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The omega probability distribution and its applications in reliability theory

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Abstract

A new three-parameter probability distribution called the omega probability distribution is introduced, and its connection with the Weibull distribution is discussed. We show that the asymptotic omega distribution is just the Weibull distribution and point out that the mathematical properties of the novel distribution allow us to model bathtub-shaped hazard functions in two ways. On the one hand, we demonstrate that the curve of the omega hazard function with special parameter settings is bathtub shaped and so it can be utilized to describe a complete bathtub-shaped hazard curve. On the other hand, the omega probability distribution can be applied in the same way as the Weibull probability distribution to model each phase of a bathtub-shaped hazard function. Here, we also propose two approaches for practical statistical estimation of distribution parameters. From a practical perspective, there are two notable properties of the novel distribution, namely, its simplicity and flexibility. Also, both the cumulative distribution function and the hazard function are composed of power functions, which on the basis of the results from analyses of real failure data, can be applied quite effectively in modeling bathtub-shaped hazard curves.

KEYWORDS
hazard function modeling, omega distribution, reliability analysis, Weibull distribution

1 | INTRODUCTION

In 1951, Weibull published a study on an extension of the exponential probability distribution, which later became known as the Weibull probability distribution.[1] This date of publication may be considered as a momentous cornerstone in the progress of this distribution in statistical theory as well as in applied statistics. Up to the end of 1950s, lifetime in engineering sciences was nearly always modeled by the exponential distribution,[2] which then was gradually substituted by the more flexible Weibull distribution.[1] The intense interest towards the Weibull distribution is due to its multiple special features and its ability to fit data from various fields, ranging from life data to observations made in economics and business administration or in the engineering sciences.[1,3,4]

The two-parameter probability density function $f(x; \beta, \lambda)$ of the random variable, which has a Weibull probability distribution, is generally given by

$$f(x;\beta,\lambda)=\begin{cases}0, & \text{if } x\le 0\\ \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\mathrm{e}^{-(x/\lambda)^{\beta}}, & \text{if } x>0,\end{cases} \tag{1}$$

where $\beta, \lambda \in \mathbb{R}$ and $\beta, \lambda > 0$ are the shape and scale parameters of the distribution, respectively.[5,6] By applying the $\alpha = \lambda^{-\beta}$ substitution, (1) may be written in the form

$$f^{(\alpha,\beta)}(x)=\begin{cases}0, & \text{if } x\le 0\\ \alpha\beta x^{\beta-1}\mathrm{e}^{-\alpha x^{\beta}}, & \text{if } x>0,\end{cases} \tag{2}$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha, \beta > 0$. Hereafter, we will use this alternative definition of the two-parameter probability density function of the random variable that has a Weibull probability distribution.

From a managerial point of view, it is helpful to have a model that is reasonably simple and suitable for the whole product life cycle when making overall managerial decisions.[7-9] Furthermore, for complex systems, both the decreasing and increasing parts of the failure rate fall into the ordinary product lifetime.[7] On the basis of this fact, several models were proposed to model bathtub-shaped failure rates from the very beginning, which apply a variety of methods for estimating and testing including the method of moments, least squares, and maximum likelihood.[7,8,10-14] Comprehensive overviews of bathtub-shaped failure rate functions are provided by Rajarshi and Rajarshi[8] and Lai et al.[15] Models that present bathtub-shaped failure rates are also extremely useful in survival analysis.[16] Much research has been carried out recently with the aim of serving the needs of reliability engineers and practitioners, most of them presenting new lifetime distributions that have bathtub-shaped failure rate functions.[17] To satisfy all these needs, the Weibull distributions have also been proven to be very flexible in modeling various types of lifetime distributions. The general usefulness of the Weibull probability distribution enhances its applicability in a wide range of reliability analyses, especially in the theory and practice of reliability management.[1,18,19] Almalki and Nadarajah[9] provide a detailed literature review of some discrete and continuous versions of the modifications of the Weibull distribution.

When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, the Weibull distribution does not provide a reasonable parametric fit for modeling phenomenon with nonmonotone failure rates such as the bathtub-shaped failure rates.[20] A number of studies have been published on modifications, generalizations, and approximations to the Weibull probability distribution with the number of parameters ranging from two to five, with the purpose of enhancing its capability of modeling bathtub-shaped failure rate curves.[4,21-25] In the last few years, the relevant literature is extremely rich in providing recent results of this area.[9] Cordeiro et al[26] provided a five-parameter extension of the Weibull distribution to model both monotone and nonmonotone failure rates. Khan[27] introduced a five-parameter modified beta Weibull probability distribution for analyzing positive data having a bathtub and upside-down bathtub hazard rate function. Nadarajah et al[28] gave a review of the exponentiated Weibull (EW) distribution to accommodate nonmonotone hazard rates. Almalki and Nadarajah[29] introduced a new three-parameter discrete distribution on a recent modification of the continuous Weibull distribution. Pu et al[30] proposed a new class of five-parameter gamma-exponentiated or generalized modified Weibull (GEMW) distribution. Nassar et al[31] define a new lifetime distribution referred to as the alpha power Weibull distribution with the capability of modeling both monotone and nonmonotone failure rate functions. He and others[32] studied a five-parameter lifetime distribution to model bathtub-shaped hazard rate data. Bagheri et al[33] proposed a new distribution with increasing, decreasing, bathtub-shaped and unimodal failure rate curves called the GEMW power series distribution. With the same aim, Afify et al[34] introduced the Marshall-Olkin additive Weibull distribution with a variable-shaped hazard rate.

In this paper, a new probability distribution, namely, the omega probability distribution, is introduced and its application in reliability theory is discussed. This novel probability distribution is founded on the so-called omega function, which just like the exponential function $f(x) = \mathrm{e}^{-\alpha x^{\beta}}$, may be deduced from the generalized exponential differential equation that we introduce here. The omega probability distribution has three parameters, namely, α, β, and d. The parameters α and β have similar meanings to those of the Weibull probability distribution given by the density function (2), while the parameter d determines the domain $(0, d)$ where the omega function is defined ($d > 0$). Next, it is shown that the asymptotic omega probability distribution is just the Weibull probability distribution, which means in practice that the two-parameter Weibull probability distribution with the parameters α and β can be substituted by the omega probability distribution that has parameters α, β, and d.

These results lay the foundations for two novel bathtub-shaped hazard function (HF) models that we call the piecewise model and the all-in-one model. On the one hand, since the omega probability distribution may be viewed as an alternative to the Weibull probability distribution and the latter can be utilized for modeling each phase of a bathtub-shaped HF, the omega probability distribution can be applied in the same way. Here, we show that the asymptotic omega HF is just the Weibull HF. On the other hand, we demonstrate that the curve of the omega HF, with special parameter settings, is bathtub shaped and so it can be utilized to describe a complete bathtub-shaped hazard curve in one go.

We also show how the omega probability distribution can be applied to model the probability distribution of the time-to-first-failure random variable if its HF is bathtub shaped. Since the omega HF has three parameters, it is compared with the HFs of the well-known three-parameter modifications of the Weibull distribution. These distributions are the modified Weibull (MW) distribution proposed by Lai et al,[35] Mudholkar and Srivastava's

EW distribution,[13] the generalized Weibull family (GWF) distribution first introduced by Mudholkar and Kollia,[36] the generalized power Weibull (GPW) distribution discussed by Nikulin and Haghighi,[37] the modified Weibull extension (MWEX) distribution proposed by Xie et al,[38] the odd Weibull (ODDW) distribution presented by Cooray,[39] and the reduced modified Weibull (RNMW) distribution introduced in Almalki's[40] paper. Our results are in line with the results of Almalki and Nadarajah[9] as several models in the literature are not able to follow a bathtub shape if the second constant phase of the failure rate time series is not long enough. Another important feature is that the omega HF does not contain any exponential term. Similar to the GWF and GPW models, the omega HF is composed of power functions and it has a very simple form. Moreover, while the exponential function tends to infinity over an unbounded domain, the omega function does so over the bounded domain $(0, d)$, which means that the omega HF can more appropriately follow sudden changes $(d > 0)$.

The remaining part of the paper is organized as follows. In Section 2, the omega probability distribution and its connection with the Weibull distribution is introduced and discussed. In Section 3, we introduce two novel models of bathtub-shaped failure rate functions, and through a practical example, we demonstrate how the omega probability distribution can be applied in reliability theory. Lastly, we draw some key conclusions about the new probability distribution and make some suggestions for future research.

2 | THE OMEGA PROBABILITY DISTRIBUTION

In the last 60 years, the Weibull distribution has become a very popular distribution for modeling lifetime data and phenomena with a monotone failure rate. The Weibull probability distribution can be utilized to model the probability distribution of the time-to-first-failure (or the time between failures) random variable in each of the three characteristic phases of a bathtub-shaped failure rate curve. Because of its interesting properties, the Weibull distribution has been widely used for modeling different phases of product and system lifetimes.[9,41-43]

Here, we will introduce the omega probability distribution and show how it is connected with the two-parameter Weibull probability distribution. This novel distribution is founded on an auxiliary function that we call the omega function, the appropriate linear transformation of which is the generator function of certain unary operators in continuous-valued logic.[44] Firstly, we will introduce the omega function.

Definition 1. The omega function $\omega_d^{(\alpha,\beta)}(x)$ is given by

$$\omega_d^{(\alpha,\beta)}(x) = \left(\frac{d^\beta + x^\beta}{d^\beta - x^\beta}\right)^{\frac{\alpha d^\beta}{2}}, \tag{3}$$

where $\alpha, \beta, d \in \mathbb{R}$, $\beta, d > 0$, $x \in (0, d)$.

Later we will explain why this formula is so useful. Utilizing the omega function, the density function of the omega probability distribution is given as follows.

Definition 2. The continuous random variable ξ has an omega probability distribution with the parameters $\alpha, \beta, d > 0$, if the probability density function $f_d^{(\alpha,\beta)}(x)$ of ξ is given by

$$f_d^{(\alpha,\beta)}(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \alpha\beta x^{\beta-1}\frac{d^{2\beta}}{d^{2\beta}-x^{2\beta}}\omega_d^{(-\alpha,\beta)}(x), & \text{if } 0 < x < d \\ 0, & \text{if } x \geq d, \end{cases} \tag{4}$$

where

$$\omega_d^{(-\alpha,\beta)}(x) = \left(\frac{d^\beta + x^\beta}{d^\beta - x^\beta}\right)^{\frac{-\alpha d^\beta}{2}}. \tag{5}$$

In order to demonstrate that the function $f_d^{(\alpha,\beta)}(x)$ given in (4) is in fact a probability density function, we will prove Lemma 1 for which we will utilize the first derivative of the omega function given by Equation 5:

$$\frac{\mathrm{d}\omega_d^{(-\alpha,\beta)}(x)}{\mathrm{d}x} = -\alpha\beta x^{\beta-1}\frac{d^{2\beta}}{d^{2\beta} - x^{2\beta}}\omega_d^{(-\alpha,\beta)}(x). \tag{6}$$

Lemma 1. *The function $f_d^{(\alpha,\beta)}(x)$ has the following properties:*

1. $f_d^{(\alpha,\beta)}(x) \geq 0$ *for any* $x \in \mathbb{R}$;
2. $\int_{-\infty}^{\infty} f_d^{(\alpha,\beta)}(x)\mathrm{d}x = 1.$

Proof. The first property of $f_d^{(\alpha,\beta)}(x)$ trivially follows. Utilizing Equation 6, the second property of $f_d^{(\alpha,\beta)}(x)$ can be demonstrated as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} f_d^{(\alpha,\beta)}(x)\mathrm{d}x &= \int_0^d \alpha\beta x^{\beta-1}\frac{d^{2\beta}}{d^{2\beta} - x^{2\beta}}\omega_d^{(-\alpha,\beta)}(x)\mathrm{d}x = \\ &= \left[-\omega_d^{(-\alpha,\beta)}(x)\right]_0^d = 1.\end{aligned} \tag{7}$$

□

Exploiting the above results, it can be shown that the probability distribution function $F_d^{(\alpha,\beta)}(x)$ of the random variable ξ that has an omega probability distribution with parameters $\alpha, \beta, d > 0$ is

$$F_d^{(\alpha,\beta)}(x) = \int_{-\infty}^{x} f_d^{(\alpha,\beta)}(t)\mathrm{d}t = \begin{cases} 0, & \text{if } x \leq 0 \\ 1 - \omega_d^{(-\alpha,\beta)}(x), & \text{if } 0 < x < d \\ 1, & \text{if } x \geq d. \end{cases} \tag{8}$$

It is worth pointing out that the same auxiliary omega function is utilized in the omega probability density and distribution functions. Note that from here on, a probability distribution function always means a cumulative distribution function (CDF). We will show that the omega probability distribution may also be viewed as an alternative to the Weibull probability distribution. For this purpose, first of all, we will discuss the main properties of the omega function.

2.1 | Main properties of the omega function

Here, we state the most important properties of the omega function, namely, differentiability, monotonicity, limits, and convexity.

Differentiability. $\omega_d^{(\alpha,\beta)}(x)$ is a differentiable function in the interval $(0, d)$.

Monotonicity.

- If $\alpha > 0$, then $\omega_d^{(\alpha,\beta)}(x)$ is strictly monotonously increasing.
- If $\alpha < 0$, then $\omega_d^{(\alpha,\beta)}(x)$ is strictly monotonously decreasing.
- If $\alpha = 0$, then $\omega_d^{(\alpha,\beta)}(x)$ has a constant value of 1

in the interval $(0, d)$.

Limits.

$$\lim_{x \to d^-} \omega_d^{(\alpha,\beta)}(x) = \begin{cases} \infty, & \text{if } \alpha > 0 \\ 0, & \text{if } \alpha < 0. \end{cases} \tag{9}$$

Convexity. It can be shown that the shape of function $\omega_d^{(\alpha,\beta)}(x)$ in the interval $(0, d)$ is as follows:

- If
$$d^{2\beta} < \frac{4\left(\beta^2 - 1\right)}{\alpha^2\beta^2}, \alpha \neq 0, \tag{10}$$
then $\omega_d^{(\alpha,\beta)}(x)$ is convex when $\alpha > 0$ and $\omega_d^{(\alpha,\beta)}(x)$ is concave when $\alpha < 0$.
- If
$$d^{2\beta} \geq \frac{4\left(\beta^2 - 1\right)}{\alpha^2\beta^2}, \alpha \neq 0, \tag{11}$$
then we can distinguish the following cases:
-if $\alpha > 0$ and $0 < \beta < 1$, then $\omega_d^{(\alpha,\beta)}(x)$ changes its shape from concave to convex at x_r;
-if $\alpha > 0$ and $\beta \geq 1$, then $\omega_d^{(\alpha,\beta)}(x)$ is convex;
-if $\alpha < 0$, $0 < \beta \leq 1$, and $x_r < d$, then $\omega_d^{(\alpha,\beta)}(x)$ changes its shape from convex to concave at x_r;
-if $\alpha < 0$, $0 < \beta \leq 1$, and $x_r \geq d$, then $\omega_d^{(\alpha,\beta)}(x)$ is convex;
-if $\alpha < 0$, $\beta > 1$, and $x_r < d$, then $\omega_d^{(\alpha,\beta)}(x)$ changes its shape from concave to convex at x_l and from convex to concave at x_r; and
-if $\alpha < 0$, $\beta > 1$, and $x_r \geq d$, then $\omega_d^{(\alpha,\beta)}(x)$ changes its shape from concave to convex at x_l,
where

$$x_l = \left(\frac{-\alpha\beta d^{2\beta} - \sqrt{\alpha^2\beta^2 d^{4\beta} - 4(\beta^2 - 1)d^{2\beta}}}{2(\beta + 1)}\right)^{1/\beta}, \tag{12}$$

$$x_r = \left(\frac{-\alpha\beta d^{2\beta} + \sqrt{\alpha^2\beta^2 d^{4\beta} - 4(\beta^2 - 1)d^{2\beta}}}{2(\beta + 1)}\right)^{1/\beta}. \tag{13}$$

Figure 1 shows some concrete examples of the omega function curve.

