1, \text{ or equivalently } n_d(x) < n(x) < n_c(x)$$

holds for all $x \in (0, 1)$*, where* f_c *and* f_d *are the normalized generator functions of the conjunction and disjunction, and* n_c, n_d *are the natural negations.*

The following example shows the existence of consistent bounded systems.

Example 9. *(See also Table 1.) The connective system generated by*

$$f_c(x) := 1 - x^\alpha, f_d(x) := 1 - (1 - x)^\alpha, n(x) := 1 - x, \quad \alpha \in (1, \infty]$$

is a consistent bounded system.

Proof. Applying (17) from Proposition 12, we obtain: $f_c(n(x)) = 1-(1-x)^\alpha = f_d(x)$, which means that the De Morgan law holds. It is easy to see that $n_c(x) = \sqrt[\alpha]{1-x^\alpha}$, $n_d(x) = 1-\sqrt[\alpha]{1-(1-x)^\alpha}$, i.e.

$$n_d(x) < n(x) < n_c(x),$$

which means that the classification property is also true (see Figure 2).

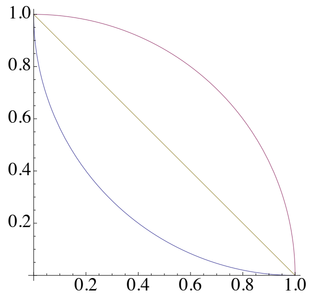


Figure 2: $n_d(x) < n(x) < n_c(x)$ for $\alpha = 2$

For the normalized generator functions we have $f_c(x) + f_d(x) > 1$ for all $x \in (0,1)$. □

Remark 15. *In Example 9 for $\alpha = 1$ we get $n_d(x) = n(x) = n_c(x)$, i.e. $f_c(x) + f_d(x) = 1$.*

Proposition 17. *In a connective system (c,d,n), the following statements are equivalent:*

$$f_c(x) + f_d(x) > 1 \quad \text{for all } x \in (0,1), \tag{24}$$

$$f_d\left(f_c^{-1}(x)\right) > 1-x \quad \text{for all } x \in (0,1), \tag{25}$$

$$f_c\left(f_d^{-1}(x)\right) > 1-x \quad \text{for all } x \in (0,1), \tag{26}$$

where f_c and f_d are the normalized generator functions of c and d.

Proof. From $n_d(x) < n(x) < n_c(x)$ we have $f_d^{-1}(1-f_d(x)) < f_c^{-1}(f_d(x))$. Substituting x by $f_d(x)$ we get $f_d^{-1}(1-x) < f_c^{-1}(x)$, i.e. $f_c\left(f_d^{-1}(x)\right) > 1-x$, which is also equivalent to $f_c\left(f_d^{-1}(1-x)\right) > x$. □

Next we consider the case of the rational family of the normalized generator functions introduced by Dombi in [6].

Proposition 18. *For the Dombi functions (see also Equation (5) and Proposition 6)*

$$f_n(x) = \frac{1}{1+\frac{\nu}{1-\nu}\frac{1-x}{x}}, \quad f_n(0) = 0,$$

$$f_c(x) = \frac{1}{1+\frac{\nu_c}{1-\nu_c}\frac{x}{1-x}}, \quad f_d(0) = 0,$$

$$f_d(x) = \frac{1}{1+\frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}, \quad f_c(1) = 0,$$

the following statements are equivalent:

1. *The connective system generated by the Dombi functions in Proposition 18 satisfies the De Morgan law.*

2. *For parameters ν_d and ν_c in the normalized generator functions and for parameter ν in the negation function the following equation holds:*

$$\left(\frac{1-\nu}{\nu}\right)^2 = \frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}. \tag{27}$$

Proof. By Proposition 12, the De Morgan law holds iff:

$$f_c(n(x)) = f_d(x). \tag{28}$$

From Proposition 6 for $\alpha = -1$ we know that

$$n(x) = \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2 \frac{x}{1-x}}, \tag{29}$$

so

$$f_c(n(x)) = \frac{1}{1+(\frac{\nu_c}{1-\nu_c})(\frac{\nu}{1-\nu})^2\frac{1-x}{x}} = \frac{1}{1+\frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}.$$

This means that the equality (28) holds if and only if the parameters on the left and the right hand side are equal, i.e.:

$$\left(\frac{1-\nu}{\nu}\right)^2 = \frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}. \tag{30}$$

□

Remark 16. *From (30) we get that the De Morgan law holds iff*

$$\nu = \frac{1}{1+\sqrt{\frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}}}. \tag{31}$$

Proposition 19. *For the natural negations derived from the Dombi functions defined in Proposition 18, the following statements are equivalent for $x \in (0,1)$:*

$$n_d(x) < n(x) < n_c(x), \tag{32}$$

$$\nu_d < \nu < \nu_c. \tag{33}$$

Proof.

