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The λ-additive Measure in a New Light

The Q_ν measure and its connections with belief, probability, plausibility, rough sets, multiattribute utility functions and fuzzy operators

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Abstract The aim of this paper is twofold. On the one hand, the λ-additive measure (Sugeno λ-measure) is revisited and a state-of-the-art summary of its most important properties is provided. On the other hand, the so-called ν-additive measure as an alternatively parameterized λ-additive measure is introduced. Here, the advantages of the ν-additive measure are discussed and it is demonstrated that these two measures are closely related to various areas of science. The motivation for introducing the ν-additive measure lies in the fact that its parameter $\nu \in (0,1)$ has an important semantic meaning as it is the fix point of the complement operation. Here, by utilizing the ν-additive measure, some well-known results concerning the λ-additive measure are put into a new light and rephrased in more advantageous forms. It is discussed here how the ν-additive measure is connected with the belief-, probability- and plausibility measures. Next, it is also shown that two ν-additive measures, with the parameters ν_1 and ν_2, are a dual pair of belief- and plausibility measures if and only if $\nu_1 + \nu_2 = 1$. Furthermore, it is demonstrated how a ν-additive measure (or a λ-additive measure) can be transformed to a probability measure and vice versa. Lastly, it is discussed here how the ν-additive measures are connected with rough sets, multi-attribute utility functions and with certain operators of fuzzy logic.

Keywords belief · probability · plausibility · λ-additive measure · rough sets · multi-attribute utility functions

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1 Introduction

It is an acknowledged fact that the λ-additive measure (Sugeno λ-measure) (Sugeno 1974) is one of the most widely applied monotone measure (fuzzy measure). The usefulness, versatility and applicability of λ-additive measures has inspired numerous theoretical and practical researches since Sugeno's original results were published in 1974 (see, e.g. Magadum and Bapat (2018); Mohamed and Xiao (2003); Chiţescu (2015); Chen et al. (2016); Singh (2018)).

The aim of the present study is twofold. On the one hand, we will revisit the λ-additive measure and give a state-of-the-art summary of its most important properties. On the other hand, we will introduce the so-called ν-additive measure as an alternatively parameterized λ-additive measure, demonstrate the advantages of the ν-additive measure and point out that these two measures are closely related to various areas of science. The motivation for introducing the ν-additive measure lies in the fact that its parameter $\nu \in (0,1)$ has an important semantic meaning. Namely, ν is the fix point of the complement operation; that is, if the ν additive measure of a set has the value ν, then the ν-additive measure of its complement set has the value ν as well. It should be added that by utilizing the ν-additive measure, some well-known results concerning the λ-additive measure can be put into a new light and rephrased in more advantageous forms. Here, we will discuss how the ν-additive measure is connected with the belief-, probability- and plausibility measures (see, e.g. Wang and Klir (2013); Höhle (1987); Dubois and Prade (1980); Spohn (2012); Feng et al. (2014)). Also, we will demonstrate that a ν-additive measure is a

(1) belief measure if and only if $0 < \nu \leq 1/2$
(2) probability measure if and only if $\nu = 1/2$
(3) plausibility measure if and only if $1/2 \leq \nu < 1$.

Next, we will show that two ν-additive measures, with the parameters ν_1 and ν_2, are a dual pair of belief- and plausibility measures if and only if $\nu_1 + \nu_2 = 1$. Furthermore, we will discuss how a ν-additive measure (or a λ-additive measure) can be transformed to a probability measure and vice versa. Moreover, we will also discuss how the ν-additive measures are connected with rough sets (see, e.g. Dubois and Prade (1990); Yao and Lingras (1998); Wu et al. (2002); Polkowski (2013)), multi-attribute utility functions (see, e.g. Sarin (2013); Greco et al. (2016); Keeney and Raiffa (1993)), and with certain operators of continuous-valued logic (see, e.g. Dombi (1982, 2008)).

The rest of this paper is structured as follows. In Section 2, we give an overview of the monotone (fuzzy) measures including the belief-, probability- and plausibility measures. In Section 3, the ν-additive measure is introduced and its key properties are discussed. In Section 4, we demonstrate how the ν-additive measure is related to the belief-, probability- and plausibility measures, and in Section 5, we show how a ν-additive measure can be transformed to a probability measure and vice versa. Section 6 reveals some areas of science which the ν-additive (λ-additive) measures are connected with. Lastly, in Section 7, we give a short summary of our findings and highlight our future research plans including the possible application of ν-additive measure in network science.

In this study, we will use the common notations $\cap$ and $\cup$ for the intersection and union operations over sets, respectively. Also, will use the notation $\overline{A}$ for the complement of set A.

2 Monotone measures

Now, we will introduce the monotone measures and give a short overview of them that covers the probability-, belief- and plausibility measures.

Definition 1 Let Σ be a σ-algebra on the set X. Then the function $g : \Sigma \to [0,1]$ is a monotone measure on the measurable space (X,Σ) iff g satisfies the following requirements:

(1) $g(\emptyset) = 0$, $g(X) = 1$
(2) if $B \subseteq A$, then $g(B) \le g(A)$ for any $A,B \in \Sigma$ (monotonicity)
(3) if $\forall i \in \mathbb{N}, A_i \in \Sigma$ and (A_i) is monotonic ($A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$ or $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \ldots$), then

$$\lim_{i\to\infty} g(A_i) = g\left(\lim_{i\to\infty} A_i\right) \text{ (continuity).}$$

If X is a finite set, then the continuity requirement in Definition 1 can be disregarded and the monotone measure is defined as follows.

Definition 2 The function $g : \mathscr{P}(X) \to [0,1]$ is a monotone measure on the finite set X iff g satisfies the following requirements:

(1) $g(\emptyset) = 0$, $g(X) = 1$
(2) if $B \subseteq A$, then $g(B) \le g(A)$ for any $A,B \in \mathscr{P}(X)$ (monotonicity).

Note that the monotone measures given by Definition 1 and Definition 2 are known as fuzzy measures, which were originally defined by Choquet (Choquet 1954) and Sugeno (Sugeno 1974).

2.1 Some examples of monotone measures

2.1.1 Dirac measure

Definition 3 The function $\delta_{x_0} : \mathscr{P}(X) \to [0,1]$ is a Dirac measure on the set X, iff $\forall A \in \mathscr{P}(X)$:

$$\delta_{x_0}(A) = \begin{cases} 1, & \text{if } x_0 \in A \\ 0, & \text{otherwise.} \end{cases}$$

2.1.2 Probability measure

Definition 4 Let Σ be a σ-algebra over the set X. Then the function $Pr : \Sigma \to [0,1]$ is a probability measure on the space (X,Σ) iff Pr satisfies the following requirements:

(1) $\forall A \in \Sigma : Pr(A) \ge 0$
(2) $Pr(X) = 1$
(3) $\forall A_1, A_2, \ldots \in \Sigma$, if $A_i \cap A_j = \emptyset, \forall i \ne j$, then

$$Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} Pr(A_i).$$

Remark 1 If X is a finite set, then requirement (3) in Definition 4 can be reduced to the following requirement: for any disjoint $A,B \in \mathscr{P}(X)$, $Pr(A \cup B) = Pr(A) + Pr(B)$.

2.1.3 Belief measure and plausibility measure

Definition 5 The function $Bl : \mathscr{P}(X) \to [0,1]$ is a belief measure on the finite set X, iff Bl satisfies the following requirements:

(1) $Bl(\emptyset) = 0, Bl(X) = 1$
(2) for any $A_1, A_2, \ldots, A_n \in \mathscr{P}(X)$,

$$Bl(A_1 \cup A_2 \cup \cdots \cup A_n) \ge$$
$$\ge \sum_{k=1}^{n} \sum_{\substack{1 \le i_1 < i_2 \cdots \\ \cdots < i_k \le n}} (-1)^{k-1} Bl\left(A_{i_1} \cap A_{i_1} \cap \cdots \cap A_{i_k}\right). \tag{1}$$

Here, $Bl(A)$ is interpreted as a grade of belief in that a given element of X belongs to A.

Lemma 1 *If Bl is a belief measure on the finite set X, then for any $A \in \mathscr{P}(X)$,*

$Bl(A) + Bl(\overline{A}) \leq 1.$

Proof Noting Definition 5, we have

$$1 = Bl(A \cup \overline{A}) \geq Bl(A) + Bl(\overline{A}) - Bl(A \cap \overline{A}) = Bl(A) + Bl(\overline{A}).$$

□

The inequality $Bl(A) + Bl(\overline{A}) \leq 1$ means that a lack of belief in $x \in A$ does not imply a strong belief in $x \in \overline{A}$. In particular, total ignorance is modeled by the belief function Bl_i such that $Bl_i(A) = 0$ if $A \neq X$ and $Bl_i(A) = 1$ if $A = X$.

The following proposition is about the monotonicity of belief measures.

Proposition 1 *If X is a finite set, Bl is a belief measure on X, $A, B \in \mathscr{P}(X)$ and $B \subseteq A$, then $Bl(B) \leq Bl(A)$.*

Proof Let $B \subseteq A$. Hence, there exists $C \in \mathscr{P}(X)$ such that $A = B \cup C$ and $B \cap C = \emptyset$. Now, by utilizing the definition of the belief measure and the fact that $B \cap C = \emptyset$, we get

$$Bl(A) = Bl(B \cup C) \geq Bl(B) + Bl(C) \geq Bl(B).$$

□

Corollary 1 *The belief measure given by Definition 5 is a monotone measure.*

Proof Let Bl be a belief measure. It follows from Definition 5 that Bl satisfies criterion (1) for a monotone measure given in Definition 2. Moreover, the monotonicity of Bl was proven in Proposition 1; that is, Bl also satisfies criterion (2) in Definition 2. □

Definition 6 The function $Pl : \mathscr{P}(X) \to [0, 1]$ is a plausibility measure on the finite set X, iff Pl satisfies the following requirements:

(1) $Pl(\emptyset) = 0, Pl(X) = 1$
(2) for any $A_1, A_2, \dots, A_n \in \mathscr{P}(X)$,

$$\begin{aligned} &Pl(A_1 \cap A_2 \cap \cdots \cap A_n) \leq \\ &\leq \sum_{k=1}^{n} \sum_{\substack{1 \leq i_1 < i_2 \cdots \\ \cdots < i_k \leq n}} (-1)^{k-1} Pl\left(A_{i_1} \cup A_{i_2} \cdots \cup A_{i_k}\right). \end{aligned} \tag{2}$$

Here, $Pl(A)$ is interpreted as the plausibility of A.

Lemma 2 *If Pl is a plausibility measure on the finite set X, then for any $A \in \mathscr{P}(X)$,*

$Pl(A) + Pl(\overline{A}) \geq 1.$

Proof Noting Definition 6, we have

$$\begin{aligned} 0 = Pl(A \cap \overline{A}) &\leq Pl(A) + Pl(\overline{A}) - Pl(A \cup \overline{A}) = \\ &= Pl(A) + Pl(\overline{A}) - 1, \end{aligned}$$

from which $Pl(A) + Pl(\overline{A}) \geq 1$ follows. □

This result can be interpreted so that the plausibility of $x \in A$ does not imply a strong plausibility of $x \in \overline{A}$.

The following proposition is about the monotonicity of plausibility measures.

Proposition 2 *If X is a finite set, Pl is a plausibility measure on X, $A, B \in \mathscr{P}(X)$ and $B \subseteq A$, then $Pl(B) \leq Pl(A)$.*

Proof Let $B \subseteq A$. Let $C \in \mathscr{P}(X)$ such that $A \cap C = B$ and $A \cup C = X$. Now, by utilizing the definition of plausibility measure, and the facts that $A \cup C = X$ and $Pl(C) \leq 1$, we get

$$\begin{aligned} Pl(B) = Pl(A \cap C) &\leq Pl(A) + Pl(C) - Pl(A \cup C) = \\ &= Pl(A) + Pl(C) - 1 \leq Pl(A). \end{aligned}$$

□

Corollary 2 *The plausibility measure given by Definition 6 is a monotone measure.*

Proof Let Pl be a plausibility measure. It follows from Definition 6 that Pl satisfies criterion (1) for a monotone measure given in Definition 2. Next, the monotonicity of Pl was proven in Proposition 2; that is, Pl also satisfies criterion (2) in Definition 2. □

The plausibility of a subset A of the finite set X was defined by Shafer (Shafer 1976) as

$Pl(A) = 1 - Bl(\overline{A}),$

where Bl is a belief function. The following proposition states an interesting connection between the belief measure and the plausibility measure.

Proposition 3 *Let X be a finite set and let $\mu_1, \mu_2 : \mathscr{P}(X) \to [0, 1]$ be two monotone measures on X such that*

$$\mu_2(A) = 1 - \mu_1(\overline{A}) \tag{3}$$

holds for any $A \in \mathscr{P}(X)$. Then, either (1) μ_1 is a belief measure on X if and only if μ_2 is a plausibility measure on X, or (2) μ_1 is a plausibility measure on X if and only if μ_2 is a belief measure on X.