2.1.1 | The generalized exponential differential equation

Now, we introduce the generalized exponential differential equation and show how it is connected with the exponential function $f(x) = \mathrm{e}^{\alpha x^\beta}$, $(\alpha, \beta \in \mathbb{R}, \beta > 0)$ and with the omega function.

Definition 3. We define the generalized exponential differential equation as

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \alpha\beta x^{\beta-1}\left(\frac{d^{2\beta}}{d^{2\beta} - x^{2\beta}}\right)^{\varepsilon} f(x), \tag{14}$$

where $\varepsilon \in \{0, 1\}$, $\alpha, \beta, d \in \mathbb{R}$, $\beta, d > 0$, $x \in (0, d)$, $f(x) > 0$.

Lemma 2. *The solutions of the generalized exponential differential equation are*

$$f(x) = \begin{cases} C\mathrm{e}^{\alpha x^\beta}, & \text{if } \varepsilon = 0 \\ C\left(\frac{d^\beta + x^\beta}{d^\beta - x^\beta}\right)^{\frac{\alpha d^\beta}{2}}, & \text{if } \varepsilon = 1, \end{cases} \tag{15}$$

where $C \in \mathbb{R}, C > 0$.

Proof. If $\varepsilon = 0$, then the differential equation in (14) may be written as

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \alpha\beta x^{\beta-1} f(x). \tag{16}$$

Separating the variables in (16) and integrating both sides lead to

$$\int \frac{1}{f(x)} \mathrm{d}f(x) = \int \alpha\beta x^{\beta-1} \mathrm{d}x, \tag{17}$$

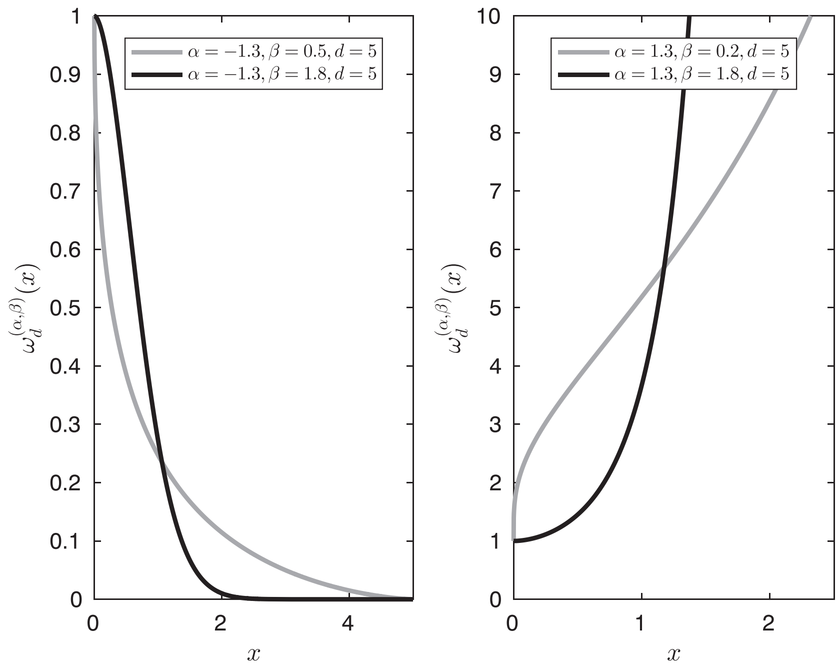

FIGURE 1 Examples of omega function curves

$$\ln|f(x)| = \alpha x^\beta + \ln C, \tag{18}$$

where $C > 0$. Utilizing the fact that $f(x) > 0$, the last equation may be written as

$$f(x) = C\mathrm{e}^{\alpha x^\beta}. \tag{19}$$

If $\varepsilon = 1$, then the differential equation in (14) becomes

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \alpha\beta x^{\beta-1}\frac{d^{2\beta}}{d^{2\beta}-x^{2\beta}}f(x). \tag{20}$$

Exploiting the fact that

$$\frac{1}{d^{2\beta}-x^{2\beta}} = \frac{1}{2d^\beta}\left(\frac{1}{d^\beta+x^\beta}+\frac{1}{d^\beta-x^\beta}\right), \tag{21}$$

separating the variables in (20), and integrating both sides give

$$\int\frac{1}{f(x)}\mathrm{d}f(x) = \alpha\beta\frac{d^\beta}{2}\left(\int\frac{x^{\beta-1}}{d^\beta+x^\beta}\mathrm{d}x+\int\frac{x^{\beta-1}}{d^\beta-x^\beta}\mathrm{d}x\right), \tag{22}$$

$$\ln|f(x)| = \frac{\alpha d^\beta}{2}\left(\ln|d^\beta+x^\beta|-\ln|d^\beta-x^\beta|\right)+\ln C, \tag{23}$$

where $C > 0$. Since $f(x) > 0$ and $x \in (0, d)$, the last equation may be written as

$$f(x) = C\left(\frac{d^\beta+x^\beta}{d^\beta-x^\beta}\right)^{\frac{\alpha d^\beta}{2}}. \tag{24}$$

□

2.1.2 | Connections between the exponential and omega functions

Lemma 2 suggests that there is an important connection between the exponential function $f(x) = \mathrm{e}^{\alpha x^\beta}$ and the omega function. Namely, the solution of the generalized exponential differential equation for $\varepsilon = 0$ and $C = 1$ is simply the exponential function $f(x) = \mathrm{e}^{\alpha x^\beta}$, while the solution of (14) for $\varepsilon = 1$, $C = 1$ is the omega function. Furthermore, if d is much greater than x, then

$$\frac{d^{2\beta}}{d^{2\beta}-x^{2\beta}} \approx 1, \tag{25}$$

and the generalized exponential differential equation for $\varepsilon = 1$ becomes the following approximate equation:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} \approx \alpha\beta x^{\beta-1}f(x), \tag{26}$$

which is nearly the generalized exponential differential equation with $\varepsilon = 0$, the solution of which is the exponential function $f(x) = \mathrm{e}^{\alpha x^\beta}$. The following theorem provides the theoretical basis for this result.

Theorem 1. *For any $x \in (0, d)$ and $\beta > 0$,*

$$\lim_{d\to\infty}\omega_d^{(\alpha,\beta)}(x) = \mathrm{e}^{\alpha x^\beta}. \tag{27}$$

Proof. Let x have a fixed value, where again $x \in (0, d)$.

$$
\begin{aligned}
\lim_{d\to\infty}\omega_d^{(\alpha,\beta)}(x) &= \lim_{d\to\infty}\left(\frac{d^\beta + x^\beta}{d^\beta - x^\beta}\right)^{\frac{\alpha d^\beta}{2}} \\
&= \lim_{d\to\infty}\left(\left(\frac{d^\beta - x^\beta + 2x^\beta}{d^\beta - x^\beta}\right)^{d^\beta}\right)^{\frac{\alpha}{2}} = \\
&= \lim_{d\to\infty}\left(\left(1 + \frac{2x^\beta}{d^\beta - x^\beta}\right)^{d^\beta}\right)^{\frac{\alpha}{2}}.
\end{aligned} \tag{28}
$$

Since x is fixed, if $d \to \infty$, then $\Delta = d^\beta - x^\beta \to \infty$ and so the previous calculation can be continued as follows:

$$
\begin{aligned}
\lim_{d\to\infty}\left(\left(1+\frac{2x^\beta}{d^\beta - x^\beta}\right)^{d^\beta}\right)^{\frac{\alpha}{2}} &= \lim_{\Delta\to\infty}\left(\left(1+\frac{2x^\beta}{\Delta}\right)^{\Delta + x^\beta}\right)^{\frac{\alpha}{2}} = \\
&= \left(\lim_{\Delta\to\infty}\left(1+\frac{2x^\beta}{\Delta}\right)^{\Delta}\lim_{\Delta\to\infty}\left(1+\frac{2x^\beta}{\Delta}\right)^{x^\beta}\right)^{\frac{\alpha}{2}} \\
&= \left(e^{2x^\beta}\right)^{\frac{\alpha}{2}} \cdot 1^{\frac{\alpha}{2}} = e^{\alpha x^\beta}.
\end{aligned} \tag{29}
$$

□

On the basis of Theorem 1, it can be stated that the asymptotic omega function is just the exponential function $f(x) = e^{\alpha x^\beta}$. Actually, if $x \ll d$, then $\omega_d^{(\alpha,\beta)}(x) \approx e^{\alpha x^\beta}$; that is, if d is sufficiently large, then the omega function suitably approximates the exponential function $f(x) = e^{\alpha x^\beta}$.

2.2 | An approximation to the Weibull probability distribution

If the random variable η has a two-parameter Weibull probability distribution with the parameters $\alpha, \beta > 0$, then the probability density function $f^{(\alpha,\beta)}(x)$ of η is given by

$$
f^{(\alpha,\beta)}(x) = \begin{cases} 0, & \text{if } x \le 0 \\ \alpha\beta x^{\beta-1}e^{-\alpha x^\beta}, & \text{if } x > 0. \end{cases} \tag{30}
$$

The next lemma tells us how the omega probability distribution is connected with the Weibull probability distribution.

Lemma 3. *For any $x \in \mathbb{R}$ and $\alpha, \beta > 0$, if $d \to \infty$, then*

$$
f_d^{(\alpha,\beta)}(x) \to f^{(\alpha,\beta)}(x). \tag{31}
$$

Proof. Let $x \in \mathbb{R}$ be fixed. We will now distinguish the following two cases.

- If $x \le 0$ or $x \ge d$, then $f_d^{(\alpha,\beta)}(x) = f^{(\alpha,\beta)}(x) = 0$ holds by definition.
- If $x \in (0, d)$, $d > 0$, then

$$
f_d^{(\alpha,\beta)}(x) = \alpha\beta x^{\beta-1}\frac{d^{2\beta}}{d^{2\beta} - x^{2\beta}}\omega_d^{(-\alpha,\beta)}(x). \tag{32}
$$

If $d \to \infty$, then

$$
\frac{d^{2\beta}}{d^{2\beta} - x^{2\beta}} \to 1, \tag{33}
$$

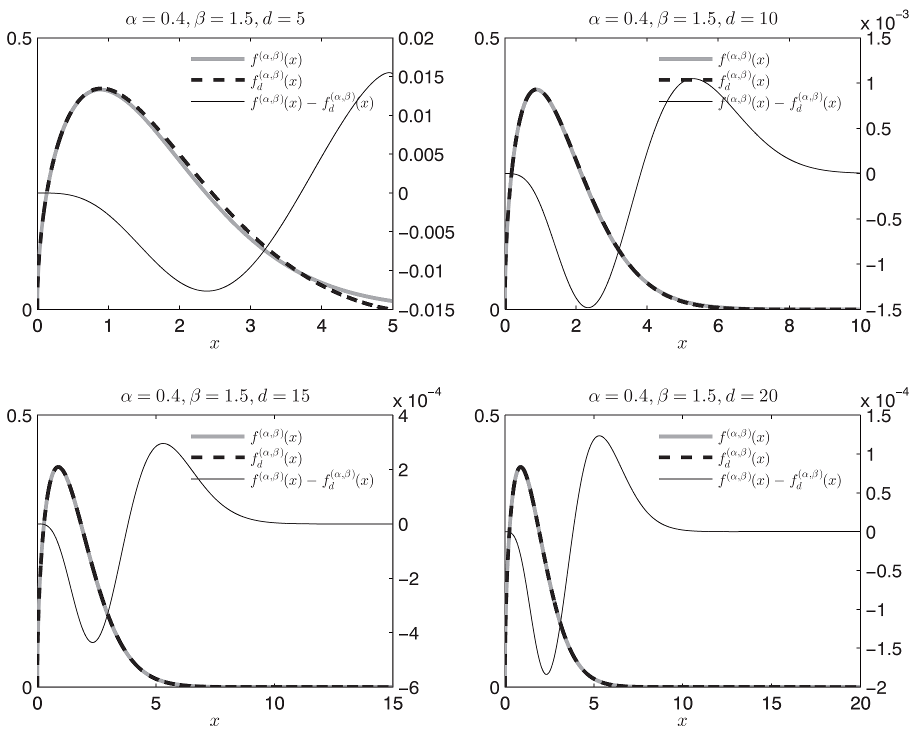

FIGURE 2 Plots of Weibull and omega probability density functions

and following Theorem 1,

$$\omega_d^{(-\alpha,\beta)}(x) \to \mathrm{e}^{-\alpha x^{\beta}}. \tag{34}$$

That is, if $d \to \infty$, then

$$f_d^{(\alpha,\beta)}(x) \to \alpha\beta x^{\beta-1}\mathrm{e}^{-\alpha x^{\beta}} = f^{(\alpha,\beta)}(x). \tag{35}$$

□

Figure 2 shows some examples of how the omega probability density function can approximate the Weibull probability density function. In each subplot of Figure 2, the left-hand side scale is associated with functions $f^{(\alpha,\beta)}(x)$ (gray line) and $f_d^{(\alpha,\beta)}(x)$ (dashed black line), while the right-hand side scale is connected with the difference function $f^{(\alpha,\beta)}(x) - f_d^{(\alpha,\beta)}(x)$ (thin black line). Table 1 shows the maximum and the mean of absolute differences between $f^{(\alpha,\beta)}(x)$ and $f_d^{(\alpha,\beta)}(x)$ for the plots shown in Figure 2. We can see that, in line with Lemma 3, the goodness of approximation improves as d increases.

