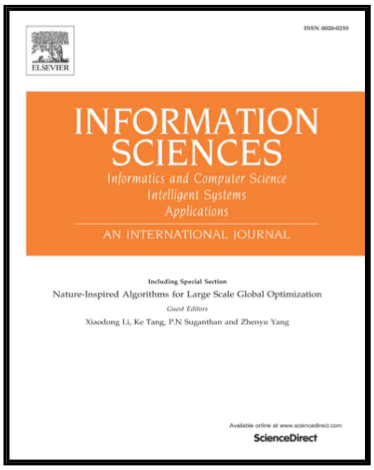
# Accepted Manuscript




The General Poincaré Formula for $\lambda$-additive Measures

József Dombi, Tamás Jónás

PII: S0020-0255(19)30273-7
DOI: https://doi.org/10.1016/j.ins.2019.03.059
Reference: INS 14399

To appear in: *Information Sciences*

Received date: 5 December 2018
Revised date: 7 March 2019
Accepted date: 24 March 2019

# The General Poincaré Formula for $\lambda$-additive Measures

József Dombi[a], Tamás Jónás[b]

[a] *Institute of Informatics, University of Szeged, Szeged, Hungary*
[b] *Institute of Business Economics, Eötvös Loránd University, Budapest, Hungary*

**Abstract**

In this study, the general formula for $\lambda$-additive measure of union of $n$ sets is introduced. Here, it is demonstrated that the well-known Poincaré formula of probability theory may be viewed as a limit case of our general formula. Moreover, it is also explained how this novel formula along with an alternatively parameterized $\lambda$-additive measure can be applied in theory of belief- and plausibility measures, and in theory of rough sets.

*Keywords:* $\lambda$-additive measure, plausibility, probability, belief, rough sets

## 1. Introduction

In many situations, the application of traditional additive measures is not sufficient to describe the uncertainty appropriately. Therefore, new demands have arisen for not necessarily additive, but monotone (fuzzy) measures. Since these measures play an important role in describing various phenomena, there has been an increasing interest in them (see, e.g. [13, 14, 24, 10, 8]). Undoubtedly, one of the most widely applied class of monotone measures is the class of $\lambda$-additive measures (Sugeno $\lambda$-measures) [22]. Although there are many theoretical and practical articles (see, e.g. [11, 12, 2, 1, 18]) that discuss the $\lambda$-additive measure, its properties and its applicability, there are no papers dealing with the general form of $\lambda$-additive measure of union of $n$ sets. Our study seeks to fill this gap. Namely, here, we will prove that if $X$ is a finite set, $A_1, \ldots, A_n \in \mathcal{P}(X)$, $n \geq 2$, $Q_\lambda$ is a $\lambda$-additive measure on $X$ and $\lambda \neq 0$, then

$$Q_\lambda\left(\bigcup_{i=1}^{n} A_i\right) = \\ = \frac{1}{\lambda}\left(\prod_{k=1}^{n}\left(\prod_{1 \leq i_1 < \cdots < i_k \leq n}\left(1 + \lambda Q_\lambda\left(A_{i_1} \cap \cdots \cap A_{i_k}\right)\right)\right)^{(-1)^{k-1}} - 1\right), \tag{1}$$

*Email addresses:* dombi@inf.u-szeged.hu (József Dombi), jonas@gti.elte.hu (Tamás Jónás)

where $\mathcal{P}(X)$ denotes the power set of $X$. Taking into account the fact that the fuzzy measures and the fuzzy measure related aggregation are important topics, it is worth mentioning that the formula in Eq. (1) may also be viewed as an aggregation related to the $\lambda$-additive measure, which is a fuzzy measure.

The well-known Poincaré formula of probability theory is

$$Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n} Pr\left(A_{i_1}\cap\cdots\cap A_{i_k}\right), \quad (2)$$

where $Pr$ is a probability measure on $X$ and $A_1,\ldots,A_n \in \mathcal{P}(X)$. Here, we will demonstrate that the Poincaré formula of probability theory given in Eq. (2) may be viewed as a limit case of the general formula of $\lambda$-additive measure of union of $n$ sets given in Eq. (1).

