International Journal of Approximate Reasoning 81 (2017) 115–127

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Self-dual operators and a general framework for weighted nilpotent operators

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ARTICLE INFO

Article history:
Received 2 April 2016
Received in revised form 11 July 2016
Accepted 7 November 2016
Available online 17 November 2016

Keywords:
Nilpotent operators
Bounded system
Aggregative operator
Weighted aggregation
Uninorm
Preference modeling

ABSTRACT

The main purpose of this paper is to consider generated nilpotent operators in an integrative frame and to examine the nilpotent aggregative operator. As a starting point, instead of associativity, we focus on the necessary and sufficient condition of the self-dual property. A parametric form of the generated operator o_ν is given by using a shifting transformation of the generator function. The parameter has an important semantical meaning as a threshold of expectancy (decision level). Nilpotent conjunctive, disjunctive, aggregative and negation operators can be obtained by changing the parameter value. The properties (De Morgan property, commutativity, self-duality, fulfillment of the boundary conditions, bisymmetry) of the weighted general operator are examined and the formula of the commutative self-dual generated operator, the so-called weighted aggregative operator is given. It is proved that the two-variable operator with weights $w_1 = w_2 = 1$ $\forall i$ is conjunctive for low input values, disjunctive for high ones, and averaging otherwise; i.e. a high input can compensate for a lower one.

1. Introduction

One of the most significant problems of fuzzy set theory is the proper choice of set-theoretic operations [29,32]. Triangular norms and conorms have been thoroughly examined in the literature [14,15,19,22], and are often used as conjunctions and disjunctions in logical structures [18,27].

The most well-characterized class of t-norms is the so-called representable t-norms. t-norms generated by continuous additive generators were described by Mostert and Shield [26]. The two main types of representable t-norms are the strict and non-strict or nilpotent t-norms. The nilpotent operators have some nice properties which make them more useful when constructing logical structures. Among these properties are the fulfillment of the law of contradiction and the excluded middle, and the coincidence of the residual and the S-implication [11,31]. In [8], Dombi and Csiszár showed that a consistent connective system generated by nilpotent operators is not necessarily isomorphic to the Łukasiewicz-system. Using more than one generator function, consistent nilpotent connective systems (so-called bounded systems) can be obtained in a significantly different way with three naturally derived negation operators. Due to the fact that all continuous Archimedean (i.e. representable) nilpotent t-norms are isomorphic to the Łukasiewicz t-norm [19], the previously studied nilpotent systems were all isomorphic to the well-known Łukasiewicz-logic. In [9] and in [10], Dombi and Csiszár examined the implications and equivalence operators in bounded systems.

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http://dx.doi.org/10.1016/j.ijar.2016.11.006
0888-613X/

In human thinking, averaging operators, where a high input can compensate for a lower one, play a significant role. The aggregative operator was first introduced in 1982 by Dombi [7], by selecting a set of minimal concepts that must be fulfilled by an evaluation-like operator. The concept of uninorms was introduced in [33], as a generalization of both t-norms and t-conorms. By adjusting its neutral element ν, a uninorm is a t-norm if $\nu = 1$ and a t-conorm if $\nu = 0$. Uninorms have turned out to be useful in many areas like expert systems [6], aggregation [3,34] and the fuzzy integral [4,21].

The main difference in the definition of the uninorms and aggregative operators is that the self-duality requirement does not appear in uninorms, and the neutral element property is not in the definition for the aggregative operators. The representation theorem for strict, continuous on $[0,1] \times [0,1] \backslash \{(0,1),(1,0)\}$ uninorms (or representable uninorms) was given by Fodor et al. [16] (see also Klement et al. [20]). Such uninorms are called representable uninorms and they were previously introduced as aggregative operators [7]. Recently, a characterization of the class of uninorms with a strict underlying t-norm and t-conorm was presented in [13]. In [24], the authors show that uninorms with nilpotent underlying t-norm and t-conorm belong to U_{min} or U_{max}. Further results on uninorms with fixed values along their borders can be found in [5].

Our main purpose here is to consider generated nilpotent operators in an integral frame and to examine the nilpotent self-dual generated operators. A general parametric framework for the nilpotent conjunctive, disjunctive, aggregative and negation operators is given and it is demonstrated how the nilpotent generated operator can be applied for preference modeling.

The article is organized as follows. After a preliminary discussion in Section 2, a general parametric operator $o_\nu(\mathbf{x})$ of nilpotent systems is given in Section 3. The parameter has an important semantical meaning as the threshold of expectancy. In Section 4, the weighted form of this operator, $a_{\nu,\mathbf{w}}(\mathbf{x})$ is examined. In Section 5, the properties (De Morgan property, commutativity, self-duality, fulfillment of the boundary conditions, bisymmetry) of the weighted general operator are examined. Here, the formula for the commutative self-De Morgan operator, the so-called weighted aggregative operator is presented. Then in Section 6 we focus on the two-variable case, where it is proved that the two-variable operator with weights $w_1 = w_2 = 1$ is conjunctive for low input values, disjunctive for high ones, and averaging otherwise; i.e. a high input can compensate for a lower one. In Section 7, the main results are summarized and a possible direction of future work is mentioned.

2. Preliminaries

2.1. Negations, t-norms and t-conorms

First, we recall some basic notations and results regarding negation operators, t-norms and t-conorms that will be useful in the sequel.

Definition 1. A unary operation $n : [0,1] \to [0,1]$ is called a negation if it is non-increasing and compatible with classical logic; i.e. $n(0) = 1$ and $n(1) = 0$.

A negation is strict if it is also strictly decreasing and continuous.

A negation is strong, if it is also involutive; i.e. $n(n(x)) = x$.

The well-known representation theorem for strong negations was obtained by Trillas in [30]:

Proposition 1. *$n(x) : [0,1] \to [0,1]$ is a strong negation if and only if there exists an increasing bijection $f_n(x) : [0,1] \to [0,1]$ such that*

$$n(x) = f_n^{-1}(1 - f_n(x)).$$

Remark 1. In Proposition 1, the bijection may also be decreasing (see Dombi and Csiszár [8]).

