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REFLECTIONS ON AND OF MINOR-CLOSED CLASSES OF MULTISORTED OPERATIONS

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Abstract. The minor relation of functions is generalized to multisorted functions. Pippenger's Galois theory for minor-closed sets of functions is extended to multisorted functions and multisorted relation pairs. Reflections of minor-closed sets are again minor-closed, and the effect of reflections on the invariant relation pairs of minor-closed sets of multisorted functions is described.

1. Introduction

A function $f\colon A^n \to B$ is called a minor of a function $g\colon A^m \to B$, if f can be obtained from g by permuting arguments, introducing fictitious arguments, and identifying arguments. Formation of minors is a way of building new functions from given ones; in fact, they are substitution instances of functions in which variables are substituted for variables. As such, they arise naturally in universal algebra as particular term operations of an algebra. The minor relation is a quasiorder on the set of all functions of several arguments from A to B, and it induces a partial order (the so-called "minor poset") on the equivalence classes.

Minors of functions have been investigated by several authors from different points of view. In the current paper, our aim is to extend this line of research to the setting of multisorted functions, which are briefly recalled in Section 2. We define minors and make a few initial observations on the structure of the minor poset of multisorted functions in Section 3, focusing on the minimal and maximal elements, ascending chains, and principal filters and ideals. The generalization is rather straightforward in itself, but, as we will see, there are some interesting phenomena that do not arise in the one-sorted case.

Minor-closed sets of functions, or "minions", as recently coined by Opršal, were characterized by Pippenger [6] in terms of a Galois connection induced by the so-called preservation relation between functions and relation pairs. In Section 4, we extend Pippenger's Galois theory in a natural way to multisorted functions and multisorted relation pairs. A few technical complications arise due to the fact that some of the components of a multisorted universe may be empty, but these are treated quite efficiently by our formalism. Nullary relations are nevertheless needed, in contrast to the classical case.

Motivated by considerations of the complexity of constraint satisfaction problems, Barto, Opršal and Pinsker [1] introduced an algebraic construction called reflection. Given sets A and B, an operation $f\colon A^n \to A$, and maps $h\colon B \to A$ and $h'\colon A \to B$, they defined the (h,h')-reflection of f as the operation $f_{(h,h')}\colon B^n \to B$ given by the rule

$$f_{(h,h')}(b_1,\dots,b_n) = h'(f(h(b_1),\dots,h(b_n))),$$

for all $b_1,\dots,b_n \in B$. As proposed by the current authors [4], the notion of reflection extends to multisorted functions with little modifications in the definition. Observing first that reflections of minor-closed sets of functions are again minor-closed, we

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describe in the final section, Section 5, how the invariant relation pairs of multisorted operations are affected by reflections.

2. Multisorted operations

We will start with recalling the definitions of basic concepts in the theory of multisorted sets and multisorted operations. We will mainly follow the notation and terminology used in the book by Wechler [9].

Definition 2.1. We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{N}_+$ the set of positive integers. For $n \in \mathbb{N}$, let $[n] := \{1, \dots, n\}$. Note that $[0] = \emptyset$.

Definition 2.2. We write tuples $(a_1, a_2, \dots, a_n)$ interchangeably as words $a_1 a_2 \dots a_n$. The set of all words over a set S is denoted by $W(S)$. The empty word is denoted by ε. The *length* of a word $w \in W(S)$ is the number of letters in w and it is denoted by $|w|$. Thus, $|w_1 w_2 \dots w_n| = n$ for $w_1, w_2, \dots, w_n \in S$. For $s \in S$, the number of occurrences of s in w is denoted by $|w|_s$.

Since a word $w = w_1 \dots w_n$ is formally a map $w\colon [n] \to S$, it makes sense to speak of the *image* of w, namely, the set $\operatorname{Im} w = \{w_1, \dots, w_n\}$ of values, or entries of w. For $u, w \in W(S)$, we write $u \subseteq w$ if $\operatorname{Im} u \subseteq \operatorname{Im} w$.

Definition 2.3. Let S be a set of elements called *sorts.* An S-indexed family of sets is called an *S-sorted set.* The usual set-theoretical relations and operations are carried over to S-sorted sets by componentwise definitions. Thus, given S-sorted sets $A = (A_s)_{s \in S}$ and $B = (B_s)_{s \in S}$, we say that A is an (*S-sorted*) *subset* of B and we write $A \subseteq B$ if $A_s \subseteq B_s$ for all $s \in S$. The *union* and *intersection* of S-sorted sets A and B are $A \cup B := (A_s \cup B_s)_{s \in S}$ and $A \cap B := (A_s \cap B_s)_{s \in S}$. More generally, the union $\bigcup \mathcal{C}$ and the intersection $\bigcap \mathcal{C}$ of an arbitrary family $\mathcal{C}$ of S-sorted sets are given by $(\bigcup \mathcal{C})_s := \bigcup \{A_s \mid A \in \mathcal{C}\}$ and $(\bigcap \mathcal{C})_s := \bigcap \{A_s \mid A \in \mathcal{C}\}$, for each $s \in S$. For any subset $S' \subseteq S$, we denote by $A|_{S'}$ the S-sorted subset of A given by

$$(A|_{S'})_s := \begin{cases} A_s, & \text{if } s \in S', \\ \emptyset, & \text{if } s \notin S'. \end{cases}$$

When we make statements such as "let A be an S-sorted set", it is understood that the member of the family A indexed by $s \in S$ is denoted by A_s.

Definition 2.4. Let A be an S-sorted set. If $A_s \neq \emptyset$, then we say that sort s is *essential* in A; otherwise sort s is *inessential* in A. Let $S_A := \{s \in S \mid A_s \neq \emptyset\}$ be the set of essential sorts in A. It follows immediately from the definitions that $A|_{S_A} = A$ and $S_{A|_{S'}} = S_A \cap S'$ for any $S' \subseteq S$.

Definition 2.5. Let A and B be S-sorted sets. An *S-sorted mapping* f from A to B, denoted by $f\colon A \to B$, is a family $(f_s)_{s \in S}$ of maps $f_s\colon A_s \to B_s$. If $x \in A_s$ and there is no risk of confusion about the sort s, we may write $f(x)$ instead of $f_s(x)$.

Definition 2.6. For an S-sorted set $A = (A_s)_{s \in S}$ and a word $w = w_1 w_2 \dots w_n \in W(S)$, let $A_w := A_{w_1} \times A_{w_2} \times \dots \times A_{w_n}$. Note that $A_\varepsilon = \{\emptyset\}$.

Definition 2.7. A pair $(w, s) \in W(S) \times S$ is called a *declaration* over S. Let A be an S-sorted set. A declaration (w, s) with $w = w_1 \dots w_n$ is *reasonable* in A if $A_s = \emptyset$ implies $A_{w_i} = \emptyset$ for some i, or, equivalently, if $A_w \neq \emptyset$ implies $A_s \neq \emptyset$. Note that the declaration (ε, s) is reasonable in A if and only if $A_s \neq \emptyset$.

An *S-sorted operation* on A is any function $f\colon A_w \to A_s$ for some declaration (w, s) that is reasonable in A. The word w is called the *arity* of f and the element s is called the (*output*) *sort* of f. The elements of S occurring in the word w are called the *input sorts* of f. We denote the declaration, arity, sort, and the set of input sorts of f by $\operatorname{dec}(f)$, $\operatorname{ar}(f)$, $\operatorname{sort}(f)$, and $\operatorname{inp}(f)$, respectively. If $|w| = n$, then we also say that f has *numerical arity* n, or that f is *n-ary.*

Note that if $w = w_1 \dots w_n$ and $A_{w_i} = \emptyset$ for some $i \in [n]$, then $A_w = \emptyset$ and $f\colon A_w \to A_s$ is the empty function $\emptyset \to A_s$. Even though, in pure set-theoretical terms, any empty function is equal to the empty set, we will nevertheless distinguish between empty functions of different declarations.

Definition 2.8. We denote the set of all S-sorted operations of declaration (w, s) on A by $\mathcal{F}_A^{(w,s)}$. Let $\mathcal{F}_A$ be the set of all S-sorted operations on A, i.e.,

$$\mathcal{F}_A := \bigcup \{\mathcal{F}_A^{(w,s)} \mid (w, s) \in W(S) \times S\}.$$

Remark 2.9. The special case when $|S| = 1$ corresponds to the usual, one-sorted operations on a set A. In this case, the declaration of a function is completely specified by the numerical arity, and we may simply speak of n-ary operations on A.

Another important special case are the functions of several arguments from A_1 to A_2, where A_1 and A_2 are possibly different sets, i.e., functions $f\colon A_1^n \to A_2$ for some $n \in \mathbb{N}$. These can be seen as S-sorted operations on $A = (A_s)_{s \in S}$, with $S = \{1, 2\}$, such that the only input sort is 1 and the output sort is 2.