Proof We will prove case (1), and the proof of case (2) is similar. Firstly, we will show that if μ_1 is a belief measure on X and $\mu_2(A)$ is given as $\mu_2(A) = 1 - \mu_1(\overline{A})$ for any $A \in \mathscr{P}(X)$, then μ_2 is a plausibility measure on X. Let μ_1 be a belief measure on X and $\mu_2(A) = 1 - \mu_1(\overline{A})$

for any $A \in \mathscr{P}(X)$. Then, $\mu_2(\emptyset) = 0$ and $\mu_2(X) = 1$ trivially follow from the fact that μ_1 is a belief measure and $\mu_2(A) = 1 - \mu_1(\overline{A})$. That is, function μ_2 satisfies requirement (1) for a plausibility measure given in Definition 6. Furthermore, since function μ_1 is a belief measure, the inequality

$$\begin{aligned} &\mu_1(A_1 \cup A_2 \cup \cdots \cup A_n) \geq \\ &\geq \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq n} (-1)^{k-1} \mu_1\left(A_{i_1} \cap A_{i_1} \cap \cdots \cap A_{i_k}\right). \end{aligned} \tag{4}$$

holds for any $A_1, A_2, \ldots, A_n \in \mathscr{P}(X)$. From the condition $\mu_2(A) = 1 - \mu_1(\overline{A})$, we also have that $\mu_1(A) = 1 - \mu_2(\overline{A})$. Next, applying the inequality in Eq. (4) to the complement sets $\overline{A_1}, \overline{A_2}, \ldots, \overline{A_n} \in \mathscr{P}(X)$ and utilizing the fact that $\mu_1(A) = 1 - \mu_2(\overline{A})$, we get

$$\begin{aligned} &1 - \mu_2(\overline{\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}}) \geq \\ &\geq 1 - \mu_2(\overline{\overline{A_1}}) + 1 - \mu_2(\overline{\overline{A_2}}) + \cdots + 1 - \mu_2(\overline{\overline{A_n}}) - \\ &-(1 - \mu_2(\overline{\overline{A_1} \cap \overline{A_2}})) - \cdots - (1 - \mu_2(\overline{\overline{A_{n-1}} \cap \overline{A_n}})) + \cdots \\ &\cdots + (-1)^{n+1}\left(1 - \mu_2(\overline{\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}})\right) = \\ &= -\mu_2(\overline{\overline{A_1}}) - \mu_2(\overline{\overline{A_2}}) - \cdots - \mu_2(\overline{\overline{A_n}}) + \\ &+\mu_2(\overline{\overline{A_1} \cap \overline{A_2}}) + \cdots + \mu_2(\overline{\overline{A_{n-1}} \cap \overline{A_n}}) + \cdots \\ &\cdots + (-1)^n \mu_2(\overline{\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}}) + \\ &+\binom{n}{1} - \binom{n}{2} + \cdots + \binom{n}{n}(-1)^{n+1}. \end{aligned}$$

Noting the fact that

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} = 1, \tag{5}$$

the previous inequality can be written as

$$\begin{aligned} &1 - \mu_2(\overline{\overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_n}) \geq \\ &\geq 1 - \mu_2(\overline{\overline{A_1}}) - \mu_2(\overline{\overline{A_2}}) - \cdots - \mu_2(\overline{\overline{A_n}}) + \\ &+\mu_2(\overline{\overline{A_1} \cap \overline{A_2}}) + \cdots + \mu_2(\overline{\overline{A_{n-1}} \cap \overline{A_n}}) + \cdots \\ &\cdots + (-1)^n \mu_2(\overline{\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}}). \end{aligned}$$

Now, applying the De Morgan law to the last inequality, we get

$$\begin{aligned} &\mu_2(A_1 \cap A_2 \cap \cdots \cap A_n) \leq \\ &\leq \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq n} (-1)^{k-1} \mu_2\left(A_{i_1} \cup A_{i_1} \cup \cdots \cup A_{i_k}\right), \end{aligned}$$

which means that function μ_2 is a plausibility measure.

Secondly, we will demonstrate that if μ_2 is a plausibility measure on X and $\mu_2(A)$ is given as $\mu_2(A) = 1 - \mu_1(\overline{A})$ for any $A \in \mathscr{P}(X)$, then μ_1 is a belief measure on X. Let μ_2 be a plausibility measure on X and $\mu_2(A) = 1 - \mu_1(\overline{A})$ for any $A \in \mathscr{P}(X)$. These conditions trivially imply that $\mu_1(\emptyset) = 0$ and $\mu_1(X) = 1$; that is, function μ_1 satisfies requirement (1) for a belief measure given in Definition 5. Next, because function μ_2 is a plausibility measure, the inequality

$$\begin{aligned} &\mu_2(A_1 \cap A_2 \cap \cdots \cap A_n) \leq \\ &\leq \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq n} (-1)^{k-1} \mu_2\left(A_{i_1} \cup A_{i_2} \cdots \cup A_{i_k}\right) \end{aligned} \tag{6}$$

holds for any $A_1, A_2, \ldots, A_n \in \mathscr{P}(X)$. Then, applying the inequality in Eq. (6) to the complement sets $\overline{A_1}, \overline{A_2}, \ldots, \overline{A_n} \in \mathscr{P}(X)$ and utilizing the condition that $\mu_2(A) = 1 - \mu_1(\overline{A})$, we get

$$\begin{aligned} &1 - \mu_1(\overline{\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}}) \leq \\ &\leq 1 - \mu_1(\overline{\overline{A_1}}) + 1 - \mu_1(\overline{\overline{A_2}}) + \cdots + 1 - \mu_1(\overline{\overline{A_n}}) - \\ &-(1 - \mu_1(\overline{\overline{A_1} \cup \overline{A_2}})) - \cdots - (1 - \mu_1(\overline{\overline{A_{n-1}} \cup \overline{A_n}})) + \cdots \\ &\cdots + (-1)^{n+1}\left(1 - \mu_1(\overline{\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}})\right) = \\ &= -\mu_1(\overline{\overline{A_1}}) - \mu_1(\overline{\overline{A_2}}) - \cdots - \mu_1(\overline{\overline{A_n}}) + \\ &+\mu_1(\overline{\overline{A_1} \cup \overline{A_2}}) + \cdots + \mu_1(\overline{\overline{A_{n-1}} \cup \overline{A_n}}) + \cdots \\ &\cdots + (-1)^n \mu_1(\overline{\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}}) + \\ &+\binom{n}{1} - \binom{n}{2} + \cdots + \binom{n}{n}(-1)^{n+1}. \end{aligned}$$

Again, taking into account Eq. (5), the previous inequality can be written as

$$\begin{aligned} &1 - \mu_1(\overline{\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_n}) \leq \\ &\leq 1 - \mu_1(\overline{\overline{A_1}}) - \mu_1(\overline{\overline{A_2}}) - \cdots - \mu_1(\overline{\overline{A_n}}) + \\ &+\mu_1(\overline{\overline{A_1} \cup \overline{A_2}}) + \cdots + \mu_1(\overline{\overline{A_{n-1}} \cup \overline{A_n}}) + \cdots \\ &\cdots + (-1)^n \mu_1(\overline{\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}}). \end{aligned}$$

Now, applying the De Morgan law to the last inequality, we get

$$\begin{aligned} &\mu_1(A_1 \cup A_2 \cup \cdots \cup A_n) \geq \\ &\geq \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq n} (-1)^{k-1} \mu_1\left(A_{i_1} \cap A_{i_1} \cap \cdots \cap A_{i_k}\right). \end{aligned}$$

Hence, μ_1 is a belief measure. □

Later, we will use the concept of dual pair of belief- and plausibility measures.

Definition 7 Let *Bl* and *Pl* be a belief measure and a plausibility measure, respectively, on set X. Then *Bl* and *Pl* are said to be a dual pair of belief- and plausibility measures iff

$$Pl(A) = 1 - Bl(\overline{A})$$

holds for any $A \in \mathscr{P}(X)$.

In the Dempster-Shafer theory of evidence, a belief mass is assigned to each element of the power set $\mathscr{P}(X)$, where X is a finite set. The belief mass is given by the so-called basic probability assignment m from $\mathscr{P}(X)$ to $[0,1]$ that is defined as follows.

Definition 8 The function $m : \mathscr{P}(X) \to [0,1]$ is a basic probability assignment (mass function) on the finite set X, iff m satisfies the following requirements:

(1) $m(\emptyset) = 0$
(2) $\sum_{A\in\mathscr{P}(X)} m(A) = 1$.

The subsets A of X for which $m(A) > 0$ are called the focal elements of m. Let $x \in A$ and $A \in \mathscr{P}(X)$. Then, the mass $m(A)$ can be interpreted as the probability of knowing $x \in A$ given the available evidence. Utilizing a given basic probability assignment m, the belief $Bl(A)$ for the set A is

$$Bl(A) = \sum_{B|B\subseteq A} m(B),$$

and the plausibility $Pl(A)$ is

$$Pl(A) = \sum_{B|B\cap A\neq\emptyset} m(B).$$

A basic probability assignment m can be represented by its belief function Bl as

$$m(B) = \sum_{A\subseteq B} (-1)^{|B\setminus A|} Bl(A),$$

where $B \in \mathscr{P}(X)$. Here, m is the basic probability assignment of the belief measure Bl. Note that plausibility measures and belief functions were introduced by Dempster Dempster (1967) under the names upper and lower probabilities, induced by a probability measure by a multivalued mapping.

Remark 2 The monotonicity of the plausibility measure Pl can also be demonstrated by utilizing the duality $Pl(A) = 1 - Bl(\overline{A})$ and the monotonicity of the belief measure Bl. Namely, if $B \subseteq A$, then $\overline{A} \subseteq \overline{B}$ and so

$$Bl(\overline{A}) \le Bl(\overline{B}),$$

from which

$$1 - Bl(\overline{A}) \ge 1 - Bl(\overline{B}),$$

which means that

$$Pl(A) \ge Pl(B).$$

3 Introduction to the Q_ν measure

Relaxing the additivity property of the probability measure, the λ-additive measures were proposed by Sugeno in 1974 (Sugeno 1974).

Definition 9 The function $Q_\lambda : \mathscr{P}(X) \to [0,1]$ is a λ-additive measure (Sugeno λ-measure) on the finite set X, iff Q_λ satisfies the following requirements:

(1) $Q_\lambda(X) = 1$
(2) for any $A, B \in \mathscr{P}(X)$ and $A \cap B = \emptyset$,

$$Q_\lambda(A\cup B) = Q_\lambda(A) + Q_\lambda(B) + \lambda Q_\lambda(A) Q_\lambda(B), \qquad (7)$$

where $\lambda \in (-1, \infty)$.

Note that if X is an infinite set, then the continuity of function Q_λ is also required. Here, we will show that the λ-additive measures are monotone measures as well.

Proposition 4 *Every λ-additive measure is a monotone measure.*

Proof Let Q_λ be a λ-additive measure on the set X. Then $Q_\lambda(X) = 1$ holds by definition. Next, by utilizing Eq. (7), we get $Q_\lambda(X) = Q_\lambda(X\cup\emptyset) = Q_\lambda(X) + Q_\lambda(\emptyset)(1 + \lambda Q_\lambda(X))$, which implies that $Q_\lambda(\emptyset) = 0$. Thus, Q_λ satisfies criterion (1) of a monotone measure given in Definition 2.

Next, let $A, B \in \mathscr{P}(X)$ and let $B \subseteq A$. Then there exists a $C \in \mathscr{P}(X)$ such that $A = B \cup C$ and $B \cap C = \emptyset$. Now, by utilizing Eq. (7) and the fact that $\lambda > -1$, we get

$$\begin{aligned} Q_\lambda(A) = Q_\lambda(B\cup C) = \\ = Q_\lambda(B) + Q_\lambda(C)(1 + \lambda Q_\lambda(B)) \ge Q_\lambda(B). \end{aligned}$$

It means that Q_λ also satisfies the monotonicity criterion of a monotone measure. $\square$

Remark 3 The requirement $\lambda \ge -1$ instead of the requirement $\lambda > -1$ would be sufficient to ensure the monotonicity of Q_λ (see Proposition 4). The requirement $\lambda \ge -1$ also ensures that for any $A, B \in \mathscr{P}(X)$ and $A \cap B = \emptyset$ the $Q_\lambda(A\cup B)$ quantity is non-negative. Namely, since $Q_\lambda(A), Q_\lambda(B) \in [0,1]$, the inequality

$$Q_\lambda(A)Q_\lambda(B) \le \sqrt{Q_\lambda(A)Q_\lambda(B)} \le Q_\lambda(A) + Q_\lambda(B)$$

holds, and so if $\lambda \ge -1$, then

$$\begin{aligned} 0 \le (1+\lambda)Q_\lambda(A)Q_\lambda(B) = \\ = Q_\lambda(A)Q_\lambda(B) + \lambda Q_\lambda(A)Q_\lambda(B) \le \\ \le Q_\lambda(A) + Q_\lambda(B) + \lambda Q_\lambda(A)Q_\lambda(B) = Q_\lambda(A\cup B). \end{aligned}$$

However, the requirement $\lambda > -1$ is given in the definition of λ-additive measures. Later, we will see that certain properties of λ-additive measures hold only if $\lambda > -1$.