Definition 2. Let $o(x, y) : [0,1]^2 \to [0,1]$, and let $n(x)$ be the negation generated by $f(x) : [0,1] \to [0,1]$. The operator $o(x, y)$ satisfies the self-De Morgan property if it satisfies the following equation for all $x, y \in [0,1]$:

$$n(o(x, y) = o(n(x), n(y)).$$

A triangular norm (*t-norm* for short) T is a binary operation on the closed unit interval $[0,1]$ such that $([0,1], T)$ is an abelian semigroup with neutral element 1 which is totally ordered; i.e., for all $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \le x_2$ and $y_1 \le y_2$, we have $T(x_1, y_1) \le T(x_2, y_2)$, where $\le$ is the natural order on $[0,1]$.

A triangular conorm (*t-conorm* for short) S is a binary operation on the closed unit interval $[0,1]$ such that $([0,1], S)$ is an abelian semigroup with neutral element 0 which is totally ordered.

A continuous t-norm T is said to be *Archimedean* if $T(x,x) < x$ holds for all $x \in (0,1)$, *strict* if T is strictly increasing on $(0,1]^2$; i.e. $T(x,y) < T(x,z)$ whenever $x \in (0,1]$ and $y < z$, and *nilpotent* if each $a \in (0,1)$ is a nilpotent element; i.e. $\exists n \in \{1,2,...\}$ such that $T(\underbrace{a,a,...a}_{n\text{-times}}) = 0$ for any $a \in (0,1)$.

From the duality between t-norms and t-conorms we can readily obtain the similar properties for t-conorms as well.

Proposition 2. *[25] A function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm if and only if it has a continuous additive generator; i.e. there exists a continuous strictly decreasing function $t : [0,1] \to [0,\infty)$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x,y) = t^{-1}(min(t(x) + t(y), t(0))), \quad x, y \in [0,1]. \tag{1}$$

Proposition 3. *[25] A function $S : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-conorm if and only if it has a continuous additive generator; i.e. there exists a continuous strictly increasing function $s : [0,1] \to [0,\infty]$ with $s(0) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$S(x,y) = s^{-1}(min(s(x) + s(y), s(1))), \quad x, y \in [0,1]. \tag{2}$$

Proposition 4. *[19]*

A t-norm T is strict if and only if $t(0) = \infty$ holds for each continuous additive generator t of T.
A t-norm T is nilpotent if and only if $t(0) < \infty$ holds for each continuous additive generator t of T.
A t-conorm S is strict if and only if $s(1) = \infty$ holds for each continuous additive generator s of S.
A t-conorm S is nilpotent if and only if $s(1) < \infty$ holds for each continuous additive generator s of S.

In both of Propositions 2 and 3 above we can permit the generator functions to be strictly increasing *or* strictly decreasing, which will mean that they can be determined up to a (not necessarily positive) multiplicative constant. In this case we have $t(0) = \pm\infty$ and $s(1) = \pm\infty$ for strict norms and, similarly, $t(0) < \infty$ *or* $t(0) > -\infty$ and $s(1) < \infty$ *or* $s(1) > -\infty$ for the nilpotent ones.

From the definitions of t-norms and t-conorms it follows immediately that t-norms are conjunctive, while t-conorms are disjunctive aggregation functions. This is why they are widely used as conjunctions and disjunctions in multivalued logical structures.

Since the generator functions of the nilpotent t-norms and t-conorms are bounded and determined up to a multiplicative constant (see Propositions 2 and 3), they can be normalized (see [8]). Let us use the following notations for the uniquely defined normalized generator functions:

$$f_c(x) := \frac{t(x)}{t(0)}, \qquad f_d(x) := \frac{s(x)}{s(1)}.$$

Next, we define the so-called cutting function in order to simplify the notations.

Definition 3. (See Dombi and Csiszár [8], Sabo and Strezo [28]) Let us define the cutting operation [ ] by

$$[x] = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 < x \end{cases}$$

and let the notation [ ] also act as brackets when writing the argument of an operator. Then we can write $f[x]$ instead of $f([x])$.

Definition 4. Let us define the generalized cutting operation $[\ ]_a^b$ by

$$[x]_a^b := \max(a, \min(b, x)),$$

where $a, b \in \mathbb{R}, a < b$.

Remark 2. For $a = 0$ and $b = 1$, we get the cutting function defined in Definition 3.

Proposition 5. *(See Dombi and Csiszár [8]) With the help of the cutting operator, we can write the conjunction and disjunction operators in the following form, where $f_c(x)$ and $f_d(x)$ are decreasing and increasing normalized generator functions, respectively.*

$$c(x,y) = f_c^{-1}[f_c(x) + f_c(y)], \tag{3}$$

$$d(x,y) = f_d^{-1}[f_d(x) + f_d(y)]. \tag{4}$$

Remark 3. Note that we use the notation $c(x, y)$ and $d(x, y)$ for the conjunction and disjunction to emphasize the use of the normalized generator functions.

To construct a logical system, we need to define the appropriate logical operators. As in [8] and [9], we consider connective systems where the conjunction and disjunction operators are special types of t-norms and t-conorms, respectively.

Definition 5. [8] The triple (c, d, n), where c is a continuous Archimedean t-norm, d is a continuous Archimedean t-conorm and n is a strong negation, is called a connective system.

Definition 6. [8] A connective system is nilpotent if the conjunction c is a nilpotent t-norm, and the disjunction d is a nilpotent t-conorm.

Definition 7. An operator $o(x, y) : [0, 1]^2 \to [0, 1]$ is bisymmetric if it satisfies the following equation for all $x_i \in [0, 1]$:

$$o(o(x_1, x_2), o(x_3, x_4)) = o(o(x_1, x_3), o(x_2, x_4)).$$

Definition 8. An operator $o(\mathbf{x}) : [0, 1]^n \to [0, 1]$ is idempotent if for all $\mathbf{x} = (x, \ldots x)$, $x \in [0, 1]$:

$$o(\mathbf{x}) = x.$$

The concept of aggregative operators and uninorms will play an important role in the sequel.