It is an acknowledged fact that the $\lambda$-additive measure is strongly connected with the belief- and plausibility measures of Dempster-Shafer theory (see, e.g. [24, 9, 5, 21, 7, 3, 17]), and with the theory of rough sets (see, e.g. [6, 26, 25, 16, 19, 20]). Hence, our formula for the $\lambda$-additive measure of union of $n$ sets may play an important role in these areas of computer science.

The rest of this paper is structured as follows. In Section 2, we will introduce our new result regarding the $\lambda$-additive measure of union of $n$ sets. In Section 3, we will discuss how our new formula along with an alternatively parameterized $\lambda$-additive measure can be applied in theory of belief- and plausibility measures and in theory of rough sets. Lastly, in Section 4, we will give a short summary of our findings and highlight our future research plans including the possible application of our results in network science.

In this study, we will use the common notations $\cap$ and $\cup$ for the intersection and union operations over sets, respectively. Also, will use the notation $\overline{A}$ for the complement of set $A$.

## 2. The general Poincaré formula

Relaxing the additivity property of the probability measure, the $\lambda$-additive measures were proposed by Sugeno in 1974 [22].

**Definition 1.** *The function $Q_\lambda : \mathcal{P}(X) \to [0,1]$ is a $\lambda$-additive measure (Sugeno $\lambda$-measure) on the finite set $X$, iff $Q_\lambda$ satisfies the following requirements:*

*(1)* $Q_\lambda(X) = 1$
*(2) for any* $A, B \in \mathcal{P}(X)$ *and* $A \cap B = \emptyset$,

$$Q_\lambda(A\cup B) = Q_\lambda(A) + Q_\lambda(B) + \lambda Q_\lambda(A)Q_\lambda(B), \quad (3)$$

*where* $\lambda \in (-1,\infty)$ *and* $\mathcal{P}(X)$ *is the power set of* $X$.

Note that if $X$ is an infinite set, then the continuity of function $Q_\lambda$ is also required. From now on, $\mathcal{P}(X)$ will denote the power set of the finite set $X$ and $Q_\lambda$ will always denote a $\lambda$-additive measure on $X$.

In order to prove the next results, let us consider a fixed $\lambda \in (-1, \infty)$ and the corresponding strictly increasing bijection $h_\lambda : [0,1] \to [0,1]$, given via

$$h_\lambda(x) = \begin{cases} \dfrac{(1+\lambda)^x - 1}{\lambda}, & \text{if } \lambda \neq 0 \\ x, & \text{if } \lambda = 0 \end{cases}$$

with inverse $\theta_\lambda = h_\lambda^{-1} : [0,1] \to [0,1]$, given via

$$\theta_\lambda(y) = \begin{cases} \dfrac{\ln(1+\lambda y)}{\ln(1+\lambda)}, & \text{if } \lambda \neq 0 \\ y, & \text{if } \lambda = 0. \end{cases}$$

One can see that, for a fixed $x \in [0,1]$ (respectively $y \in [0,1]$), the function $\lambda \longmapsto h_\lambda(x)$ (respectively $\lambda \longmapsto \theta_\lambda(y)$) is continuous. The continuity of $\lambda \longmapsto h_\lambda(x)$ at $\lambda = 0$ means that

$$\lim_{\lambda \to 0} \frac{(1+\lambda)^x - 1}{\lambda} = x. \tag{4}$$

Now, let us consider some fixed $\lambda \in (-1, \infty)$, $\lambda \neq 0$ and a fixed $\lambda$-additive measure $Q_\lambda : \mathcal{P}(X) \to [0,1]$. According to [2], $Q_\lambda$ is representable. More precisely, one has $Q_\lambda = h_\lambda \circ \mu$ for a uniquely determined additive measure $\mu : \mathcal{P}(X) \to [0,1]$. Having this in mind, we can prove the following theorem.