Definition 2.10. Let $f\colon A_w \to A_s$ be an n-ary operation, and let $i \in [n]$. The i-th argument is *essential* in f, if there exist tuples $\mathbf{a}, \mathbf{b} \in A_w$ such that $a_j = b_j$ for all $j \in [n] \setminus \{i\}$ and $f(\mathbf{a}) \neq f(\mathbf{b})$. An argument that is not essential is *inessential* or *fictitious.* Let

$$\operatorname{Ess} f := \{i \in [n] \mid \text{the } i\text{-th argument is essential in } f\},$$

and $\operatorname{ess} f := |\operatorname{Ess} f|$. The quantity $\operatorname{ess} f$ is called the *essential arity* of f.

3. Minors of multisorted operations

New functions can be built from a given function $f\colon A^n \to B$ of several arguments by manipulation of its arguments: permutation of arguments, identification of arguments, introduction of inessential arguments. The functions that can be formed in this way are called minors of f. We shall extend the notion of minor to multisorted operations. Multisorted operations differ markedly from one-sorted operations in that arguments cannot be identified arbitrarily; it is only possible to identify arguments of the same sort.

Definition 3.1. Recall that a tuple $\mathbf{a} = (a_1, \dots, a_n) \in A_w$, with $w = w_1 \dots w_n$, is a mapping $\mathbf{a}\colon [n] \to A$ satisfying $\mathbf{a}(i) = a_i \in A_{w_i}$ for all $i \in [n]$. As such, it makes perfect sense to compose tuples with other maps. In particular, for any map $\lambda\colon [m] \to [n]$, the composite $\mathbf{a} \circ \lambda$ is a map $[m] \to A$, i.e., an m-tuple given by $\mathbf{a} \circ \lambda = (a_{\lambda(1)}, \dots, a_{\lambda(m)})$, and it is an element of A_u, where $u = w_{\lambda(1)} \dots w_{\lambda(m)}$. We will write briefly $\mathbf{a}\lambda$ for $\mathbf{a} \circ \lambda$.

Definition 3.2. Let $f\colon A_w \to A_s$ and $g\colon A_u \to A_s$ be S-sorted operations on A, with $w = w_1 \dots w_n$, $u = u_1 \dots u_m$. We say that f is a *minor* of g, or that g is a *major* of f, and we write $f \leq g$, if there exists a map $\lambda\colon [m] \to [n]$ such that $u_i = w_{\lambda(i)}$ for all $i \in [m]$ and $f(\mathbf{a}) = g(\mathbf{a}\lambda)$ (i.e., $f(a_1, \dots, a_n) = g(a_{\lambda(1)}, \dots, a_{\lambda(m)})$) for all $\mathbf{a} \in A_w$. (Note that the existence of such a map λ implies $u \subseteq w$.)

Given an S-sorted operation $g\colon A_u \to A_s$ with $|u| = m$, a word $w = w_1 \dots w_n \in W(S)$ such that $u \subseteq w$ and a map $\lambda\colon [m] \to [n]$ satisfying $u_i = w_{\lambda(i)}$ for all $i \in [m]$, define the function $g_\lambda^w\colon A_w \to A_s$ of declaration (w, s) on A by the rule $g_\lambda^w(\mathbf{a}) = g(\mathbf{a}\lambda)$, for all $\mathbf{a} = (a_1, \dots, a_n) \in A_w$. The function g_λ^w is a minor of g. Conversely, every minor of g is of the form g_λ^w for some suitable w and λ.

Remark 3.3. In the case of usual, one-sorted operations, Definition 3.2 becomes somewhat simpler, since the condition $u_i = w_{\lambda(i)}$ for all $i \in [m]$ is automatically satisfied by every map $\lambda\colon [m] \to [n]$. Thus $f\colon A^n \to A$ is a minor of $g\colon A^m \to A$ if there exists a map $\lambda\colon [m] \to [n]$ such that $f(\mathbf{a}) = g(\mathbf{a}\lambda)$ for all $\mathbf{a} \in A^n$.

The minor relation $\leq$ is a quasiorder on $\mathcal{F}_A$. As for all quasiorders, it induces an equivalence relation $\equiv$ on $\mathcal{F}_A$ by the rule $f \equiv g$ if and only if $f \leq g$ and $g \leq f$. Moreover, $\leq$ induces a partial order on the quotient $\mathcal{F}_A/{\equiv}$ by the rule $f/{\equiv} \leq g/{\equiv}$ if and only if $f \leq g$. We will refer to $(\mathcal{F}_A/{\equiv}, \leq)$ as the *minor poset* of multisorted operations on A. Its elements are equivalence classes of operations, but, for the sake of notational simplicity, we will denote an equivalence class by any of its representatives.

Remark 3.4. Informally speaking, f is a minor of g if f can be obtained from g by permutation of arguments, introduction of inessential arguments, and identification of arguments of the same sort. Note that formation of minors allows of introducing fictitious arguments of any sort, even of a sort that is not among the input sorts of a given function. On the other hand, it is never possible to get rid of all arguments of any input sort. (In the case of one-sorted operations, we can permute and identify arguments and introduce fictitious arguments, but it is never possible to get a nullary function from a non-nullary one.)

The equivalence of functions could be described as follows. A *reduced form* of a function $f\colon A_w \to A_s$ is a function obtained from f by deleting as many inessential arguments as possible while retaining at least one argument of each input sort. Two functions are equivalent if and only if their reduced forms are the same up to permutation of arguments.

It is part of the folklore of the theory of minors of one-sorted operations that $f \leq g$ implies $\operatorname{ess} f \leq \operatorname{ess} g$. It is easy to see that this holds as well for multisorted operations.

Remark 3.5. The minors of a nullary operation $f\colon A_\varepsilon \to A_s$, $f(\emptyset) = c \in A_s$ are all constant operations taking value c of any declaration (w, s) with $w \in W(S)$. On the other hand, no non-nullary operation has a nullary minor.

We shall establish a few basic facts about the structure of the minor poset of multisorted operations on A. In particular, we are going to describe the minimal and maximal elements, as well as the finite principal filters and ideals. Let us start with the minimal elements.

Lemma 3.6. *Let $f\colon A_w \to A_s$. If $\operatorname{Im} w \neq S$, then there exists $g \in \mathcal{F}_A$ such that $g < f$.*

Proof. Let $t \in S \setminus \operatorname{Im} w$, and define $g\colon A_{wt} \to A_s$ by the rule

$$g(x_1, \ldots, x_n, x_{n+1}) := f(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_n, x_{n+1}) \in A_{wt}$. Then clearly $g < f$. □

Proposition 3.7. *Let A be an S-sorted set.*

(i) *If S is infinite, then the minor poset $(\mathcal{F}_A/{\equiv}, \leq)$ has no minimal elements.*
(ii) *If S is finite, then the minimal elements of $(\mathcal{F}_A/{\equiv}, \leq)$ are precisely the operations that are "unary at each sort", i.e., operations $f\colon A_w \to A_s$ where $|w|_s = 1$ for every $s \in S$. Moreover, every operation is bounded below by a unique minimal element.*

Proof. (i) Follows immediately from Lemma 3.6.

(ii) Assume first that $f\colon A_w \to A_s$ is minimal, $|w| = n$. Lemma 3.6 implies that $\operatorname{Im} w = S$. Let $u \in W(S)$ be a word such that $|u|_s = 1$ for every $s \in S$, $|u| = m$. Then there exists a (unique) map $\lambda\colon [n] \to [m]$ with $w_i = u_{\lambda(i)}$ for all $i \in [n]$. By definition $f^u_\lambda \leq f$. Since f is minimal, we must have $f^u_\lambda \equiv f$, so $f/{\equiv} = f^u_\lambda/{\equiv}$.

For the converse, let $f\colon A_w \to A_s$ with $|w| = n$ and $|w|_s = 1$ for every $s \in S$. Suppose $g\colon A_u \to A$, $|u| = m$, satisfies $g \leq f$. By the definition of minor, there exists $\lambda\colon [n] \to [m]$ such that $w_i = u_{\lambda(i)}$ for all $i \in [n]$ and $g(\mathbf{a}) = f(\mathbf{a}\lambda)$ for all $\mathbf{a} \in A_u$. Then there exists a (unique) map $\sigma\colon [m] \to [n]$ with $u_i = w_{\sigma(i)}$ for all $i \in [m]$, and it holds that $\sigma\lambda = \mathrm{id}_{[n]}$. Consequently, $g(\mathbf{a}\sigma) = f(\mathbf{a}\sigma\lambda) = f(\mathbf{a})$ for all $\mathbf{a} \in A_w$, that is, $f \leq g$. Thus $f \equiv g$, so f is minimal.

Concerning the last claim, it is easy to see that for any operation $f\colon A_w \to A_s$, we obtain a minor of f that is unary at each sort by introducing fictitious arguments of every sort, if necessary, and then identifying all arguments of the same sort. This minor is unique, up to permutation of arguments. □

Remark 3.8. In the case of one-sorted functions, the minimal elements of the minor poset are precisely the unary functions.