Definition 9. (See Dombi [7].) An aggregative operator is a function $a : [0, 1] \to [0, 1]$ with the following properties:

1. Continuous on $[0, 1]^2 \setminus \{(0, 1), (1, 0)\}$;
2. $a(x, y) < a(x, y')$ if $y < y', x \neq 0, x \neq 1, a(x, y) < a(x', y)$ if $x < x', y \neq 0, y \neq 1$;
3. $a(0, 0) = 0$ and $a(1, 1) = 1$ (boundary conditions);
4. There exists a strong negation η such that $a(x, y) = \eta(a(\eta(x), \eta(y)))$ (the self-De Morgan identity) if $\{x, y\} \neq \{0, 1\}$;
5. $a(1, 0) = a(0, 1) = 0$ or $a(1, 0) = a(0, 1) = 1$.

Definition 10. (See Yager and Rybalov [33].) A mapping $U : [0, 1] \times [0, 1] \to [0, 1]$ is a uninorm, if it is symmetric, associative, nondecreasing and there exists an $e \in [0, 1]$ such that $U(e, x) = x$ for all $x \in [0, 1]$.

The structure of uninorms was first examined by Fodor et al. in [16].

Proposition 6. *(See Fodor et al. [16].) Let $U : [0, 1]^2 \to [0, 1]$ be a function and $\nu \in]0, 1[$. The following statements are equivalent:*

1. *U is a uninorm with neutral element ν which is strictly monotonic on $]0, 1[^2$ and continuous on $[0, 1]^2 \setminus \{(0, 1), (1, 0)\}$.*
2. *There exists an increasing bijection $u : [0, 1] \to (-\infty, \infty)$ with $u(\nu) = 0$ such that for all $(x, y) \in [0, 1]^2$, we have*

$$U(x, y) = u^{-1}(u(x) + u(y)), \tag{5}$$

where, in the case of a conjunctive uninorm U, we use the convention $\infty + (-\infty) = -\infty$, while, in the disjunctive case, we use $\infty + (-\infty) = \infty$.

If (5) holds, the function u is uniquely determined by U up to a positive multiplicative constant, and it is called an additive generator of the uninorm U.

3. Shifting transformations on the generator functions – a general parametric formula

From now on, we consider nilpotent logical systems. First we show that by shifting the generator function of a disjunction, we can get a conjunction and also operators that fulfill the self-De Morgan property. We provide a general parametric formula for these operators, where the conjunction, disjunction and the so-called aggregative operator differ only in one single parameter. See Fig. 1.

Definition 11. Let $f : [0, 1] \to [0, 1]$ be an increasing bijection, $\nu \in [0, 1]$, and $\mathbf{x} = (x_1, \ldots, x_n)$, where $x_i \in [0, 1]$ and let us define the general operator by

$$o_\nu(\mathbf{x}) = f^{-1}\left[\sum_{i=1}^{n}(f(x_i) - f(\nu)) + f(\nu)\right] = f^{-1}\left[\sum_{i=1}^{n} f(x_i) - (n-1)f(\nu)\right]. \tag{6}$$

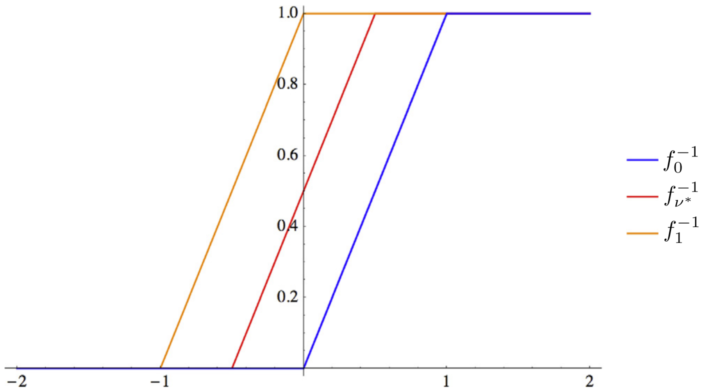


Fig. 1. Shifting transformation in the linear case, $f_\nu^{-1}(x)$ for $\nu = 0$, $\nu = \nu^*$, $\nu = 1$; where $\nu^* = f^{-1}(\frac{1}{2})$.

Remark 4. Note that ν is a neutral element of $o_\nu(\mathbf{x})$ and that $o_\nu(\mathbf{x})$ can be generated by $g(x) = f(x) - f(\nu)$, since in this case $g^{-1}(x) = f^{-1}(x + f(\nu))$.

Aggregation functions generated in a similar way as (6) were also discussed by Kolesárová and Komorníková [23].

Proposition 7. *The general operator in* (6)

1. *For* $\nu = 1$ *is* $o_1(\mathbf{x}) = c(\mathbf{x})$*, a conjunction.*
2. *For* $\nu = 0$ *is* $o_0(\mathbf{x}) = d(\mathbf{x})$*, a disjunction.*

Proof. Since $f(1) = 1$ and $f(0) = 0$, the proof is trivial. □

Remark 5. A conjunction and a disjunction differ only in one parameter of the general operator in (6). The parameter has the semantical meaning of the level of expectancy. Generalized conjunction and disjunction functions (GCD) were also examined by Dujmović and Larsen in [12].

Next, a more general, weighted form of this operator will be examined.

4. The weighted general operator

If a weighted operator $o_{\mathbf{w}}(\mathbf{x}) : [0,1]^n \to [0,1]$ with $\mathbf{w} = (w_1, \ldots, w_n)$, $w_i > 0$ real parameters is represented by $o_{\mathbf{w}}(\mathbf{x}) = f^{-1}\left[\sum_{i=1}^{n} w_i f(x_i)\right]$, where $f : [0,1] \to [0,1]$ is a bijection, then it can also be written as $o_{\mathbf{w}}(\mathbf{x}) = f^{-1}\left[\sum_{i=1}^{n} f(x'_i)\right]$, where x'_i is got via a so-called weighting transformation: $x'_i = f^{-1}(w_i f(x_i))$.

Below, we apply this weighting transformation to the arguments of the operator in (6) to get the so-called weighted general operator.