**Theorem 1.** *If $X$ is a finite set, $Q_\lambda$ is a $\lambda$-additive measure on $X$, $\lambda \in (-1, \infty)$, $\lambda \neq 0$ and, if $A_1, \ldots, A_n \in \mathcal{P}(X)$, $n \geq 2$, one has*

$$Q_\lambda\left(\bigcup_{i=1}^{n} A_i\right) = \\ = \frac{1}{\lambda}\left(\prod_{k=1}^{n}\left(\prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(1 + \lambda Q_\lambda\left(A_{i_1} \cap \cdots \cap A_{i_k}\right)\right)\right)^{(-1)^{k-1}} - 1\right). \tag{5}$$

*Proof.* In view of the Poincaré formula, one has

$$\mu(A) = \sum_{k=1}^{n} (-1)^{k-1} a_k, \tag{6}$$

where $A \overset{def}{=} A_1 \cup A_2 \cup \cdots \cup A_n$ and $a_k \overset{def}{=} \sum\limits_{1 \leq i_1 < \cdots < i_k \leq n} \mu\left(A_{i_1} \cap \cdots \cap A_{i_k}\right)$. Applying $h_\lambda$ in both members of Eq. (6), we get

$$h_\lambda(\mu(A)) = Q_\lambda(A) = h_\lambda\left(\sum_{k=1}^{n} (-1)^{k-1} a_k\right) = \\ = \frac{(1+\lambda)^{\sum_{k=1}^{n} (-1)^{k-1} a_k} - 1}{\lambda} = \frac{\prod\limits_{k=1}^{n} \left((1+\lambda)^{a_k}\right)^{(-1)^{k-1}} - 1}{\lambda}. \tag{7}$$

It can be seen that

$$(1+\lambda)^{a_k} = \prod_{1\le i_1<\cdots<i_k\le n} (1+\lambda)^{\mu\left(A_{i_1}\cap\cdots\cap A_{i_k}\right)} =$$
$$= \prod_{1\le i_1<\cdots<i_k\le n} \left(1+\lambda Q_\lambda\left(A_{i_1}\cap\cdots\cap A_{i_k}\right)\right)$$

because of the identity (valid for any $B \subset X$)

$$(1+\lambda)^{\mu(B)} = 1+\lambda Q_\lambda(B).$$

Applying this to Eq. (7), we get Eq. (5) (or Eq. (1)). □

**Remark 1.** *In the particular case when the sets $A_1, A_2, \ldots, A_n$ are mutually disjoint, Theorem 1 gives (for $\lambda \neq 0$)*

$$Q_\lambda\left(\bigcup_{i=1}^n A_i\right) = \frac{1}{\lambda}\left(\prod_{i=1}^n (1+\lambda Q_\lambda(A_i)) - 1\right) \tag{8}$$

*because only the factors for $k = 1$ can be different from 1 in Eq.* (5).

**Remark 2.** *The Poincaré formula can be viewed as the limit case of the formula in Eq.* (5) *when $\lambda$ tends to zero. Namely, in view of Eq.* (4)*, one has for any $A = A_1 \cup A_2 \cup \cdots \cup A_n$:*

$$\lim_{\lambda\to 0} Q_\lambda(A) = \lim_{\lambda\to 0} h_\lambda(\mu(A)) = \mu(A).$$

## 3. Some applications of the results

Now, we will show how our formula for the $\lambda$-additive measure of union of $n$ sets can be used in some areas of computer science. Namely, we will discuss how our results can be applied in theory of belief- and plausibility measures and in theory of rough sets.

### *3.1. Application to belief- and plausibility measures*

In the theory of belief functions (Dempster-Shafer theory), the belief- and plausibility measures are defined as follows [3, 17].

**Definition 2.** *The function $Bl : \mathcal{P}(X) \to [0,1]$ is a belief measure on the finite set $X$, iff $Bl$ satisfies the following requirements:*

*(1)* $Bl(\emptyset) = 0, Bl(X) = 1$
*(2) for any $A_1, \ldots, A_n \in \mathcal{P}(X)$,*

$$Bl(A_1 \cup \cdots \cup A_n) \geq$$
$$\geq \sum_{k=1}^n \sum_{1\le i_1<\cdots<i_k\le n} (-1)^{k-1} Bl\left(A_{i_1}\cap\cdots\cap A_{i_k}\right).$$

**Definition 3.** *The function* $Pl : \mathcal{P}(X) \to [0,1]$ *is a plausibility measure on the finite set* $X$*, iff* $Pl$ *satisfies the following requirements:*

*(1)* $Pl(\emptyset) = 0, Pl(X) = 1$
*(2) for any* $A_1, \ldots, A_n \in \mathcal{P}(X)$*,*

$$Pl(A_1 \cap \cdots \cap A_n) \leq$$
$$\leq \sum_{k=1}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^{k-1} Pl\left(A_{i_1} \cup \cdots \cup A_{i_k}\right).$$

The next proposition (see, e.g. [5]) highlights an important connection between the $\lambda$-additive measure and the belief-, probability- and plausibility measures.