Now we turn our attention to maximal elements. With the exception of a few "pathological cases" (described in Lemma 3.10 below), almost every function has proper majors and is hence non-maximal.

Lemma 3.9. *Let A be an S-sorted set, let $f\colon A_w \to A_s$ for some $w \in W(S)$, $s \in S$, and assume that $|A_w| > 1$ and $|A_s| > 1$. Then there exists $g \in \mathcal{F}_A$ such that $f < g$.*

Proof. Since $|A_w| > 1$, there exists an i such that $|A_{w_i}| > 1$. Without loss of generality, assume that $i = n$. Choose a point $\mathbf{a} \in A_w$ and an element $c \in A_s$ such that $f(\mathbf{a}) \neq c$. Define $g\colon A_{ww_n} \to A_s$ as

$$g(x_1, \dots, x_n, x_{n+1}) = \begin{cases} f(x_1, \dots, x_n), & \text{if } x_n = x_{n+1}, \\ c, & \text{if } x_n \neq x_{n+1}. \end{cases}$$

We clearly have $f \leq g$, because we can obtain f from g by identifying the last two arguments. In order to prove that $g \not\leq f$, we will show that $\operatorname{ess} f < \operatorname{ess} g$. Assume that the i-th argument is essential in f, and let the tuples $(b_1, \dots, b_{i-1}, b_i, b_{i+1}, \dots, b_n)$ and $(b_1, \dots, b_{i-1}, b'_i, b_{i+1}, \dots, b_n)$ witness this fact. If $i \neq n$, then

$$\begin{aligned} g(b_1, \dots, b_i, \dots, b_n, b_n) &= f(b_1, \dots, b_i, \dots, b_n) \\ &\neq f(b_1, \dots, b'_i, \dots, b_n) = g(b_1, \dots, b'_i, \dots, b_n, b_n), \end{aligned}$$

so the i-th argument is essential in g. Since $|A_{w_n}| > 1$, there exists an element $a'_n \in A_{w_n}$ distinct from a_n. By the choice of $\mathbf{a}$, we also have

$$g(a_1, \dots, a_{n-1}, a_n, a_n) = f(\mathbf{a}) \neq c = \begin{cases} g(a_1, \dots, a_{n-1}, a'_n, a_n) \\ g(a_1, \dots, a_{n-1}, a_n, a'_n), \end{cases}$$

which shows that the n-th and $(n+1)$-st arguments are essential in g as well. Consequently, $\operatorname{ess} f < \operatorname{ess} g$, and we conclude that g is a proper major of f. □

It still remains to deal with the cases when $|A_w| \leq 1$ or $|A_s| \leq 1$. Note that $A_s = \emptyset$ implies $A_w = \emptyset$, so we have the three exceptional cases described in the following lemma.

Lemma 3.10. *Let A be an S-sorted set, and let $f\colon A_w \to A_s$ and $g\colon A_u \to A_s$ for some $w, u \in W(S)$, $s \in S$.*

(i) *If $|A_s| = 1$, then $f \leq g$ if and only if $u \subseteq w$.*
(ii) *If $|A_w| = 1$, then there exists $c \in A_s$ such that $f(\mathbf{a}) = c$ for all $\mathbf{a} \in A_w$; moreover $f \leq g$ if and only if $u \subseteq w$ and $g(\mathbf{a}) = c$ for all $\mathbf{a} \in A_u$.*
(iii) *If $A_w = \emptyset$, then $f \leq g$ if and only if $u \subseteq w$.*

Proof. Let $n = |w|$, $m = |u|$. Assume first that $|A_s| = 1$, and let c be the unique element of A_s. Then every function $h\colon A_v \to A_s$, $v \in W(S)$, satisfies $h(\mathbf{a}) = c$ for all $\mathbf{a} \in A_v$. If $f \leq g$, then $u \subseteq w$ by the definition of minor. Conversely, if $u \subseteq w$, then there exists a map $\lambda\colon [m] \to [n]$ with $u_i = w_{\lambda(i)}$ for all $i \in [m]$, and we have $f(\mathbf{a}) = c = g(\mathbf{a}\lambda)$ for all $\mathbf{a} \in A_w$, that is, $f \leq g$.

Assume then that $|A_w| = 1$. Then $|A_{w_i}| = 1$ for every $i \in [n]$, and it clearly holds that $|A_v| = 1$ for every $v \in W(S)$ such that $v \subseteq w$ (this holds even for $v = \varepsilon$). Then there exists $c \in A_s$ such that $f(\mathbf{a}) = c$ for every (in fact, the unique) element $\mathbf{a}$ of A_w. If $f \leq g$, then $u \subseteq w$ and there exists a map $\lambda\colon [m] \to [n]$ with $u_i = w_{\lambda(i)}$ for all $i \in [m]$ such that $g(\mathbf{a}\lambda) = f(\mathbf{a}) = c$ for all $\mathbf{a} \in A_w$. Since $|A_u| = |A_w| = 1$, it holds that $\{\mathbf{a}\lambda \mid \mathbf{a} \in A_w\} = A_u$; hence g takes value c at every point in A_u. Conversely, assume that $u \subseteq w$ and $g(\mathbf{a}) = c$ for all $\mathbf{a} \in A_u$. Then there exists a map $\lambda\colon [m] \to [n]$ with $u_i = w_{\lambda(i)}$ for all $i \in [m]$, and we have $f(\mathbf{a}) = c = g(\mathbf{a}\lambda)$ for all $\mathbf{a} \in A_w$, that is, $f \leq g$.

Finally, assume that $A_w = \emptyset$. If $f \leq g$, then $u \subseteq w$ by the definition of minor. Conversely, if $u \subseteq w$, then there exists a map $\lambda\colon [m] \to [n]$ with $u_i = w_{\lambda(i)}$ for all

$i \in [m]$, and the condition that $f(\mathbf{a}) = g(\mathbf{a}\lambda)$ for all $\mathbf{a} \in A_w$ is vacuously true; hence $f \leq g$. □

Proposition 3.11. *Let A be an S-sorted set. The maximal elements of the minor poset $(\mathcal{F}_A/{\equiv}, \leq)$ are precisely the nullary operations and the operations of the form $f\colon A_t \to A_s$, where $s, t \in S$ and $A_s = A_t = \emptyset$.*

Proof. Follows immediately from Lemmas 3.10 and 3.9. (Note that there exists no nullary operation $A_\varepsilon \to A_s$ if A_s is empty.) □

Remark 3.12. In the case of one-sorted operations, the maximal elements of the minor poset are the nullary operations if $A \neq \emptyset$. Every non-nullary operation on a set with at least one element has proper majors, and the construction of Lemma 3.9 provides one whenever $|A| > 1$. However, if A is empty, then there is no nullary operation on A, and for each $n \geq 1$ there is just one n-ary operation on A, namely the empty operation. The latter are all equivalent to each other, hence the minor poset has only one element, which is maximal and minimal at the same time.

The previous results can be applied to characterize the finite principal filters of the minor poset.

Definition 3.13. Let $\uparrow f$ be the principal filter generated by $f/{\equiv}$ in $(\mathcal{F}_A/{\equiv}, \leq)$, that is, $\uparrow f := \{g/{\equiv} \mid f \leq g\}$.

Proposition 3.14. *Let A be an S-sorted set. Let $f\colon A_w \to A_s$ for some $w = w_1 \dots w_n \in W(S)$, $s \in S$.*

(i) *The principal filter $\uparrow f$ contains an infinite chain if and only if $|A_s| > 1$ and $|A_{w_i}| > 1$ for some $i \in [n]$.*
(ii) *If $A_s = \emptyset$, then $\uparrow f \cong (G, \supseteq)$, where $G := \mathcal{P}(\operatorname{Im} w) \setminus \mathcal{P}(\operatorname{Im} w \cap S_A)$.*
(iii) *If $|A_s| = 1$, then $\uparrow f \cong (\mathcal{P}(\operatorname{Im} w), \supseteq)$.*
(iv) *If $A_s \neq \emptyset$ and $|A_{w_i}| \leq 1$ for all $i \in [n]$, then $\uparrow f \cong (G \cup H, \leq)$, where G is as in (ii) and $H := \mathcal{P}(\operatorname{Im} w \cap S_A) \times A_s$, and the partial order relation is defined as follows: $X \leq Y$ if and only if*
   - *$X, Y \in G$ and $X \supseteq Y$, or*
   - *$X, Y \in H$ with $X = (B, b)$, $Y = (C, c)$ and $B \supseteq C$ and $b = c$, or*
   - *$X \in G$, $Y \in H$ with $Y = (C, c)$ and $X \supseteq C$.*

Proof. For statement (i), assume first that $|A_s| > 1$ and $|A_{w_i}| > 1$ for some $i \in [n]$. If $A_w = \emptyset$, then we need to first remove some letters from w to obtain a word u such that $w_i \in \operatorname{Im} u$, $u \subseteq w$ and $A_u \neq \emptyset$, and we let $g\colon A_u \to A_s$ be any function; then $f < g$ by Lemma 3.10(iii). (If $A_w \neq \emptyset$, such preprocessing is not necessary.) Repeated application of Lemma 3.9 then yields an infinite ascending chain above f.