Definition 12. Let $\mathbf{w} = (w_1, \ldots, w_n)$ and $w_i > 0$ be real parameters, $f : [0,1] \to [0,1]$ an increasing bijection with $\nu \in [0,1]$. The weighted general operator is defined by

$$a_{\nu,\mathbf{w}}(\mathbf{x}) := f^{-1}\left[\sum_{i=1}^{n} w_i(f(x_i) - f(\nu)) + f(\nu)\right]. \quad (7)$$

5. Properties of the general and the weighted general operator

5.1. De Morgan property

The question that immediately arises is: for which parameter values does the above-defined general operator satisfy the generalized De Morgan property concerning the negation generated by $f(x)$ (the generator function of $o(\mathbf{x})$). That is, for which values of ν_1, ν_2 satisfy the following equation for all $\mathbf{x} \in [0,1]^n$:

$$n(o_{\nu_1}(\mathbf{x})) = o_{\nu_2}(n(\mathbf{x})).$$

For $\nu_1 = 0, \nu_2 = 1$ or $\nu_1 = 1, \nu_2 = 0$, we get the classical De Morgan law.

Proposition 8. *Let $f : [0,1] \to [0,1]$ be an increasing bijection, $\nu_i \in [0,1]$ and $\mathbf{x} = (x_1, \ldots, x_n)$, where $x_i \in [0,1]$, $n(x) = f^{-1}(1 - f(x))$ and $o_{\nu_i}(\mathbf{x})$ the general operator. Then*

$$n(o_{\nu_1}(\mathbf{x})) = o_{\nu_2}(n(\mathbf{x}))$$

holds for all $\mathbf{x} = (x_1, \ldots, x_n)$, where $x_i \in [0,1]$ if and only if $f(\nu_1) + f(\nu_2) = 1$.

Proof. Using the fact that $n(x) = f^{-1}(1 - f(x))$, we get

$$f^{-1}\left(1 - \left[\sum_{i=1}^{n} f(x_i) - (n-1)f(\nu_1)\right]\right) = f^{-1}\left[\sum_{i=1}^{n} (1 - f(x_i)) - (n-1)f(\nu_2)\right].$$

1. First, we show that $f(\nu_1) + f(\nu_2) = 1$ is necessary.
   Using the notations $A := \sum_{i=1}^{n} f(x_i)$, $B := -(n-1)f(\nu_1)$, we get
   $$[A - (n-1)f(\nu_2)] = 1 - [A + B]. \tag{8}$$
   Let us consider the following cases:
   (a) First let us assume that $\nu_1 \neq 0; 1$. (8) must hold for all $\mathbf{x} \in [0,1]^n$, in particular for $\mathbf{x} = (\nu_1, \ldots, \nu_1)$. In this case $0 < A + B = f(\nu_1) < 1$, so the cutting function can be omitted, and we get $B = (1-n) + (n-1)f(\nu_2)$, from which $f(\nu_1) + f(\nu_2) = 1$.
   (b) Next, we show that the cutting function can also be omitted, if $\nu_1 = 0$ (i.e. $B = 0$), $\nu_2 \neq 1$. This means that we have to show that
      i. $n - A - (n-1)f(\nu_2) \leq 0$ and $A + B = A \geq 1$, or
      ii. $n - A - (n-1)f(\nu_2) \geq 1$ and $A + B = A \leq 0$ cannot hold for all $\mathbf{x} \in [0,1]^n$.
      For example, for $\mathbf{x} = (x, \ldots, x)$, where $x = f^{-1}\left(\frac{1}{n}\right) \neq 0$, we get $A = 1$, and $(n-1)(1 - f(\nu_2)) > 0$.
   (c) Next, we show that the cutting function can also be omitted if $\nu_1 = 1$ (i.e. $B = 1 - n$) and $\nu_2 \neq 0$. This means that we have to show that
      i. $n - A - (n-1)f(\nu_2) \leq 0$ and $A + B = A + 1 - n \geq 1$, or
      ii. $n - A - (n-1)f(\nu_2) \geq 1$ and $A + B = A + 1 - n \leq 0$ cannot hold for all $\mathbf{x} \in [0,1]^n$.
      Since $A \leq n$, the first condition in 1(c)i holds only for $\mathbf{x} = \mathbf{1}$, not for all $\mathbf{x} \in [0,1]^n$. The condition in 1(c)ii does not hold for $\mathbf{x} = \mathbf{1}$ and $A = n$, say.
   (d) For $\nu_1 = 0, \nu_2 = 1$ or $\nu_1 = 1, \nu_2 = 0$, the self-De Morgan property trivially holds.
2. Next, we prove that $f(\nu_1) + f(\nu_2) = 1$ is also sufficient.
   If $f(\nu_1) + f(\nu_2) = 1$ holds, than $f(\nu_1) = 1 - f(\nu_2)$, so we have to prove the following equation:
   $$f^{-1}(1 - [A - n + 1 + C]) = f^{-1}[n - A - C],$$
   where $A := \sum_{i=1}^{n} f(x_i)$ and $C := (n-1)f(\nu_2)$. Since $1 - [A - n + 1 + C] = [1 - A + n - 1 - C]$, the statement is trivial. □

Remark 6. For $\nu_1 = \nu_2$, the only solution is $\nu_1 = \nu_2 = f^{-1}\left(\frac{1}{2}\right)$; i.e. the self-De Morgan property holds if and only if the parameter ν is the fixpoint of the negation; i.e. $\nu = f^{-1}\left(\frac{1}{2}\right) = \nu^*$.

Remark 7. For $\nu_1 = \nu_2$, we find that the operator $o_\nu(x, y)$ fulfills the self-De Morgan property if and only if it has the following form:

$$f^{-1}\left[\sum_{i=1}^{n} f(x_i) - \frac{n-1}{2}\right]. \tag{9}$$

In particular, for two variables:

$$f^{-1}\left[f(x) + f(y) - \frac{1}{2}\right]. \tag{10}$$

Proposition 9. *Let $f : [0,1] \to [0,1]$ be an increasing bijection, $\nu \in [0,1]$ and $\mathbf{x} = (x_1, \ldots, x_n)$, where $x_i \in [0,1]$, $\mathbf{w} = (w_1, \ldots, w_n)$, $w_i > 0$, $n(x) = f^{-1}(1 - f(x))$. The weighted general operator $a_{\nu,\mathbf{w}}(\mathbf{x})$ satisfies the self-De Morgan property, if and only if $\sum_{i=1}^{n} w_i = 1$, or $\nu = f^{-1}\left(\frac{1}{2}\right) = \nu^*$.*