**Proposition 1.** *Let* $X$ *be a finite set and let* $Q_\lambda$ *be a* $\lambda$*-additive measure on* $X$*. Then, on set* $X$*,* $Q_\lambda$ *is a*

*(1) plausibility measure if and only if* $-1 < \lambda \leq 0$
*(2) probability measure if and only if* $\lambda = 0$
*(3) belief measure if and only if* $\lambda \geq 0$*.*

*Proof.* See [5]. □

*3.1.1. An application connected with the* $\nu$*-additive measure*

Adapting the enunciation and the proof of Theorem 4.7 from [24], we get the following proposition.

**Proposition 2.** *Assume* $Q_\lambda : \mathcal{P}(X) \to [0,1]$ *is a* $\lambda$*-additive measure,* $A_1, A_2, \ldots, A_n \in \mathcal{P}(X)$ *are such that* $\bigcup_{i=1}^{n} A_i = X$*,* $A_i \cap A_j = \emptyset$*, if* $i \neq j$*, and we know the values* $Q_\lambda(A_i)$*,* $i = 1, 2, \ldots, n$*. Assume supplementarily that* $n \geq 2$*,* $Q_\lambda(A_i) < 1$ *for all* $i = 1, 2, \ldots, n$ *and at least two distinct values* $Q_\lambda(A_i)$ *are not null. Write* $S = \sum_{i=1}^{n} Q_\lambda(A_i)$*.*

*Then* $\lambda$ *is uniquely determined, namely:*

*(1) If* $S < 1$*, then* $\lambda > 0$*, hence* $Q_\lambda$ *is a belief measure.*
*(2) If* $S = 1$*, then* $\lambda = 0$*, hence* $Q_\lambda$ *is a probability measure.*
*(3) If* $S > 1$*, then* $\lambda < 0$*, hence* $Q_\lambda$ *is a plausibility measure.*

From practical point of view, the number $\lambda$ from above is the unique solution of the equation (generated by Eq. (8) for $\bigcup_{i=1}^{n} A_i = X$)

$$\lambda + 1 = \prod_{i=1}^{n} (1 + \lambda Q_\lambda(A_i)). \tag{8'}$$

This equation can be numerically solved for $\lambda$. The value of parameter $\lambda$ informs us about the 'plausibilitiness' or 'beliefness' of the $Q_\lambda$ measure.

**Interpretation.** *Let $A_1, \ldots, A_n$ be $n$ pairwise disjoint groups of people and let $X = \bigcup_{i=1}^{n} A_i$ be the universe of groups. Furthermore, let us assume that we have the value of $Q_\lambda(A_i)$ for all $i = 1, \ldots, n$, and the $\lambda$-additive measure is a performance measure; that is, $Q_\lambda(A_i)$ represents the performance of group $A_i$. Here, if $\sum_{i=1}^{n} Q_\lambda(A_i) = 1$, then $Q_\lambda$ is a probability measure. If $\sum_{i=1}^{n} Q_\lambda(A_i) \neq 1$, then the solution of equation Eq.* (8') *for $\lambda$ informs us about the 'plausibilitiness' or 'beliefness' of the measure $Q_\lambda$. Namely, if $\lambda \gg 0$, then $Q_\lambda$ is a 'strong' belief measure, which indicates that uniting all the groups into one results in a better performing group. Similarly, if $-1 < \lambda \ll 0$, then $Q_\lambda$ is a 'strong' plausibility measure, which tells us that merging all the groups into one results in a worse performing group.*

It should be added here that the value of parameter $\lambda$ of a $\lambda$-additive measure is in the interval $(-1, \infty)$. Since this interval is unbounded (from the right hand side) and the zero value of $\lambda$ does not divide it into two symmetric domains, it is difficult to judge the 'plausibilitiness' or 'beliefness' of a $\lambda$-additive measure based on the value of parameter $\lambda$. Now, we will demonstrate how the application of an alternatively parameterized $\lambda$-additive measure, which we will call the $\nu$-additive measure, can be utilized to characterize the 'plausibilitiness' or 'beliefness' on a normalized scale. We will also outline how the application of the $\nu$-additive measure can simplify the numerical solution of Eq. (8').