The converse implication is established in the remaining statements of the current proposition, which can be proved with straightforward verification using Lemma 3.10. □

Example 3.15. In order to illustrate the posets appearing in Proposition 3.14, let $S = \{1, 2, 3, 4, 5\}$, $A_1 = A_2 = \emptyset$, $A_3 = \{0, 1, 2\}$, $A_4 = \{0, 1, 2, 3\}$, $A_5 = \{0\}$, and let f, g, h be S-sorted operations on A with $\operatorname{dec}(f) = (1234, 5)$, $\operatorname{dec}(g) = (1234, 1)$, $\operatorname{dec} h = (1234, 3)$. These are all empty operations, since $A_{1234} = A_1 \times A_2 \times A_3 \times A_4 = \emptyset$. However, the principal filters $\uparrow f$, $\uparrow g$ and $\uparrow h$ are quite distinct; they are shown in Figure 1, which is a Hasse diagram with some shorthand notation for easier readability. The diagram comprises several copies of a diamond (the big disks) connected by thick lines. Each thick line between a pair of diamonds represents four edges, each connecting a vertex of one diamond to its corresponding vertex in the other diamond.

In fact, Figure 1 gives three Hasse diagrams at once: each one of the posets $\uparrow f$, $\uparrow g$ and $\uparrow h$ is represented by the part of the diagram inside the polygonal frame labeled as such. The vertices bear labels indicating a representative of each equivalence class, and they can be interpreted as follows. Let s be the output sort of the function (f, g or h) being considered. A label of the form $w \in W(S)$ designates the empty function

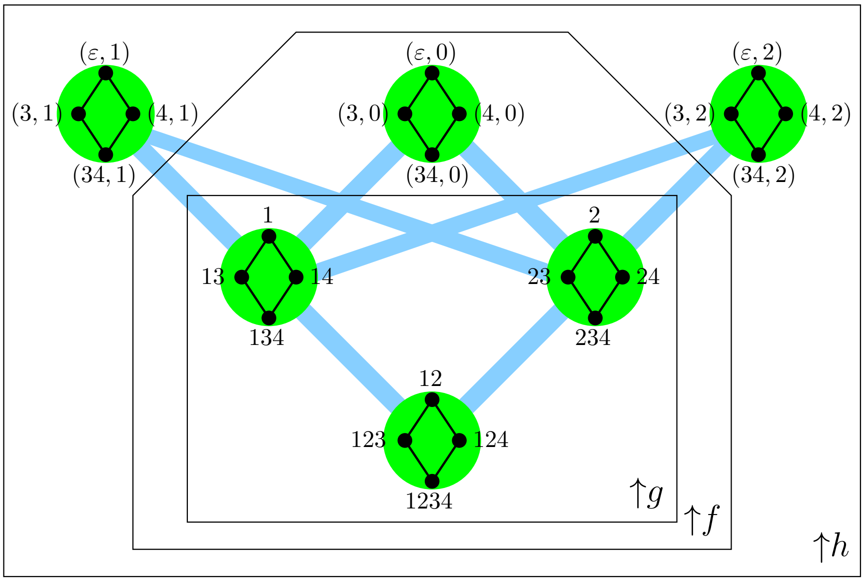


FIGURE 1. Principal filters generated by the functions f, g, h of Example 3.15.

of declaration (w, s). A label of the form $(w, a) \in W(S) \times A_s$ designates the constant function of declaration (w, s) taking value a everywhere. For example, the vertex labeled 14 represents (the equivalence class of) the empty function of declaration $(14, 5)$ in the diagram of $\uparrow f$, the empty function of declaration $(14, 1)$ in $\uparrow g$, and the empty function of declaration $(14, 3)$ in $\uparrow h$.

Proposition 3.14 describes, fully and accurately, all finite principal filters of the minor poset. In contrast, description of the finite principal ideals seems quite a challenging task, even for one-sorted functions. A function-free characterization of finite principal ideals in terms of quotients of partition lattices was obtained by Lehtonen and Waldhauser [5], but the condition is rather intricate, and we do not even know whether there exists any finite bounded poset that does not satisfy the condition. This remains a topic of further investigation. (Note that a slightly different terminology is used in [5]: "minor poset" refers there to principal ideals of the minor poset $(\mathcal{F}_A/{\equiv}, \leq)$.)

4. GALOIS THEORY OF MINOR-CLOSED CLASSES OF FUNCTIONS AND RELATION PAIRS

We say that a class $F \subseteq \mathcal{F}_A$ of S-sorted operations on A is *minor-closed* if all minors of members of F are members of F. Denote by $\langle F \rangle_{\mathrm{mc}}$ the *minor-closure* of F, i.e., the smallest minor-closed class containing F. The minor-closed classes are exactly the downsets (order ideals) of the quasiordered set $(\mathcal{F}_A; \leq)$. Let $\mathcal{M}$ be the set of all minor-closed subsets of $\mathcal{F}_A$, and order it by inclusion.

Proposition 4.1. *For any family $\mathcal{C}$ of minor-closed subsets of $\mathcal{F}_A$, the union $\bigcup \mathcal{C}$ and the intersection $\bigcap \mathcal{C}$ are minor-closed. Consequently, the set $\mathcal{M}$ of all minor-closed subsets constitutes a complete sublattice of the power set lattice $(\mathcal{P}(\mathcal{F}_A), \subseteq)$. The least and greatest elements of $\mathcal{M}$ are $\emptyset$ and $\mathcal{F}_A$, respectively.*

Proof. Let $\mathcal{C}$ be a family of minor-closed subsets of $\mathcal{F}_A$. Let $g \in \bigcup \mathcal{C}$ and $f \leq g$. Then $g \in C$ for some $C \in \mathcal{C}$. Since C is minor-closed, we have $f \in C$, so $f \in \bigcup \mathcal{C}$. We conclude that $\bigcup \mathcal{C}$ is minor-closed. The proof that $\bigcap \mathcal{C}$ is minor-closed is similar.

The statement about the least and greatest elements is obvious, because both $\emptyset$ and $\mathcal{F}_A$ are minor-closed. □

Pippenger's [6] Galois theory of minor-closed classes of functions and invariant relation pairs (constraints) can be translated to the setting of multisorted functions. We will develop the theory under the assumption that all components of the underlying S-sorted set A are finite (possibly empty). The set S of sorts may be finite or infinite.

Definition 4.2. Let $A := (A_s)_{s \in S}$ be an S-sorted set. For $m \in \mathbb{N}$, an *m-ary S-sorted relation on A* is a family $(R_s)_{s \in S}$ of m-ary relations $R_s \subseteq A_s^m$. An *m-ary S-sorted relation pair on A* is a pair (R, R'), where R and R' are m-ary S-sorted relations on A. The relations R and R' are called the *antecedent* and the *consequent* of the relation pair, respectively.

We say that a relation pair (R, R') has *finite support* if the set $S_R = \{s \in S \mid R_s \neq \emptyset\}$ is finite. (The set $S_{R'}$ may nevertheless be infinite.) In this paper, we will only consider relation pairs with finite support. This does not impose a significant restriction, because for any S-sorted relation pair (R, R'), it holds that

$$\mathrm{mPol}(R, R') = \mathrm{mPol}\{(R|_{S'}, R') \mid S' \subseteq S,\ S' \text{ finite}\}.$$

We denote by $\mathcal{Q}_A^{(m)}$ the set of all m-ary S-sorted relation pairs on A with finite support, and we denote by $\mathcal{Q}_A$ the set of all S-sorted relation pairs on A with finite support.

An n-tuple $(\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^n)$ of m-tuples can be viewed as an $m \times n$ matrix with columns $\mathbf{a}^1$, $\mathbf{a}^2$, ..., $\mathbf{a}^n$. The i-th row of this matrix is $(\mathbf{a}^1(i), \mathbf{a}^2(i), \dots, \mathbf{a}^n(i))$. With this viewpoint in mind, we will often think of tuples belonging to a relation as columns, and we refer to their components as rows.

Note that $B^0 = \{\emptyset\}$ for any set B, either empty or nonempty. Hence, there exist exactly two nullary relations on any set B, namely, $\emptyset$ and $\{\emptyset\}$. In contrast, for any $n \geq 1$, the only n-ary relation on the empty set $\emptyset$ is the empty relation $\emptyset$.