$$\frac{1}{1+(\frac{1-\nu_d}{\nu_d})^2\frac{x}{1-x}} < \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2\frac{x}{1-x}}$$

(see Table 5) if and only if $\nu_d < \nu$. Similarly, we can prove the other side of the inequality as well. □

Remark 17. *Note that if the De Morgan property holds,*

$$f_c(x) + f_d(x) > 1 \tag{34}$$

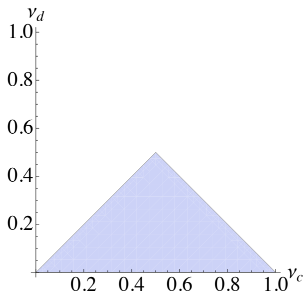
is also equivalent to (32) and (33).

Proposition 20. *For the Dombi functions defined in Proposition 18, the followings are equivalent for $x \in (0,1)$:*

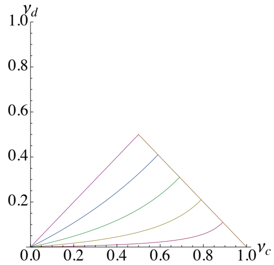
$$f_c(x) + f_d(x) > 1, \tag{35}$$

$$\nu_c + \nu_d < 1. \tag{36}$$

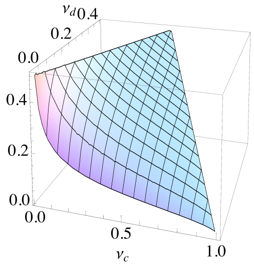
Figure 3: The relationship between ν, ν_c and ν_d in consistent rational systems



(a) The relationship between ν_c and ν_d



(b) The relationship of ν_c and ν_d for different fixed values of ν



(c) ν as a function of ν_c and ν_d

Proof.

$$\frac{1}{1+\left(\frac{\nu_c}{1-\nu_c}\frac{x}{1-x}\right)} > 1 - \frac{1}{1+\left(\frac{\nu_d}{1-\nu_d}\frac{1-x}{x}\right)} = \frac{1}{1+\left(\frac{1-\nu_d}{\nu_d}\frac{x}{1-x}\right)}$$

if and only if

$$\frac{\nu_c}{1-\nu_c} < \frac{\nu_d}{1-\nu_d},$$

which is equivalent to $\nu_c + \nu_d < 1$. □

Remark 18. *Note that if the De Morgan property holds,*

$$n_d(x) < n(x) < n_c(x) \tag{37}$$

is also equivalent to (35) and (36).

The relationship between ν_c and ν_d from Propositions 19 and 20 can be seen in Figure 3a. In Figure 3b we can see the possible values of ν_c and ν_d for fixed values of ν. The values of ν as a function of ν_c and ν_d can be seen on Figure 3c.

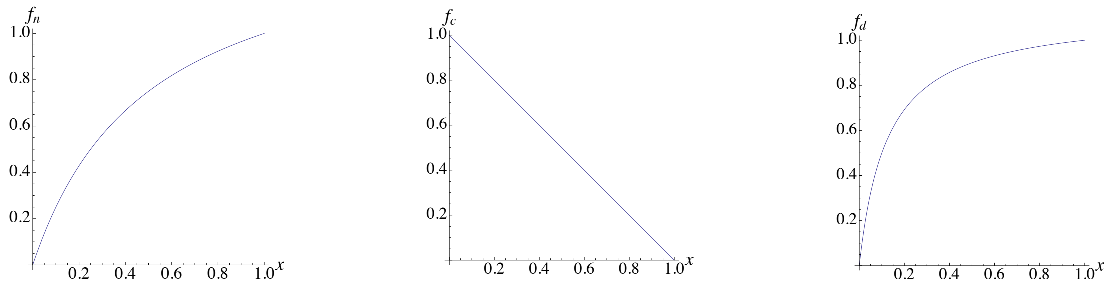
Remark 19. *By using (37), (36) and (31) we obtain that in a consistent system with $f_c(x) + f_d(x) > 1$, $\nu < \frac{1}{2}$ always holds.*

Remark 20. *For $\nu = \frac{1}{2}$ we get $\sqrt{\frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}} = 1$, so $\nu_c = \nu_d = \nu = \frac{1}{2}$.*

Example 10. *For $\nu_c = 0.5$ and $\nu_d = 0.1$ $\nu = 0.25$, $\nu_c + \nu_d < 1$ and $n_d(x) < n(x) < n_c(x)$. See Figure 4.*