Proof. The self-De Morgan property means that

$$n(a_{\nu,\mathbf{w}}(\mathbf{x})) = a_{\nu,\mathbf{w}}(n(x))$$

holds for all $\mathbf{x}$; i.e.

$$f^{-1}\left(1 - \left[\sum_{i=1}^{n} w_i\,(f(x_i) - f(\nu)) + f(\nu)\right]\right) = f^{-1}\left[\sum_{i=1}^{n} w_i\,(1 - f(x_i) - f(\nu)) + f(\nu)\right].$$

Let $A := \sum_{i=1}^{n} w_i f(x_i)$ and $B := \sum_{i=1}^{n} w_i$. Since $f(x)$ is strictly increasing, we have to show that

$$1 - [A - f(\nu)(B-1)] = [B - A - Bf(\nu) + f(\nu)].$$

1. First, we show that this condition is sufficient. If $B = 1$, then we get $1 - [A] = [1 - A]$, which always holds. If $f(\nu) = \frac{1}{2}$, then we get $1 - \left[A - \frac{B-1}{2}\right] = \left[B - A - \frac{B}{2} + \frac{1}{2}\right]$. Using the fact that $1 - [x] = [x]$ always holds, we can readily see that the two sides are equal.
2. Second, we show that this condition is also necessary.
   (a) First, let us assume that $\nu \neq 0; 1$. For $\mathbf{x} = (\nu, \ldots, \nu)$, $A = f(\nu)B$, so on the left hand side we get $1 - [f(\nu)]$, which means that the cutting function can be omitted. Thus $2f(\nu)(B-1) = B - 1$, from which $B = \sum_{i=1}^{n} w_i = 1$, or $f(\nu) = \frac{1}{2}$.
   (b) For $\nu = 0$, we get $1 - [A] = [B - A]$. For $\mathbf{x}_0 = (x_0, \ldots, x_0)$, where $0 < x_0 < 1$, $A = f(x_0)B$; i.e. $1 - [f(x_0)B] = [(1 - f(x_0))B]$, where the cutting function can be omitted, since $f(x_0)B > 0$ and $(1 - f(x_0))B > 0$. Thus $B = \sum_{i=1}^{n} w_i = 1$.
   (c) For $\nu = 1$, we get $1 - [A - B + 1] = [-A + 1]$. For $\mathbf{x}_0 = (x_0, \ldots, x_0)$, $0 < x_0 < 1$, $A = f(x_0)B$, so $1 - [f(x_0)B - B + 1] = [-f(x_0)B + 1]$; i.e. $[B - f(x_0)B] = [1 - f(x_0)B]$ must hold.
   - If $B = \sum_{i=1}^{n} w_i \leq 1$, then the cutting function can be omitted, and we get $B = \sum_{i=1}^{n} w_i = 1$.
   - If $B = \sum_{i=1}^{n} w_i \geq 1$, then let $f(x_0) := \frac{1}{\sum_{i=1}^{n} w_i} = \frac{1}{B}$. So we get $B \leq 1$, and $B = \sum_{i=1}^{n} w_i = 1$ must hold. □

Proposition 10. *The weighted general operator $a_{\nu,\mathbf{w}}(\mathbf{x})$ is commutative, if and only if $w_1 = w_2 = \cdots = w_n$.*

Proof. Trivial. □

Corollary 1. *A commutative weighted general operator fulfills the self-De Morgan property if and only if $w = \frac{1}{n}$ or $\nu = \nu^*$, where $f(\nu^*) = \frac{1}{2}$; i.e. it has one of the following forms:*

$$f^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right] \tag{11}$$

or

$$f^{-1}\left[w\left(\sum_{i=1}^{n} f(x_i) - \frac{n}{2}\right) + \frac{1}{2}\right]. \tag{12}$$

Remark 8. Note that (11) is a special case of (12) for $w = \frac{1}{n}$.

Remark 9. If $a_{\nu,\mathbf{w}}$ is commutative and satisfies the self-De Morgan operator, then it is independent of the parameter ν. Therefore the lower index ν can be omitted, and we will refer to this case simply as a_w.

As we have seen, the weighted general operator of the form $f^{-1}\left[w\left(\sum_{i=1}^{n} f(x_i) - \frac{n}{2}\right) + \frac{1}{2}\right]$, is commutative and satisfies the self-De Morgan property. With such nice properties it is a good idea to give it a distinctive name.

Definition 13. The operator

$$a_w(\mathbf{x}) = f^{-1}\left[w\left(\sum_{i=1}^{n} f(x_i) - \frac{n}{2}\right) + \frac{1}{2}\right], \tag{13}$$

where $w > 0$, is called the weighted aggregative operator.

Proposition 11. *The weighted general operator* $a_{\nu,\mathbf{w}}(\mathbf{x})$ *satisfies*

1. *The boundary condition* $a_{\nu,\mathbf{w}}(\mathbf{0}) = 0$*, if and only if* $\nu = 0$ *or* $\sum_{i=1}^{n} w_i \geq 1$ *(for a commutative operator:* $w \geq \frac{1}{n}$*);*
2. *The boundary condition* $a_{\nu,\mathbf{w}}(1, \ldots, 1) = 1$*, if and only if* $\nu = 1$ *or* $\sum_{i=1}^{n} w_i \geq 1$ *(for a commutative operator:* $w \geq \frac{1}{n}$*);*
3. *Both of the above-mentioned boundary conditions if and only if* $\sum_{i=1}^{n} w_i \geq 1$ *(for a commutative operator:* $w \geq \frac{1}{n}$*);*
4. $a_{\nu,\mathbf{w}}(\nu, \ldots, \nu) = \nu$.

Proof. Let $B := \sum_{i=1}^{n} w_i$.