If the random variable η has a two-parameter Weibull probability distribution with the parameters $\alpha, \beta > 0$, then the probability distribution function $F^{(\alpha,\beta)}(x)$ of η is given by

$$F^{(\alpha,\beta)}(x) = \begin{cases} 0, & \text{if } x \le 0 \\ 1 - \mathrm{e}^{-\alpha x^{\beta}}, & \text{if } x > 0. \end{cases} \tag{36}$$

Theorem 2. *For any $x \in \mathbb{R}$, $\alpha, \beta, d > 0$, if the random variable ξ has an omega probability distribution with the parameters α, β, d and the η random variable has a Weibull probability distribution with parameters α, β, then*

$$\lim_{d\to\infty} P(\xi < x) = P(\eta < x). \tag{37}$$

Proof. Since $F_d^{(\alpha,\beta)}(x) = P(\xi < x)$ and $F^{(\alpha,\beta)}(x) = P(\eta < x)$ for any $x \in \mathbb{R}$, using the definitions of $F_d^{(\alpha,\beta)}(x)$ and $F^{(\alpha,\beta)}(x)$, this theorem follows from Theorem 1. □

Some examples of Weibull probability distribution functions and their approximations by the omega probability distribution functions are shown in Figure 3. Similar to Figure 2, in each subplot of Figure 3, the left-hand side scale is associated with functions $F^{(\alpha,\beta)}(x)$ (gray line) and $F_d^{(\alpha,\beta)}(x)$ (dashed black line), while the right-hand side scale is connected with the difference function $F^{(\alpha,\beta)}(x) - F_d^{(\alpha,\beta)}(x)$ (thin black line). Table 2 shows the maximum and the mean of absolute differences between $F^{(\alpha,\beta)}(x)$ and $F_d^{(\alpha,\beta)}(x)$ for the examples in Figure 3. We can see that the goodness of approximation improves as d increases.

TABLE 1 Errors of approximations to Weibull probability density functions

d	$\max_{x\in(0,d)} \left\| f^{(\alpha,\beta)}(x) - f_d^{(\alpha,\beta)}(x) \right\|$	$\frac{1}{d}\int_0^d \left\| f^{(\alpha,\beta)}(x) - f_d^{(\alpha,\beta)}(x) \right\| \mathrm{d}x$
5	1.5498e-02	7.0973e-03
10	1.4812e-03	5.6295e-04
15	4.3621e-04	1.1082e-04
20	1.8376e-04	3.5031e-05

On the basis of Theorem 2, it can be stated that the asymptotic omega probability distribution is just the Weibull probability distribution. Thus, in practical applications, the Weibull probability distribution with parameters $\alpha, \beta > 0$ can be substituted by the omega probability distribution that has the parameters $\alpha, \beta, d > 0$, if $x \ll d$. It is worth mentioning that while the Weibull probability distribution function is a transcendental function, the omega probability distribution function is a power function. This means that from a computational point of view, the omega probability distribution function is more convenient than the Weibull probability distribution function. This feature of the omega probability distribution further enhances its applicability in problems where computation time is a critical factor.

From here on, we will use the notations $\xi \sim \omega(\alpha, \beta, d)$ and $\eta \sim W(\alpha, \beta)$ to indicate that ξ has an omega probability distribution with the parameters $\alpha, \beta, d > 0$ and η has a Weibull probability distribution with the parameters $\alpha, \beta > 0$, respectively.

2.3 | Asymptotic properties of the omega probability distribution

The next corollary summarizes the main asymptotic characteristics of the random variable that has an omega probability distribution.

Corollary 1. *If $\xi \sim \omega(\alpha, \beta, d)$, then*

$$\lim_{d\to\infty} E(\xi) = \alpha^{-\frac{1}{\beta}}\Gamma_1, \tag{38}$$

$$\lim_{d\to\infty} Mo(\xi) = \begin{cases} \alpha^{-\frac{1}{\beta}}\left(\frac{\beta-1}{\beta}\right)^{\frac{1}{\beta}}, & \text{if } \beta > 1 \\ 0, & \text{if } 0 < \beta \le 1, \end{cases} \tag{39}$$

$$\lim_{d\to\infty} Me(\xi) = \alpha^{-\frac{1}{\beta}}(\ln 2)^{\frac{1}{\beta}}, \tag{40}$$

$$\lim_{d\to\infty} Var(\xi) = \alpha^{-\frac{2}{\beta}}\left(\Gamma_2 - \Gamma_1^2\right), \tag{41}$$

$$\lim_{d\to\infty} m_n = \alpha^{-\frac{n}{\beta}}\Gamma_n, \tag{42}$$

$$\lim_{d\to\infty} F^{-1}(p) = \alpha^{-\frac{1}{\beta}}\left(\ln\frac{1}{1-p}\right)^{\frac{1}{\beta}}, \tag{43}$$

$$\lim_{d\to\infty} \gamma_1 = \frac{2\Gamma_1^3 - 3\Gamma_1\Gamma_2 + \Gamma_3}{\left(\Gamma_2 - \Gamma_1^2\right)^{\frac{3}{2}}}, \tag{44}$$

$$\lim_{d\to\infty} \gamma_2 = \frac{-6\Gamma_1^4 + 12\Gamma_1^2\Gamma_2 - 3\Gamma_2^2 - 4\Gamma_1\Gamma_3 + \Gamma_4}{\left(\Gamma_2 - \Gamma_1^2\right)^2}, \tag{45}$$

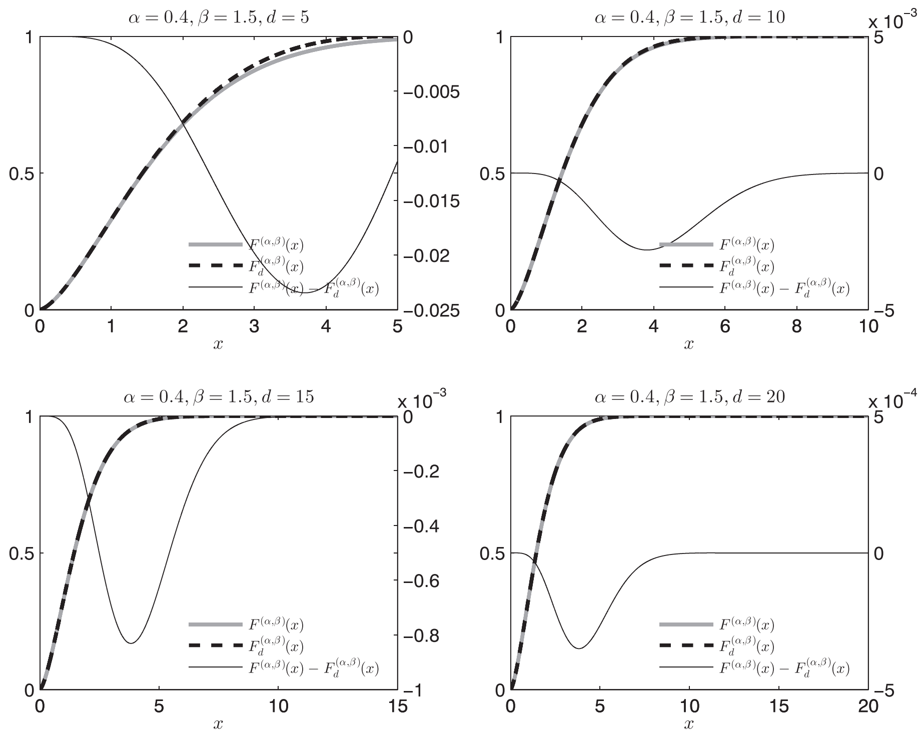

FIGURE 3 Examples of Weibull and omega probability distribution functions

TABLE 2 Errors of approximations to Weibull probability distribution functions

d	$\max\limits_{x\in(0,d)} \left\lvert F^{(\alpha,\beta)}(x) - F_d^{(\alpha,\beta)}(x)\right\rvert$	$\frac{1}{d}\int_0^d \left\lvert F^{(\alpha,\beta)}(x) - F_d^{(\alpha,\beta)}(x)\right\rvert \mathrm{d}x$
5	2.3459e-02	1.1677e-02
10	2.8164e-03	9.9133e-04
15	8.3117e-04	2.0060e-04
20	3.5031e-04	6.3862e-05

where $E(\xi)$, $Mo(\xi)$, $Me(\xi)$, $Var(\xi)$, m_n, $F^{-1}(p)$, γ_1, and γ_2 are the mean, mode, median, variance, nth raw moment, quantile function, skewness, and kurtosis excess of the random variable ξ, respectively, and

$$\Gamma_i = \Gamma\left(1 + \frac{i}{\beta}\right). \tag{46}$$

Here, $0 < p < 1$, and Γ denotes Euler's gamma function.

Proof. On the basis of Theorem 2, if $d \to \infty$, then $P(\xi < x) = P(\eta < x)$, where the random variable η has a Weibull distribution with the parameters $\alpha, \beta > 0$. The corollary can be proven by utilizing this result and the characteristics of the Weibull probability distribution. □

It should be added that the analytic calculations of the main characteristics of the omega probability distribution including the mean, mode, median, variance, nth raw moment, quantile function, skewness, and kurtosis excess lead to complicated integrals and formulas that are difficult to deal with. At the same time, in practical reliability engineering applications, the omega probability distribution is typically deployed with a value of parameter d that is sufficiently large to utilize the result of Theorem 2 and Corollary 1, which allow us to substitute the above-mentioned characteristics of the probability distribution with their asymptotic values. For example, if we have weekly failure data for a year, then the time horizon of the analyses is 52 weeks, and so $d \geq 52$. In such a case, the asymptotic values of the characteristics can be utilized instead of their exact values.

2.4 | Interpretation of parameters

The omega probability distribution has three parameters, namely, the parameters α, β, and d, which are all positive. On the basis of the definition of the probability density function $f_d^{(\alpha,\beta)}(x)$ of the omega probability distribution, the parameter d specifies the support of $f_d^{(\alpha,\beta)}(x)$; that is, $f_d^{(\alpha,\beta)}(x)$ is positive only if $x \in (0, d)$. By Theorem 2, we have demonstrated that if $d \to \infty$, then the omega probability distribution is identical with the two-parameter Weibull probability distribution given in (2). This result also indicates that if the value of parameter d is sufficiently large, then the role of the parameters α and β of the omega probability distribution is very similar to those of the cor-

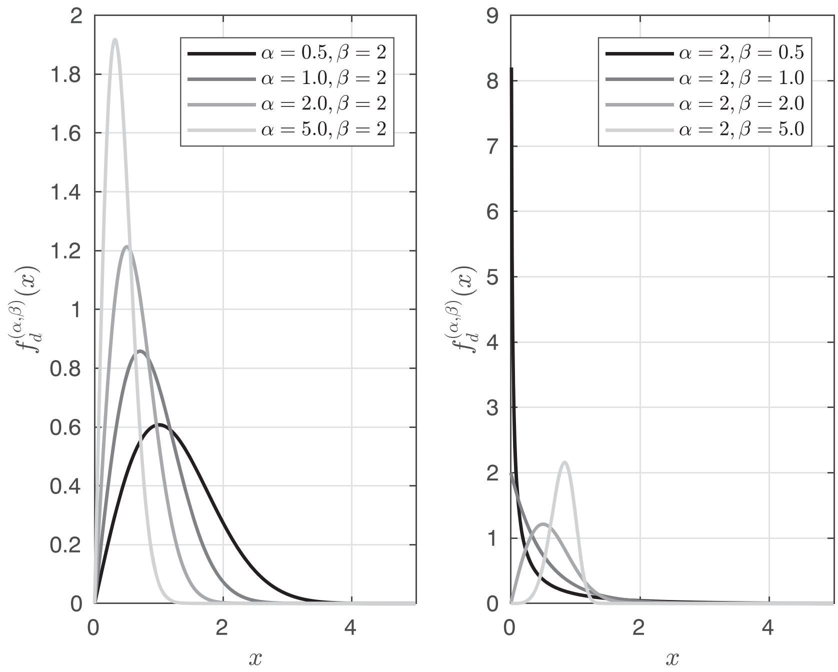

FIGURE 4 Interpretation of parameters

responding α and β parameters of the Weibull probability distribution. Note that even for $d \geq 5$, the omega probability distribution approximates the Weibull probability distribution quite well (see Figures 2 and 3).

Figure 4 shows how the parameters α and β affect the characteristics of the omega probability density function. The parameter α may be viewed as the scale parameter of the distribution; that is, the greater the value of α is, the greater the maximum value of the density function is. The parameter β affects the shape of the density function in the following way:

- If $0 < \beta < 1$, then $\lim_{x \to 0^+} f_d^{(\alpha,\beta)}(x) = \infty$ and $f_d^{(\alpha,\beta)}(x)$ is strictly monotonously decreasing for $x > 0$.
- If $\beta = 1$, then $f_d^{(\alpha,\beta)}(0) = \alpha$ and $f_d^{(\alpha,\beta)}(x)$ is strictly monotonously decreasing for $x > 0$.
- If $\beta > 1$, then $f_d^{(\alpha,\beta)}(x)$ has its maximum at $x \approx \alpha^{-1/\beta}((\beta - 1)/\beta)^{1/\beta}$.

It is also worth mentioning that the asymptotic skewness and the asymptotic kurtosis excess of the omega probability distribution given by (44) and (45), respectively, depend only on the parameter β. Hence, the parameter β may be viewed as the shape parameter of the omega probability distribution.

2.5 | Statistical estimation of parameters

Here, we will discuss two methods for the parameter estimation of the omega distribution. Firstly, we will provide the log-likelihood function and propose a method to maximize it. Secondly, we will discuss how the parameters of the omega distribution can be estimated by fitting its CDF to an empirical CDF. Here, we assume that the random variable τ represents the time-to-first-failure of a component or system and τ has an omega probability distribution with the parameters $\alpha, \beta, d > 0$. It should be highlighted that the above-mentioned two methods utilize different data set types. In the case of maximum likelihood estimation, we assume that independent and identically distributed $t_1, t_2, \ldots, t_n$ observations are available on the random variable τ. However, in many cases of practical reliability engineering, the exact $t_1, t_2, \ldots, t_n$ time-to-first-failure data are not available, rather we have frequency data indicating the number of components or systems that have failed in given time periods. In such cases, the empirical CDF of τ can be directly produced, and the second method, which estimates the parameters by fitting the omega CDF to the empirical CDF of τ, can be applied.