Definition 4.3. Let $A := (A_s)_{s \in S}$ be an S-sorted set. Let f be an S-sorted operation of declaration (w, s) on A ($w = w_1 \dots w_n$), and let $(R, R') \in \mathcal{Q}_A^{(m)}$ be an m-ary S-sorted relation pair on A with $R = (R_s)_{s \in S}$ and $R' = (R'_s)_{s \in S}$. We write $\mathbf{M} \prec_w R$ if $\mathbf{M} := (\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^n)$ is an $m \times n$ matrix such that $\mathbf{a}^j \in R_{w_j}$ for all $j \in [n]$. For a matrix $\mathbf{M} = (\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^n)$ such that $\mathbf{a}^j \in A_{w_j}^m$ for all $j \in [n]$, we write $f(\mathbf{M})$ to denote the m-tuple in A_s^m whose i-th entry is $f(\mathbf{a}^1(i), \mathbf{a}^2(i), \dots, \mathbf{a}^n(i))$, for $i \in [m]$. In other words, if $\mathbf{M} = (a_{ij})$, then

$$f(\mathbf{M}) := \begin{pmatrix} f(a_{11}, a_{12}, \dots, a_{1n}) \\ f(a_{21}, a_{22}, \dots, a_{2n}) \\ \vdots \\ f(a_{m1}, a_{m2}, \dots, a_{mn}) \end{pmatrix}.$$

We say that f *preserves* (R, R'), or that f is a *polymorphism* of (R, R'), or (R, R') is *invariant* under f, and we write $f \rhd (R, R')$, if for all $m \times n$ matrices $\mathbf{M}$, the condition $\mathbf{M} \prec_w R$ implies $f(\mathbf{M}) \in R'_s$.

This notation extends to sets of S-sorted operations and sets of S-sorted relation pairs in the obvious way: for any $F \subseteq \mathcal{F}_A$ and $Q \subseteq \mathcal{Q}_A$, we write $F \rhd Q$ to mean that $f \rhd (R, R')$ holds for all $f \in F$ and for all $(R, R') \in Q$. Furthermore, we simplify the notation for singletons and write $f \rhd Q$ for $\{f\} \rhd Q$ and $F \rhd (R, R')$ for $F \rhd \{(R, R')\}$.

Let us point out three special cases.

- If $m = 0$, then the condition $\mathbf{M} \prec_w R$ asserts that $\mathbf{M} = (\emptyset, \emptyset, \dots, \emptyset)$, and $f(\mathbf{M}) = \emptyset \in A_s^0$. Consequently, $f \rhd (R, R')$ if and only if

  $$(\forall i \in [n] \colon R_{w_i} = \{\emptyset\}) \implies R'_s = \{\emptyset\}.$$

  Thus, whether a function f preserves a nullary relation pair depends only on the declaration of f.
- If $w = \varepsilon$ and $m \geq 1$, then $f \rhd (R, R')$ if and only if $(c, \dots, c) \in R'_s$, where c is the constant value taken by f.

- If $A_w = \emptyset$ and $m \geq 1$, then $R_{w_i} = \emptyset$ for some $i \in [n]$, so there is no matrix $\mathbf{M}$ such that $\mathbf{M} \prec_w R$. Hence the implication in the definition of preservation holds vacuously. Thus, every empty function preserves every relation pair of arity at least 1.

Definition 4.4. The preservation relation induces a Galois connection between multisorted operations and multisorted relation pairs on A. For any set $F \subseteq \mathcal{F}_A$ of S-sorted operations and for any set $Q \subseteq \mathcal{Q}_A$ of S-sorted relation pairs, we write

$$\begin{aligned} \operatorname{mInv} F &:= \{(R, R') \in \mathcal{Q}_A \mid \forall f \in F \colon f \rhd (R, R')\}, \\ \operatorname{mPol} Q &:= \{f \in \mathcal{F}_A \mid \forall (R, R') \in Q \colon f \rhd (R, R')\}. \end{aligned}$$

Definition 4.5. Assume that $A = (A_s)_{s \in S}$ is an S-sorted set in which every component A_s is finite. For any $w = w_1 \dots w_n \in W(S)$, let $\mathbf{X}_w = (\mathbf{x}^w_1, \dots, \mathbf{x}^w_n)$ be the $N \times n$ matrix whose rows are all the n-tuples in A_w in some fixed order, where $N = |A_w|$. Then each column $\mathbf{x}^w_i$ is a tuple in $A^N_{w_i}$. Let $\chi_w = (\chi_{w,s})_{s \in S}$ be the N-ary S-sorted relation in which the component $\chi_{w,s}$ of sort $s \in S$ comprises those columns $\mathbf{x}^w_i$ of $\mathbf{X}_w$ for which $w_i = s$, i.e., $\chi_{w,s} := \{\mathbf{x}^w_i \mid i \in [n],\ w_i = s\}$.

Let us point out two special cases.

- If $w = \varepsilon$, then $N = 1$ and $n = 0$, and $\chi_\varepsilon = (\chi_{\varepsilon,s})_{s \in S}$ is the unary S-sorted relation with $\chi_{\varepsilon,s} = \emptyset$ for all $s \in S$.
- If $A_w = \emptyset$, then $N = 0$, $n = |w|$, and $\chi_w = (\chi_{w,s})_{s \in S}$ is the nullary S-sorted relation with $\chi_{w,s} = \{\emptyset\}$ if $s = w_i$ for some $i \in [n]$, and $\chi_{w,s} = \emptyset$ otherwise.

Definition 4.6. For an S-sorted relation σ and a set $F \subseteq \mathcal{F}_A$ of S-sorted operations on A, let

$$\Gamma_F(\sigma) := \bigcap \{R' \mid F \rhd (\sigma, R')\}.$$

Remark 4.7. The relation pair $(\sigma, \Gamma_F(\sigma))$ is an invariant of F, since it is the intersection of the relation pairs $(\sigma, R') \in \operatorname{mInv} F$. In fact, $\Gamma_F(\sigma)$ is the least relation R' such that $F \rhd (\sigma, R')$.

Lemma 4.8. *Let $A = (A_s)_{s \in S}$ be an S-sorted set in which every set A_s is finite. Assume that $F \subseteq \mathcal{F}_A$ is minor-closed. Then for every word $w \in W(S)$ we have $\Gamma_F(\chi_w) = (\{f(\mathbf{X}_w) \mid f \in F^{(w,s)}\})_{s \in S}$.*

Proof. Write $\gamma_s := \{f(\mathbf{X}_w) \mid f \in F^{(w,s)}\}$ and let $\gamma = (\gamma_s)_{s \in S}$. In order to prove the inclusion $\Gamma_F(\chi_w) \subseteq \gamma$, we show that $(\chi_w, \gamma) \in \operatorname{mInv} F$. Let $f \in F$, say with $\operatorname{dec}(f) = (u, s)$, $|u| = m$, and let $\mathbf{M} \prec_u \chi_w$. Then there exists $\lambda \colon [m] \to [n]$ such that $\mathbf{M} = (\mathbf{x}^w_{\lambda(1)}, \dots, \mathbf{x}^w_{\lambda(m)})$ and thus $u_i = w_{\lambda(i)}$ for all $i \in [m]$. Then $f^w_\lambda \in F^{(w,s)}$, because F is minor-closed, and consequently $f(\mathbf{M}) = f^w_\lambda(\mathbf{X}_w) \in \gamma_s$.

In order to prove the converse inclusion $\gamma \subseteq \Gamma_F(\chi_w)$, we show that $\gamma \subseteq R'$ for every R' such that $(\chi_w, R') \in \operatorname{mInv} F$. Indeed, let $r \in \gamma_s$. Then there exists $f \in F^{(w,s)}$ such that $r = f(\mathbf{X}_w)$. Since $\mathbf{X}_w \prec_w \chi_w$ and $f \rhd (\chi_w, R')$, we must have $r = f(\mathbf{X}_w) \in R'_s$, and this proves that $\gamma \subseteq R'$. From the definition of $\Gamma_F(\chi_w)$, we conclude that $\gamma \subseteq \Gamma_F(\chi_w)$. □

Lemma 4.9. *Let $A = (A_s)_{s \in S}$ be an S-sorted set in which every set A_s is finite. Let $F \subseteq \mathcal{F}_A$ be a minor-closed class and $w \in W(S)$. Then the following statements hold.*

(i) *For any $s \in S$ and $f, g \in \mathcal{F}^{(w,s)}_A$, $f = g$ if and only if $f(\mathbf{X}_w) = g(\mathbf{X}_w)$.*
(ii) *For any $f \in \mathcal{F}_A$ satisfying $\operatorname{ar}(f) = w$ we have $f \in F$ if and only if $f \rhd (\chi_w, \Gamma_F(\chi_w))$.*
(iii) *$\Gamma_F(\chi_w) = \Gamma_{F'}(\chi_w)$, where $F' := \{f \in F \mid \operatorname{ar}(f) = w\}$.*
(iv) *$F \rhd (\chi_w, \Gamma_F(\chi_w))$.*
(v) *$F = \operatorname{mPol}\{(\chi_w, \Gamma_F(\chi_w)) \mid w \in W(S)\}$.*

Proof. (i) Obvious, because the rows of $\mathbf{X}_w$ are all the n-tuples of the set A_w and hence $f(\mathbf{X}_w)$ is the tuple listing the values of f at each point in its domain.