3.1 The λ-additive complement and the Dombi form of negation

Proposition 5 *If X is a finite set and Q_λ is a λ-additive measure on X, then for any $A \in \mathscr{P}(X)$ the Q_λ measure of the complement set $\overline{A} = X \setminus A$ is*

$$Q_\lambda(\overline{A}) = \frac{1 - Q_\lambda(A)}{1 + \lambda Q_\lambda(A)}. \tag{8}$$

Proof Since $A \cap \overline{A} = \emptyset$, we can write

$$\begin{aligned} 1 = Q_\lambda(X) = Q_\lambda(A \cup \overline{A}) = \\ = Q_\lambda(A) + Q_\lambda(\overline{A}) + \lambda Q_\lambda(A) Q_\lambda(\overline{A}) = \\ = Q_\lambda(A) + Q_\lambda(\overline{A})(1 + \lambda Q_\lambda(A)), \end{aligned}$$

from which we get

$$Q_\lambda(\overline{A}) = \frac{1 - Q_\lambda(A)}{1 + \lambda Q_\lambda(A)}.$$

□

Remark 4 For any $A \in \mathscr{P}(X)$, we have

$$\begin{aligned} Q_\lambda(A) + Q_\lambda(\overline{A}) = Q_\lambda(A) + \frac{1 - Q_\lambda(A)}{1 + \lambda Q_\lambda(A)} = \\ = \frac{1 + \lambda Q_\lambda^2(A)}{1 + \lambda Q_\lambda(A)} = 1 - \lambda Q_\lambda(A) Q_\lambda(\overline{A}). \end{aligned} \tag{9}$$

It can be seen from Eq. (9) that

$$\begin{array}{lll} 0 < Q_\lambda(A) + Q_\lambda(\overline{A}) \le 1 & \text{if} & \lambda \in (0, \infty) \\ Q_\lambda(A) + Q_\lambda(\overline{A}) = 1 & \text{if} & \lambda = 0 \\ 1 \le Q_\lambda(A) + Q_\lambda(\overline{A}) < 2 & \text{if} & \lambda \in (-1, 0). \end{array}$$

We have shown in Proposition 5 that if X is a finite set and Q_λ is a λ-additive measure on X, then for any $A \in \mathscr{P}(X)$ the Q_λ measure of the complement set $\overline{A} = X \setminus A$ is

$$Q_\lambda(\overline{A}) = \frac{1 - Q_\lambda(A)}{1 + \lambda Q_\lambda(A)}. \tag{10}$$

Now, let us assume that $0 \le Q(A) < 1$. Then, Eq. (10) can be written as

$$Q_\lambda(\overline{A}) = \frac{1 - Q_\lambda(A)}{1 + \lambda Q_\lambda(A)} = \frac{1}{1 + (1 + \lambda) \frac{Q_\lambda(A)}{1 - Q_\lambda(A)}}. \tag{11}$$

In continuous-valued logic, the Dombi form of negation with the neutral value $\nu \in (0, 1)$ is given by the operator $n_\nu : [0, 1] \to [0, 1]$ as follows:

$$n_\nu(x) = \begin{cases} \frac{1}{1 + \left(\frac{1-\nu}{\nu}\right)^2 \frac{x}{1-x}} & \text{if } x \in [0, 1) \\ 0 & \text{if } x = 1, \end{cases} \tag{12}$$

where $x \in [0, 1]$ is a continuous-valued logic variable (Dombi 2008). Note that the Dombi form of negation is the unique Sugeno's negation (Sugeno 1993) with the fix point $\nu \in (0, 1)$. Also, for $Q_\lambda(A) \in [0, 1)$, the formula of λ-additive measure of $Q_\lambda(\overline{A})$ in Eq. (11) is the same as the formula of the Dombi form of negation in Eq. (12) with $x = Q_\lambda(A)$ and

$$\left(\frac{1 - \nu}{\nu}\right)^2 = 1 + \lambda.$$

Based on the definition of λ-additive measures, $\lambda > -1$, and since

$$\lambda = \left(\frac{1 - \nu}{\nu}\right)^2 - 1$$

is a bijection between $(0, 1)$ and $(-1, \infty)$, the λ-additive measure of the complement set $\overline{A}$ can be alternatively re-defined as

$$Q_\lambda(\overline{A}) = \begin{cases} \frac{1}{1 + \left(\frac{1-\nu}{\nu}\right)^2 \frac{Q_\lambda(A)}{1 - Q_\lambda(A)}} & \text{if } Q_\lambda(A) \in [0, 1) \\ 0 & \text{if } Q_\lambda(A) = 1, \end{cases} \tag{13}$$

where $\left(\frac{1-\nu}{\nu}\right)^2 = 1 + \lambda$, $\nu \in (0, 1)$.

Following this line of thinking, here, we will introduce the ν-additive measure and state some of its properties.

Definition 10 The function $Q_\nu : \mathscr{P}(X) \to [0, 1]$ is a ν-additive measure on the finite set X, iff Q_ν satisfies the following requirements:

(1) $Q_\nu(X) = 1$
(2) for any $A, B \in \mathscr{P}(X)$ and $A \cap B = \emptyset$,

$$\begin{aligned} Q_\nu(A \cup B) = Q_\nu(A) + Q_\nu(B) + \\ + \left(\left(\frac{1 - \nu}{\nu}\right)^2 - 1 \right) Q_\nu(A) Q_\nu(B), \end{aligned} \tag{14}$$

where $\nu \in (0, 1)$.

Note that if X is an infinite set, then the continuity of function Q_ν is also required. Here, we state a key proposition that we will frequently utilize later on.

Proposition 6 *Let X be a finite set, and let Q_λ and Q_ν be a λ-additive and a ν-additive measure on X, respectively. Then,*

$$Q_\lambda(A) = Q_\nu(A) \tag{15}$$

for any $A \in \mathscr{P}(X)$, if and only if

$$\lambda = \left(\frac{1 - \nu}{\nu}\right)^2 - 1, \tag{16}$$

where $\lambda > -1$, $\nu \in (0, 1)$.

Proof This proposition immediately follows from the definitions of the λ-additive measure and ν-additive measure.

□

If Q_ν is a ν-additive measure on the finite set X, then, by utilizing Eq. (13), the Q_ν measure of the complement set $\overline{A}$ is

$$Q_\nu(\overline{A}) = \begin{cases} \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2 \frac{Q_\nu(A)}{1-Q_\nu(A)}} & \text{if } Q_\nu(A) \in [0,1) \\ 0 & \text{if } Q_\nu(A) = 1. \end{cases} \tag{17}$$

Moreover, as the ν parameter is the neutral value of the Dombi negation operator (see Eq. (12)), the following property of the ν-additive measure holds as well.

Proposition 7 *Let X be a finite set, Q_ν a ν-additive measure on X and let the set A_ν be given as*

$$A_\nu = \{A \in \mathscr{P}(X) | Q_\nu(A) = \nu\},$$

where $\nu \in (0,1)$. Then for any $A \in A_\nu$ the Q_ν measure of the complement set $\overline{A}$ is equal to ν; that is, $Q_\nu(\overline{A}) = \nu$.

Proof If $A \in A_\nu$, then $Q_\nu(A) = \nu$ and utilizing the ν-additive negation given by Eq. (17), we have

$$Q_\nu(\overline{A}) = \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2 \frac{\nu}{1-\nu}} = \nu.$$

□

This result means that the ν-additive complement operation may be viewed as a complement operation characterized by its fix point ν.

3.2 Main properties of the ν-additive (λ-additive) measures

It is worth mentioning that the definition of the ν-additive measure is the same as that of the λ-additive measure with an alternative parametrization. Thus, utilizing the fact that any ν-additive measure is a λ-additive measure with $\lambda = \left(\frac{1-\nu}{\nu}\right)^2 - 1$, some of the properties of λ-additive measures can be expressed in terms of ν-additive measures and vice versa. In this section, we will discuss the main properties of these two measures. In many cases, to make the calculations simpler, we will use the λ-additive form to demonstrate some properties and then we will state them in terms of the ν-additive measure as well. We will follow this approach from now on, and Q_λ will always denote a λ-additive measure with the parameter $\lambda \in (-1,\infty)$ and Q_ν will always denote a ν-additive measure with the parameter $\nu \in (0,1)$.

3.2.1 ν-additive (λ-additive) measure of collection of disjoint sets

Here, we will outline the computation of the ν-additive (λ-additive) measure of collection of pairwise disjoint sets.

Proposition 8 *If X is a finite set, Q_λ is a λ-additive measure on X and $A_1, A_2, \ldots, A_n \in \mathscr{P}(X)$ are pairwise disjoint sets, then*

$$Q_\lambda\left(\bigcup_{i=1}^n A_i\right) = \\ = \begin{cases} \sum_{i=1}^n Q_\lambda(A_i), & \text{if } \lambda = 0 \\ \frac{1}{\lambda}\left(\prod_{i=1}^n (1+\lambda Q_\lambda(A_i)) - 1\right), & \text{if } \lambda > -1, \lambda \neq 0. \end{cases} \tag{18}$$

Proof Here, we will discuss the two possible cases: (1) $\lambda = 0$, (2) $\lambda > -1$ and $\lambda \neq 0$.

(1) In this case, the proposition trivially follows from the definition of the λ-additive measures.
(2) Here, we will apply induction. By utilizing the definition of the λ-additive measures, the associativity of the union operation over sets and simple calculations, it can be shown that

$$Q_\lambda\left(\bigcup_{i=1}^n A_i\right) = \frac{1}{\lambda}\left(\prod_{i=1}^n (1+\lambda Q_\lambda(A_i)) - 1\right) \tag{19}$$

holds for $n = 2$ and $n = 3$, where $A_1, A_2, A_3 \in \mathscr{P}(X)$, $\lambda > -1$, $\lambda \neq 0$. Now, let us assume that Eq. (19) holds for any $A_1, A_2, \ldots, A_n \in \mathscr{P}(X)$, $\lambda > -1$, $\lambda \neq 0$. Let G_n be defined as follows:

$$G_n = \prod_{i=1}^n (1+\lambda Q_\lambda(A_i)).$$

With this notation, $G_{n+1} = G_n(1+\lambda Q_\lambda(A_{n+1}))$, and the equality that we seek to prove is

$$Q_\lambda\left(\bigcup_{i=1}^{n+1} A_i\right) = \frac{1}{\lambda}(G_{n+1} - 1).$$

By utilizing the definition of the λ-additive measures and the associativity of the union operation over sets, we get

$$Q_\lambda\left(\bigcup_{i=1}^{n+1} A_i\right) = Q_\lambda\left(\bigcup_{i=1}^{n} A_i\right) + Q_\lambda(A_{n+1}) + \\ + \lambda Q_\lambda\left(\bigcup_{i=1}^{n} A_i\right) Q_\lambda(A_{n+1}).$$

Now, utilizing the inductive condition, the last equation can be written as

$$Q_\lambda\left(\bigcup_{i=1}^{n+1} A_i\right) = \frac{1}{\lambda}(G_n - 1) + Q_\lambda(A_{n+1}) +$$
$$+\lambda\frac{1}{\lambda}(G_n - 1)Q_\lambda(A_{n+1}) =$$
$$= \frac{1}{\lambda}(G_n - 1)(1+\lambda Q_\lambda(A_{n+1})) + Q_\lambda(A_{n+1}) =$$
$$= \frac{1}{\lambda}G_n(1+\lambda Q_\lambda(A_{n+1})) - \frac{1}{\lambda} =$$
$$= \frac{1}{\lambda}(G_n(1+\lambda Q_\lambda(A_{n+1})) - 1) = \frac{1}{\lambda}(G_{n+1} - 1).$$

□

Remark 5 Note that in Eq. (18), the case $\lambda = 0$ may be viewed as a special case of $\lambda > -1$ and $\lambda \neq 0$. Namely, the right hand side of Eq. (19) can be written as

$$\frac{1}{\lambda}\left(\prod_{i=1}^{n}(1+\lambda Q_\lambda(A_i)) - 1\right) = \sum_{i=1}^{n} Q_\lambda(A_i) +$$
$$+\sum_{k=2}^{n}\lambda^{k-1}\sum_{\substack{1\le i_1<i_2\cdots\\ \cdots<i_k\le n}} Q_\lambda(A_{i_1})Q_\lambda(A_{i_2})\cdots Q_\lambda(A_{i_k})$$

from which

$$\lim_{\lambda\to 0}\left(\frac{1}{\lambda}\left(\prod_{i=1}^{n}(1+\lambda Q_\lambda(A_i)) - 1\right)\right) = \sum_{i=1}^{n} Q_\lambda(A_i).$$

Proposition 8 can be stated in terms of the ν-additive measure as follows.

Proposition 9 *If X is a finite set, Q_ν is a ν-additive measure on X and $A_1, A_2, \ldots, A_n \in \mathscr{P}(X)$ are pairwise disjoint sets, then*

$$Q_\nu\left(\bigcup_{i=1}^{n} A_i\right) = \\ = \begin{cases} \sum_{i=1}^{n} Q_\nu(A_i), & \text{if } \nu = 1/2 \\ y & \text{if } \nu \neq 1/2, \end{cases} \tag{20}$$

where $\nu \in (0,1)$,

$$y = \frac{1}{\left(\frac{1-\nu}{\nu}\right)^2 - 1}\left(\prod_{i=1}^{n}(1+\left(\left(\frac{1-\nu}{\nu}\right)^2 - 1\right)Q_\nu(A_i)) - 1\right).$$

Proof Recalling Proposition 6, this proposition directly follows from Proposition 8. □

3.2.2 General forms for the ν-additive (λ-additive) measure of union and intersection of two sets

The calculations of the λ-additive measure and ν-additive measure of two disjoint sets are given in Definition 9 and Definition 10, respectively. Here, we will show how the ν-additive (λ-additive) measure of two sets can be computed when these sets are not disjoint. We will also discuss how the ν-additive (λ-additive) measure of intersection of two sets can be computed.