The next well-known proposition (see Theorem 4.6 (3) in [24]) tells us how the $\lambda$-additive measure of a complement set can be computed.

**Proposition 3.** *If $X$ is a finite set and $Q_\lambda$ is a $\lambda$-additive measure on $X$, then for any $A \in \mathcal{P}(X)$ the $Q_\lambda$ measure of the complement set $\overline{A} = X \setminus A$ is*

$$Q_\lambda(\overline{A}) = \frac{1 - Q_\lambda(A)}{1 + \lambda Q_\lambda(A)}. \tag{9}$$

*Proof.* See the proof of Theorem 4.6 in [24], or the proof of Theorem 2.27 in [8]. □

Now, let us assume that $0 \leq Q(A) < 1$. Then, Eq. (9) can be written as

$$Q_\lambda(\overline{A}) = \frac{1 - Q_\lambda(A)}{1 + \lambda Q_\lambda(A)} = \frac{1}{1 + (1+\lambda)\frac{Q_\lambda(A)}{1 - Q_\lambda(A)}}. \tag{10}$$

In continuous-valued logic, the Dombi form of negation with the neutral value $\nu \in (0, 1)$ is given by the operator $n_\nu : [0, 1] \to [0, 1]$ as follows:

$$n_\nu(x) = \begin{cases} \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2 \frac{x}{1-x}} & \text{if } x \in [0, 1) \\ 0 & \text{if } x = 1, \end{cases} \tag{11}$$

where $x \in [0, 1]$ is a continuous-valued logic variable [4]. Note that the Dombi form of negation is the unique Sugeno's negation [23] with the fix point $\nu \in (0, 1)$. Also, for $Q_\lambda(A) \in [0, 1)$, the formula of $\lambda$-additive measure of $Q_\lambda(\overline{A})$ in Eq.

(10) is the same as the formula of the Dombi form of negation in Eq. (11) with $x = Q_\lambda(A)$ and

$$\left(\frac{1-\nu}{\nu}\right)^2 = 1+\lambda.$$

Based on the definition of $\lambda$-additive measures, $\lambda > -1$, and since

$$\lambda = \left(\frac{1-\nu}{\nu}\right)^2 - 1$$

is a bijection between $(0,1)$ and $(-1,\infty)$, the $\lambda$-additive measure of the complement set $\overline{A}$ can be alternatively redefined as

$$Q_\lambda(\overline{A}) = \begin{cases} \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2 \frac{Q_\lambda(A)}{1-Q_\lambda(A)}} & \text{if } Q_\lambda(A) \in [0,1) \\ 0 & \text{if } Q_\lambda(A) = 1, \end{cases} \tag{12}$$

where $\left(\frac{1-\nu}{\nu}\right)^2 = 1+\lambda$, $\nu \in (0,1)$.

Following this line of thinking, here, we will introduce the $\nu$-additive measure and state some of its properties.

**Definition 4.** *The function $Q_\nu : \mathcal{P}(X) \to [0,1]$ is a $\nu$-additive measure on the finite set $X$, iff $Q_\nu$ satisfies the following requirements:*

*(1)* $Q_\nu(X) = 1$

*(2) for any $A, B \in \mathcal{P}(X)$ and $A \cap B = \emptyset$,*

$$Q_\nu(A \cup B) = Q_\nu(A) + Q_\nu(B) + \left(\left(\frac{1-\nu}{\nu}\right)^2 - 1\right) Q_\nu(A) Q_\nu(B), \tag{13}$$

*where $\nu \in (0,1)$.*

Note that if $X$ is an infinite set, then the continuity of function $Q_\nu$ is also required. Here, we state a key proposition that we will utilize later on.