(ii) If $f \in F$, then $f \rhd (\chi_w, \Gamma_F(\chi_w))$ by Remark 4.7. Since $\mathbf{X}_w \prec_w \chi_w$, it follows that $f(\mathbf{X}_w) \in \Gamma_F(\chi_w)$, so by Lemma 4.8 there is $f' \in F$ with $\operatorname{dec}(f') = \operatorname{dec}(f)$ such that $f(\mathbf{X}_w) = f'(\mathbf{X}_w)$. By (i) we obtain $f = f' \in F$.

(iii) Since $F' \subseteq F$, we have $\operatorname{mInv} F \subseteq \operatorname{mInv} F'$, whence $\Gamma_{F'}(\chi_w) \subseteq \Gamma_F(\chi_w)$. For the converse inclusion, note that $F' = \bigcup_{s \in S} F^{(w,s)}$. Since $\mathbf{X}_w \prec_w \chi_w$, we must have $(\{f(\mathbf{X}_w) \mid f \in F^{(w,s)}\})_{s \in S} \subseteq R'$ for every R' such that $(\chi_w, R') \in \operatorname{mInv} F'$. Lemma 4.8 and the definition of $\Gamma_{F'}(\chi_w)$ then yield $\Gamma_F(\chi_w) \subseteq \Gamma_{F'}(\chi_w)$.

(iv) Follows immediately from Remark 4.7.

(v) Follows immediately from items (ii) and (iv). □

Theorem 4.10. *Let $A := (A_s)_{s \in S}$ be an S-sorted set, and assume that the sets A_s are all finite. Let $F \subseteq \mathcal{F}_A$ be a set of S-sorted operations on A. Then $F = \operatorname{mPol} Q$ for some $Q \subseteq \mathcal{Q}_A$ if and only if F is minor-closed. Consequently, $\langle F \rangle_{\mathrm{mc}} = \operatorname{mPol} \operatorname{mInv} F$ for any $F \subseteq \mathcal{F}_A$.*

Proof. The "if" part is given by Lemma 4.9. For the converse, assume that $F = \operatorname{mPol} Q$. Let $f \in F^{(w,s)}$, $w = w_1 \dots w_n$. Let $u := u_1 \dots u_m \in W(S)$ be such that $\{w_1, \dots, w_n\} \subseteq \{u_1, \dots, u_m\}$, and let $\lambda\colon [n] \to [m]$ be a map satisfying $w_i = u_{\lambda(i)}$ for all $i \in [n]$. We need to show that $f^u_\lambda \in F$. Let $(R, R') \in Q$ be a q-ary relation pair. Let $\mathbf{M} := (\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^m)$ be a $q \times m$ matrix with columns $\mathbf{a}^j$ ($j = 1, \dots, m$) and assume that $\mathbf{M} \prec_u R$. Then $(\mathbf{a}^{\lambda(1)}, \dots, \mathbf{a}^{\lambda(n)}) \prec_w R$, so we have $f^u_\lambda(\mathbf{M}) = f(\mathbf{a}^{\lambda(1)}, \dots, \mathbf{a}^{\lambda(n)}) \in R'_s$, because $f \rhd (R, R')$. We conclude that $f^u_\lambda \in \operatorname{mPol} Q = F$, so F is minor-closed.

We have shown that F is minor-closed if and only if $F = \operatorname{mPol} Q$ for some $Q \subseteq \mathcal{Q}_A$. By the general properties of Galois connections, the latter is equivalent to $F = \operatorname{mPol} \operatorname{mInv} F$. Thus we see that the closed classes corresponding to the closure operators $F \mapsto \operatorname{mPol} \operatorname{mInv} F$ and $F \mapsto \langle F \rangle_{\mathrm{mc}}$ are the same, therefore the two closure operators coincide: $\langle F \rangle_{\mathrm{mc}} = \operatorname{mPol} \operatorname{mInv} F$ for all $F \subseteq \mathcal{F}_A$. □

We are now going to describe the Galois closed sets of relation pairs. We follow the approach taken by Lau [3, Section II.2] and Pöschel and Kalužnin [8, Sections 1.1–1.2] for describing the Galois closed sets of relations in the classical theory of clones and relational clones. The notions and ideas present in these pieces of literature can be translated in a straightforward way to the realm of S-sorted operations and relation pairs.

For an arbitrary equivalence relation ϱ on $[m]$, let $\delta^m_\varrho = (\delta^m_{\varrho,s})_{s \in S}$, where

$$\delta^m_{\varrho,s} := \{(a_1, \dots, a_m) \in A^m_s \mid (i,j) \in \varrho \implies a_i = a_j\}.$$

We write simply δ_ϱ when m is clear from the context. Relation pairs of the form $(\delta^m_\varrho, \delta^m_\varrho)$ are called *diagonal relation pairs.* (Note that $\delta^0_{\varrho,s} = \{\emptyset\}$.)

Remark 4.11. It is easy to verify that every S-sorted operation in $\mathcal{F}_A$ preserves every diagonal relation pair $(\delta^m_\varrho, \delta^m_\varrho)$.

Recall the "elementary operations" ζ, τ, pr, $\times$ and $\wedge$ on relations (see Lau [3, Section II.2.3]). Let R and $\tilde{R}$ be m-ary and m'-ary relations on a set B, respectively. Then $\zeta R = \tau R = R$ for $m \leq 1$, $\operatorname{pr} R = R$ for $m = 0$, $R \wedge \tilde{R} = R$ for $m \neq m'$, and

$$\begin{aligned}
\zeta R &:= \{(a_2, a_3, \dots, a_m, a_1) \mid (a_1, a_2, \dots, a_m) \in R\} \quad (m \geq 2), \\
\tau R &:= \{(a_2, a_1, a_3, \dots, a_m) \mid (a_1, a_2, \dots, a_m) \in R\} \quad (m \geq 2), \\
\operatorname{pr} R &:= \{(a_2, \dots, a_m) \mid (a_1, a_2, \dots, a_m) \in R\} \quad (m \geq 1), \\
R \times \tilde{R} &:= \{(a_1, \dots, a_m, b_1, \dots, b_{m'}) \mid (a_1, \dots, a_m) \in R, (b_1, \dots, b_{m'}) \in \tilde{R}\}, \\
R \wedge \tilde{R} &:= \{(a_1, \dots, a_m) \mid (a_1, \dots, a_m) \in R \cap \tilde{R}\} \quad (m = m').
\end{aligned}$$

The operation ζ is called *cyclic shift of rows,* τ is called *transposition of first two rows,* pr is called *deletion of first row,* $\times$ is called *Cartesian product,* and $\wedge$ is called *intersection.* Observe that if R is an empty relation (of arbitrary arity), then $\operatorname{pr} R = \emptyset$ and $R \times \tilde{R} = \emptyset$ for every relation $\tilde{R}$. If R is a nonempty unary relation, then we have $\operatorname{pr} R = \{\emptyset\}$, and if R is the nonempty nullary relation (i.e., $R = \{\emptyset\}$), then $R \times \tilde{R} = \tilde{R}$.

We define analogous elementary operations of S-sorted relation pairs. For $(R, R') \in \mathcal{Q}_A^{(m)}$, we set $\zeta(R, R') := (\zeta R, \zeta R')$, where $\zeta R := (\zeta R_s)_{s \in S}$ and $\zeta R' := (\zeta R'_s)_{s \in S}$. Similarly, $\tau(R, R')$, $\operatorname{pr}(R, R')$, $(R, R') \times (\tilde{R}, \tilde{R}')$ and $(R, R') \wedge (\tilde{R}, \tilde{R}')$ are defined componentwise and in parallel in each sort.

A relation pair (R, R') is a *relaxation* of $(\tilde{R}, \tilde{R}')$ if $R \subseteq \tilde{R}$ and $R' \supseteq \tilde{R}'$. We say that (R, R') is obtained from $(\tilde{R}, R')$ by *restricting the antecedent* if $R \subseteq \tilde{R}$, and we say that (R, R') is obtained from $(R, \tilde{R}')$ by *extending the consequent* if $R' \supseteq \tilde{R}'$.

Definition 4.12. Following Pippenger [6], we say that a set $Q \subseteq \mathcal{Q}_A$ of relation pairs is *minor-closed* if it contains the diagonal relation pairs and is closed under the elementary operations ζ, τ, pr, $\times$, $\wedge$, relaxations and arbitrary intersections.

Remark 4.13. The closure under arbitrary intersections subsumes the closure under $\wedge$. If the set S of sorts is finite and every component of the S-sorted set A is finite, then closure under arbitrary intersections can be omitted from Definition 4.12.