Proposition 10 *If X is a finite set and Q_λ is a λ-additive measure on X, then for any $A, B \in \mathscr{P}(X)$,*

$$Q_\lambda(A\cup B) = \\ = \frac{Q_\lambda(A) + Q_\lambda(B) + \lambda Q_\lambda(A)Q_\lambda(B) - Q_\lambda(A\cap B)}{1+\lambda Q_\lambda(A\cap B)}.$$

Proof Since $A\cap(\overline{A}\cap B) = \emptyset$ and $A\cup(\overline{A}\cap B) = A\cup B$, applying Eq. (7) gives us

$$\begin{aligned} &Q_\lambda(A\cup B) = \\ &= Q_\lambda(A) + Q_\lambda(\overline{A}\cap B) + \lambda Q_\lambda(A)Q_\lambda(\overline{A}\cap B) = \\ &= Q_\lambda(A) + Q_\lambda(\overline{A}\cap B)(1+\lambda Q_\lambda(A)). \end{aligned} \tag{21}$$

Next, since $(A\cap B)\cap(\overline{A}\cap B) = \emptyset$ and $(A\cap B)\cup(\overline{A}\cap B) = B$, applying Eq. (7) again gives

$$\begin{aligned} &Q_\lambda(B) = Q_\lambda(A\cap B) + Q_\lambda(\overline{A}\cap B) + \\ &+\lambda Q_\lambda(A\cap B)Q_\lambda(\overline{A}\cap B) = \\ &= Q_\lambda(A\cap B) + Q_\lambda(\overline{A}\cap B)(1+\lambda Q_\lambda(A\cap B)). \end{aligned} \tag{22}$$

Now, by expressing $Q_\lambda(\overline{A}\cap B)$ in terms of (22), we get

$$Q_\lambda(\overline{A}\cap B) = \frac{Q_\lambda(B) - Q_\lambda(A\cap B)}{1+\lambda Q_\lambda(A\cap B)}$$

and substituting this into (21), we get

$$\begin{aligned} &Q_\lambda(A\cup B) = \\ &= Q_\lambda(A) + \frac{Q_\lambda(B) - Q_\lambda(A\cap B)}{1+\lambda Q_\lambda(A\cap B)}(1+\lambda Q_\lambda(A)) = \\ &= \frac{Q_\lambda(A) + Q_\lambda(B) + \lambda Q_\lambda(A)Q_\lambda(B) - Q_\lambda(A\cap B)}{1+\lambda Q_\lambda(A\cap B)}. \end{aligned} \tag{23}$$

Hence, we have the general form of the λ-additive measure of the union of two sets. □

Remark 6 Notice that if $\lambda = 0$, then Eq. (23) reduces to $Q_\lambda(A\cup B) = Q_\lambda(A) + Q_\lambda(B) - Q_\lambda(A\cap B)$, which has the same form as the probability measure of union of two sets. Later, we will discuss how the λ-additive (ν-additive) measure is related to the probability measure.

Remark 7 Note that Eq. (23) can be written in the following equivalent forms:

$$Q_\lambda(A\cup B)+Q_\lambda(A\cap B)+\lambda Q_\lambda(A\cup B)Q_\lambda(A\cap B)= \\ =Q_\lambda(A)+Q_\lambda(B)+\lambda Q_\lambda(A)Q_\lambda(B)$$

or for $\lambda \neq 0$

$$\frac{1}{\lambda}\left((1+Q_\lambda(A\cup B))(1+\lambda Q_\lambda(A\cap B))-1\right)= \\ =\frac{1}{\lambda}\left((1+Q_\lambda(A))(1+\lambda Q_\lambda(B))-1\right).$$

Corollary 3 *If X is a finite set and Q_λ is a λ-additive measure on X, then for any $A,B\in\mathscr{P}(X)$,*

$$Q_\lambda(A\cap B)= \\ =\frac{Q_\lambda(A)+Q_\lambda(B)+\lambda Q_\lambda(A)Q_\lambda(B)-Q_\lambda(A\cup B)}{1+\lambda Q_\lambda(A\cup B)}. \tag{24}$$

Proof By expressing $Q_\lambda(A\cap B)$ in Eq. (23), we get Eq. (24). □

3.2.3 Other properties of the ν-additive (λ-additive) measure of union and the intersection of two sets

The following results are related to the ν-additive measure of union and the intersection of two sets.

Proposition 11 *Let X be a finite set, Q_ν be a ν-additive measure on X and let $A,B\in\mathscr{P}(X)$. Then*

(1) if $A\cup B=X$ (complementing case), then

$$Q_\nu(A\cap B)=Q_\nu(A)Q_\nu(B)- \\ -\left(\frac{\nu}{1-\nu}\right)^2(1-Q_\nu(A))(1-Q_\nu(B)) \tag{25}$$

(2) if $A\cap B=\emptyset$ (disjoint case), then

$$Q_\nu(A\cup B)= \\ =1-\Bigg((1-Q_\nu(A))(1-Q_\nu(B))- \\ -\left(\frac{1-\nu}{\nu}\right)^2 Q_\nu(A)Q_\nu(B)\Bigg). \tag{26}$$

Proof (1) Since the ν-additive measure Q_ν is identical to the λ-additive measure Q_λ with $\lambda=\left(\frac{1-\nu}{\nu}\right)^2-1$, $Q_\nu(A\cap B)=Q_\lambda(A\cap B)$. Now, utilizing the fact that $Q_\nu(A\cup B)$ = $Q_\lambda(A\cup B)$ = 1 and Eq. (24), $Q_\lambda(A\cap B)$ can be written as

$$Q_\lambda(A\cap B)=\frac{Q_\lambda(A)+Q_\lambda(B)+\lambda Q_\lambda(A)Q_\lambda(B)-1}{1+\lambda}= \\ =\frac{(1+\lambda)Q_\lambda(A)Q_\lambda(B)}{1+\lambda}- \\ -\frac{1-Q_\lambda(A)-Q_\lambda(B)+Q_\lambda(A)Q_\lambda(B)}{1+\lambda}= \\ =Q_\lambda(A)Q_\lambda(B)-\frac{1}{1+\lambda}(1-Q_\lambda(A))(1-Q_\lambda(B)).$$

And by using the equation $\lambda=\left(\frac{1-\nu}{\nu}\right)^2-1$, we get

$$Q_\nu(A\cap B)= \\ =Q_\nu(A)Q_\nu(B)-\left(\frac{\nu}{1-\nu}\right)^2(1-Q_\nu(A))(1-Q_\nu(B)).$$

(2) Since $A\cap B=\emptyset$, applying the definition of the ν-additive measure gives

$$Q_\nu(A\cup B)= \\ =Q_\nu(A)+Q_\nu(B)+\left(\left(\frac{1-\nu}{\nu}\right)^2-1\right)Q_\nu(A)Q_\nu(B)= \\ =1-(1-Q_\nu(A)-Q_\nu(B)+Q_\nu(A)Q_\nu(B))+ \\ +\left(\frac{1-\nu}{\nu}\right)^2 Q_\nu(A)Q_\nu(B)= \\ =1-\Bigg((1-Q_\nu(A))(1-Q_\nu(B))- \\ -\left(\frac{1-\nu}{\nu}\right)^2 Q_\nu(A)Q_\nu(B)\Bigg).$$

□

Note that the term $\left(\frac{\nu}{1-\nu}\right)^2(1-Q_\nu(A))(1-Q_\nu(B))$ in Eq, (25) may be regarded as the corrective term of the intersection; that is, if $\nu\to 0$, then $Q_\nu(A\cap B)=Q_\nu(A)Q_\nu(B)$. Similarly, the term $\left(\frac{1-\nu}{\nu}\right)^2 Q_\nu(A)Q_\nu(B)$ in Eq. (26) may be interpreted as the corrective term of the union; that is, if $\nu\to 1$, then $Q_\nu(A\cup B)=1-(1-Q_\nu(A))(1-Q_\nu(B))$.

3.2.4 Characterization by independent variables

We have demonstrated (see Proposition 8) that if X is a finite set, Q_λ is a λ-additive measure on X, $\lambda>-1$, $\lambda\neq 0$, $A_1,A_2,\ldots,A_n\in\mathscr{P}(X)$ are pairwise disjoint sets, and

$$A=\bigcup_{i=1}^{n}A_i,$$

then

$$Q_\lambda(A)=Q_\lambda\left(\bigcup_{i=1}^{n}A_i\right)=\frac{1}{\lambda}\left(\prod_{i=1}^{n}(1+\lambda Q_\lambda(A_i))-1\right).$$

It means that the value of $Q_\lambda(A)$ can be readily calculated from the independent values $Q_\lambda(A_i)$, where $i=1,2,\ldots,n$. If $X=\{A_1,A_2,\ldots,A_n\}$, then

$$Q_\lambda(X)=Q_\lambda\left(\bigcup_{i=1}^{n}A_i\right)= \\ =\frac{1}{\lambda}\left(\prod_{i=1}^{n}(1+\lambda Q_\lambda(A_i))-1\right)=1. \tag{27}$$

The following proposition demonstrates that Eq. (27) has only one root in the interval $(-1,0)\cup(0,\infty)$.

Proposition 12 *If X is a finite set, Q_λ is a λ-additive measure on X, $\lambda > -1$, $\lambda \neq 0$, $A_1, A_2, \dots, A_n \in \mathscr{P}(X)$ are pairwise disjoint sets such that $Q_\lambda(A_i) < 1$, $i \in \{1,2,\dots,n\}$, then the equation*

$$\frac{1}{\lambda}\left(\prod_{i=1}^{n}(1+\lambda Q_\lambda(A_i)) - 1\right) = 1 \tag{28}$$

has only one root in the interval $(-1,0)\cup(0,\infty)$.

Proof This proof is based on the proof of a theorem connected with the multiplicative utility functions described by Keeney in (Keeney 1974, Appendix B). Since $\lambda \neq 0$, Eq. (28) can be written as

$$\lambda + 1 = \prod_{i=1}^{n}(1+\lambda z_i), \tag{29}$$

where $z_i = Q_\lambda(A_i)$, $i = 1,2,\dots,n$. Now, let $S = \sum_{i=1}^{n} z_i$ and let the polynomial $f(q)$ be given as

$$f(q) = q + 1 - \prod_{i=1}^{n}(1+qz_i), \tag{30}$$

where $-1 \leq q < \infty$. From Eq. (29) and Eq. (30), we get the following results:

$$f(\lambda) = 0,\ f(0) = 0,\ f(-1) = -\prod_{i=1}^{n}(1-z_i) < 0.$$

The first derivative of function f is

$$f'(q) = \frac{\mathrm{d}f(q)}{\mathrm{d}q} = 1 - \sum_{i=1}^{n} z_i \prod_{i\neq j}(1+z_j q),$$

from which we can see that $f'(q)$ is decreasing (with respect to q) in the interval $(-1,\infty)$,

$$f'(0) = 1 - \sum_{i=1}^{n} z_i = 1 - S \tag{31}$$

and

$$\lim_{q\to\infty} f'(q) = -\infty. \tag{32}$$

Here, we will distinguish three cases: (1) $S < 1$; (2) $S = 1$; (3) $S > 1$.

(1) Eq. (31) implies that if $S < 1$, then $f'(0) > 0$. Since $f'(0) > 0$ and $f'(q)$ is decreasing in the interval $(-1,\infty)$, $f'(q)$ is positive in $(-1,0)$. Therefore, $f'(q) = 0$ has no root in $(-1,0)$. Based on Eq. (32), $f'(\infty) = -\infty$, and so $f'(q) = 0$ has a unique root q^* in $(0,\infty)$. Since $f(0) = 0$ and $f'(q) > 0$ in $(0,q^*)$, $f(q) = 0$ has no root in $(0,q^*)$. As $f(q^*) > 0$ and $f'(q)$ is negative and decreasing to $-\infty$ in (q^*,∞), $f(q) = 0$ has a unique root q_0 in (q^*,∞). Moreover, $f(q) > 0$ in $(0,q_0)$ and $f(q) < 0$ in (q_0,∞); that is, the unique root q_0 is in $(0,\infty)$.

(2) It follows from Eq. (31) that if $S = 1$, then $f'(0) = 0$. Since $f'(0) = 0$ and $f'(q)$ is decreasing in the interval $(-1,\infty)$, $f'(q)$ is positive in the interval $(-1,0)$ and it is negative in the interval $(0,\infty)$. Thus, $q = 0$ is the only root of $f'(q) = 0$ in the interval $(-1,\infty)$. Moreover, since $f(0) = 0$, $q = 0$ is the only root of $f(q) = 0$. It means that if $S = 1$, then the only solution of Eq. (29) is $\lambda = 0$. Recall that $\lambda \neq 0$; that is, in this case we do not get any solution to the equation in (28).