2.5.1 | Maximum likelihood estimation

Let $t_1, t_2, \ldots, t_n$ be independent and identically distributed observations on the random variable τ and $\tau \sim \omega(\alpha, \beta, d)$. Utilizing the definition of the omega probability density function $f_d^{(\alpha,\beta)}(x)$ given by (4), the likelihood function $L(\alpha, \beta, d)$ is

$$L(\alpha, \beta, d) = \prod_{i=1}^{n} f_d^{(\alpha,\beta)}(t_i) = \\ = \alpha^n \beta^n \prod_{i=1}^{n} \left(t_i^{\beta-1} \frac{d^{2\beta}}{d^{2\beta} - t_i^{2\beta}} \left(\frac{d^\beta + t_i^\beta}{d^\beta - t_i^\beta} \right)^{\frac{-\alpha d^\beta}{2}} \right). \tag{47}$$

The log-likelihood function $l(\alpha,\beta,d) = \ln(L(\alpha,\beta,d))$ is

$$l(\alpha,\beta,d) = n\ln\alpha + n\ln\beta + (\beta-1)\sum_{i=1}^{n}\ln t_i + \\ + \sum_{i=1}^{n}\ln\frac{d^{2\beta}}{d^{2\beta}-t_i^{2\beta}} - \frac{\alpha d^{\beta}}{2}\sum_{i=1}^{n}\ln\frac{d^{\beta}+t_i^{\beta}}{d^{\beta}-t_i^{\beta}}. \tag{48}$$

Notice that according to Definition 2, the parameter d specifies the support of $f_d^{(\alpha,\beta)}(x)$; that is, $f_d^{(\alpha,\beta)}(x)$ is positive only if $x \in (0,d)$. It means that the value of parameter d needs to satisfy the condition $d > \max_{i=1,\dots,n}(t_i)$. So the maximum likelihood estimations of the parameters α, β, and d can be obtained by solving the following minimization problem:

$$\begin{aligned} &-l(\alpha,\beta,d) \to \min \\ &\alpha,\beta,d > 0 \\ &d > \max_{i=1,\dots,n}(t_i). \end{aligned} \tag{49}$$

There is no closed form solution for this minimization problem; it can be solved by utilizing a global optimization method. We propose the application of the so-called GLOBAL method, which is a stochastic global optimization procedure introduced by Csendes et al.[45,46]

It is worth mentioning that there is an interesting connection between the log-likelihood functions of the Weibull and omega probability distributions. On the one hand, the log-likelihood function of the Weibull probability distribution given by the probability density function in (2) is

$$l(\alpha,\beta) = n\ln\alpha + n\ln\beta + (\beta-1)\sum_{i=1}^{n}\ln t_i - \alpha\sum_{i=1}^{n}t_i^{\beta}. \tag{50}$$

On the other hand,

$$\lim_{d\to\infty}\left(\sum_{i=1}^{n}\ln\frac{d^{2\beta}}{d^{2\beta}-t_i^{2\beta}}\right) = 0, \tag{51}$$

and on the basis of Theorem 1,

$$\lim_{d\to\infty}\left(\frac{\alpha d^{\beta}}{2}\sum_{i=1}^{n}\ln\frac{d^{\beta}+t_i^{\beta}}{d^{\beta}-t_i^{\beta}}\right) = \alpha\sum_{i=1}^{n}t_i^{\beta}. \tag{52}$$

Hence, if $d \to \infty$, then the log-likelihood function of the omega probability distribution in (48) is identical with the log-likelihood function of the Weibull probability distribution given by (50).

2.5.2 | Fitting the cumulative probability distribution function

Let $N(t)$ denotes the number of components or systems that have survived up to time t from the number of components or systems $N(0)$ that were initially put into operation. Let Δt denotes the length of a time period, and let $t = i\Delta t$, where $i = 0,1,\dots,n$ and $\Delta t > 0$. If $\Delta t = 1$, then the $N(t) - N(t+\Delta t)$ difference, which represents the number of components or systems that fail in the time interval $(t, t+\Delta t]$, is $N(i) - N(i+1)$, where $i = 0,1,\dots,n-1$. For example, if $\Delta t = 1$ week, then the difference $N(3) - N(4)$ represents the number of components or systems that failed on the fourth week. As noted before, there are cases in practice, when the exact $t_1, t_2, \dots, t_n$ time-to-first-failure data are not available, rather frequency data are available indicating the number of components or systems that have failed in given time periods. In such cases, when the $N(0), N(1), \dots, N(n)$ data are available, the empirical CDF $\hat{F}(t)$ of the time-to-first-failure random variable τ can be computed as

$$\hat{F}(i) = 1 - \frac{N(i)}{N(0)}, \tag{53}$$

where $i = 0,1,\dots,n$. Next, the parameters α, β, and d of the omega probability distribution can be identified by fitting the omega CDF $F_d^{(\alpha,\beta)}(t)$ to the empirical CDF $\hat{F}(t)$. For this purpose, we need to minimize the following sum of squares:

$$S(\alpha,\beta,d) = \sum_{i=1}^{n}\left(F_d^{(\alpha,\beta)}(i) - \hat{F}(i)\right)^2, \tag{54}$$

with the constraints $\alpha,\beta,d > 0$. This minimization problem can be solved by utilizing the GLOBAL method that we referenced in Section 2.5.1. In our demonstrative example (Section 3.2), we will show how the CDF fitting method can be applied in practice.

3 | NOVEL MODELS OF BATHTUB-SHAPED HAZARD CURVES

Now, potential applications of the omega probability distribution for failure rate function modeling will be discussed. We will demonstrate that the so-called omega HF can be viewed as a suitable model of bathtub-shaped failure rate functions.

Here again, let the random variable τ be the time-to-first-failure of a component or system. It is well known that a typical HF curve of a component or a system is bathtub shaped; that is, it can be divided into three distinct phases called the infant mortality period, useful life, and wear-out period. It is also typical that the probability distribution of τ is different in the three characteristic phases of the bathtub-shaped HF. If τ has a Weibull probability distribution with the parameters $\alpha,\beta > 0$, then the HF $h^{(\alpha,\beta)}(t)$ of τ, which we call the Weibull HF, is

$$h^{(\alpha,\beta)}(t) = \alpha\beta t^{\beta-1}. \tag{55}$$

We can see from (55) a notable property of the HF $h^{(\alpha,\beta)}(t)$. Namely,

- if $0 < \beta < 0$, then $h^{(\alpha,\beta)}(t)$ is decreasing,
- if $\beta = 1$, then $h^{(\alpha,\beta)}(t)$ is constant with the value of $\alpha\beta$,
- if $\beta > 1$, then $h^{(\alpha,\beta)}(t)$ is increasing

with respect to time. This property of the Weibull HF indicates that each of the three characteristic phases of a bathtub-shaped hazard curve can be described by an appropriate Weibull HF. That is, three Weibull HFs, a decreasing, a constant, and an increasing, connected to each other can exhibit a bathtub-shaped hazard curve. This universality of the Weibull probability distribution makes it suitable for modeling the probability distribution of time-to-first-failure random variable in a wide range of reliability analyses.

3.1 | The omega hazard function

Now, let us assume that τ has an omega probability distribution with the parameters $\alpha, \beta, d > 0$. In this case, the HF $h_d^{(\alpha,\beta)}(t)$ of τ, which we will call the omega HF, is

$$h_d^{(\alpha,\beta)}(t) = \frac{f_d^{(\alpha,\beta)}(t)}{1 - F_d^{(\alpha,\beta)}(t)} = \frac{\alpha\beta t^{\beta-1}\frac{d^{2\beta}}{d^{2\beta}-t^{2\beta}}\omega_d^{(-\alpha,\beta)}(t)}{\omega_d^{(-\alpha,\beta)}(t)} = \alpha\beta t^{\beta-1}\frac{d^{2\beta}}{d^{2\beta} - t^{2\beta}}, \tag{56}$$

if $0 < t < d$. Utilizing (55) and (56), the omega HF $h_d^{(\alpha,\beta)}(t)$ may be written as

$$h_d^{(\alpha,\beta)}(t) = h^{(\alpha,\beta)}(t) g_d^{(\beta)}(t), \tag{57}$$

where

$$g_d^{(\beta)}(t) = \frac{d^{2\beta}}{d^{2\beta} - t^{2\beta}}, \tag{58}$$

and $\alpha, \beta, d > 0$, $t \in (0, d)$. That is, the omega HF may be viewed as the Weibull HF multiplied by the corrector function $g_d^{(\beta)}(t)$.

The omega HF $h_d^{(\alpha,\beta)}(t)$ has some important properties that make it suitable for modeling bathtub-shaped failure rate curves.

3.1.1 | Piecewise modeling

The following lemma states a key property of the omega HF $h_d^{(\alpha,\beta)}(t)$. It allows us to utilize the omega HF as an alternative to the Weibull HF.

Lemma 4. *For any $t \in (0, d)$, if $d \to \infty$, then $h_d^{(\alpha,\beta)}(t) \to h^{(\alpha,\beta)}(t)$, where $\alpha, \beta, d > 0$.*

Proof. If $t \in (0, d)$ is fixed and $d \to \infty$, then $g_d^{(\beta)}(t) \to 1$ and so

$$h_d^{(\alpha,\beta)}(t) = h^{(\alpha,\beta)}(t) g_d^{(\beta)}(t) \to h^{(\alpha,\beta)}(t). \tag{59}$$

□

The practical implication of this result is as follows. Since the Weibull HF can be utilized as a model for each phase of a bathtub-shaped failure rate curve and $h_d^{(\alpha,\beta)}(t) \approx h^{(\alpha,\beta)}(t)$, if t is small compared with d, the omega HF can also model each phase of the same bathtub-shaped failure rate curve, if d is sufficiently large. That is, the omega

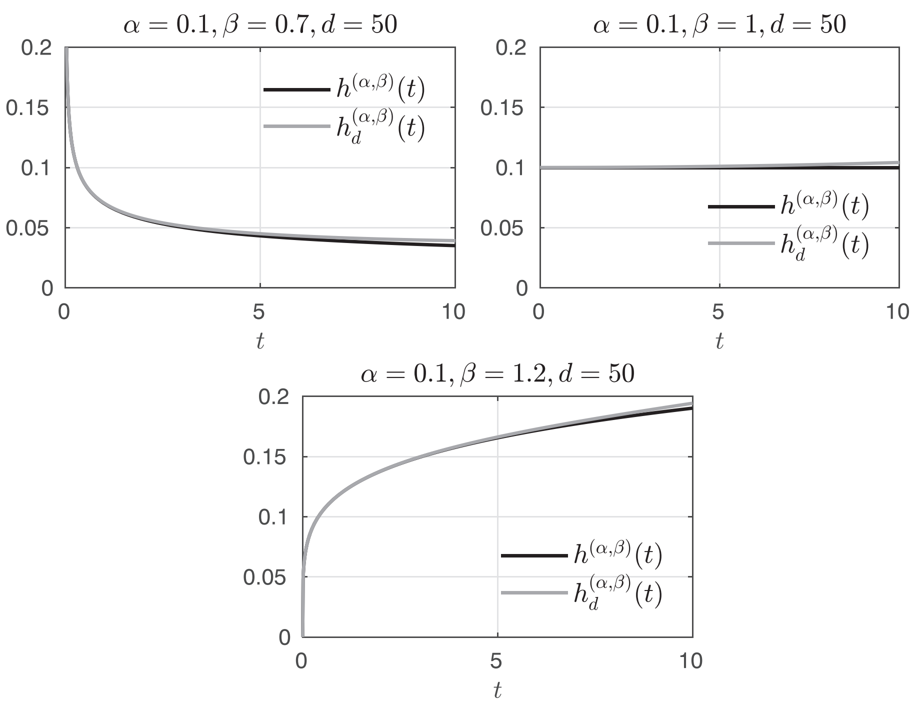

FIGURE 5 Plots of Weibull and omega hazard function curves

HF as an alternative to the Weibull HF can be utilized as a phase-by-phase model of a bathtub-shaped failure rate curve.

Figure 5 shows how the Weibull and omega HFs can model each characteristic phase of a failure rate curve. The plots in Figure 5 demonstrate the results of the previous lemma; that is, if $t \ll d$, then the omega HF approximates quite well the Weibull HF.

3.1.2 | The all-in-one model

The Weibull HF $h^{(\alpha,\beta)}(t)$ is either monotonic or constant; that is, its curve cannot be bathtub shaped. However, lifetime data of a component or system typically require nonmonotonic shapes like the bathtub shape. There have been many modifications developed to the Weibull probability distribution in order to achieve nonmonotonic shapes. See, for example, the publications of Kies,[47] Phani,[48] Mudholkar et al,[49] Xie and Lai,[7] Zhang and Xie,[50] Ghitany et al,[51] Ghitany et al,[52] Lai et al,[35] Sarhan and Apaloo,[53] Silva et al,[54] Bebbington et al,[55] and Cordeiro et al.[56] A comprehensive review of the known modifications of the Weibull probability distribution can be found in the paper of Almalki and Nadarajah.[9] The following lemma suggests that the omega HF $h_d^{(\alpha,\beta)}(t)$ can be utilized as a model for all the three phases of a bathtub-shaped failure rate curve.

Lemma 5. *If $0 < \beta < 1$, then $h_d^{(\alpha,\beta)}(t)$ is strictly convex in the interval $(0,d)$ and $h_d^{(\alpha,\beta)}(t)$ has its minimum at*

$$t_0 = d\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2\beta}}. \tag{60}$$

Proof. The lemma follows from the elementary properties of the omega function $h_d^{(\alpha,\beta)}(t)$ by using its first and second derivatives. Namely,

$$\frac{\mathrm{d}h_d^{(\alpha,\beta)}(t)}{\mathrm{d}t} = h_d^{(\alpha,\beta)}(t)\frac{g_d^{(\beta)}(t)}{t}\left((\beta+1)\frac{t^{2\beta}}{d^{2\beta}} + (\beta-1)\right) \tag{61}$$

and recalling that $h_d^{(\alpha,\beta)}(t) > 0$, $\beta, d, t > 0$ and $t \in (0,d)$, if $0 < \beta < 1$, then the first derivative in (61) changes its sign from negative to positive at t_0. That is, if $0 < \beta < 0$, then $h_d^{(\alpha,\beta)}(t)$ has its single minimum point at t_0. Next, it can be shown that

$$\frac{\mathrm{d}^2 h_d^{(\alpha,\beta)}(t)}{\mathrm{d}t^2} = h_d^{(\alpha,\beta)}(t)\left(\frac{g_d^{(\beta)}(t)}{t}\right)^2\left((\beta+1)(\beta+2)\frac{t^{4\beta}}{d^{4\beta}} + \right.$$
$$\left. + \left(6\beta^2-4\right)\frac{t^{2\beta}}{d^{2\beta}} + (\beta-1)(\beta-2)\right) \tag{62}$$

is positive for any $t \in (0,d)$, if $0 < \beta < 1$; that is, the omega HF is strictly convex in the interval $(0,d)$. □