1. $a_{\nu,\mathbf{w}}(\mathbf{0}) = f^{-1}\left[\left(\sum_{i=1}^{n} w_i\,(-f(\nu))\right) + f(\nu)\right] = 0$, if and only if $f(\nu)(1-B) \leq 0$; i.e. $\nu = 0$ or $\sum_{i=1}^{n} w_i \geq 1$.
2. $a_{\nu,\mathbf{w}}(1, \ldots, 1) = f^{-1}\left[\left(\sum_{i=1}^{n} w_i\,(1 - f(\nu))\right) + f(\nu)\right] = 1$, if and only if $(1 - f(\nu))B + f(\nu) \geq 1$; i.e. $(1-B)(f(\nu) - 1) \geq 0$, so $\nu = 1$ or $\sum_{i=1}^{n} w_i \geq 1$.
3. It follows from the previous two statements.
4. $a_{\nu,\mathbf{w}}(\nu, \ldots, \nu) = f^{-1}[f(\nu)] = \nu$. □

Remark 10. Note that for commutative operators, the condition $\sum_{i=1}^{n} w_i \geq 1$ is equivalent to $w \geq \frac{1}{n}$.

5.2. Bisymmetry

An important property of aggregation functions concerns the grouping character; i.e. whether it is possible to build a partial aggregation for subgroups of input values, and then to get the overall value by combining these partial results. A strong form of such a condition is associativity, which allows us to start with the aggregation process before knowing all inputs to be aggregated. However, associativity is a rather restrictive property. Associativity and idempotency together cancel the effect of repeating arguments in the aggregation procedure, so it is not possible to simulate the presence of weights by repeating arguments. A weaker condition is bisymmetry, which expresses the fact that the aggregation of the elements of any matrix can be performed first on the rows, then on the columns, or conversely. This natural property means that in the case of n judges and m candidates, say, the overall score of the candidates can be calculated by first aggregating the scores of each candidate, and then aggregating these overall values; or an alternative way is to first aggregate the scores given by each judge and then aggregate these values. The following propositions characterize bisymmetric and associative functions (see Aczél [1,2]).

Proposition 12. *An operator* $o : [0,1]^n \to \mathbb{R}$ *is continuous, strictly increasing, idempotent, and bisymmetric if and only if it represents a quasi-linear mean; i.e. there is a continuous and strictly monotonic function* $f : [0,1] \to \mathbb{R}$ *such that*

$$o(\mathbf{x}) = f^{-1}\left(\sum_{i=1}^{n} w_i f(x_i)\right),$$

where $w_i > 0$, $\sum_{i=1}^{n} w_i = 1$.

Proposition 13. *An operator* $o : [0,1]^n \to \mathbb{R}$ *is continuous, strictly increasing and bisymmetric if and only if it represents a quasi-linear function; i.e. there is a continuous and strictly monotonic function* $f : [0,1] \to \mathbb{R}$ *such that*

$$o(\mathbf{x}) = f^{-1}\left(\sum_{i=1}^{n} w_i f(x_i) + b\right),$$

where $w_i > 0$, $b \in \mathbb{R}$.

If instead of bisymmetry the function satisfies the stronger conditions of commutativity and associativity, then we have the following corollary when $w_i = 1$.

Proposition 14. *An operator $o : [0,1]^n \to \mathbb{R}$ is continuous, strictly increasing, commutative and associative if and only if it represents a quasi-linear function with $w_i = 1$; i.e. there is a continuous and strictly monotonic function $f : [0,1] \to \mathbb{R}$ such that*

$$o(\mathbf{x}) = f^{-1}\left[\sum_{i=1}^{n} f(x_i) + b\right],$$

$b \in \mathbb{R}$.

Proposition 15. *The weighted aggregative operator with weights $w \le \frac{1}{n}$ is bisymmetric.*

Proof. 1. Since $0 \le f(x) \le 1$, $0 \le \sum_{i=1}^{n} f(x_i) \le n$. Therefore, $0 \le w\left(\sum_{i=1}^{n} f(x_i) - \frac{n}{2}\right) + \frac{1}{2} \le 1$, so in (13), the cutting function can be omitted, and so the operator has the form of the function in Proposition 13, which means it is bisymmetric. □

6. The two-variable general and weighted aggregative operator

Now, we examine the weighted aggregative operator of two variables.

Corollary 2. *A commutative weighted general operator $a_{\nu,\mathbf{w}}$ fulfills the self-De Morgan property (see Definition 2) if and only if $w = \frac{1}{2}$ or $\nu = \nu^*$, where $f(\nu^*) = \frac{1}{2}$; i.e. the weighted aggregative operator of two variables has the following form:*

$$f^{-1}\left[w(f(x) + f(y) - 1) + \frac{1}{2}\right]. \tag{14}$$

Proof. It follows directly from Proposition 9. □

Remark 11. Note that for $w = \frac{1}{2}$, (14) has the following form:

$$f^{-1}\left[\frac{f(x) + f(y)}{2}\right]. \tag{15}$$

This is the so-called general arithmetic mean, where the cutting function can be omitted.

Corollary 3. *A two-variable weighted aggregative operator a_w,*

$$n(a_w(n(x), x)) = a_w(n(x), x) = \nu^*,$$

and, in particular, $a_w(0,1) = a_w(1,0) = \nu^*$, where $\nu^* = f^{-1}\left(\frac{1}{2}\right)$.

Corollary 4. *A two-variable commutative weighted general operator $a_{\nu,\mathbf{w}}$ satisfies the boundary conditions*

1. *$a_{\nu,\mathbf{w}}(0,0) = 0$, if and only if $w \ge \frac{1}{2}$ or $\nu = 0$;*
2. *$a_{\nu,\mathbf{w}}(1,1) = 1$, if and only if $w \ge \frac{1}{2}$ or $\nu = 1$.*