**Proposition 4.** *Let $X$ be a finite set, and let $Q_\lambda$ and $Q_\nu$ be a $\lambda$-additive and a $\nu$-additive measure on $X$, respectively. Then,*

$$Q_\lambda(A) = Q_\nu(A) \tag{14}$$

*for any $A \in \mathcal{P}(X)$, if and only if*

$$\lambda = \left(\frac{1-\nu}{\nu}\right)^2 - 1, \tag{15}$$

*where $\lambda > -1$, $\nu \in (0,1)$.*

*Proof.* This proposition immediately follows from the definitions of the $\lambda$-additive measure and $\nu$-additive measure. □

If $Q_\nu$ is a $\nu$-additive measure on the finite set $X$, then, by utilizing Eq. (12), the $Q_\nu$ measure of the complement set $\overline{A}$ is

$$Q_\nu(\overline{A}) = \begin{cases} \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2 \frac{Q_\nu(A)}{1-Q_\nu(A)}} & \text{if } Q_\nu(A) \in [0,1) \\ 0 & \text{if } Q_\nu(A) = 1. \end{cases} \tag{16}$$

Moreover, as the $\nu$ parameter is the neutral value of the Dombi negation operator (see Eq. (11)), the following property of the $\nu$-additive measure holds as well.

**Proposition 5.** *Let $X$ be a finite set, $Q_\nu$ a $\nu$-additive measure on $X$ and let the set $A_\nu$ be given as*

$$A_\nu = \{A \in \mathcal{P}(X) | Q_\nu(A) = \nu\},$$

*where $\nu \in (0,1)$. Then for any $A \in A_\nu$ the $Q_\nu$ measure of the complement set $\overline{A}$ is equal to $\nu$; that is, $Q_\nu(\overline{A}) = \nu$.*

*Proof.* If $A \in A_\nu$, then $Q_\nu(A) = \nu$ and utilizing the $\nu$-additive complement given by Eq. (16), we have

$$Q_\nu(\overline{A}) = \frac{1}{1 + \left(\frac{1-\nu}{\nu}\right)^2 \frac{\nu}{1-\nu}} = \nu.$$

□

This result means that the $\nu$-additive complement operation may be viewed as a complement operation characterized by its fix point $\nu$. Notice that the value of parameter $\nu$ of a $\nu$-additive measure is in the bounded interval $(0,1)$, while the value of parameter $\lambda$ of the corresponding $\lambda$-additive measure is in the unbounded interval $(-1,\infty)$. It means that the parameter $\nu$ characterizes the 'plausibilitiness' or 'beliefness' of the $\nu$-additive measure on a normalized scale. Moreover, $\nu = 0.5$, which corresponds to a probability measure, divides the interval $(0,1)$ symmetrically to the parameter domains of belief- ($\nu \in (0,0.5)$) and plausibility ($\nu \in (0.5,1)$) measures. It should be also noted that when we seek to numerically solve Eq. (8'), then we need to search for the solution in the interval $(-1,\infty)$. However, if we utilize the corresponding $\nu$-additive measure, then we need to search for the value of $\nu$ in the interval $(0,1)$, which considerably simplifies the numerical computation.

### *3.2. Application to rough sets*

It is a well-known fact that the belief- and plausibility measures are connected with the rough set theory (see [6, 26, 25]). Here, we will show how the $\lambda$-additive measure is connected with the rough set theory, and how the general formula for $\lambda$-additive measure of union of $n$ sets can be utilized in this area of computer science.

Later, we will use the concept of dual pair of belief- and plausibility measures.

**Definition 5.** *Let $Bl$ and $Pl$ be a belief measure and a plausibility measure, respectively, on the finite set $X$. Then $Bl$ and $Pl$ are said to be a dual pair of belief- and plausibility measures iff*

$$Pl(A) = 1 - Bl(\overline{A})$$

*holds for any $A \in \mathcal{P}(X)$.*

The concept of a rough set was introduced by Pawlak [15] as follows.

**Definition 6.** *Let $X$ be a finite set, and let $R \subseteq X \times X$ be a binary equivalence relation on $X$. The pair $(\underline{R}(A), \overline{R}(A))$ is said to be the the rough set of $A \subseteq X$ in the approximation space $(X, R)$ if*

$$\underline{R}(A) = \{x \in X | [x]_R \subseteq A\}$$
$$\overline{R}(A) = \{x \in X | [x]_R \cap A \neq \emptyset\},$$