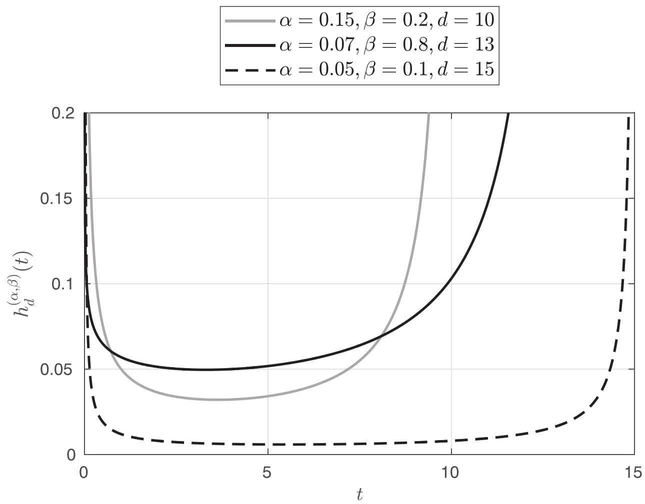

FIGURE 6 Some examples of bathtub-shaped omega hazard function plots

Figure 6 shows some typical plots of the omega HF $h_d^{(\alpha,\beta)}(t)$ with $0 < \beta < 1$. On the basis of the above-discussed properties of the omega HF, we may conclude that the omega probability distribution can be employed to describe the probability distribution of the time-to-first-failure random variable in each characteristic phase of a bathtub-shaped failure rate curve. Moreover, we have two possibilities for modeling a bathtub-shaped failure rate curve. Namely, either we piecewise describe each phase by an omega HF or we apply one omega HF that models the entire failure rate curve. It is worth mentioning that the omega probability distribution with the parameter setting $\beta = 1$ gives the so-called epsilon probability distribution, which can be utilized to approximate the exponential probability distribution and to model the second and third phases of bathtub-shaped hazard curves.[57]

3.2 | A demonstrative example

Now, we will demonstrate how the omega probability distribution can be utilized to model the probability distribution of the time-to-first-failure random variable if its HF is bathtub shaped. In our example, real-life empirical failure data of a laptop motherboard type were analyzed. Here, the examined motherboard type was taken into account as a component, the typical lifetime of which is between 4 and 6 years. In our case, the initially released number of motherboards was $N(0) = 21000$, while the $N(t)$ values were available on weekly basis; that is, $N(i)$ represents the number of components that have survived up to the end of the ith week, $i = 0, 1, \ldots, 295$. Using the $N(i)$ values, the empirical CDF of the time-to-first-failure random variable was calculated according to (53). Utilizing the life table method (see, eg, Saunders[58]), the empirical HF $\hat{h}(t)$

can be computed as

$$\hat{h}(t) = \frac{N(t) - N(t + \Delta t)}{N(t)\Delta t}, \tag{63}$$

and utilizing the fact that $t = i\Delta t$, where $\Delta t = 1$, we get

$$\hat{h}(i) = \frac{N(i) - N(i + 1)}{N(i)}, \tag{64}$$

where $i = 0, 1, \ldots, 294$. The $N(i)$ values and the computed values of the empirical CDF and empirical HF are listed in Tables A1 to A3. Notice that from week $i = 295$, the number of functioning components $N(i)$ is zero and so $\hat{h}(i)$ could be computed for $i = 0, 1, \ldots, 294$.

3.2.1 | Estimating the parameters of the omega distribution

Here, the parameters α, β, and d of the omega probability distribution were identified by fitting the omega CDF $F_d^{(\alpha,\beta)}(t)$ to the empirical CDF $\hat{F}(t)$. For this purpose, we solved the minimization problem

$$S(\alpha, \beta, d) = \sum_{i=1}^{n} \left(F_d^{(\alpha,\beta)}(i) - \hat{F}(i) \right)^2 \to \min, \tag{65}$$

by utilizing the GLOBAL method referenced in Section 2.5.1 (in our case, $n = 295$). The GLOBAL method was implemented, and the following analyses were done in the MATLAB 2017b numerical computing environment. In order to determine parameter constraints for the minimization problem in (54), the following properties of the empirical data and of the omega CDF were taken into consideration.

The examined empirical HF $\hat{h}(t)$ is bathtub shaped (see Figure 9), and on the basis of the results of Section 3.1.2, the omega HF $h_d^{(\alpha,\beta)}(t)$ is bathtub shaped only if $0 < \beta < 1$. That is, we can set the constraint $0 < \beta < 1$ to find the minimum of the function $S(\alpha, \beta, d)$.

Since $F_d^{(\alpha,\beta)}(t) = 1$ if $t \geq d$, and the smallest i for which $\hat{F}(i) = 1$ holds is $i = n$, we may expect that the optimal value of parameter d is not much greater than n and it is not much less than n. Therefore, it is rational setting the constraint $n/2 \leq d \leq 2n$ for the parameter d in the minimization problem given by (54). Note that although this constraint setting is somewhat arbitrary, it proves to be valid in practice.

Here, the following heuristic was utilized to identify boundaries for the parameter α. On the basis of Lemma 5, if $\beta \in (0, 1)$, then the omega HF $h_d^{(\alpha,\beta)}(t)$ is minimal in the interval $(0, d)$ at

$$t_0 = d\left(\frac{1 - \beta}{1 + \beta}\right)^{\frac{1}{2\beta}}, \tag{66}$$

and so the minimum value of $h_d^{(\alpha,\beta)}(t)$ in $(0, d)$ is

$$\begin{aligned} \min_{t\in(0,d)} \left(h_d^{(\alpha,\beta)}(t) \right) &= h_d^{(\alpha,\beta)}(t_0) = \alpha\beta t_0^{\beta-1} \frac{d^{2\beta}}{d^{2\beta} - t_0^{2\beta}} = \\ &= \alpha \frac{d^{\beta-1}}{2} (1 - \beta)^{\frac{\beta-1}{2\beta}} (1 + \beta)^{\frac{\beta+1}{2\beta}}. \end{aligned} \tag{67}$$

By utilizing the values of the empirical HF $\hat{h}(t)$, we can empirically identify an h_l lower boundary and an h_u upper boundary for the minimum value of the HF $h_d^{(\alpha,\beta)}(t)$; that is,

$$h_l \leq \alpha \frac{d^{\beta-1}}{2} (1 - \beta)^{\frac{\beta-1}{2\beta}} (1 + \beta)^{\frac{\beta+1}{2\beta}} \leq h_u. \tag{68}$$

We will utilize the results of the following lemma to identify boundaries for the parameter α.

Lemma 6. *For any $\beta \in (0, 1)$ and $d > 1$,*

$$\frac{1}{d} < d^{\beta-1} < 1, \tag{69}$$

$$1 < (1 - \beta)^{\frac{\beta-1}{2\beta}} < \sqrt{\mathrm{e}}, \tag{70}$$

$$\sqrt{\mathrm{e}} < (1 + \beta)^{\frac{\beta+1}{2\beta}} < 2. \tag{71}$$

Proof. The inequalities in (69) trivially follow from the conditions $\beta \in (0, 1)$ and $d > 1$.

Since

$$(1 - \beta)^{\frac{\beta-1}{2\beta}} \tag{72}$$

is continuous and strictly monotonously decreasing in $(0, 1)$, to prove the inequalities in (70), it is sufficient to show that

$$\lim_{\beta\to 0^+} (1 - \beta)^{\frac{\beta-1}{2\beta}} = \sqrt{\mathrm{e}} \tag{73}$$

and

$$\lim_{\beta\to 1^-} (1 - \beta)^{\frac{\beta-1}{2\beta}} = 1. \tag{74}$$

Let $\beta' = 1/\beta$. Then

$$\begin{aligned} \lim_{\beta\to 0^+} (1 - \beta)^{\frac{\beta-1}{2\beta}} &= \left(\lim_{\beta'\to\infty} \left(1 - \frac{1}{\beta'}\right)^{1-\beta'} \right)^{\frac{1}{2}} = \\ &= \left(\lim_{\beta'\to\infty} \left(1 - \frac{1}{\beta'}\right) \right)^{\frac{1}{2}} \left(\lim_{\beta'\to\infty} \left(1 - \frac{1}{\beta'}\right)^{\beta'} \right)^{-\frac{1}{2}} = 1 \cdot \sqrt{\mathrm{e}}. \end{aligned} \tag{75}$$

Now, let $\beta' = 1 - \beta$. Utilizing this substitution and the L'Hospital rule gives

$$\lim_{\beta\to 1^-} (1 - \beta)^{\frac{\beta-1}{2\beta}} = \left(\lim_{\beta'\to 0^+} \mathrm{e}^{-\frac{\beta'}{1-\beta'} \ln \beta'} \right)^{\frac{1}{2}} = \left(\mathrm{e}^0\right)^{\frac{1}{2}} = 1. \tag{76}$$

As

$$(1 + \beta)^{\frac{\beta+1}{2\beta}} \tag{77}$$

is continuous and strictly monotonously increasing in $(0, 1)$, to prove the inequalities in (71), it is sufficient to show that

$$\lim_{\beta\to 0^+} (1 + \beta)^{\frac{\beta+1}{2\beta}} = \sqrt{\mathrm{e}} \tag{78}$$

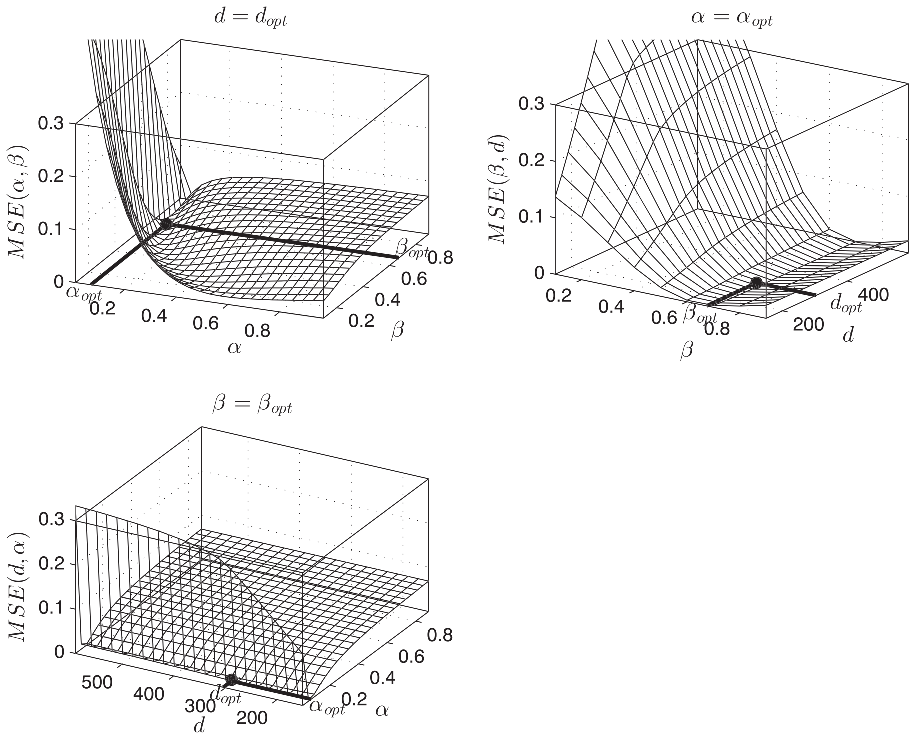

FIGURE 7 Mean squared error (MSE) vs parameters of the fitted omega cumulative distribution function (CDF)

and

$$\lim_{\beta \to 1^-} (1+\beta)^{\frac{\beta+1}{2\beta}} = 2. \tag{79}$$

Let $\beta' = 1/\beta$. Then

$$\lim_{\beta \to 0^+} (1+\beta)^{\frac{\beta+1}{2\beta}} = \left(\lim_{\beta' \to \infty} \left(1 + \frac{1}{\beta'}\right)^{(1+\beta')} \right)^{\frac{1}{2}} =$$
$$= \left(\lim_{\beta' \to \infty} \left(1 + \frac{1}{\beta'}\right) \right)^{\frac{1}{2}} \left(\lim_{\beta' \to \infty} \left(1 + \frac{1}{\beta'}\right)^{\beta'} \right)^{\frac{1}{2}} = \sqrt{e}. \tag{80}$$

Since (79) trivially holds, the lemma has been proven. □

In our case, $0 < \beta < 1$ and $d > 1$, and so the results of Lemma 6 and the inequalities in (68) lead to

$$h_l \le \alpha \frac{d^{\beta-1}}{2} (1-\beta)^{\frac{\beta-1}{2\beta}} (1+\beta)^{\frac{\beta+1}{2\beta}} < \alpha\sqrt{e}, \tag{81}$$

$$h_u \ge \alpha \frac{d^{\beta-1}}{2} (1-\beta)^{\frac{\beta-1}{2\beta}} (1+\beta)^{\frac{\beta+1}{2\beta}} > \frac{\alpha}{2} \frac{1}{d} \sqrt{e}. \tag{82}$$

Next, utilizing the condition that $d \le 2n$, from (81) and from (82), we get the following boundaries for α:

$$h_l e^{-\frac{1}{2}} < \alpha < 2h_u d e^{-\frac{1}{2}} \le 4nh_u e^{-\frac{1}{2}}. \tag{83}$$

In summary, the following parameter constraints were set in the minimization problem given by (54):

$$\begin{gathered} 0 < \beta < 1 \\ \frac{n}{2} \le d \le 2n \\ h_l e^{-\frac{1}{2}} < \alpha < 4nh_u e^{-\frac{1}{2}}. \end{gathered} \tag{84}$$

In the case of our empirical data, h_l and h_u could be set as $h_l = 0.01$ and $h_u = 0.02$, and since $n = 295$, we had the following parameter constraints:

$$\begin{gathered} 0 < \beta < 1 \\ \frac{295}{2} \le d \le 590 \\ 0.0061 < \alpha < 14.3141. \end{gathered} \tag{85}$$

The optimal values α_{opt}, β_{opt}, and d_{opt} of the parameters α, β, and d, respectively, identified by using the GLOBAL method are $\alpha_{opt} = 0.069240$, $\beta_{opt} = 0.674587$, and $d_{opt} = 304.12$.

The three-dimensional surfaces of projections of the mean squared error (MSE) function $MSE(\alpha, \beta, d) = S(\alpha, \beta, d)/n$, in which one of the three parameters is always fixed at its optimal value, and the projections of the global minimum of $MSE(\alpha, \beta, d)$ are shown in Figure 7.

These three-dimensional plots can graphically inform us about the sensitivity of the MSE to the values of parameters α, β, and d quite well. Utilizing Figure 7, the following general conclusions can be drawn about the sensitivity of function $MSE(\alpha, \beta, d)$. From the right upper and left lower subplots of Figure 7, we can see that the MSE of the fitted omega CDF is much less sensitive to the value of the parameter d than to the values of parameters α and β. At the same time, we can see from the left upper subplot of Figure 7 that at $\alpha = \alpha_{opt}$, $\beta = \beta_{opt}$, $d = d_{opt}$, the function $MSE(\alpha, \beta, d)$ is more sensitive to the value of parameter α than to the value of parameter β.

3.2.2 | Comparisons with some well-known three-parameter modifications of the Weibull distribution

Our method was compared with the well-known modifications of the Weibull distribution having three parameters. Namely, the CDFs of the MW distribution,[35] the EW distribution,[13] the GWF distribution,[36] the GPW distribution,[37] the MWEX distribution,[38] the ODDW distribution,[39] and the RNMW distribution[40] were also fitted to the same empirical CDF. Tables 3 and 4 show the CDFs and the HFs of the examined three-parameter modifications of the Weibull distribution, respectively. Note that here we apply the same parameter notations as Almalki and Nadarajah did in their review paper.[9] The parameters of each of these distributions were identified like those of the omega distribution; that is, the sum of squared differences between the parametric CDF and the empirical CDF was minimized by using the GLOBAL minimization method. The optimal parameter values and the MSE value for each CDF fitting are summarized in Table 5.