(3) Eq. (31) implies that if $S > 1$, then $f'(0) < 0$. Since $f'(0) < 0$ and $f'(q)$ is decreasing in the interval $(-1,\infty)$, $f'(q)$ is negative in $(0,\infty)$. As $f(0) = 0$ and $f'(q)$ is negative in $(0,\infty)$, $f(q) = 0$ has no root in $(0,\infty)$. On the one hand, as $f(0) = 0$ and $f'(0) < 0$, $f(q) > 0$ immediately to the left of zero. On the other hand, $f(-1) < 0$. It means that there must be at least one root q_0 of $f(q) = 0$ in $(-1,0)$. Since $f'(q)$ is decreasing and $f(0) = 0$, q_0 is the unique root of $f(q) = 0$ in $(0,1)$. □

Proposition 12 tells us that Eq. (27) can be solved numerically for λ in the interval $(-1,0)$ or in the interval $(0,\infty)$. Hence, the λ-additive measure Q_λ can be unambiguously characterized by n independent variables.

3.3 Dual ν-additive (λ-additive) measures and their properties

Later, we will utilize the concept of the dual pair of λ-additive measures and the concept of the dual pair of ν-additive measures.

Definition 11 Let Q_{λ_1} and Q_{λ_2} be two λ-additive measures on the finite set X. Then, Q_{λ_1} and Q_{λ_2} are said to be a dual pair of λ-additive measures iff

$$Q_{\lambda_1}(A) + Q_{\lambda_2}(\overline{A}) = 1$$

holds for any $A \in \mathscr{P}(X)$.

Definition 12 Let Q_{ν_1} and Q_{ν_2} be two ν-additive measures on the finite set X. Then, Q_{ν_1} and Q_{ν_2} are said to be a dual pair of ν-additive measures iff

$$Q_{\nu_1}(A) + Q_{\nu_2}(\overline{A}) = 1$$

holds for any $A \in \mathscr{P}(X)$.

Later, we will utilize the following proposition.

Proposition 13 *Let Q_{λ_1} and Q_{λ_2} be two λ-additive measures on the finite set X and let*

$$\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}. \tag{33}$$

Then, for any $A \in \mathscr{P}(X)$

$$Q_{\lambda_2}(A) > 1 - Q_{\lambda_1}(\overline{A}), \tag{34}$$

if and only if

$$Q_{\lambda_2}(\overline{A}) < 1 - Q_{\lambda_1}(A). \quad (35)$$

Proof Firstly, we will show that if $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$ and $Q_{\lambda_2}(A) > 1 - Q_{\lambda_1}(\overline{A})$ holds for any $A \in \mathscr{P}(X)$, then $Q_{\lambda_2}(\overline{A}) < 1 - Q_{\lambda_1}(A)$ holds as well. By utilizing the formula for the λ-additive measure of complementer set given by Eq. (8), we get

$$Q_{\lambda_2}(A) = \frac{1 - Q_{\lambda_2}(\overline{A})}{1 + \lambda_2 Q_{\lambda_2}(\overline{A})}$$

and

$$Q_{\lambda_1}(\overline{A}) = \frac{1 - Q_{\lambda_1}(A)}{1 + \lambda_1 Q_{\lambda_1}(A)}$$

for any $A \in \mathscr{P}(X)$. Next, based on the condition $Q_{\lambda_2}(A) > 1 - Q_{\lambda_1}(\overline{A})$, we have the following inequality:

$$\frac{1 - Q_{\lambda_2}(\overline{A})}{1 + \lambda_2 Q_{\lambda_2}(\overline{A})} > 1 - \frac{1 - Q_{\lambda_1}(A)}{1 + \lambda_1 Q_{\lambda_1}(A)}. \quad (36)$$

From Eq. (36), via simple calculations, we get

$$\begin{aligned} &1 - Q_{\lambda_1}(A) - Q_{\lambda_2}(\overline{A}) > \\ &> Q_{\lambda_1}(A) Q_{\lambda_2}(\overline{A})(\lambda_1 + \lambda_2 + \lambda_1\lambda_2). \end{aligned} \quad (37)$$

From the condition $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$, we have the equation $\lambda_1 + \lambda_2 + \lambda_1\lambda_2 = 0$, and so the inequality relation in Eq. (37) can be written as

$$1 - Q_{\lambda_1}(A) - Q_{\lambda_2}(\overline{A}) > 0,$$

which is equivalent to that stated in Eq. (35).

Secondly, we will show that if $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$ and $Q_{\lambda_2}(\overline{A}) < 1 - Q_{\lambda_1}(A)$ holds for any $A \in \mathscr{P}(X)$, then $Q_{\lambda_2}(A) > 1 - Q_{\lambda_1}(\overline{A})$ holds as well. By utilizing the formula for the λ-additive measure of complementer set given by Eq. (8), we get

$$Q_{\lambda_2}(\overline{A}) = \frac{1 - Q_{\lambda_2}(A)}{1 + \lambda_2 Q_{\lambda_2}(A)}$$

and

$$Q_{\lambda_1}(A) = \frac{1 - Q_{\lambda_1}(\overline{A})}{1 + \lambda_1 Q_{\lambda_1}(\overline{A})}$$

for any $A \in \mathscr{P}(X)$. Next, based on the condition $Q_{\lambda_2}(\overline{A}) < 1 - Q_{\lambda_1}(A)$, we have the following inequality:

$$\frac{1 - Q_{\lambda_2}(A)}{1 + \lambda_2 Q_{\lambda_2}(A)} < 1 - \frac{1 - Q_{\lambda_1}(\overline{A})}{1 + \lambda_1 Q_{\lambda_1}(\overline{A})}. \quad (38)$$

From Eq. (38), by direct calculations, we get

$$\begin{aligned} &1 - Q_{\lambda_1}(\overline{A}) - Q_{\lambda_2}(A) < \\ &< Q_{\lambda_1}(\overline{A}) Q_{\lambda_2}(A)(\lambda_1 + \lambda_2 + \lambda_1\lambda_2). \end{aligned} \quad (39)$$

Since the condition $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$ is equivalent to the equation $\lambda_1 + \lambda_2 + \lambda_1\lambda_2 = 0$, the inequality relation in Eq. (39) can be written as

$$1 - Q_{\lambda_1}(\overline{A}) - Q_{\lambda_2}(A) < 0,$$

which is equivalent to that stated in Eq. (34). □

Here, we will demonstrate some key properties of the ν-additive (λ-additive) measure related to a dual pair of ν-additive (λ-additive) measures.

Proposition 14 *Let Q_{λ_1} and Q_{λ_2} be two λ-additive measures on the finite set X. Then Q_{λ_1} and Q_{λ_2} are a dual pair of λ-additive measures if and only if*

$$\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}.$$

Proof Firstly, we will show that if Q_{λ_1} and Q_{λ_2} are a dual pair of λ-additive measures on the finite set X, then $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$. Let Q_{λ_1} and Q_{λ_2} be a dual pair of λ-additive measures on X. It means that $Q_{\lambda_2}(A) = 1 - Q_{\lambda_1}(\overline{A})$ holds for any $A \in \mathscr{P}(X)$. Next, let $A, B \in \mathscr{P}(X)$ such that $A \cap B = \emptyset$. Then, $X = \overline{A \cap B} = \overline{A} \cup \overline{B}$. Now, noting that $Q_{\lambda_2}(A) = 1 - Q_{\lambda_1}(\overline{A})$, the formula for the λ-additive measure of the intersection of two sets given by Eq. (24) and the fact that $Q_{\lambda_1}(\overline{A} \cup \overline{B}) = Q_{\lambda_1}(X) = 1$, we get

$$\begin{aligned} &Q_{\lambda_2}(A \cup B) = 1 - Q_{\lambda_1}(\overline{A \cup B}) = \\ &= 1 - Q_{\lambda_1}(\overline{A} \cap \overline{B}) = 1 - \frac{Q_{\lambda_1}(\overline{A}) + Q_{\lambda_1}(\overline{B})}{1 + \lambda_1 Q_{\lambda_1}(\overline{A} \cup \overline{B})} - \\ &- \frac{\lambda_1 Q_{\lambda_1}(\overline{A}) Q_{\lambda_1}(\overline{B}) - Q_{\lambda_1}(\overline{A} \cup \overline{B})}{1 + \lambda_1 Q_{\lambda_1}(\overline{A} \cup \overline{B})} = \\ &= 1 - \frac{1 - Q_{\lambda_2}(A) + 1 - Q_{\lambda_2}(B)}{1 + \lambda_1} - \\ &- \frac{\lambda_1(1 - Q_{\lambda_2}(A))(1 - Q_{\lambda_2}(B)) - 1}{1 + \lambda_1} = \\ &= Q_{\lambda_2}(A) + Q_{\lambda_2}(B) - \frac{\lambda_1}{1 + \lambda_1} Q_{\lambda_2}(A) Q_{\lambda_2}(B). \end{aligned} \quad (40)$$

Moreover, since Q_{λ_2} is a λ-additive measure and $A \cap B = \emptyset$, the equation

$$Q_{\lambda_2}(A \cup B) = Q_{\lambda_2}(A) + Q_{\lambda_2}(B) + \lambda_2 Q_{\lambda_2}(A) Q_{\lambda_2}(B) \quad (41)$$

holds. Thus, from Eq. (40) and Eq. (41) we get that $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$.

Secondly, we will show that if $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$, then Q_{λ_1} and Q_{λ_2} are a dual pair of λ-additive measures on X. Let $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$. Here, we seek to show that Q_{λ_1} and Q_{λ_2} are a dual pair of λ-additive measures; that is, $Q_{\lambda_2}(A) = 1 - Q_{\lambda_1}(\overline{A})$ holds for any $A \in \mathscr{P}(X)$. Now, we will give an indirect proof of this. Let us assume that $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$, but either (1)

$Q_{\lambda_2}(A) > 1 - Q_{\lambda_1}(\overline{A})$, or (2) $Q_{\lambda_2}(A) < 1 - Q_{\lambda_1}(\overline{A})$ holds for any $A \in \mathscr{P}(X)$. We will show that this assumption leads to contradictions. Let $A, B \in \mathscr{P}(X)$ such that $A \cap B = \emptyset$. Then, $X = \overline{A \cap B} = \overline{A} \cup \overline{B}$.

(1) Here, as $Q_{\lambda_2}(A) > 1 - Q_{\lambda_1}(\overline{A})$ holds for any $A \in \mathscr{P}(X)$, $Q_{\lambda_1}(\overline{A}) > 1 - Q_{\lambda_2}(A)$ holds as well, and applying it to $\overline{A} \cap \overline{B}$, we get

$$Q_{\lambda_1}(A \cup B) = Q_{\lambda_1}(\overline{\overline{A} \cap \overline{B}}) > 1 - Q_{\lambda_2}(\overline{A} \cap \overline{B}). \tag{42}$$

Utilizing the formula for the λ-additive measure of the intersection of two sets given by Eq. (24) and the fact that $Q_{\lambda_2}(\overline{A} \cup \overline{B}) = Q_{\lambda_2}(X) = 1$, the right hand side of Eq. (42) can be expressed as

$$\begin{aligned} &1 - Q_{\lambda_2}(\overline{A} \cap \overline{B}) = \\ &= 1 - \frac{Q_{\lambda_2}(\overline{A}) + Q_{\lambda_2}(\overline{B}) + \lambda_2 Q_{\lambda_2}(\overline{A}) Q_{\lambda_2}(\overline{B}) - 1}{1 + \lambda_2}. \end{aligned} \tag{43}$$

Now, applying the inequality $Q_{\lambda_2}(A) > 1 - Q_{\lambda_1}(\overline{A})$ to set B, we have $Q_{\lambda_2}(B) > 1 - Q_{\lambda_1}(\overline{B})$, and so utilizing the fact that $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$, $Q_{\lambda_2}(A) > 1 - Q_{\lambda_1}(\overline{A})$ and $Q_{\lambda_2}(B) > 1 - Q_{\lambda_1}(\overline{B})$, Proposition 13 yields the inequality relations

$$Q_{\lambda_2}(\overline{A}) < 1 - Q_{\lambda_1}(A) \tag{44}$$

and

$$Q_{\lambda_2}(\overline{B}) < 1 - Q_{\lambda_1}(B). \tag{45}$$

Next, noting Eq. (44) and Eq. (45), from Eq. (43) we can further derive the result

$$\begin{aligned} 1 - &\frac{Q_{\lambda_2}(\overline{A}) + Q_{\lambda_2}(\overline{B}) + \lambda_2 Q_{\lambda_2}(\overline{A}) Q_{\lambda_2}(\overline{B}) - 1}{1 + \lambda_2} > \\ &> 1 - \frac{1 - Q_{\lambda_1}(A) + 1 - Q_{\lambda_1}(B)}{1 + \lambda_2} - \\ &- \frac{\lambda_2 (1 - Q_{\lambda_1}(A))(1 - Q_{\lambda_1}(B)) - 1}{1 + \lambda_2} = \\ &= Q_{\lambda_1}(A) + Q_{\lambda_1}(B) - \frac{\lambda_2}{1 + \lambda_2} Q_{\lambda_1}(A) Q_{\lambda_1}(B). \end{aligned} \tag{46}$$

On the one hand, utilizing $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$, from Eq. (42), Eq. (43) and Eq. (46), we get

$$Q_{\lambda_1}(A \cup B) > Q_{\lambda_1}(A) + Q_{\lambda_1}(B) + \lambda_1 Q_{\lambda_1}(A) Q_{\lambda_1}(B).$$

On the other hand, as $A \cap B = \emptyset$ and Q_{λ_1} is a λ-additive measure, we have

$$Q_{\lambda_1}(A \cup B) = Q_{\lambda_1}(A) + Q_{\lambda_1}(B) + \lambda_1 Q_{\lambda_1}(A) Q_{\lambda_1}(B).$$

Thus, the assumption that $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$ and $Q_{\lambda_2}(A) > 1 - Q_{\lambda_1}(\overline{A})$ leads to a contradiction.