Moreover, we would like to mention that the closure of sets of S-sorted relation pairs under the operations ζ, τ, pr, $\times$, $\wedge$ is equivalent to the closure with respect to pp-formulas (more precisely, with respect to logical operations on relations defined by primitive positive first-order formulas) as it is known from the one-sorted case (cf., e.g., [7, Remark 1.6]).

Using the argument provided by Lau [3, Section II.2.5], one can show that a minor-closed set Q is also closed under operations derivable from the elementary operations, such as permutation of rows, projection onto rows $i_1, i_2, \dots, i_t$ (denoted by $\operatorname{pr}_{i_1, i_2, \dots, i_t}$), identification of rows, repetition of rows, introduction of fictitious rows, relational product. We denote by $[Q]_{\mathrm{mc}}$ the *minor-closure* of a set $Q \subseteq \mathcal{Q}_A$, i.e., the smallest minor-closed set of relation pairs containing Q.

It follows immediately from the definitions that $\operatorname{mInv} F$ is minor-closed for every $F \subseteq \mathcal{F}_A$.

Lemma 4.14. *Let $A = (A_s)_{s \in S}$ be an S-sorted set in which every set A_s is finite. Let $(R, R') \in \mathcal{Q}_A$ and $F \subseteq \mathcal{F}_A$. Assume that $(R, R') \in \operatorname{mInv} F$. Then there exists $(R, R'') \in \operatorname{mInv} F$ such that $R'' \subseteq R'$, and there are a word $w \in W(S)$ and $i_1, \dots, i_m \in [q]$, $q := |A_w|$, such that $(R, R'') = \operatorname{pr}_{i_1, \dots, i_m}(\chi_w, \Gamma_F(\chi_w))$. In particular,*

$$\operatorname{mInv} F = [\{(\chi_w, \Gamma_F(\chi_w)) \mid w \in W(S)\}]_{\mathrm{mc}}.$$

Proof. Assume that R is m-ary and $S_R = \{s_1, \dots, s_t\}$. For $s \in S$, let $n_s := |R_s|$, and let $N := \sum_{s \in S} n_s$. Note that N is a well-defined integer, because S_R is finite and every set A_s is finite. Let $\mathbf{M}_R$ be the $m \times N$ matrix, whose leftmost columns are the n_{s_1} tuples in R_{s_1}, which are followed by the n_{s_2} tuples in R_{s_2}, and so on, and the rightmost columns are the n_{s_t} tuples in R_{s_t}. Let

$$w := \underbrace{s_1 \dots s_1}_{n_{s_1}} \underbrace{s_2 \dots s_2}_{n_{s_2}} \dots \underbrace{s_t \dots s_t}_{n_{s_t} \text{ times}}.$$

There exist $i_1, \dots, i_m \in [q]$, $q := |A_w|$, such that $\operatorname{pr}_{i_1, \dots, i_m}(\mathbf{X}_w) = \mathbf{M}_R$ and hence $\operatorname{pr}_{i_1, \dots, i_m}(\chi_w) = R$. Let $R'' := \operatorname{pr}_{i_1, \dots, i_m}(\Gamma_F(\chi_w))$.

We claim that $R'' \subseteq R'$. Let $r \in R''_s$. By Lemma 4.8 and Theorem 4.10, there exists $f_{\tilde{r}} \in \langle F \rangle_{\mathrm{mc}}^{(w,s)} = (\operatorname{mPol} \operatorname{mInv} F)^{(w,s)}$ such that $f_{\tilde{r}}(\mathbf{X}_w) = \tilde{r}$ and $r = \operatorname{pr}_{i_1, \dots, i_m}(\tilde{r})$. Then

$$r = \operatorname{pr}_{i_1, \dots, i_m}(\tilde{r}) = \operatorname{pr}_{i_1, \dots, i_m}(f_{\tilde{r}}(\mathbf{X}_w)) = f_{\tilde{r}}(\operatorname{pr}_{i_1, \dots, i_m}(\mathbf{X}_w)) = f_{\tilde{r}}(\mathbf{M}_R) \in R'_s,$$

because $\mathbf{M}_R \prec_w R$ and $(R, R') \in \operatorname{mInv} F$. Clearly, $(R, R'') \in \operatorname{mInv} F$ since $(\chi_w, \Gamma_F(\chi_w)) \in \operatorname{mInv} F$ by Remark 4.7 and $\operatorname{mInv} F$ is minor-closed. ☐

Lemma 4.15. *Let $A = (A_s)_{s \in S}$ be an S-sorted set in which every set A_s is finite. Let $Q \subseteq \mathcal{Q}_A$ be a minor-closed class, and let $F := \mathrm{mPol}\, Q$. Then $(\chi_w, \Gamma_F(\chi_w)) \in Q$ for every $w \in W(S)$.*

Proof. Fix $w \in W(S)$, and let $\gamma := \bigcap \{R' \mid (\chi_w, R') \in Q\}$. Then $(\chi_w, \gamma) \in Q$, because Q is minor-closed. We are going to show that $\gamma \subseteq \Gamma_F(\chi_w)$; from this it follows that $(\chi_w, \Gamma_F(\chi_w)) \in Q$, because Q is closed under extension of consequents.

Suppose, to the contrary, that $\gamma \nsubseteq \Gamma_F(\chi_w)$. Then there exists $s \in S$ and $r \in A_s^q$, $q := |A_w|$, such that $r \in \gamma_s \setminus \Gamma_F(\chi_w)_s$. Define the function $f_r \colon A_w \to A_s$ by the rule $f_r(\mathbf{X}_w) := r$. Since $r \notin \Gamma_F(\chi_w)$ and $\mathbf{X}_w \prec_w \chi_w$ we have $f_r \not\triangleright (\chi_w, \Gamma_F(\chi_w))$. The set F is minor-closed by Theorem 4.10, whence it follows by Lemma 4.9(ii) that $f_r \notin F$. Since $f_r \notin F = \mathrm{mPol}\, Q$, there exists a relation pair $(R, R') \in Q$, say, of arity m, that is not preserved by f_r, i.e., there exists $\mathbf{M} \prec_w R$ such that $f_r(\mathbf{M}) \notin R'_s$.

Let $\mathbf{N}$ be the $(q+m) \times |w|$ matrix obtained by placing $\mathbf{X}_w$ on top of $\mathbf{M}$. Let ϱ be the partition of the set $[q+m]$ in which elements $i, j \in [q+m]$ belong to the same block if and only if rows i and j of $\mathbf{N}$ are equal. Recall that the rows of $\mathbf{X}_w$ are all tuples in A_w, so each row of $\mathbf{M}$ appears also as a row of $\mathbf{X}_w$. Let $h \colon [m] \to [q]$ be the map such that for each $i \in [m]$, the i-th row of $\mathbf{M}$ equals the $h(i)$-th row of $\mathbf{X}_w$; thus $\mathbf{M} = \mathrm{pr}_{h(1),\dots,h(m)} \mathbf{X}_w$. Let

$$(\sigma, \sigma') := ((\chi_w, \gamma) \times (R, R')) \wedge (\delta_\varrho, \delta_\varrho),$$
$$(\tilde{\sigma}, \tilde{\sigma}') := \mathrm{pr}_{1,\dots,q}(\sigma, \sigma').$$

Note that both (σ, σ') and $(\tilde{\sigma}, \tilde{\sigma}')$ belong to Q because Q is closed under products, intersections and projections. Observe also that $\chi_w \subseteq \tilde{\sigma}$, because the columns of $\mathbf{N}$ belong to $(\chi_w \times R) \wedge \delta_\varrho$. Furthermore, $\tilde{\sigma} \subseteq \chi_w$ by definition, so $\tilde{\sigma} = \chi_w$ and we have $(\chi_w, \tilde{\sigma}') \in Q$. By the definition of γ, we have $(\chi_w, \gamma) \subseteq (\chi_w, \tilde{\sigma}') = (\tilde{\sigma}, \tilde{\sigma}')$.