Figure 8 shows the plots of the empirical CDF $\hat{F}(t)$ and the fitted parametric CDF $F(t)$ for each of the examined three-parameter probability distributions. In each subplot of Figure 8, the left-hand side scale belongs to functions $\hat{F}(t)$ and $F(t)$, while the right-hand side scale is connected with the difference function $\hat{F}(t) - F(t)$. Figure 9 shows the plot of the empirical HF $\hat{h}(t)$ and the HF plots of the examined distributions. Furthermore, for each of the CDFs fitted, the three-dimensional surfaces of the projections of the MSE of fit as function of the model parameters and the optimal model parameter values are shown in Figures A1 to A7. These three-dimensional plots express graphically the sensitivity of the MSE to the model parameters quite well.

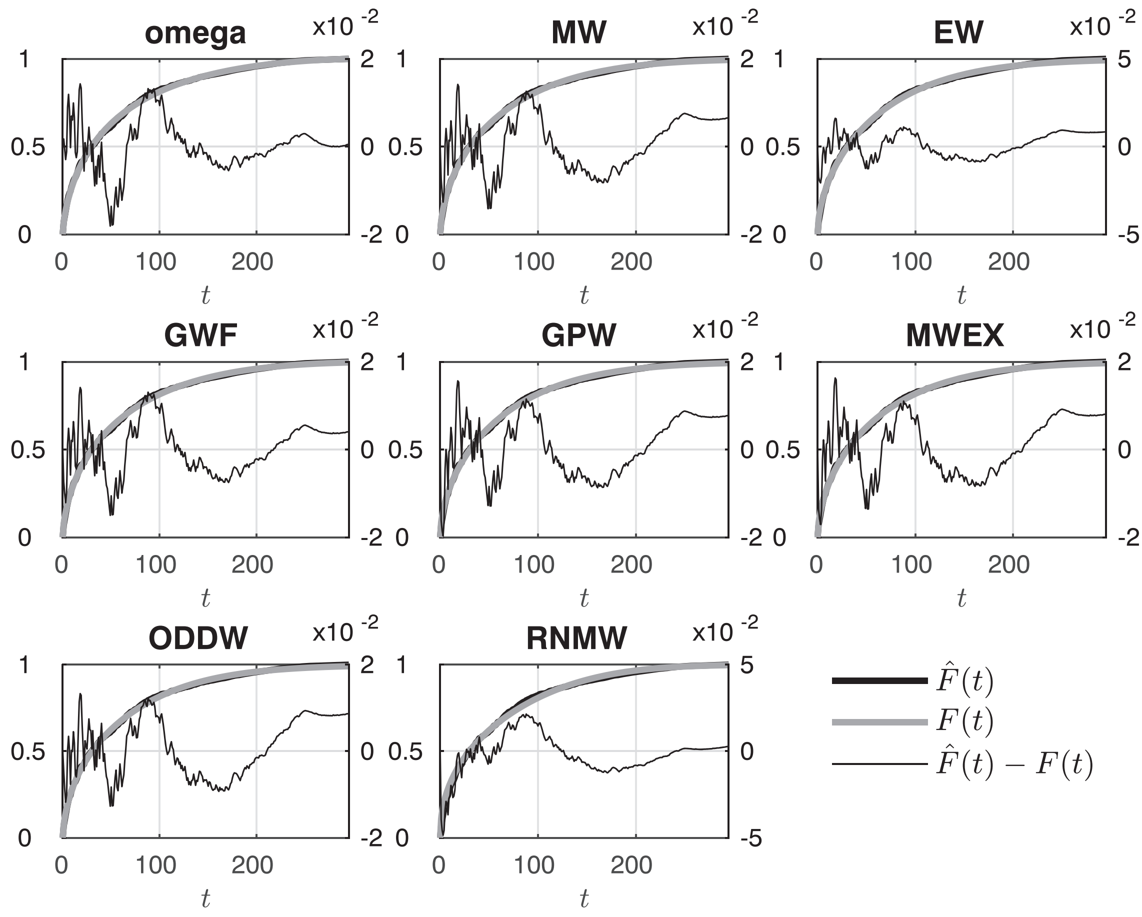

FIGURE 8 Plots of the fitted three-parameter CDFs. CDF, cumulative distribution function; EW, exponentiated Weibull; GPW, generalized power Weibull; GWF, generalized Weibull family; MW, modified Weibull; MWEX, modified Weibull extension; ODDW, odd Weibull; RNMW, reduced modified Weibull

TABLE 3 The examined three-parameter modifications of the Weibul CDF

Model	CDF	Parameter Domains
Omega	$F(t)=\begin{cases}1-\left(\frac{d^\beta+t^\beta}{d^\beta-t^\beta}\right)^{\frac{-\alpha d^\beta}{2}}, & \text{if } 0<t<d\\ 1, & \text{if } t\geq d\end{cases}$	$\alpha,\beta,d>0$
MW	$F(t)=1-\mathrm{e}^{-\beta t^\gamma \mathrm{e}^{\lambda t}}$	$\beta>0,\gamma,\lambda\geq 0$
EW	$F(t)=\left(1-\mathrm{e}^{-\alpha t^\theta}\right)^\lambda$	$\alpha,\theta,\lambda>0$
GWF	$F(t)=1-\left(1-\alpha\lambda t^\theta\right)^{1/\lambda}$	$\alpha,\theta>0,-\infty<\lambda<\infty$
GPW	$F(t)=1-\mathrm{e}^{1-(1+\alpha t^\theta)^{1/\lambda}}$	$\alpha,\theta,\lambda>0$
MWEX	$F(t)=1-\mathrm{e}^{\lambda\alpha^{-1/\theta}\left(1-\mathrm{e}^{\alpha t^\theta}\right)}$	$\alpha,\theta,\lambda>0$
ODDW	$F(t)=1-\left[1+\left(\mathrm{e}^{\alpha t^\theta}-1\right)^\lambda\right]^{-1}$	$\alpha,\theta,\lambda>0$
RNMW	$F(t)=1-\mathrm{e}^{-\alpha\sqrt{t}-\beta\sqrt{t}\mathrm{e}^{\lambda t}}$	$\alpha,\beta,\lambda>0$

Abbreviations: CDF, cumulative distribution function; EW, exponentiated Weibull; GPW, generalized power Weibull; GWF, generalized Weibull family; MW, modified Weibull; MWEX, modified Weibull extension; ODDW, odd Weibull; RNMW, reduced modified Weibull.

$t>0$; for the GWF model, if $\lambda>0$, then $t\in(0,(\alpha\lambda)^{-1/\theta})$.

TABLE 4 Hazard functions of the examined three-parameter modifications of the Weibul distribution

Model	Hazard Function	Restriction of Parameters if $h(t)$ Is Bathtub Shaped
Omega	$h(t)=\alpha\beta t^{\beta-1}\frac{d^{2\beta}}{d^{2\beta}-t^{2\beta}}$	$0<\beta<1$
MW	$h(t)=\beta(\gamma+\lambda t)t^{\gamma-1}\mathrm{e}^{\lambda t}$	$0<\theta<1$
EW	$h(t)=\alpha\theta\lambda t^{\theta-1}\mathrm{e}^{-\alpha t^\theta}(1-\mathrm{e}^{-\alpha t^\theta})^{-1}$	$\theta>1;\theta\lambda<1$
GWF	$h(t)=\alpha\theta t^{\theta-1}(1-\alpha\lambda t^\theta)^{-1}$	$\theta<1;0<\lambda$
GPW	$h(t)=\alpha\theta\lambda^{-1}(1+\alpha t^\theta)^{1/\lambda-1}$	$0<\lambda<\theta<1$
MWEX	$h(t)=\lambda\alpha^{((\theta-1)/\theta)}t^{\theta-1}\mathrm{e}^{\alpha t^\theta}$	$\theta<1$
ODDW	$h(t)=\alpha\theta\lambda t^{\theta-1}\mathrm{e}^{\alpha t^\theta}(\mathrm{e}^{\alpha t^\theta}-1)^{\lambda-1}(1+(\mathrm{e}^{\alpha t^\theta}-1)^\lambda)^{-1}$	$\theta>1;\theta\lambda<1$
RNMW	$h(t)=\frac{1}{2\sqrt{t}}(\alpha+\beta(1+2\lambda t)\mathrm{e}^{\lambda t})$	

Abbreviations: EW, exponentiated Weibull; GPW, generalized power Weibull; GWF, generalized Weibull family; MW, modified Weibull; MWEX, modified Weibull extension; ODDW, odd Weibull; RNMW, reduced modified Weibull.

$t>0$; for the GWF model, if $\lambda>0$, then $t\in(0,(\alpha\lambda)^{-1/\theta})$.

TABLE 5 CDF fitting results

Function	Parameters			MSE
Omega	$\alpha=0.069240$	$\beta=0.674587$	$d=304.121895$	3.22703e-05
MW	$\beta=0.078857$	$\gamma=0.618880$	$\lambda=2.153\text{e-}03$	3.86417e-05
EW	$\alpha=2.045\text{e-}03$	$\theta=1.326312$	$\lambda=0.396102$	4.78888e-05
GWF	$\alpha=0.077903$	$\theta=0.607756$	$\lambda=0.343967$	3.34385e-05
GPW	$\alpha=1.33\text{e-}04$	$\theta=0.534145$	$\lambda=1.571\text{e-}03$	4.50262e-05
MWEX	$\alpha=0.072205$	$\theta=0.550252$	$\lambda=9.651\text{e-}03$	4.45713e-05
ODDW	$\alpha=0.020835$	$\theta=1.027860$	$\lambda=0.657597$	4.57777e-05
RNMW	$\alpha=1\text{e-}05$	$\beta=0.113847$	$\lambda=3.727\text{e-}03$	1.25777e-04

Abbreviations: CDF, cumulative distribution function; EW, exponentiated Weibull; GPW, generalized power Weibull; GWF, generalized Weibull family; MSE, mean squared error; MW, modified Weibull; MWEX, modified Weibull extension; ODDW, odd Weibull; RNMW, reduced modified Weibull.

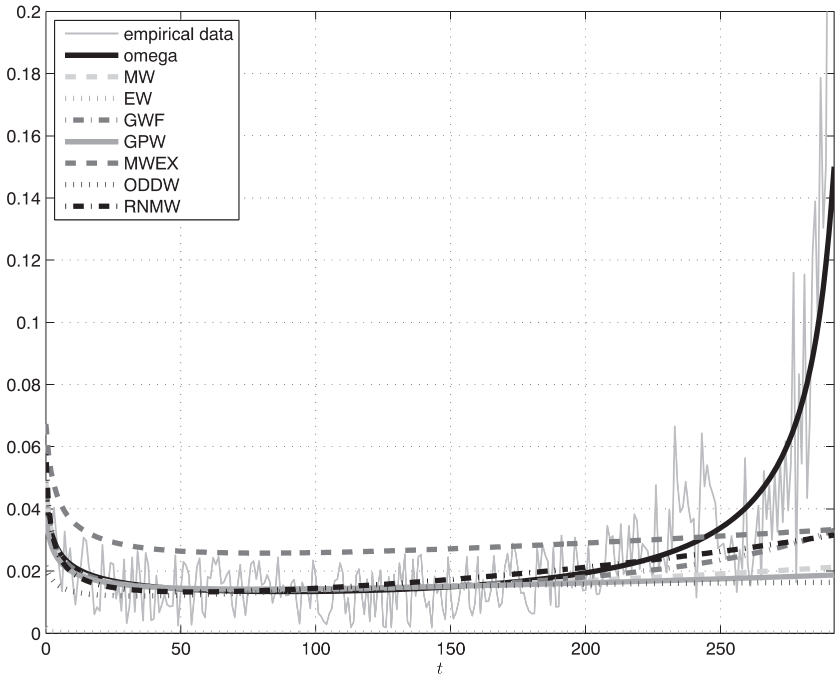

FIGURE 9 Plots of the hazard functions of the fitted three-parameter distributions. EW, exponentiated Weibull; GPW, generalized power Weibull; GWF, generalized Weibull family; MW, modified Weibull; MWEX, modified Weibull extension; ODDW, odd Weibull; RNMW, reduced modified Weibull

3.2.3 | Discussion

The CDF fitting results show that the examined CDFs match quite well the empirical CDF. Expect for the RNMW CDF, which has an order of magnitude greater MSE value than the other models, the MSE values of fitted parametric CDFs all have order of magnitude of −5. Note that among the CDFs examined, the omega and the GWF CDFs give the best fitting results. Their MSE values are very close, 3.22703e-05 and 3.34385e-05, respectively. From the HF plots of the fitted distributions (Figure 9), it can be concluded that the majority of the HFs do not exhibit complete bathtub shapes. The HFs of the majority of the examined probability distributions match the first phase and the second phase of the empirical HF quite well, but only the HFs of the omega, GWF and RNMW distributions show increasing third phases. It should be emphasized here that the HFs of the fitted GWF and RNMW distributions just slightly increase in the third phase of the empirical HF, while the HF of the fitted omega distribution is able to match the third rapidly increasing phase of the empirical HF quite well. These results are in line with Almalki and Nadarajah's findings[9] who carried out an excellent review of the literature on modifications of the Weibull probability distribution. In their article, the possible shapes of the functions were examined with various parameter values. Both our results and their figures suggest that several models proposed in the literature are not able to follow a bathtub shape if the second phase of the failure rate time series is not long enough.

At this point, it should also be mentioned that neither the CDF nor the HF of the omega probability distribution contains any exponential term, these functions being composed of power terms. This property of our novel model is advantageous in situations where computation time is a critical factor. Notice that among the probability distributions examined, apart from the omega distribution, only the GWF distribution possesses a CDF and a HF having no exponential terms in them. The flexibility of our model is because the exponential function tends to infinity over an unbounded domain, while the omega function does so over the bounded $(0, d)$ domain $(d > 0)$, and so the omega function can more appropriately follow sudden changes.

4 | CONCLUSIONS AND FUTURE WORK

We introduced the omega probability distribution that is given by the probability density function

$$f_d^{(\alpha,\beta)}(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \alpha\beta x^{\beta-1}\frac{d^{2\beta}}{d^{2\beta}-x^{2\beta}}\omega_d^{(-\alpha,\beta)}(x), & \text{if } 0 < x < d \\ 0, & \text{if } x \geq d, \end{cases} \tag{86}$$

where

$$\omega_d^{(-\alpha,\beta)}(x) = \left(\frac{d^\beta + x^\beta}{d^\beta - x^\beta}\right)^{\frac{-\alpha d^\beta}{2}}, \tag{87}$$

$\alpha, \beta, d > 0$. We showed that with the d parameter, the asymptotic omega probability distribution is just the Weibull probability distribution. Afterwards, we demonstrated that the HF $h_d^{(\alpha,\beta)}(t)$ of the omega probability distribution is

$$h_d^{(\alpha,\beta)}(t) = \alpha\beta t^{\beta-1} \frac{d^{2\beta}}{d^{2\beta} - t^{2\beta}}, \tag{88}$$

if $t \in (0, d)$. It was also proven here that for any $t \in (0, d)$, the omega HF $h_d^{(\alpha,\beta)}(t)$ tends to the Weibull HF if $d \to \infty$. Moreover, we pointed out that the omega HF can be utilized for modeling bathtub-shaped failure rate functions. This means that the stated properties of the omega HF allow us to utilize the omega probability distribution not only to describe the probability distribution of the time-to-first-failure random variable in each phase of the bathtub curve similar to the Weibull distribution but also to use the omega HF to model the whole bathtub-shaped failure rate curve.