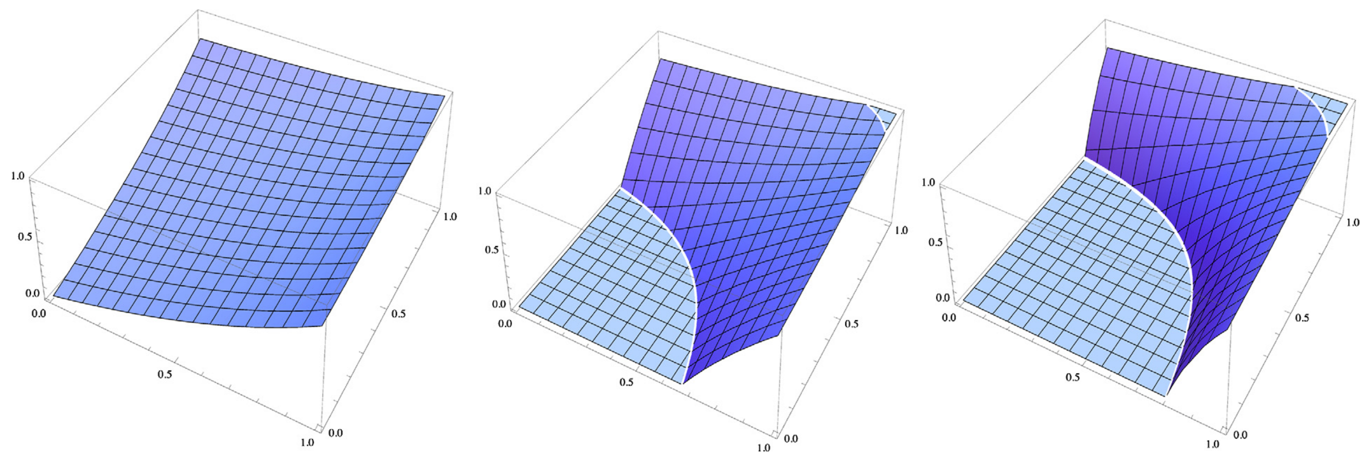
Proof. It follows directly from Proposition 11. □

Corollary 5. *A two-variable commutative weighted aggregative operator a_w satisfies the boundary conditions $a_w(0,0) = 0$ and $a_w(1,1) = 1$, if and only if $w \ge \frac{1}{2}$.*

Corollary 6. *A weighted aggregative operator $a_w(x,y)$ satisfies the boundary conditions $a_w(0,0) = 0$ and $a_w(1,1) = 1$, if and only if it has the following form:*

$$a_w(x,y) = f^{-1}\left[w(f(x) + f(y) - 1) + \frac{1}{2}\right], \tag{16}$$

where $w \ge \frac{1}{2}$.



(a) $w = 0.5$ (b) $w = 0.75$ (c) $w = 1$

Fig. 2. The weighted aggregative operator a_w for $f(x) = \dfrac{1}{1 + \frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}$, $\nu_d = 0.8$.

Proposition 16. *A weighted aggregative operator $a_w(x, y)$, which satisfies the boundary conditions, has the following property (see Fig. 2):*

1. *If $x, y \le \nu$, then $a_w(x, y) \le \nu$,*
2. *If $x, y \ge \nu$, then $a_w(x, y) \ge \nu$.*

Proof. A weighted aggregative operator $a_w(x, y)$, which satisfies the boundary conditions has the following form:

$$a_w(x, y) = f^{-1}\left[w(f(x) + f(y) - 1) + \frac{1}{2}\right],$$

where $w \ge \frac{1}{2}$.

1. First, we consider the case where ν is the fix point of the negation; i.e. $\nu = \nu^*$, $f(\nu) = \frac{1}{2}$.
   (a) If $x, y \le \nu$, then from the increasing property of $f(x)$, we find that $f(x), f(y) \le \frac{1}{2}$; i.e. $w(f(x) + f(y) - 1) + \frac{1}{2} \le \frac{1}{2}$, so $a_w(x, y) \le \nu$.
   (b) If $x, y \ge \nu$, then from the increasing property of $f(x)$, we find that $f(x), f(y) \ge \frac{1}{2}$; i.e. $w(f(x) + f(y) - 1) + \frac{1}{2} \ge \frac{1}{2}$, so $a_w(x, y) \ge \nu$.
2. Second, we consider the case where $w = \frac{1}{2}$. From $x, y \le \nu$ follows that $f(x), f(y) \le f(\nu)$. Therefore, $f^{-1}\left[\frac{f(x)+f(y)}{2}\right] \le f^{-1}[f(\nu)] = \nu$. □

Proposition 17. *A weighted aggregative operator, with $w_1 = w_2 = 1$, has the following properties (see Fig. 3):*

1. *If $x, y \le \nu^*$, then $a_1(x, y)$ is conjunctive; i.e. $\forall x, y$ $a_1(x, y) \le min(x, y)$.*
2. *If $x, y \ge \nu^*$, then $a_1(x, y)$ is disjunctive; i.e. $\forall x, y$ $a_1(x, y) \ge max(x, y)$.*
3. *If $x \le \nu^* \le y$, or $y \le \nu^* \le x$ then $a_1(x, y)$ is averaging; i.e. $\forall x, y$ $min(x, y) \le a_1(x, y) \le max(x, y)$, where $\nu^* = f^{-1}\left(\frac{1}{2}\right)$.*

Proof. The operator has the following form:

$$a_1(x, y) = f^{-1}\left[(f(x) + f(y) - 1) + \frac{1}{2}\right] = f^{-1}\left[f(x) + f(y) - \frac{1}{2}\right].$$

1. Let us assume that $x \le y \le \nu^*$. From the increasing property of $f(x)$, we see that $f(x) \le f(y) \le f(\nu^*) = \frac{1}{2}$; i.e. $a_1(x, y) = f^{-1}\left[(f(x) + f(y) - 1) + \frac{1}{2}\right] \le x = \min(x, y)$.
2. Let us assume that $\nu^* \le x \le y$. From the increasing property of $f(x)$, we see that $\frac{1}{2} = f(\nu^*) \le f(x) \le f(y)$; i.e. $a_1(x, y) = f^{-1}\left[(f(x) + f(y) - 1) + \frac{1}{2}\right] \ge y = \max(x, y)$.
3. Let us assume that $x \le \nu^* \le y$. If $x \le \nu^* \le y$, then $f(x) \le \frac{1}{2} \le f(y)$, so $\min(x, y) = x \le a_1(x, y) = f^{-1}[(f(x) + f(y) - 1) + \frac{1}{2}] \le y = \max(x, y)$. □

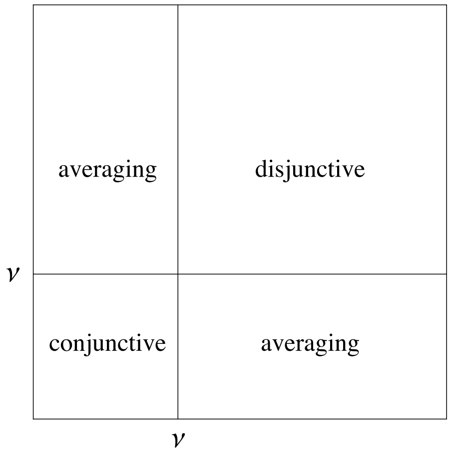


Fig. 3. Uninorm-like property of the weighted aggregative operator $a_1(x, y)$.