(2) Following the same steps as in case (1), the assumption that $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$ and $Q_{\lambda_2}(A) < 1 - Q_{\lambda_1}(\overline{A})$ leads to the inequality

$$Q_{\lambda_1}(A \cup B) < Q_{\lambda_1}(A) + Q_{\lambda_1}(B) + \lambda_1 Q_{\lambda_1}(A) Q_{\lambda_1}(B),$$

which contradicts the fact that Q_{λ_1} is a λ-additive measure.

Based on case (1) and case (2), we may conclude that assuming that $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$ and $Q_{\lambda_2}(A) \neq 1 - Q_{\lambda_1}(\overline{A})$ leads to contradictions. That is, we have proven that if $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$, then $Q_{\lambda_2}(A) = 1 - Q_{\lambda_1}(\overline{A})$ holds. It means that the equation $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$ implies that Q_{λ_1} and Q_{λ_2} are a dual pair of λ-additive measures on X. □

Proposition 14 can be stated in terms of the ν-additive measure as follows.

Proposition 15 *Let Q_{ν_1} and Q_{ν_2} be two ν-additive measures on the finite set X. Then, Q_{ν_1} and Q_{ν_2} are a dual pair of ν-additive measures if and only if*

$$\nu_1 + \nu_2 = 1.$$

Proof Utilizing Proposition 6, this proposition immediately follows from Proposition 14. □

Utilizing the definition of the dual pair of λ-additive measures, the following corollary can be stated.

Corollary 4 *Let Q_{λ_1} and Q_{λ_2} be a dual pair of λ-additive measures on the finite set X. Then, $\lambda_1 \in (-1, 0]$ if and only if $\lambda_2 \in [0, \infty)$.*

Proof Since $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$ is a bijection from $(-1, 0]$ to $[0, \infty)$, this corollary follows from Proposition 14. □

Corollary 4 can be stated in terms of the ν-additive measure as follows.

Corollary 5 *Let Q_{ν_1} and Q_{ν_2} be a dual pair of ν-additive measures on the finite set X. Then, $\nu_1 \in [1/2, 1)$ if and only if $\nu_2 \in (0, 1/2]$.*

Proof Taking into account Proposition 6, this corollary immediately follows from Corollary 4. □

It should be mentioned here that one of the λ parameters of a dual pair of λ-additive measures is always in the unbounded interval $[0, \infty)$. At the same time, the ν parameters of a dual pair of ν-additive measures are both in a bounded interval; namely, one of them is in the interval $(0, 1/2]$ and the other one is in the interval $[1/2, 1)$.

3.3.1 The decomposition property of the λ-additive measure

The following proposition reveals an interesting property of the λ-additive measures.

Proposition 16 *If X is a finite set and Q_λ is a λ-additive measure on X, $A_1,\dots,A_n,B_1,\dots,B_m \in \mathscr{P}(X)$, $A_i \cap A_j = \emptyset$, $B_i \cap B_j = \emptyset$ for all $i \neq j$, $A_i \cap B_j = \emptyset$ for all i,j, $\lambda > -1$, $\lambda \neq 0$ and*

$$A = \bigcup_{i=1}^{n} A_i; B = \bigcup_{i=1}^{m} B_i,$$

then

$$Q_\lambda(A \cup B) = \\ = \frac{1}{\lambda}\left(\prod_{i=1}^{n}(1+\lambda Q_\lambda(A_i))\prod_{i=1}^{m}(1+\lambda Q_\lambda(B_i)) - 1\right).$$

Proof Since A and B are two disjoint sets and Q_λ is a λ-additive measure,

$$Q_\lambda(A \cup B) = Q_\lambda(A) + Q_\lambda(B) + \lambda Q_\lambda(A) Q_\lambda(B)$$

holds ($\lambda > -1, \lambda \neq 0$). Next, utilizing the conditions $\lambda \neq 1$, $A_i \cap A_j = \emptyset$, $B_i \cap B_j = \emptyset$ for all $i \neq j$,

$$A = \bigcup_{i=1}^{n} A_i; B = \bigcup_{i=1}^{m} B_i,$$

and the result of Proposition 8, $Q_\lambda(A \cup B)$ can be written as

$$Q_\lambda(A \cup B) = \\ = \frac{1}{\lambda}(G-1) + \frac{1}{\lambda}(H-1) + \lambda\frac{1}{\lambda}(G-1)\frac{1}{\lambda}(H-1), \tag{47}$$

where

$$G = \prod_{i=1}^{n}(1+\lambda Q_\lambda(A_i))$$
$$H = \prod_{i=1}^{m}(1+\lambda Q_\lambda(B_i)).$$

Here, Eq. (47) can be written as

$$Q_\lambda(A \cup B) = \\ = \frac{1}{\lambda}(G-1) + \frac{1}{\lambda}(H-1) + \frac{1}{\lambda}(G-1)(H-1) = \\ = \frac{1}{\lambda}(GH-1).$$

Hence,

$$Q_\lambda(A \cup B) = \\ = \frac{1}{\lambda}\left(\prod_{i=1}^{n}(1+\lambda Q_\lambda(A_i))\prod_{i=1}^{m}(1+\lambda Q_\lambda(B_i)) - 1\right).$$

□

4 Connection with belief-, probability- and plausibility measures

Here, we will discuss some important properties of the ν-additive (λ-additive) measure and how it is connected to the belief-, probability- and plausibility measures.

Proposition 17 *Let X be a finite set and let Q_λ be a λ-additive measure on X. Then, on set X, Q_λ is a*

(1) plausibility measure if and only if $-1 < \lambda \leq 0$
(2) probability measure if and only if $\lambda = 0$
(3) belief measure if and only if $\lambda \geq 0$.

Proof See Dubois and Prade (1980) and Banon (1978)

Note that in terms of the ν-additive measure, Proposition 17 can be stated as follows.

Proposition 18 *Let X be a finite set and let Q_ν be a ν-additive measure on X. Then, on set X, Q_ν is a*

(1) belief measure if and only if $0 < \nu \leq 1/2$
(2) probability measure if and only if $\nu = 1/2$
(3) plausibility measure if and only if $1/2 \leq \nu < 1$.

Proof Taking into account Proposition 6, this proposition immediately follows from Proposition 17. □

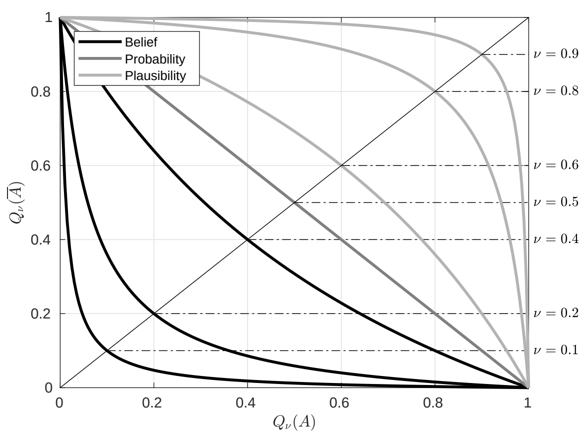

Fig. 1 ν-additive measures of set A vs. ν-additive measures of complement of A

Figure 1 shows the connection between $Q_\nu(\overline{A})$ and $Q_\nu(A)$ for various values of parameter ν of the ν-additive measure Q_ν. From this figure, in accordance with Proposition 18, we notice the following. If $\nu = 1/2$, then Q_ν is a probability measure and so $Q_\nu(\overline{A}) = 1 - Q_\nu(A)$. If $0 < \nu \leq 1/2$, then Q_ν is a belief measure and $Q_\nu(\overline{A}) \leq 1 - Q_\nu(A)$. If $1/2 \leq \nu < 1$, then Q_ν is a plausibility measure and $Q_\nu(\overline{A}) \geq 1 - Q_\nu(A)$. Moreover, in accordance with Eq. (17), for a given set A, $Q_\nu(\overline{A})$ increases with the value of parameter ν. That is, the smaller the value of parameter ν, the stronger the complement operation is. It also means that any belief measure of a complement set is always less than or equal to any plausibility measure of the same complement set.

Proposition 19 *Let Q_{λ_1} and Q_{λ_2} be two λ-additive measures on the finite set X. Then, Q_{λ_1} and Q_{λ_2} are a dual pair of belief- and plausibility measures on X if and only if they are a dual pair of λ-additive measures on X.*

Proof Firstly, we will show that if the condition of the proposition is satisfied and Q_{λ_1} and Q_{λ_2} are a dual pair of belief- and plausibility measures on X, then Q_{λ_1} and Q_{λ_2} are a dual pair of λ-additive measures on X. Let Q_{λ_1} and Q_{λ_2} be a dual pair of belief- and plausibility measures on X. Since, Q_{λ_1} and Q_{λ_2} are a dual pair; that is, $Q_{\lambda_2}(A) = 1 - Q_{\lambda_1}(\overline{A})$ holds for any $A \in \mathscr{P}(X)$, and Q_{λ_1} and Q_{λ_2} are λ-additive measures on X, they are also a dual pair of λ-additive measures on X.

Secondly, we will show that if Q_{λ_1} and Q_{λ_2} are a dual pair of λ-additive measures on X, then Q_{λ_1} and Q_{λ_2} are a dual pair of belief- and plausibility measures on X. Let Q_{λ_1} and Q_{λ_2} be a dual pair of λ-additive measures on X. Then, based on Corollary 4, either $\lambda_1 \in (-1,0]$ and $\lambda_2 \in [0,\infty)$, or $\lambda_1 \in [0,\infty)$ and $\lambda_2 \in (-1,0]$ holds. Now, utilizing Proposition 17, we get that either Q_{λ_1} is a plausibility measure and Q_{λ_2} is a belief measure, or Q_{λ_1} is a belief measure and Q_{λ_2} is a plausibility measure. Thus, noting that Q_{λ_1} and Q_{λ_2} are a dual pair of λ-additive measures on X, we may conclude that they are also a dual pair of belief- and plausibility measures on X. □

Proposition 19 can be stated in terms of the ν-additive measure as follows.

Proposition 20 *Let Q_{ν_1} and Q_{ν_2} be two ν-additive measures on the finite set X. Then, Q_{ν_1} and Q_{ν_2} are a dual pair of belief- and plausibility measures on X if and only if they are a dual pair of ν-additive measures on X.*

Proof Taking into account Proposition 6, this proposition directly follows from Proposition 19. □

Proposition 21 *Let Q_{λ_1} and Q_{λ_2} be two λ-additive measures on the finite set X. Then, Q_{λ_1} and Q_{λ_2} are a dual pair of belief- and plausibility measures on X if and only if*

$$\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}.$$

Proof Following Proposition 19, if Q_{λ_1} and Q_{λ_2} are two λ-additive measures on the finite set X, then Q_{λ_1} and Q_{λ_2} are a dual pair of belief- and plausibility measures on X if and only if they are a dual pair of λ-additive measures on X. Furthermore, based on Proposition 14, if Q_{λ_1} and Q_{λ_2} are two λ-additive measures on the finite set X, then Q_{λ_1} and Q_{λ_2} are a dual pair of λ-additive measures if and only if $\lambda_2 = -\frac{\lambda_1}{1+\lambda_1}$. Hence, this proposition follows from Proposition 19 and Proposition 14. □

Proposition 21 can be stated in terms of the ν-additive measure as follows.

Proposition 22 *Let Q_{ν_1} and Q_{ν_2} be two ν-additive measures on the finite set X. Then Q_{ν_1} and Q_{ν_2} are a dual pair of belief- and plausibility measures on X if and only if*

$$\nu_1 + \nu_2 = 1.$$

Proof Based on Proposition 6, this proposition immediately follows from Proposition 21. □

It should be added here that a ν-additive measure may be supermodular or submodular depending on the value of its parameter ν.

Definition 13 The set function $f : \mathscr{P}(X) \to \mathbb{R}$ on the finite set X is said to be submodular if

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

holds for any $A, B \in \mathscr{P}(X)$.

Definition 14 The set function $f : \mathscr{P}(X) \to \mathbb{R}$ on the finite set X is said to be supermodular if

$$f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$$

holds for any $A, B \in \mathscr{P}(X)$.

Corollary 6 *A ν-additive measure is supermodular if $\nu \in (0,1/2]$, and it is submodular if $\nu \in [1/2,1)$.*

Proof Since every belief measure is supermodular and every plausibility measure is submodular, this corollary immediately follows from Proposition 18. □

5 A transformation between a ν-additive (λ-additive) measure and a probability measure

Here, we will demonstrate that the ν-additive (λ-additive) measures can be utilized for generating probability measures; and, conversely, ν-additive (λ-additive) measures can be generated from probability measures.