On the basis of an empirical failure data set of a laptop motherboard type, the application of the omega probability distribution was demonstrated to model the probability distribution of the time-to-first-failure random variable when the empirical HF is bathtub shaped. As the CDF of the proposed omega probability distribution has three parameters, its performance was compared with the CDFs of other three-parameter modifications of the Weibull distribution. The CDF fitting results show that although the examined CDFs match quite well the empirical CDF, the majority of the corresponding HFs do not exhibit complete bathtub shapes. The HFs of the majority of the examined probability distributions match the first phase and the second phase of the empirical HF quite well, but only the HFs of the omega, GWF and RNMW distributions show increasing third phases. It should be emphasized here that the HFs of the fitted GWF and RNMW distributions just slightly increase in the third phase of the empirical HF, while the HF of the fitted omega distribution is able to match the third rapidly increasing phase of the empirical HF quite well. These findings agree with the general conclusions drawn by Almalki and Nadarajah[9] that several proposed models are not able to follow a bathtub shape if the second constant phase of the failure rate time series is not long enough.

From a practical point of view, it is worth noting that the omega probability distribution has two important features, namely, its simplicity and flexibility. Firstly, its simplicity is because both the CDF and the HF include power functions and lack exponential terms. Secondly, its flexibility arises from the fact that while the exponential function tends to infinity over an unbounded domain, the omega function does so over the bounded domain $(0, d)$, which means that the omega HF can more appropriately follow sudden changes ($d > 0$).

Finally, the advantages of utilizing the omega probability distribution to model the time-to-first-failure random variable of a component or system can be summarized as follows.

- The CDF and the HF of the omega probability distribution have simple formulas.
- The formulas of the CDF and the HF of the omega probability distribution do not include any exponential term; these are composed of power functions.
- The omega HF can exhibit monotonic and constant shapes, and so it can model each of the three phases of the hazard curve.
- The omega HF can exhibit bathtub shapes; that is, it can be utilized to describe a complete bathtub-shaped hazard curve in one go.
- The asymptotic omega probability distribution is just the Weibull probability distribution; thus, the omega probability distribution can be viewed as an alternative to the Weibull distribution.

As part of a future research, by utilizing independent and identically distributed observations on the time-to-first-failure random variable, we plan to compute the maximum likelihood estimations of the parameters of the omega probability distribution and the maximum likelihood estimations of the parameters of other well-known modifications of the Weibull distribution having three parameters. Once the maximum value of the likelihood function for each of the studied distributions has been computed, we can compare the goodness of the models by applying the Akaike information criterion (AIC) and statistical tests.

We also intend to study how the flexibility of the omega HF may further be enhanced by replacing the independent variable t in (87) with appropriate functions of t. We also plan to examine how the generalized exponential differential equation in (14) with higher ε values can be utilized to generate probability distribution functions, which are suitable for reliability modeling.

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APPENDIX

TABLE A1 Empirical data and the values of empirical cumulative distribution function (CDF) and empirical hazard function (HF) for weeks 0 to 99

i	$N(i)$	$\hat{F}(i)$	$\hat{h}(i)$	i	$N(i)$	$\hat{F}(i)$	$\hat{h}(i)$
0	21000	0.0000	0.0686	50	8018	0.6182	0.0025
1	19559	0.0686	0.0381	51	7998	0.6191	0.0139
2	18814	0.1041	0.0315	52	7887	0.6244	0.0229
3	18221	0.1323	0.0327	53	7706	0.6330	0.0218
4	17626	0.1607	0.0404	54	7538	0.6410	0.0188
5	16914	0.1946	0.0305	55	7396	0.6478	0.0027
6	16398	0.2191	0.0208	56	7376	0.6488	0.0115
7	16057	0.2354	0.0135	57	7291	0.6528	0.0214
8	15841	0.2457	0.0327	58	7135	0.6602	0.0241
9	15323	0.2703	0.0214	59	6963	0.6684	0.0069
10	14995	0.2860	0.0273	60	6915	0.6707	0.0051
11	14585	0.3055	0.0167	61	6880	0.6724	0.0182
12	14341	0.3171	0.0091	62	6755	0.6783	0.0176
13	14210	0.3233	0.0183	63	6636	0.6840	0.0243
14	13950	0.3357	0.0216	64	6475	0.6917	0.0221
15	13649	0.3500	0.0308	65	6332	0.6985	0.0246
16	13229	0.3700	0.0297	66	6176	0.7059	0.0159
17	12836	0.3888	0.0213	67	6078	0.7106	0.0165
18	12563	0.4018	0.0172	68	5978	0.7153	0.0187
19	12347	0.4120	0.0061	69	5866	0.7207	0.0201
20	12272	0.4156	0.0084	70	5748	0.7263	0.0090
21	12169	0.4205	0.0071	71	5696	0.7288	0.0051
22	12082	0.4247	0.0284	72	5667	0.7301	0.0148
23	11739	0.4410	0.0144	73	5583	0.7341	0.0220
24	11570	0.4490	0.0148	74	5460	0.7400	0.0123
25	11399	0.4572	0.0226	75	5393	0.7432	0.0057
26	11141	0.4695	0.0223	76	5362	0.7447	0.0145
27	10893	0.4813	0.0111	77	5284	0.7484	0.0223
28	10772	0.4870	0.0154	78	5166	0.7540	0.0232
29	10606	0.4950	0.0072	79	5046	0.7597	0.0174
30	10530	0.4986	0.0237	80	4958	0.7639	0.0232
31	10280	0.5105	0.0032	81	4843	0.7694	0.0155
32	10247	0.5120	0.0087	82	4768	0.7730	0.0161
33	10158	0.5163	0.0244	83	4691	0.7766	0.0188
34	9910	0.5281	0.0069	84	4603	0.7808	0.0165
35	9842	0.5313	0.0264	85	4527	0.7844	0.0097
36	9582	0.5437	0.0177	86	4483	0.7865	0.0129
37	9412	0.5518	0.0040	87	4425	0.7893	0.0244
38	9374	0.5536	0.0258	88	4317	0.7944	0.0116
39	9132	0.5651	0.0219	89	4267	0.7968	0.0155
40	8932	0.5747	0.0041	90	4201	0.8000	0.0121
41	8895	0.5764	0.0046	91	4150	0.8024	0.0123
42	8854	0.5784	0.0185	92	4099	0.8048	0.0161
43	8690	0.5862	0.0067	93	4033	0.8080	0.0174
44	8632	0.5890	0.0093	94	3963	0.8113	0.0093
45	8552	0.5928	0.0124	95	3926	0.8130	0.0031
46	8446	0.5978	0.0091	96	3914	0.8136	0.0164
47	8369	0.6015	0.0086	97	3850	0.8167	0.0135
48	8297	0.6049	0.0088	98	3798	0.8191	0.0140
49	8224	0.6084	0.0250	99	3745	0.8217	0.0080

TABLE A2 Empirical data and the values of empirical cumulative distribution function (CDF) and empirical hazard function (HF) for weeks 100 to 199

i	$N(i)$	$\hat{F}(i)$	$\hat{h}(i)$	i	$N(i)$	$\hat{F}(i)$	$\hat{h}(i)$
100	3715	0.8231	0.0234	150	2007	0.9044	0.0244
101	3628	0.8272	0.0182	151	1958	0.9068	0.0123
102	3562	0.8304	0.0065	152	1934	0.9079	0.0047
103	3539	0.8315	0.0054	153	1925	0.9083	0.0068
104	3520	0.8324	0.0037	154	1912	0.9090	0.0105
105	3507	0.8330	0.0029	155	1892	0.9099	0.0164
106	3497	0.8335	0.0063	156	1861	0.9114	0.0140
107	3475	0.8345	0.0081	157	1835	0.9126	0.0245
108	3447	0.8359	0.0168	158	1790	0.9148	0.0084
109	3389	0.8386	0.0174	159	1775	0.9155	0.0056
110	3330	0.8414	0.0225	160	1765	0.9160	0.0096
111	3255	0.8450	0.0197	161	1748	0.9168	0.0246
112	3191	0.8480	0.0063	162	1705	0.9188	0.0065
113	3171	0.8490	0.0114	163	1694	0.9193	0.0171
114	3135	0.8507	0.0019	164	1665	0.9207	0.0102
115	3129	0.8510	0.0035	165	1648	0.9215	0.0103
116	3118	0.8515	0.0186	166	1631	0.9223	0.0141
117	3060	0.8543	0.0206	167	1608	0.9234	0.0249
118	2997	0.8573	0.0127	168	1568	0.9253	0.0134
119	2959	0.8591	0.0061	169	1547	0.9263	0.0084
120	2941	0.8600	0.0173	170	1534	0.9270	0.0098
121	2890	0.8624	0.0038	171	1519	0.9277	0.0211
122	2879	0.8629	0.0195	172	1487	0.9292	0.0256
123	2823	0.8656	0.0053	173	1449	0.9310	0.0186
124	2808	0.8663	0.0153	174	1422	0.9323	0.0288
125	2765	0.8683	0.0203	175	1381	0.9342	0.0261
126	2709	0.8710	0.0022	176	1345	0.9360	0.0201
127	2703	0.8713	0.0018	177	1318	0.9372	0.0129
128	2698	0.8715	0.0182	178	1301	0.9380	0.0131
129	2649	0.8739	0.0019	179	1284	0.9389	0.0093
130	2644	0.8741	0.0189	180	1272	0.9394	0.0071
131	2594	0.8765	0.0077	181	1263	0.9399	0.0063
132	2574	0.8774	0.0225	182	1255	0.9402	0.0064
133	2516	0.8802	0.0151	183	1247	0.9406	0.0273
134	2478	0.8820	0.0254	184	1213	0.9422	0.0190
135	2415	0.8850	0.0037	185	1190	0.9433	0.0286
136	2406	0.8854	0.0021	186	1156	0.9450	0.0251
137	2401	0.8857	0.0237	187	1127	0.9463	0.0231
138	2344	0.8884	0.0119	188	1101	0.9476	0.0091
139	2316	0.8897	0.0246	189	1091	0.9480	0.0275
140	2259	0.8924	0.0252	190	1061	0.9495	0.0085
141	2202	0.8951	0.0127	191	1052	0.9499	0.0086
142	2174	0.8965	0.0083	192	1043	0.9503	0.0316
143	2156	0.8973	0.0102	193	1010	0.9519	0.0158
144	2134	0.8984	0.0019	194	994	0.9527	0.0191
145	2130	0.8986	0.0183	195	975	0.9536	0.0195
146	2091	0.9004	0.0129	196	956	0.9545	0.0146
147	2064	0.9017	0.0155	197	942	0.9551	0.0308
148	2032	0.9032	0.0079	198	913	0.9565	0.0230
149	2016	0.9040	0.0045	199	892	0.9575	0.0179

TABLE A3 Empirical data and the values of empirical cumulative distribution function (CDF) and empirical hazard function (HF) for weeks 200 to 295

i	$N(i)$	$\hat{F}(i)$	$\hat{h}(i)$	i	$N(i)$	$\hat{F}(i)$	$\hat{h}(i)$
200	876	0.9583	0.0137	250	183	0.9913	0.0273
201	864	0.9589	0.0139	251	178	0.9915	0.0281
202	852	0.9594	0.0282	252	173	0.9918	0.0289
203	828	0.9606	0.0145	253	168	0.9920	0.0238
204	816	0.9611	0.0196	254	164	0.9922	0.0244
205	800	0.9619	0.0125	255	160	0.9924	0.0313
206	790	0.9624	0.0127	256	155	0.9926	0.0258
207	780	0.9629	0.0167	257	151	0.9928	0.0132
208	767	0.9635	0.0196	258	149	0.9929	0.0268
209	752	0.9642	0.0346	259	145	0.9931	0.0207
210	726	0.9654	0.0303	260	142	0.9932	0.0563
211	704	0.9665	0.0313	261	134	0.9936	0.0224
212	682	0.9675	0.0205	262	131	0.9938	0.0153
213	668	0.9682	0.0180	263	129	0.9939	0.0233
214	656	0.9688	0.0351	264	126	0.9940	0.0317
215	633	0.9699	0.0190	265	122	0.9942	0.0410
216	621	0.9704	0.0209	266	117	0.9944	0.0171
217	608	0.9710	0.0345	267	115	0.9945	0.0522
218	587	0.9720	0.0204	268	109	0.9948	0.0367
219	575	0.9726	0.0383	269	105	0.9950	0.0476
220	553	0.9737	0.0362	270	100	0.9952	0.0300
221	533	0.9746	0.0206	271	97	0.9954	0.0515
222	522	0.9751	0.0287	272	92	0.9956	0.0326
223	507	0.9759	0.0276	273	89	0.9958	0.0562
224	493	0.9765	0.0264	274	84	0.9960	0.0357
225	480	0.9771	0.0208	275	81	0.9961	0.0617
226	470	0.9776	0.0170	276	76	0.9964	0.0395
227	462	0.9780	0.0476	277	73	0.9965	0.0548
228	440	0.9790	0.0205	278	69	0.9967	0.1159
229	431	0.9795	0.0255	279	61	0.9971	0.0164
230	420	0.9800	0.0238	280	60	0.9971	0.0833
231	410	0.9805	0.0195	281	55	0.9974	0.0545
232	402	0.9809	0.0398	282	52	0.9975	0.1154
233	386	0.9816	0.0259	283	46	0.9978	0.0435
234	376	0.9821	0.0665	284	44	0.9979	0.0682
235	351	0.9833	0.0484	285	41	0.9980	0.1220
236	334	0.9841	0.0389	286	36	0.9983	0.1389
237	321	0.9847	0.0405	287	31	0.9985	0.0968
238	308	0.9853	0.0487	288	28	0.9987	0.1786
239	293	0.9860	0.0410	289	23	0.9989	0.1304
240	281	0.9866	0.0356	290	20	0.9990	0.1500
241	271	0.9871	0.0369	291	17	0.9992	0.2941
242	261	0.9876	0.0115	292	12	0.9994	0.4167
243	258	0.9877	0.0349	293	7	0.9997	0.4286
244	249	0.9881	0.0643	294	4	0.9998	1.0000
245	233	0.9889	0.0472	295	0	1.0000	
246	222	0.9894	0.0541				
247	210	0.9900	0.0476				
248	200	0.9905	0.0450				
249	191	0.9909	0.0419				