Remark 12. The above-mentioned property holds if and only if $w = 1$. For $w > 1$, the conjunctive and disjunctive properties hold, but the averaging property does not.

Remark 13. As we have seen, $a_1(x, y)$ has a uninorm-like property (see Proposition 17) and it satisfies the self-De Morgan property as well. However, it is not associative (since $a_1(0, 1) = a_1(1, 0) = f^{-1}\left(\frac{1}{2}\right) = \nu^*$) and therefore it cannot be a uninorm. Note that aggregative operators in the strict case (see Dombi [7]) are always associative and therefore they are special uninorms.

Note that by substituting $n(x)$ and y in the commutative self-De Morgan weighted aggregative operator, the operator $a(n(x), y)$ has certain properties that are similar to those expected of a preference operator. Preference modeling is a fundamental part of several applied fields of decision-making [15]. In the classical theory, preference is a binary relation closely related to implications, with the meaning

$$xRy \iff \text{“}y \text{ is not worse than } x\text{”}.$$

Preferences between alternatives can also be described by a valued preference relation p, such that the value $p(x, y)$ is normalized, and it is understood as the degree to which the statement "y is not worse than x" is true: $p(x, y) = \text{truth of}(y \geq x)$. Here, p is a continuous function, which is strictly decreasing in the first variable and strictly increasing in the second one, and $p(x, y) = n(p(y, x))$ must also hold. Therefore it is sensible to define preference in the following way:

Definition 14. Let $w > 0$ be a real parameter and $f : [0, 1] \to [0, 1]$ be an increasing bijection. Moreover, let us define the preference operator as $p_w(x, y) = a_w(n(x), y) = f^{-1}\left[w(f(y) - f(x)) + \frac{1}{2}\right]$.

Remark 14. Note that the negation operator generated by $f(x) : [0, 1] \to [0, 1]$ can be expressed in the following way:

$$n(x) = f^{-1}\left((f(\nu^*) - f(x)) + f(\nu^*)\right), \tag{17}$$

where $\nu^* = f^{-1}\left(\frac{1}{2}.\right)$

Corollary 7. *We have shown that the general formula*

$$a_{\nu,\mathbf{w}}(\mathbf{x}) := f^{-1}\left[\sum_{i=1}^{n} w_i(f(x_i) - f(\nu)) + f(\nu)\right] \tag{18}$$

for the weighted general operator includes the following special cases:

1. *For $f(\nu) = 1$ and $w_i = 1\ \forall i$, it is a conjunction with generator $1 - f(x)$.*
2. *For $f(\nu) = 0$ and $w_i = 1\ \forall i$, it is a disjunction with generator $f(x)$.*
3. *For $f(\nu) = \frac{1}{2}$ or $\sum_{i=1}^{n} w_i = 1$, it satisfies the self-De Morgan property.*
4. *For $f(\nu) = \frac{1}{2}$ and $w_1 = \cdots = w_n$, or for $w_i = \frac{1}{n}$, it is a weighted aggregative operator (a commutative self-De Morgan operator).*

5. *For* $\nu = 0$ *or* $\sum_{i=1}^{n} w_i \geq 1$, *it satisfies the boundary condition* $a_\nu(\mathbf{0}) = 0$.
6. *For* $\nu = 1$ *or* $\sum_{i=1}^{n} w_i \geq 1$, *it satisfies the boundary condition* $a_\nu(\mathbf{1}) = 1$.
7. *In particular for two variables, with* $\frac{1}{2} \leq w_1 = w_2$ *and* $f(\nu) = \frac{1}{2}$, *it is*
   - *commutative,*
   - *satisfies the De Morgan property,*
   - *satisfies the boundary conditions (i.e.* $a(0,0) = 0$ *and* $a(1,1) = 1$*),*
   - $a(0,1) = a(1,0) = \nu$,
   - *if* $x, y \leq \nu$, *then* $a(x, y) \leq \nu$,
   - *if* $x, y \geq \nu$, *then* $a(x, y) \geq \nu$.
8. *For two variables, with* $w_1 = w_2 = 1$ *and* $f(\nu) = \frac{1}{2}$, *it is*
   - *commutative,*
   - *satisfies the De Morgan property,*
   - *satisfies the boundary conditions (i.e.* $a(0,0) = 0$ *and* $a(1,1) = 1$*),*
   - $a(0,1) = a(1,0) = \nu$,
   - *it is conjunctive for* $x, y \leq \nu$,
   - *it is disjunctive for* $x, y \geq \nu$,
   - *it is averaging for* $x \leq \nu \leq y$ *and for* $y \leq \nu \leq x$.
9. *For one variable and with* $w = -1$, *it is a negation operator with generator* $f(x)$.
10. $a_w(n(x), y) = f^{-1}\left[w(f(y) - f(x)) + \frac{1}{2}\right] = p_w(x, y)$.

7. Concluding remarks and future work

To sum up, we may conclude that the weighted general operator (obtained by shifting and weighting the generator function of a disjunction) provides a general framework for different types of operators using only one generator function. The formula contains a parameter with the semantical meaning of the threshold of expectancy. Changing the parameter values, we can obtain conjunctive, disjunctive and self-dual operators with nice properties, for one variable also a negation operator. As a starting point, instead of associativity, we focused on the necessary and sufficient condition of the self-dual property. Our results may have a considerable contribution for applications in machine learning, since the parametric formula is easy to learn. Using an adequate optimization technique, the parameter with the best fit can be found. We plan to address the thorough examination of the preference operator in our future work.

The main disadvantage of the nilpotent operator family is the lack of differentiability, since there are significant areas, where the parameters are learned by a gradient based optimization method. In this case, the lack of continuous derivatives makes the application impossible. We will concentrate in our future work on this problem. Using the so-called squashing function (see Dombi and Gera [17]) provides a solution to the above mentioned problem by a continuously differentiable approximation of the cut function. This approximation could be the next step along the path to practical applications.

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