Definition 15 Let Σ be a σ-algebra over the set X. Then the function $\mu : \Sigma \to [0,\infty)$ is a measure on the space (X,Σ) iff μ satisfies the following requirements:

(1) $\forall A \in \Sigma : \mu(A) \geq 0$
(2) $\mu(\emptyset) = 0$
(3) $\forall A_1, A_2, \ldots \in \Sigma$, if $A_i \cap A_j = \emptyset, \forall i \neq j$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Proposition 23 *If Σ is a σ-algebra over the set X, Q_λ is a λ-additive measure, which satisfies the continuity property of monotone measures, on the space (X,Σ), $\lambda > -1$, $\lambda \neq 0$, $c > 0$ and the function $\hat{Q}_{\lambda,c} : \Sigma \to [0,\infty)$ is given by*

$$\hat{Q}_{\lambda,c}(A) = c\ln(1+\lambda Q_\lambda(A))$$

for any $A \in \Sigma$, then $\hat{Q}_{\lambda,c}$ is a measure on the space (X,Σ).

Proof $\hat{Q}_{\lambda,c}(A)$ is trivially non-negative for any $A \in \Sigma$ and if $A = \emptyset$, then $\hat{Q}_{\lambda,c}(A) = 0$. That is, $\hat{Q}_{\lambda,c}$ satisfies requirements (1) and (2) of Definition 15. Next, let $A_1, A_2, \ldots \in \Sigma$ be a countable collection of pairwise disjoint sets. Now, utilizing the definition of $\hat{Q}_{\lambda,c}$, the fact that Q_λ is a λ-additive measure on (X,Σ) and Eq. (18), we get

$$\begin{aligned}\hat{Q}_{\lambda,c}\left(\bigcup_{i=1}^{\infty} A_i\right) &= c\ln\left(1+\lambda\left(Q_\lambda\left(\bigcup_{i=1}^{\infty} A_i\right)\right)\right) = \\ &= c\ln\left(1+\ \lambda\frac{1}{\lambda}\left(\prod_{i=1}^{\infty}(1+\lambda Q_\lambda(A_i))-1\right)\right) = \\ &= \sum_{i=0}^{\infty} c\ln(1+\lambda Q(A_i)) = \sum_{i=0}^{\infty}\hat{Q}_{\lambda,c}(A_i).\end{aligned}$$

It means that the function $\hat{Q}_{\lambda,c}$ satisfies requirement (3) in Definition 15 as well. □

Proposition 24 *Let Σ be a σ-algebra over the set X and let Q_λ and P_λ be two continuous functions on the space (X,Σ) such that*

$$P_\lambda(A) = \frac{\ln(1+\lambda Q_\lambda(A))}{\ln(1+\lambda)} \tag{48}$$

holds for any $A \in \Sigma$, $\lambda > -1$, $\lambda \neq 0$. Then, P_λ is a probability measure on (X,Σ) if and only if Q_λ is a λ-additive measure on (X,Σ).

Proof Firstly, we will show that if Eq. (48) holds and Q_λ is a λ-additive measure on (X,Σ), then P_λ is a probability measure on (X,Σ). Since $\forall A \in \Sigma : P_\lambda(A) = \hat{Q}_{\lambda,c}(A)$ with $c = 1/\ln(1+\lambda)$, based on Proposition 23, P_λ is a measure. Moreover, as $Q_\lambda(X) = 1$, $P_\lambda(X) = 1$ holds as well; and so the function P_λ satisfies all the requirements of a probability measure given by Definition 4.

Secondly, we will show that if Eq. (48) holds and P_λ is a probability measure on (X,Σ), then Q_λ is a λ-additive measure on (X,Σ). Let P_λ be a probability measure on (X,Σ). From Eq. (48) we have

$$Q_\lambda(A) = \frac{1}{\lambda}\left((1+\lambda)^{P_\lambda(A)}-1\right) \tag{49}$$

for any $A \in \Sigma$. Since P_λ is a probability measure on (X,Σ), $P_\lambda(X) = 1$; and so from Eq. (49) we get $Q_\lambda(X) = 1$. That is, Q_λ satisfies requirement (1) of the λ-additive measures given by Definition 9. Now, let $A, B \in \Sigma$ such that $A \cap B = \emptyset$. Then, as P_λ is a probability measure on (X,Σ), the equation

$$P_\lambda(A \cup B) = P_\lambda(A) + P_\lambda(B) \tag{50}$$

holds. Utilizing Eq. (49) and Eq. (50), $Q_\lambda(A \cup B)$ can be written as

$$\begin{aligned} Q_\lambda(A\cup B) &= \frac{1}{\lambda}\left((1+\lambda)^{P_\lambda(A\cup B)}-1\right) = \\ &= \frac{1}{\lambda}\left((1+\lambda)^{P_\lambda(A)+P_\lambda(B)}-1\right) = \\ &= \frac{1}{\lambda}\left((1+\lambda)^{P_\lambda(A)}-1\right) + \frac{1}{\lambda}\left((1+\lambda)^{P_\lambda(B)}-1\right) + \\ &+ \lambda\frac{1}{\lambda}\left((1+\lambda)^{P_\lambda(A)}-1\right)\frac{1}{\lambda}\left((1+\lambda)^{P_\lambda(B)}-1\right) = \\ &= Q_\lambda(A) + Q_\lambda(B) + \lambda Q_\lambda(A)Q_\lambda(B).\end{aligned}$$

It means that Q_λ satisfies requirement (2) of the λ-additive measures given in Definition 9 as well; that is, Q_λ meets all te requirements of a λ-additive measure. □

Remark 8 The measure P_λ is independent of the base of the logarithm because for any $A \in \Sigma$

$$\frac{\log_a(1+\lambda Q_\lambda(A))}{\log_a(1+\lambda)} = \frac{\frac{\log_s(1+\lambda Q_\lambda(A))}{\log_s(a)}}{\frac{\log_s(1+\lambda)}{\log_s(a)}} = \frac{\log_s(1+\lambda Q_\lambda(A))}{\log_s(1+\lambda)},$$

where $a, s > 0$, $a, s \neq 1$, $\lambda > -1$ and $\lambda \neq 0$. Also, if $s = 1+\lambda$, then $P_\lambda(A) = \log_{1+\lambda}(1+\lambda Q_\lambda(A))$.

Utilizing the definition of the ν-additive measure, Proposition 24 can be stated as follows.

Proposition 25 *Let Σ be a σ-algebra over the set X and let Q_ν and P_ν be two continuous functions on the space (X,Σ) such that*

$$P_\nu(A) = \frac{1}{2}\frac{\ln\left(1+\left(\left(\frac{1-\nu}{\nu}\right)^2-1\right)Q_\nu(A)\right)}{\ln\left(\frac{1-\nu}{\nu}\right)} \tag{51}$$

holds for any $A \in \Sigma$, $\nu \in (0,1)$, $\nu \neq 1/2$. Then, P_ν is a probability measure on (X,Σ) if and only if Q_ν is a ν-additive measure on (X,Σ).

Proof Taking into account Proposition 6, this corollary immediately follows from Proposition 24. □

Based on the result of Proposition 25, the formula in Eq. (51) may be viewed as a transformation between probability measures and ν-additive measures.

6 Connections of ν-additive (λ-additive) measures with other areas

6.1 Connection with rough sets

It is a well-known fact that the belief- and plausibility measures are connected with the rough set theory (see Dubois and Prade (1990); Yao and Lingras (1998); Wu et al. (2002)). Here, we will show how the ν-additive (λ-additive) measures are connected with the rough set theory.

Definition 16 Let X be a finite set, and let $R \subseteq X \times X$ be a binary equivalence relation on X. The pair $(\underline{R}(A), \overline{R}(A))$ is said to be the the rough set of $A \subseteq X$ in the approximation space (X,R) if

$$\begin{aligned} \underline{R}(A) &= \{x \in X | [x]_R \subseteq A\} \\ \overline{R}(A) &= \{x \in X | [x]_R \cap A \neq \emptyset\}, \end{aligned}$$

where $[x]_R$ is the R-equivalence class containing x.

The concept of a rough set was introduced by Pawlak (Pawlak 1982). The rough set $(\underline{R}(A), \overline{R}(A))$ can be utilized to characterize the set A by the pair of lower and upper approximations $(\underline{R}(A), \overline{R}(A))$. The lower approximation $\underline{R}(A)$ is the union of all elementary sets that are subsets of A, and the upper approximation $\overline{R}(A)$ is the union of all elementary sets that have a non-empty intersection with A. Note that the definitions of $\underline{R}(A)$ and $\overline{R}(A)$ are equivalent to the following statement: an element of X necessarily belongs to A if all of its equivalent elements belong to A, while an element of X possibly belongs to A if at least one of its equivalent elements belongs to A (Wu et al. 2002). Let the functions $\underline{q}, \overline{q} : \mathscr{P}(X) \to [0,1]$ be given as follows:

$$\underline{q}(A) = \frac{|\underline{R}(A)|}{|X|}, \; \overline{q}(A) = \frac{|\overline{R}(A)|}{|X|}$$

for any $A \subseteq X$. Skowron (Skowron 1989, 1990) showed that the functions $\underline{q}$ and $\overline{q}$ are a dual pair of belief- and plausibility measures and the corresponding basic probability assignment is $m(A^*) = |A^*|/|X|$ for all $A^* \in X/R$, and 0 otherwise. Furthermore, Yao and Lingras (Yao and Lingras 1998) demonstrated that if Pl and Bl are a dual pair of plausibility and belief functions on X and m is the basic probability assignment of Bl satisfying the conditions: (1) the set of focal elements of m is a partition of X, (2) $m(A^*) = |A^*|/|X|$ for every focal element A^* of m, then there exists an equivalence relation R on the set X, such that the induced qualities of upper and lower approximations satisfy

$$\underline{q}(A) = Bl(A), \; \overline{q}(A) = Pl(A)$$

for any $A \subseteq X$ Wu et al. (2002).

Based on these results and on our proposition findings, we will establish some connections between rough sets and ν-additive measures by using the following propositions.

Proposition 26 *Let Q_{ν_1} and Q_{ν_2} be two ν-additive measures on the finite set X, and let $R \subseteq X \times X$ be a binary equivalence relation on X. Furthermore, let $(\underline{R}(A), \overline{R}(A))$ be the rough set of $A \in \mathscr{P}(X)$ with respect to the approximation space (X,R) and let the functions $\underline{q}, \overline{q} : \mathscr{P}(X) \to [0,1]$ be given by*

$$\underline{q}(A) = \frac{|\underline{R}(A)|}{|X|}, \; \overline{q}(A) = \frac{|\overline{R}(A)|}{|X|},$$

where $\underline{R}(A)$ and $\overline{R}(A)$ are the lower- and upper approximations of A, respectively, for any $A \in \mathscr{P}(X)$. Then, if the equations

$$Q_{\nu_1}(A) = \underline{q}(A), \; Q_{\nu_2}(A) = \overline{q}(A), \tag{52}$$

hold for any $A \in \mathscr{P}(X)$, then Q_{ν_1} and Q_{ν_2} are a dual pair of ν-additive measures on X with $\nu_1 \in (0,1/2]$, $\nu_2 \in [1/2,1)$.

Proof Based on Skowron's results in (Skowron 1989, 1990), if the conditions of this proposition are satisfied, then the functions $\underline{q}$ and $\overline{q}$ are a dual pair of belief- and plausibility measures on X. Hence, the conditions that

(i) $Q_{\nu_1}(A) = \underline{q}(A)$, $Q_{\nu_2}(A) = \overline{q}(A)$ hold for any $A \in \mathscr{P}(X)$
(ii) Q_{ν_1} and Q_{ν_2} are two ν-additive measures on X

and the fact that $\underline{q}$ and $\overline{q}$ are a dual pair of belief- and plausibility measures on X together imply that Q_{ν_1} and Q_{ν_2} are also a dual pair of ν-additive measures on X. Furthermore, as $\underline{q}$ is a belief measure and $\overline{q}$ is a plausibility measure, based on Proposition 18, $\nu_1 \in (0,1/2]$ and $\nu_2 \in [1/2,1)$ hold as well. □

Proposition 27 *If Q_{ν_1} and Q_{ν_2} are a dual pair of ν-additive measures on the finite set X with $\nu_1 \in (0,1/2]$, $\nu_2 \in [1/2,1)$ and m is a basic probability assignment that satisfies the conditions:*

(1) The set of focal elements of m is a partition of X
(2) $m(A^*) = |A^*|/|X|$ for every focal element A^* of m
(3) $m(A^*) = \sum_{B \subseteq A^*} (-1)^{|A^* \setminus B|} Q_{\nu_1}(B)$ for any $A^* \in \mathscr{P}(X)$,

then there exists an equivalence relation R on the set X, such that the equations

$$Q_{\nu_1}(A) = \underline{q}(A), \; Q_{\nu_2}(A) = \overline{q}(A)$$

hold for any $A \in \mathscr{P}(X)$, where $(\underline{R}(A), \overline{R}(A))$ is the rough set of A with respect to the approximation space (X,R), $\underline{q}, \overline{q} : \mathscr{P}(X) \to [0,1]$ are given as

$$\underline{q}(A) = \frac{|\underline{R}(A)|}{|X|}, \; \overline{q}(A) = \frac{|\overline{R}(A)|}{|X|},$$

and $\underline{R}(A)$ and $\overline{R}(A)$ are the lower- and upper approximations of A, respectively.

Proof Based on result of Yao and Lingras (Yao and Lingras 1998), if Pl and Bl are a dual pair of plausibility and belief functions on X and m is the basic probability assignment of Bl satisfying the conditions: (i) the set of focal elements of m is a partition of X, (ii) $m(A^*) = |A^*|/|X|$ for every focal element A^* of m, then there exists an equivalence relation R on the set X, such that the induced qualities of upper and lower approximations satisfy

$$\underline{q}(A) = Bl(A),\ \overline{q}(A) = Pl(A)$$

for any $A \in \mathscr{P}(X)$. Therefore, it is sufficient to show that if the conditions of our proposition are satisfied, then Q_{ν_1} is a belief measure on X, Q_{ν_2} is a plausibility measure on X, and m is the basic probability assignment of the belief measure Q_{ν_1}.