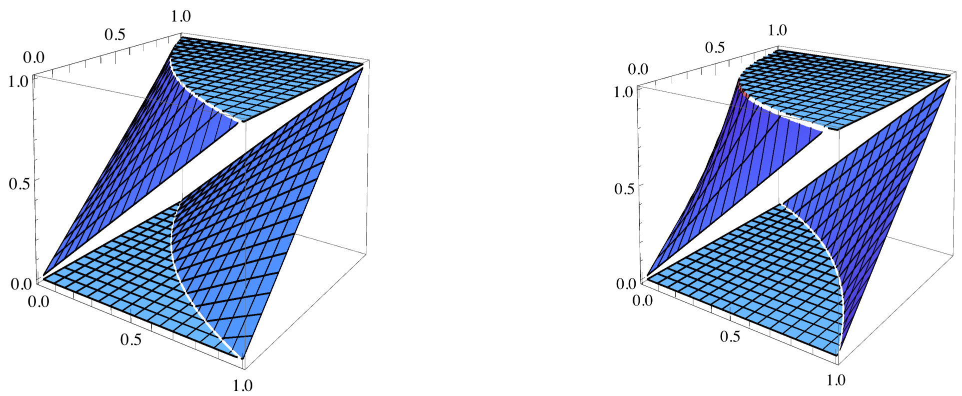
In Figure 5a and 5b examples for conjunctions and disjunctions are shown for $f_c(x) + f_d(x) = 1$ and for $f_c(x) + f_d(x) > 1$ respectively. Note that the coincidence and the separation of n_c and n_d (see their alternative definition in Proposition 9 as well) can easily be seen.

Figure 4: Normalized generators of a consistent system (Example 10)



(a) $f_n(x)$ for $\nu = 0.25$ (b) $f_c(x)$ for $\nu_c = 0.5$ (c) $f_d(x)$ for $\nu_d = 0.1$

Figure 5: Conjunction $c[x, y]$ and disjunction $d[x, y]$



(a) $\nu_c = 0.6$ and $\nu_d = 0.4$ $(\nu_c + \nu_d = 1)$ (b) $\nu_c = 0.4$ and $\nu_d = 0.3$ $(\nu_c + \nu_d < 1)$

5 Overview of examples

In this section we give an overview of the three families of normalized generator functions used in our examples and propositions, namely power, exponential and rational functions (see also (5), (6) and (7).) For the power and the rational normalized generator functions the logical connectives are also given. In the case of the rational and in a special case of the power functions we give the normalized generators in terms of the neutral values as well. Finally, we give some examples of consistent connective systems with mixed types of normalized generator functions.

	f_n	f_c	f_d	Classification	De Morgan	Remarks
Example 3	x^2	$\sqrt{1-x}$	$\sqrt{x}$	✓	–	
Example 5	x	$(1-x)^\alpha$	x^α	✓	✓	$0 < \alpha \leq 1$
Remark 11	x	$(1-x)^\alpha$	x^α	–	✓	$\alpha > 1$
Example 6	x	$(1-x)^{\frac{1}{\log_{0.5}(1-\nu_c)}}$	$x^{\log_{\nu_d} 0.5}$	iff $\nu_d \leq 0.5$	iff $\nu_c + \nu_d = 1$	Example 5 and Remark 11 in terms of the neutral value
Example 7	x^α	$1 - x^\alpha$	x^α	✓	✓	$\alpha > 0$
Example 8	x^α	$(1-x^\alpha)^{\frac{\beta}{\alpha}}$	x^β	✓	✓	$\beta \leq \alpha; \quad \alpha, \beta > 0$
Example 9	x	$1 - x^\alpha$	$1-(1-x)^\alpha$	✓	✓	$\alpha \geq 1$, $f_c + f_d > 1$ iff $\alpha > 1$

Table 1: Power functions as normalized generators

	f_n	f_c	f_d	$n(x)$	$c(x,y)$	$d(x,y)$
E 3	x^2	$\sqrt{1-x}$	$\sqrt{x}$	$\sqrt{1-x^2}$	$1 - \left[\sqrt{(1-x)} + \sqrt{(1-y)}\right]^2$	$\left[\sqrt{x} + \sqrt{y}\right]^2$
E 5	x	$(1-x)^\alpha$	x^α	$1-x$	$1 - [(1-x)^\alpha + (1-y)^\alpha]^{\frac{1}{\alpha}}$	$[x^\alpha + y^\alpha]^{\frac{1}{\alpha}}$
R 11	x	$(1-x)^\alpha$	x^α	$1-x$	$1 - [(1-x)^\alpha + (1-y)^\alpha]^{\frac{1}{\alpha}}$	$[x^\alpha + y^\alpha]^{\frac{1}{\alpha}}$
E 7	x^α	$1-x^\alpha$	x^α	$\sqrt[\alpha]{1-x^\alpha}$	$(1 - [2 - x^\alpha - y^\alpha])^{\frac{1}{\alpha}}$	$[x^\alpha + y^\alpha]^{\frac{1}{\alpha}}$
E 8	x^α	$(1-x^\alpha)^{\frac{\beta}{\alpha}}$	x^β	$\sqrt[\alpha]{1-x^\alpha}$	$\left(1 - \left[(1-x^\alpha)^{\frac{\beta}{\alpha}} + (1-y^\alpha)^{\frac{\beta}{\alpha}}\right]^{\frac{\alpha}{\beta}}\right)^{\frac{1}{\alpha}}$	$\left[x^\beta + y^\beta\right]^{\frac{1}{\beta}}$
E 9	x	$1-x^\alpha$	$1-(1-x)^\alpha$	$1-x$	$(1 - [2 - x^\alpha - y^\alpha])^{\frac{1}{\alpha}}$	$1 - [(1-x)^\alpha + (1-y)^\alpha - 1]^{\frac{1}{\alpha}}$