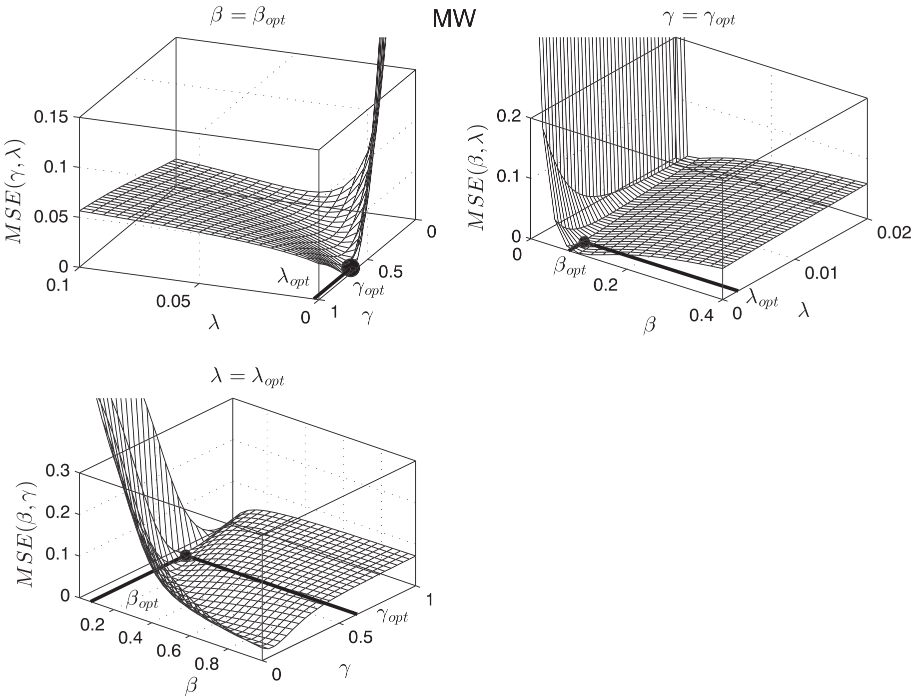

FIGURE A1 Mean squared error (MSE) of fit vs parameters of the fitted modified Weibull (MW) cumulative distribution function (CDF)

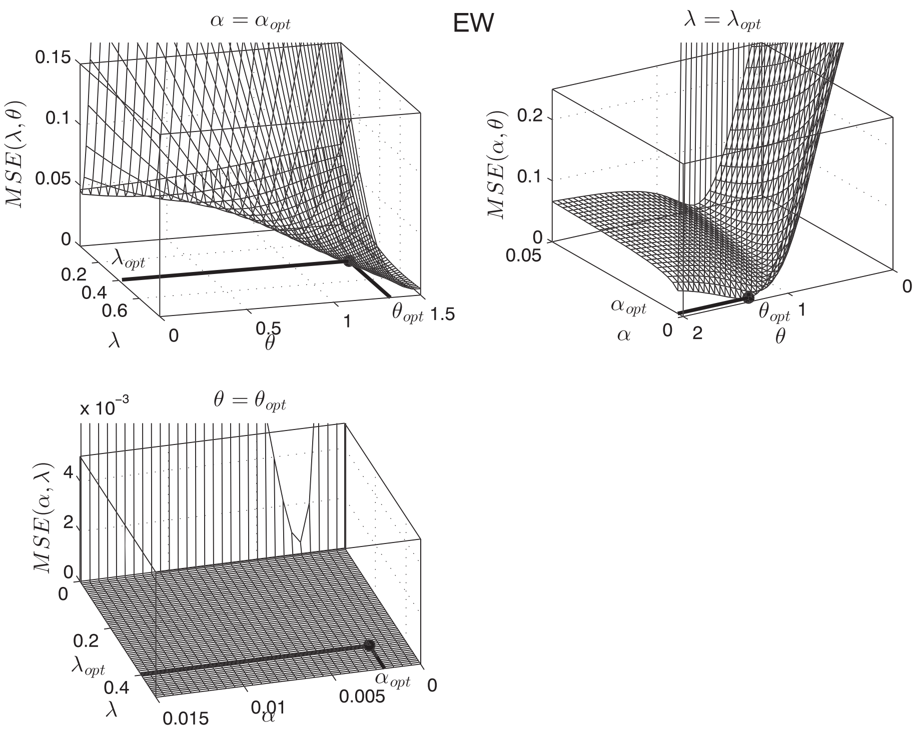

FIGURE A2 Mean squared error (MSE) of fit vs parameters of the fitted exponentiated Weibull (EW) cumulative distribution function (CDF)

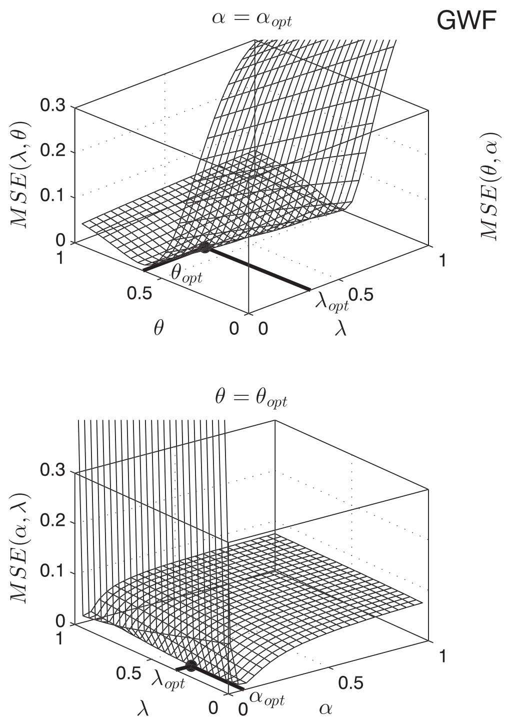

FIGURE A3 Mean squared error (MSE) of fit vs parameters of the fitted generalized Weibull family (GWF) cumulative distribution function (CDF)

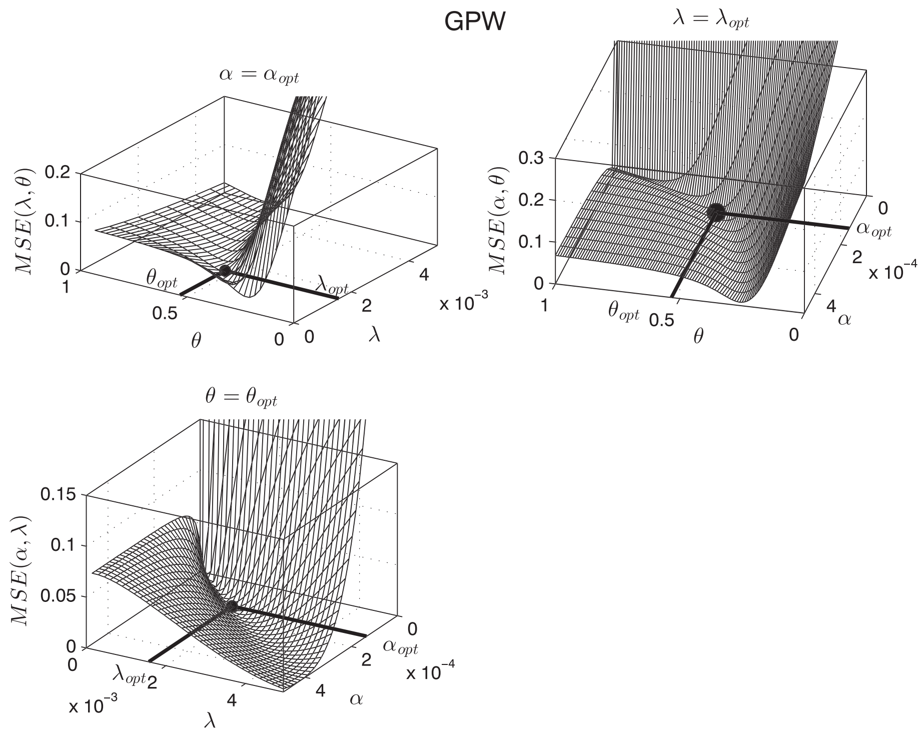

FIGURE A4 Mean squared error (MSE) of fit vs parameters of the fitted generalized power Weibull (GPW) cumulative distribution function (CDF)

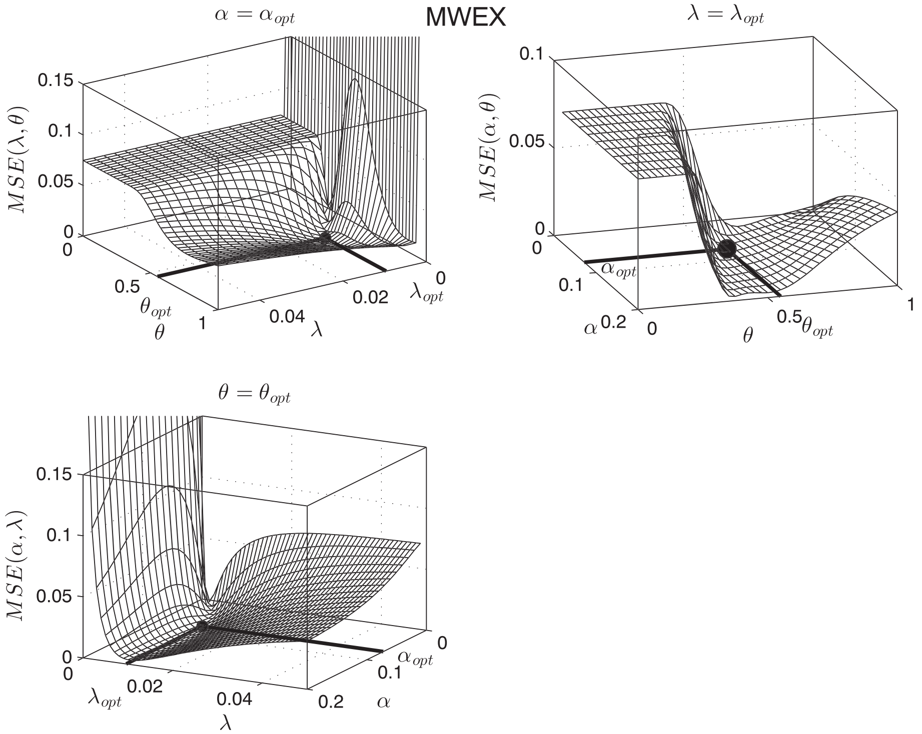

FIGURE A5 Mean squared error (MSE) of fit vs parameters of the fitted modified Weibull extension (MWEX) cumulative distribution function (CDF)

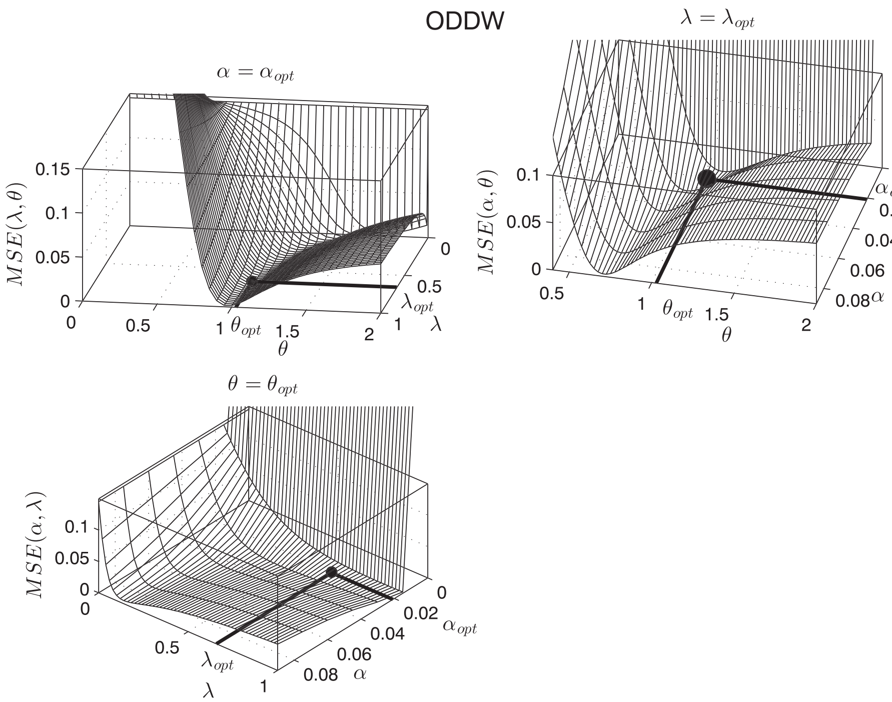

FIGURE A6 Mean squared error (MSE) of fit vs parameters of the fitted odd Weibull (ODDW) cumulative distribution function (CDF)

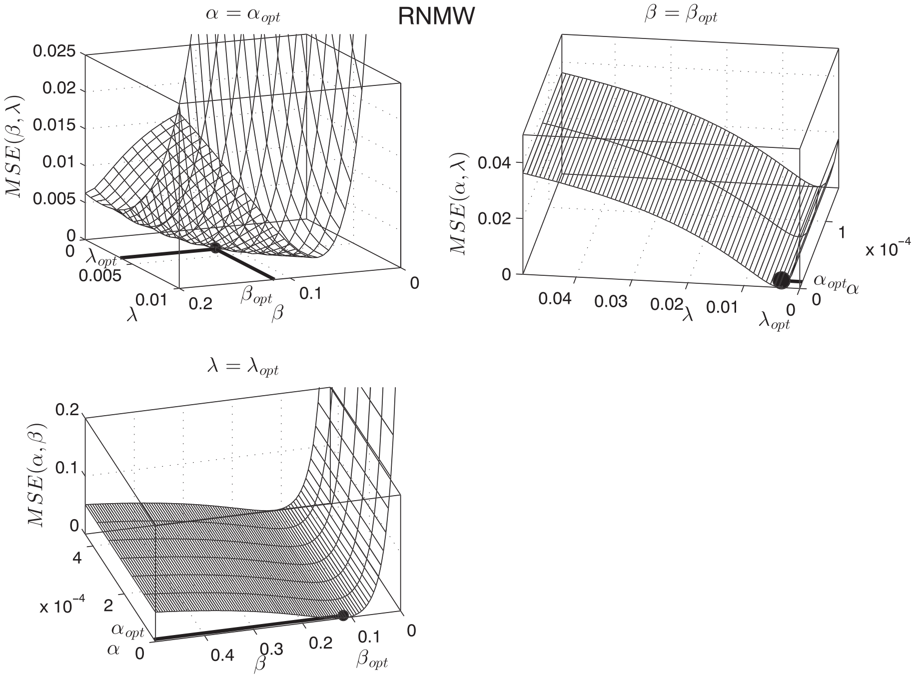

FIGURE A7 Mean squared error (MSE) of fit vs parameters of the fitted reduced modified Weibull (RNMW) cumulative distribution function (CDF)