Let us assume that the conditions of this proposition are satisfied. Then, since Q_{ν_1} and Q_{ν_2} are a dual pair of ν-additive measures on the finite set X, based on Proposition 20, Q_{ν_1} and Q_{ν_2} are a dual pair of belief- and plausibility measures on X. Furthermore, as $\nu_1 \in (0,1/2]$ and $\nu_2 \in [1/2,1)$, based on Proposition 18, Q_{ν_1} is a belief measure on X and Q_{ν_2} is a plausibility measure on X, and so condition (3) means that m is the basic probability assignment of the belief measure Q_{ν_1}. That is, we have shown that if the conditions of this proposition are satisfied, then all the conditions that are required to apply the result of Yao and Lingras (Yao and Lingras 1998) are satisfied as well. □

6.2 The λ-additive measure and the multi-attribute utility function

Here we will state interesting analogies between the λ-additive measure and the multi-attribute utility function. Let $X_1, X_2, \ldots, X_n$ be attributes, where each X_i may be either a scalar attribute or a vector of scalar attributes $(i = 1,2,\ldots,n)$. Furthermore, let the consequence space X be a rectangular subset of the n-dimensional Euclidean space. Then a specific consequence may be given by a vector $(x_1, x_2, \ldots, x_n)$, where x_i is a particular value of the attribute X_i $(i = 1,2,\ldots,n)$. The utility function $u : X \to \mathbb{R}$, which is assumed to be continuous, assigns a utility value to the consequence $(x_1, x_2, \ldots, x_n)$; that is, the utility of consequence $(x_1, x_2, \ldots, x_n)$ is $u(x_1, x_2, \ldots, x_n)$ (Keeney 1974). Here, we will utilize the concept of the utility independence of attributes (see, e.g. Keeney and Raiffa (1993)).

Definition 17 Attribute X_i is utility independent of attribute X_j if conditional preferences for lotteries over X_i given a fixed value for X_j do not depend on the particular value of X_j.

Keeney and Raiffa (Keeney and Raiffa 1993) proved the following proposition which states that the mutual utility independence of attributes implies a multiplicative multi-attribute utility function.

Proposition 28 *If $X_1, X_2, \ldots, X_n$ are mutually utility independent attributes, then*

$$u_M(x_1, x_2, \ldots, x_n) = \frac{1}{k}\left(\prod_{i=1}^{n}(1 + kk_i u_i(x_i)) - 1\right), \tag{53}$$

where $u_M : \mathbb{R}^n \to [0,1]$ is a multi-attribute utility function, $u_i : \mathbb{R} \to [0,1]$ are utility functions, k_i is the weight of attribute X_i with $0 < k_i < 1$, and $k > -1$, $k \neq 0$ is a scaling constant $(i = 1,2,\ldots,n)$.

Proof See Keeney and Raiffa (1993). □

The multi-attribute utility function u_M in Eq. (53) plays a key role in multi-attribute utility theory and can be written as

$$1 + ku_M(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n}(1 + kk_i u_i(x_i)). \tag{54}$$

If k is positive in Eq. (54), then $u^*(x_1, x_2, \ldots, x_n) = 1 + ku_M(x_1, x_2, \ldots, x_n)$ is a multi-attribute utility function, $u_i^*(x_i) = 1 + kk_i u_i(x_i)$ are utility functions and $u^*(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} u_i^*(x_i)$, where $i = 1,2,\ldots,n$. Similarly, if k is negative in Eq. (54), then $u^*(x_1, x_2, \ldots, x_n) = -(1 + ku_M(x_1, x_2, \ldots, x_n))$ is a multi-attribute utility function, $u_i^*(x_i) = -(1 + kk_i u_i(x_i))$ are utility functions and $-u^*(x_1, x_2, \ldots, x_n) = (-1)^n \prod_{i=1}^{n} u_i^*(x_i)$, where $i = 1,2,\ldots,n$. That is, Eq. (54) describes a multiplicative relationship between the multi-attribute utility function and the individual univariate utility functions. Hence, Eq. (53) is referred to as the multi-attribute multiplicative utility function.

We can see that the right hand side of Eq. (18) with $\lambda > -1$, $\lambda \neq 0$ has the same form as the right hand side of Eq. (53). It means that there is an interesting connection between the λ-additive measures and the multi-attribute multiplicative utility function. Namely, a λ-additive measure with $\lambda \neq 0$ of the union of n pairwise disjoint sets is computed in the same way as the multi-attribute utility of n mutually utility independent attributes.

Here, the formula in Eq. (53) can be written as

$$\begin{aligned} u_M(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n} k_i u_i(x_i) + \\ &+ \sum_{r=2}^{n} k^{r-1} \sum_{\substack{1 \le i_1 < \cdots \\ \cdots < i_r \le n}} k_{i_1} \cdots k_{i_r} u_{i_1}(x_{i_1}) \cdots u_{i_r}(x_{i_r}) \end{aligned} \tag{55}$$

from which

$$\lim_{k \to 0} u_M(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} k_i u_i(x_i).$$

Note that

$$u_A(x_1,x_2,\ldots,x_n)=\sum_{i=1}^{n}k_iu_i(x_i) \tag{56}$$

is the so-called multi-attribute additive utility function (Keeney 1974). We can get Eq. (56) from Eq. (55) by allowing for $k=0$.

Definition 18 Two attributes X_i and X_j are additive independent if the paired preference comparison of any two lotteries, defined by two joint probability distributions on $X_i \times X_j$, depends only on their marginal distributions.

It can be shown that if and only if the preferences over lotteries on attributes $X_1,X_2,\ldots,X_n$ depend only on their marginal probability distributions (i.e. the attributes are additive independent), then the n-attribute utility function is additive (Keeney and Raiffa 1993).

Notice that the right hand side of Eq. (18) with $\lambda=0$ has the same form as the right hand side of Eq. (56). It means that a λ-additive measure with $\lambda=0$ of the union of n pairwise disjoint sets is computed in the same way as the multi-attribute utility of n additive independent attributes.

Table 1 summarizes the analogies between the λ-additive measures and the multi-attribute utility functions.

Table 1 λ-additive measure of union of pairwise disjoint sets and utility value of consequence $(x_1,x_2,\ldots,x_n)$

Q_λ λ-additive measure of $\bigcup_{i=1}^{n} A_i$; $\lambda>-1$	
$\lambda=0$:	$\sum_{i=1}^{n} Q_\lambda(A_i)$
$\lambda\neq 0$:	$\frac{1}{\lambda}\left(\prod_{i=1}^{n}(1+\lambda Q_\lambda(A_i))-1\right)$
multi-attribute utility $u(x_1,x_2,\ldots,x_n)$; $k>-1$	
$k=0$:	$\sum_{i=1}^{n} k_iu_i(x_i)$
$k\neq 0$:	$\frac{1}{k}\left(\prod_{i=1}^{n}(1+kk_iu_i(x_i))-1\right)$

6.3 The λ-additive measure and some operators of continuous-valued logic

Here, we will state a formal connection between the λ-additive measure and certain operators of continuous-valued logic.

Definition 19 The generalized Dombi operator $o_{GD,\gamma}^{(\alpha)}:[0,1]^n\to[0,1]$ is given by

$$o_{GD,\gamma}^{(\alpha)}(\mathbf{x})= \\ =\frac{1}{1+\left(\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha}}, \tag{57}$$

where $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $x_1,x_2,\ldots,x_n$ are continuous-valued logic variables, $\alpha\in(-\infty,\infty)$ and $\gamma\in(0,\infty)$ (Dombi 2008).

It can be shown that if $\alpha>0$, then $o_{GD,\gamma}^{(\alpha)}$ is a conjunction operator, and if $\alpha<0$, then $o_{GD,\gamma}^{(\alpha)}$ is a disjunction operator (see Dombi (2008)). Moreover, the operator $o_{GD,\gamma}^{(\alpha)}$ is general because depending on its parameter values it can cover a range of familiar fuzzy conjunction and disjunction operators including the Dombi operators (Dombi 1982), the product operators (Dombi 2008), the Einstein operators (Wang and Liu 2012), the Hamacher operators (Hamacher 1978), the drastic operators (Zimmermann 2013) and the min-max operators (Zadeh 1965). Table 2 summarizes the operators that the generalized Dombi operator class can cover.

Table 2 Operators covered by the generalized Dombi opertor class

Operator	γ	conjunction value of α	disjunction
Dombi	0	$\alpha>0$	$\alpha<0$
product	1	1	-1
Einstein	2	1	-1
Hamacher	$\gamma\in(0,\infty)$	1	-1
drastic	∞	$\alpha>0$	$\alpha<0$
min-max	0	∞	$-\infty$

Here, from Eq. (57) we have

$$\left(\frac{1-o_{GD,\gamma}^{(\alpha)}(\mathbf{x})}{o_{GD,\gamma}^{(\alpha)}(\mathbf{x})}\right)^{\alpha}= \\ =\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right) \tag{58}$$

for any $o_{GD,\gamma}^{(\alpha)}(\mathbf{x})\in(0,1]$. Next, the generator function $g:(0,1]\to[0,\infty)$ of Dombi operators (Dombi 1982) is given by

$$g(x)=\left(\frac{1-x}{x}\right)^{\alpha}.$$

Utilizing this function, Eq. (58) can be written as

$$g\left(o_{GD,\gamma}^{(\alpha)}(x_1,x_2,\ldots,x_n)\right)=\frac{1}{\gamma}\left(\prod_{i=1}^{n}(1+\gamma g(x_i))-1\right). \tag{59}$$

Recall that based on Proposition 8, if X is a finite set, Q_λ is a λ-additive measure on X, $\lambda>-1$, $\lambda\neq 0$ and $A_1,A_2,\ldots,A_n\in\mathscr{P}(X)$ are pairwise disjoint sets, then

$$Q_\lambda\left(\bigcup_{i=1}^{n}A_i\right)=\frac{1}{\lambda}\left(\prod_{i=1}^{n}(1+\lambda Q_\lambda(A_i))-1\right). \tag{60}$$

From Eq. (59) and Eq. (60), we notice an interesting analogy. Namely, a λ-additive measure with $\lambda\neq 0$ of the union of n pairwise disjoint sets is computed in the same way as the value of the generator function of Dombi operator for the value of the generalized Dombi operation over n continuous-valued logic variables. It should be added that this analogy is just a formal one since $g(x_i) \in (0,\infty)$ and $Q_\lambda(A_i) \in [0,1]$, and $g(x_i)$ and $Q_\lambda(A_i)$ have different meanings.

7 Summary and future plans

In our study, we introduced the ν-additive measure as an alternatively parameterized λ-additive measure. Here, we will summarize our main findings concerning the ν-additive (λ-additive) measures.

(1) A ν-additive measure and a λ-additive measure (Sugeno λ-measure) are identical if and only if

$$\lambda = \left(\frac{1-\nu}{\nu}\right)^2 - 1,$$

where $\lambda \in (-1,\infty)$, $\nu \in (0,1)$.

(2) Two ν-additive measures are a dual pair if and only if the sum of their parameters equals 1.

(3) A ν-additive measure is a
  (a) belief measure if and only if $0 < \nu \leq 1/2$
  (b) probability measure if and only if $\nu = 1/2$
  (c) plausibility measure if and only if $1/2 \leq \nu < 1$.

(4) Two ν-additive measures are a dual pair of belief- and plausibility measures if and only if the sum of their parameters equals 1.

(5) There exists a transformation that can be utilized for transforming a ν-additive (λ-additive) measure into a probability measure; and conversely, this transformation can be utilized for transforming a probability measure into a ν-additive (λ-additive) measure.

(6) Dual pairs of ν-additive measures are strongly associated with the lower- and upper approximation pairs of rough sets.

(7) There are interesting formal connections between the λ-additive measures and the multi-attribute utility functions. Namely,
  (a) if $\lambda = 0$, then the λ-additive measure of the union of n pairwise disjoint sets is computed in the same way as the multi-attribute utility of n additive independent attributes
  (b) if $\lambda > -1$ and $\lambda \neq 0$, then the λ-additive measure of the union of n pairwise disjoint sets is computed in the same way as the multi-attribute utility of n mutually utility independent attributes.

(8) There is an interesting formal connection between the λ-additive measure and certain operators of continuous-valued logic. Namely, if $\lambda > -1$ and $\lambda \neq 0$, then the computation method of λ-additive measure of union of n pairwise disjoint sets is identical with that of the generator function of the Dombi operator at the value of the generalized Dombi operation over n continuous-valued logic variables.

As part of our future research plans, we would like to formulate a calculus of the ν-additive measure and generalize the Bayes theorem and the Poincaré formula for ν-additive measures. We also plan to study how the ν-additive measure can be utilized in the fields of computer science, engineering and economics. Especially, we aim to investigate the potential application of ν-additive measures in network science.

Compliance with Ethical Standards

- Conflict of interest: József Dombi (author) declares that he has no conflict of interest. Tamás Jónás (author) declares that he has no conflict of interest.
- Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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