Table 2: Power functions as normalized generators – logical connectives

	f_n	f_c	f_d	De Morgan law	Consistency
	$\frac{a^x-1}{a-1}$	$\frac{(a+1-a^x)^{\log_a b}-1}{b-1}$	$\frac{b^x-1}{b-1}$	✓	Consistent for e.g. $a = 0.5, b = 0.7$ or $a = 0.7, b = 0.85$

Table 3: Exponential functions as normalized generators

	f_n	f_c	f_d	Classification	De Morgan
Propositions 18 and 19	$\frac{1}{1+\frac{\nu}{1-\nu}\frac{1-x}{x}}$	$\frac{1}{1+\frac{\nu_c}{1-\nu_c}\frac{x}{1-x}}$	$\frac{1}{1+\frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}$	$\nu_d < \nu < \nu_c$	$\left(\frac{1-\nu}{\nu}\right)^2 = \frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}$ $\nu = \frac{1}{1+\sqrt{\frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}}}$
Example 3	$\nu = 0.5$	$\nu_c = 0.7$	$\nu_d = 0.3$	✓	–
Remark 11	$\nu = 0.6$	$\nu_c = 0.2$	$\nu_d = 0.36$	–	✓
Example 10	$\nu = 0.25$	$\nu_c = 0.5$	$\nu_d = 0.1$	✓	✓

Table 4: Rational functions as normalized generators

	$f(x)$ (normalized generator)	$f^{-1}(x)$	$1 - f(x)$	negation
negation	$\frac{1}{1+\frac{\nu}{1-\nu}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu}{\nu}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu}{\nu}\frac{x}{1-x}}$	$n(x) = \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2\frac{x}{1-x}}$
conjunction	$\frac{1}{1+\frac{\nu_c}{1-\nu_c}\frac{x}{1-x}}$	$\frac{1}{1+\frac{1-\nu_c}{\nu_c}\frac{x}{1-x}}$	$\frac{1}{1+\frac{1-\nu_c}{\nu_c}\frac{1-x}{x}}$	$n_c(x) = \frac{1}{1+\left(\frac{\nu_c}{1-\nu_c}\right)^2\frac{x}{1-x}}$
disjunction	$\frac{1}{1+\frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu_d}{\nu_d}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu_d}{\nu_d}\frac{x}{1-x}}$	$n_d(x) = \frac{1}{1+\left(\frac{1-\nu_d}{\nu_d}\right)^2\frac{x}{1-x}}$

Table 5: Rational functions as normalized generators – 3 negations

	f_n	f_c	f_d	De Morgan law	Consistency
Rational and power	$\frac{1}{1+\frac{\nu}{1-\nu}\frac{1-x}{x}}$	$\left(\frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2\frac{x}{1-x}}\right)^{\alpha}$	x^{α}	✓	Consistent for e.g. $\alpha = 1, \nu = 0.8$ or $\alpha = 2, \nu = 0.9$
Power and exponential	x^{α}	$\frac{a^{(1-x^{\alpha})^{\frac{1}{\alpha}}}-1}{a-1}$	$\frac{a^x-1}{a-1}$	✓	$a > 0, a \neq 1, \alpha > 0$. Consistent for e.g. $\alpha = 1, a = 0.5$

Table 6: Mixed types of normalized generator functions

6 Conclusion

After giving a characterization and a wide range of examples for negation operators, we have studied connective systems in which the conjunction, the disjunction and the negation are generated by bounded and normalized functions. Three negations can be naturally associated with the normalized generator functions, n_c, n_d and n. Necessary and sufficient conditions of the classification property (the excluded middle and the law of contradiction), the De Morgan law and consistency have been given. We thoroughly examined the question whether the three negations can differ from one another in a consistent system. The positive answer means that a consistent system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz logic. We get a system isomorphic to Łukasiewicz logic if and only if the three negations coincide. Finally, we have also given several examples for consistent systems with three different negations.

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