*where $[x]_R$ is the $R$-equivalence class containing $x$.*

The rough set $(\underline{R}(A), \overline{R}(A))$ can be utilized to characterize the set $A$ by the pair of lower and upper approximations $(\underline{R}(A), \overline{R}(A))$. The lower approximation $\underline{R}(A)$ is the union of all elementary sets that are subsets of $A$, and the upper approximation $\overline{R}(A)$ is the union of all elementary sets that have a non-empty intersection with $A$. Note that the definitions of $\underline{R}(A)$ and $\overline{R}(A)$ are equivalent to the following statement: an element of $X$ necessarily belongs to $A$ if all of its equivalent elements belong to $A$, while an element of $X$ possibly belongs to $A$ if at least one of its equivalent elements belongs to $A$ [25]. Let the functions $\underline{q}, \overline{q} : \mathcal{P}(X) \to [0, 1]$ be given as follows:

$$\underline{q}(A) = \frac{|\underline{R}(A)|}{|X|},\ \overline{q}(A) = \frac{|\overline{R}(A)|}{|X|} \tag{17}$$

for any $A \in \mathcal{P}(X)$. On the one hand, Skowron [19, 20] showed that the functions $\underline{q}$ and $\overline{q}$ are a dual pair of belief- and plausibility measures. On the other hand, based on Proposition 1, $\underline{q}$ and $\overline{q}$ can be represented by a dual pair of $\lambda$-additive measures; that is,

$$\underline{q}(A) = Q_{\lambda_l}(A),\ \overline{q}(A) = Q_{\lambda_u}(A) \tag{18}$$

for any $A \in \mathcal{P}(X)$, where $\lambda_l \geq 0$ and $-1 < \lambda_u \leq 0$. Thus, if $R \subseteq X \times X$ is a binary equivalence relation on $X$, $A_1, \ldots, A_n \in \mathcal{P}(X)$, $(\underline{R}(A_i), \overline{R}(A_i))$ is the rough set of $A_i$ in the approximation space $(X, R)$ and $i = 1, \ldots, n$, then the cardinality of the lower- and upper approximations of the set $\bigcup_{i=1}^{n} A_i$ can be computed by utilizing Eq. (17), Eq. (18) and Eq. (1) as follows:

$$|\underline{R}(A_1 \cup \cdots \cup A_n)| = |X| \underline{q}\left(\bigcup_{i=1}^{n} A_i\right) =$$

$$= \frac{|X|}{\lambda}\left(\prod_{k=1}^{n}\left(\prod_{1 \leq i_1 < \cdots < i_k \leq n}\left(1 + \lambda Q_{\lambda_l}\left(A_{i_1} \cap \cdots \cap A_{i_k}\right)\right)\right)^{(-1)^{k-1}} - 1\right)$$

$$|\overline{R}(A_1 \cup \cdots \cup A_n)| = |X|\overline{q}\left(\bigcup_{i=1}^{n} A_i\right) =$$

$$= \frac{|X|}{\lambda}\left(\prod_{k=1}^{n}\left(\prod_{1\leq i_1<\cdots<i_k\leq n}\left(1+\lambda Q_{\lambda_u}\left(A_{i_1}\cap\cdots\cap A_{i_k}\right)\right)\right)^{(-1)^{k-1}} - 1\right).$$

## 4. Summary and future plans

(1) In Eq.(1), we presented the general formula for the $\lambda$-additive measure of union of $n$ sets.

(2) We outlined how this new formula along with an alternatively parameterized $\lambda$-additive measure, which we call the $\nu$-additive measure, can be applied in theory belief- and plausibility measures and in theory of rough sets.

As part of our future research plans, we would like to formulate a calculus of the $\lambda$-additive measure and generalize the Bayes theorem for $\lambda$-additive measures. We also plan to study how the $\lambda$-additive ($\nu$-additive) measure and the generalized Poincaré formula can be utilized in the fields of computer science, engineering and economics. Especially, we aim to investigate the potential applications in network science.

## References

[1] X. Chen, Y.-A. Huang, X.-S. Wang, Z.-H. You, K. C. Chan, FMLNCSIM: fuzzy measure-based lncRNA functional similarity calculation model, Oncotarget 7 (29) (2016) 45948–45958, doi:10.18632/oncotarget.10008.

[2] I. Chiţescu, Why $\lambda$-additive (fuzzy) measures?, Kybernetika 51 (2) (2015) 246–254.

[3] A. P. Dempster, Upper and lower probabilities induced by a multivalued mapping, Annals of Mathematical Statistics 38 (1967) 325–339.

[4] J. Dombi, Towards a General Class of Operators for Fuzzy Systems, IEEE Transactions on Fuzzy Systems 16 (2) (2008) 477–484, doi: 10.1109/TFUZZ.2007.905910.

[5] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, vol. 144 of *Mathematics In Science And Engineering*, chap. 5, Academic Press, Inc., Orlando, FL, USA, 125–150, 1980.

[6] D. Dubois, H. Prade, Rough Fuzzy Sets and Fuzzy Rough Sets, International Journal of General Systems 17 (2–3) (1990) 191–209, doi: 10.1080/03081079008935107.

[7] T. Feng, J.-S. Mi, S.-P. Zhang, Belief functions on general intuitionistic fuzzy information systems, Information Sciences 271 (2014) 143 – 158, doi: https://doi.org/10.1016/j.ins.2014.02.120.

[8] M. Grabisch, Set Functions, Games and Capacities in Decision Making, Springer Publishing Company, Incorporated, 1st edn., 2016.

[9] U. Höhle, A general theory of fuzzy plausibility measures, Journal of Mathematical Analysis and Applications 127 (2) (1987) 346 – 364, doi: https://doi.org/10.1016/0022-247X(87)90114-4.

[10] L. Jin, R. Mesiar, R. R. Yager, Melting probability measure with OWA operator to generate fuzzy measure: the Crescent Method, IEEE Transactions on Fuzzy Systems doi:10.1109/TFUZZ.2018.2877605.

[11] C. Magadum, M. Bapat, Ranking of Students for Admission Process by Using Choquet Integral, International Journal of Fuzzy Mathematical Archive 15 (2) (2018) 105–113.

[12] M. A. Mohamed, W. Xiao, Q-measures: an efficient extension of the Sugeno $\lambda$-measure, IEEE Transactions on Fuzzy Systems 11 (3) (2003) 419–426.

[13] E. Pap, Null-additive set functions, vol. 337, Kluwer Academic Pub, 1995.

[14] E. Pap, Pseudo-additive measures and their applications, in: Handbook of measure theory, Elsevier, 1403–1468, 2002.

[15] Z. Pawlak, Rough sets, International Journal of Computer & Information Sciences 11 (5) (1982) 341–356, doi:10.1007/BF01001956.

[16] L. Polkowski, Rough sets in knowledge discovery 2: applications, case studies and software systems, vol. 19, Physica, 2013.

[17] G. Shafer, A mathematical theory of evidence, vol. 42, Princeton university press, 1976.

[18] A. K. Singh, Signed $\lambda$-Measures on Effect Algebras, in: Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, Springer India, 1–7, doi:10.1007/s40010-018-0510-x, 2018.

[19] A. Skowron, The relationship between the rough set theory and evidence theory, Bulletin of Polish academy of science: Mathematics 37 (1989) 87–90.

[20] A. Skowron, The Rough Sets Theory and Evidence Theory, Fundam. Inf. 13 (3) (1990) 245–262, ISSN 0169-2968.

[21] W. Spohn, The Laws of Belief: Ranking Theory and its Philosophical Applications, Oxford University Press, 2012.

[22] M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. thesis, Tokyo Institute of Technology, Tokyo, Japan, 1974.

[23] M. Sugeno, Fuzzy Measures and Fuzzy Integrals - a Survey, in: D. Dubois, H. Prade, R. R. Yager (Eds.), Readings in Fuzzy Sets for Intelligent Systems, Morgan Kaufmann, 251 – 257, doi:https://doi.org/10.1016/B978-1-4832-1450-4.50027-4, 1993.

[24] Z. Wang, G. J. Klir, Generalized Measure Theory, vol. 25 of *IFSR International Series in Systems Science and Systems Engineering*, Springer US, 2009.

[25] W.-Z. Wu, Y. Leung, W.-X. Zhang, Connections between rough set theory and Dempster-Shafer theory of evidence, International Journal of General Systems 31 (4) (2002) 405–430.

[26] Y. Yao, P. Lingras, Interpretations of belief functions in the theory of rough sets, Information Sciences 104 (1) (1998) 81 – 106, ISSN 0020-0255, doi: https://doi.org/10.1016/S0020-0255(97)00076-5.