Since $\gamma \subseteq \tilde{\sigma}'$, we have $r \in \tilde{\sigma}'_s$. This means, by the definition of $\tilde{\sigma}'$, that there exists $t \in ((\gamma \times R') \cap \delta_\varrho)_s$ such that $r = \mathrm{pr}_{1,\dots,q}(t)$; therefore $t = (r_1, \dots, r_q, r'_1, \dots, r'_m)$ for some $r' \in R'_s$. Since $t \in \delta_\varrho$, we have $r' = \mathrm{pr}_{h(1),\dots,h(m)} r$. But now

$$\begin{aligned} r' &= \mathrm{pr}_{h(1),\dots,h(m)} r = \mathrm{pr}_{h(1),\dots,h(m)} f_r(\mathbf{X}_w) \\ &= f_r(\mathrm{pr}_{h(1),\dots,h(m)} \mathbf{X}_w) = f_r(\mathbf{M}) \notin R'_s. \end{aligned}$$

This gives us the desired contradiction. □

Theorem 4.16. *Let $A = (A_s)_{s \in S}$ be an S-sorted set in which every set A_s is finite. Let $Q \subseteq \mathcal{Q}_A$. Then $[Q]_{\mathrm{mc}} = \mathrm{mInv}\, \mathrm{mPol}\, Q$. Consequently, Q is minor-closed if and only if $Q = \mathrm{mInv}\, \mathrm{mPol}\, Q$.*

Proof. For any operation f, we have that $\mathrm{mInv}\{f\}$ is minor-closed. Thus, $f \in \mathrm{mPol}\, Q$ if and only if $f \in \mathrm{mPol}[Q]_{\mathrm{mc}}$, hence $\mathrm{mPol}\, Q = \mathrm{mPol}[Q]_{\mathrm{mc}} =: F$. Applying Lemma 4.15 to the minor-closed class $[Q]_{\mathrm{mc}}$, we obtain

$$[\{(\chi_w, \Gamma_F(\chi_w)) \mid w \in W(S)\}]_{\mathrm{mc}} \subseteq [Q]_{\mathrm{mc}}.$$

On the other hand, Lemma 4.14 implies that

$$\mathrm{mInv}\, F = [\{(\chi_w, \Gamma_F(\chi_w)) \mid w \in W(S)\}]_{\mathrm{mc}}.$$

Therefore, we have $\mathrm{mInv}\, F \subseteq [Q]_{\mathrm{mc}}$. We can conclude that

$$[Q]_{\mathrm{mc}} \subseteq \mathrm{mInv}\, \mathrm{mPol}[Q]_{\mathrm{mc}} = \mathrm{mInv}\, \mathrm{mPol}\, Q = \mathrm{mInv}\, F \subseteq [Q]_{\mathrm{mc}},$$

where the first inclusion follows from the fact that mInv mPol is a closure operator. □

Remark 4.17. We developed the Galois theory of minor-closed classes of multisorted operations under the assumption that the components A_s of the S-sorted set $A = (A_s)_{s \in S}$ are finite. Should we like to abandon the finiteness assumption, it would seem necessary to introduce certain local closure conditions, much in the same way as in Couceiro and Foldes's [2] extension of Pippenger's Galois theory to arbitrary, possibly infinite sets. This remains beyond the scope of the current paper.

5. Reflections and invariant relation pairs

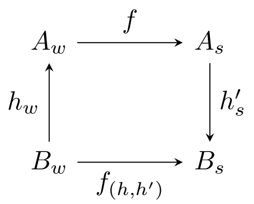
We are now going to generalize the notion of reflection (see Barto, Opršal and Pinsker [1]) to the multisorted setting.

Definition 5.1. Let A and B be S-sorted sets. A *reflection* of A into B is a pair (h, h') of S_B-sorted mappings $h = (h_s)_{s \in S_B}$, $h' = (h'_s)_{s \in S_B}$, $h_s \colon B_s \to A_s$, $h'_s \colon A_s \to B_s$. Note that reflections of A into B exist if and only if $S_B \subseteq S_A$. For, if $S_B \subseteq S_A$, then A_s and B_s are nonempty for all $s \in S_B$ and there clearly exist maps $h_s \colon B_s \to A_s$ and $h'_s \colon A_s \to B_s$. If $S_B \nsubseteq S_A$, then there is $s \in S_B \setminus S_A$, whence $A_s = \emptyset$ and $B_s \neq \emptyset$, so there is no map $h_s \colon B_s \to A_s$.

Assume that A and B are S-sorted sets with $S_B \subseteq S_A$ and (h, h') is a reflection of A into B. If $(w, s) \in W(S) \times S$ is a declaration that is reasonable in both A and B and $f \colon A_w \to A_s$, then we can define the (h, h')*-reflection* of f to be the map $f_{(h,h')} \colon B_w \to B_s$ that is the empty map if $B_w = \emptyset$ and is otherwise given by the rule

$$f_{(h,h')}(b_1, \dots, b_n) = h'_s(f(h_{w_1}(b_1), \dots, h_{w_n}(b_n)))$$

for all $(b_1, \dots, b_n) \in B_w$, which we may write in a simpler way as $f_{(h,h')}(\mathbf{b}) = h'_s(f(h_w(\mathbf{b})))$ for all $\mathbf{b} \in B_w$. This is illustrated by the commutative diagram shown below.



Let $F \subseteq \mathcal{F}_A$ be a set of S-sorted operations on A. If $\operatorname{dec}(f)$ is reasonable in B for all $f \in F$, then the (h, h')*-reflection* of F is defined as $F_{(h,h')} := \{f_{(h,h')} \mid f \in F\}$.

Proposition 5.2. *Let A and B be S-sorted sets. Let $F \subseteq \mathcal{F}_A$, and let (h, h') be a reflection of A into B such that $F_{(h,h')}$ is defined. If F is minor-closed, then $F_{(h,h')}$ is minor-closed.*

Proof. Let $g \in F_{(h,h')}$, with $\operatorname{dec}(g) = (w, s)$. Then $g = f_{(h,h')}$ for some $f \in F_A^{(w,s)}$. Any minor of g is of the form g^u_λ, where $u = u_1 \dots u_m \in W(S)$ is a word such that $\{w_1, \dots, w_n\} \subseteq \{u_1, \dots, u_m\}$ and $\lambda \colon [n] \to [m]$ is a map satisfying $w_i = u_{\lambda(i)}$ for all $i \in [n]$ (see Definition 3.2). Then for all $(b_1, \dots, b_m) \in B_u$,

$$\begin{aligned} g^u_\lambda(b_1, \dots, b_m) &= g(b_{\lambda(1)}, \dots, b_{\lambda(n)}) = f_{(h,h')}(b_{\lambda(1)}, \dots, b_{\lambda(n)}) \\ &= h'_s(f(h_{w_1}(b_{\lambda(1)}), \dots, h_{w_n}(b_{\lambda(n)}))) \\ &= h'_s(f^u_\lambda(h_{u_1}(b_1), \dots, h_{u_m}(b_m))) = (f^u_\lambda)_{(h,h')}(b_1, \dots, b_m). \end{aligned}$$

Since F is minor-closed, we have $f^u_\lambda \in F$. Hence $g^u_\lambda = (f^u_\lambda)_{(h,h')} \in F_{(h,h')}$. □

Suppose $F \subseteq \mathcal{F}_A$ is a minor-closed class. Proposition 5.2 asserts that any reflection $F_{(h,h')}$ is minor-closed. Theorem 4.10 guarantees that there exists a set $Q \subseteq \mathcal{Q}_B$ of relation pairs such that $F_{(h,h')} = \operatorname{mPol} Q$, but the obvious question is how to find such a set Q if we are given $\operatorname{mInv} F$. We are now going to describe how the invariant relation pairs of S-sorted operations are affected by reflections.

Definition 5.3. Let A and B be S-sorted sets, let $h \colon A \to B$ be an S'-sorted mapping for some $S' \subseteq S$, let R be an m-ary S-sorted relation on A, and let T be an m-ary S-sorted relation on B. The *direct image* $h(R)$ of R under h and the *inverse image* $h^{-1}(T)$ of T under h are defined as follows. If $m \geq 1$, then $h(R) := (h_s(R_s))_{s \in S}$ and $h^{-1}(T) := (h_s^{-1}(T_s))_{s \in S}$, where

$$\begin{aligned} h_s(R_s) &:= \{(h_s(a_1), \dots, h_s(a_m)) \in B_s^m \mid (a_1, \dots, a_m) \in R_s\}, \\ h_s^{-1}(T_s) &:= \{(a_1, \dots, a_m) \in A_s^m \mid (h_s(a_1), \dots, h_s(a_m)) \in T_s\}, \end{aligned}$$

for $s \in S'$, and $h_s(R_s) := \emptyset$, $h_s^{-1}(T_s) := \emptyset$ for $s \in S \setminus S'$. If $m = 0$, then $h(R) := R$ and $h^{-1}(T) := T$.

Proposition 5.4. *Let A and B be S-sorted sets, $(R, R') \in \mathcal{Q}_A$, $(T, T') \in \mathcal{Q}_B$, and let (h, h') be a reflection of A into B. Assume that (w, s) is reasonable in both A and B, and let $f \in \mathcal{F}_A^{(w,s)}$. Then the following statements hold.*

(i) *If $f \rhd (R, R')$ then $f_{(h,h')} \rhd (h^{-1}(R), h'(R'))$.*

(ii) *If $f_{(h,h')} \rhd (T, T')$ then $f \rhd (h(T), h'^{-1}(T'))$.*

(iii) *If $F \subseteq \mathcal{F}_A$ and $\operatorname{dec}(f)$ is reasonable in B for all $f \in F$, then*

$$\operatorname{mInv} F_{(h,h')} = \{(T, T') \in \mathcal{Q}_B \mid (h(T), h'^{-1}(T')) \in \operatorname{mInv} F\}.$$

Proof. (i) The claim clearly holds if (R, R') is nullary, so we assume that (R, R') has arity at least 1. Assume that $f \rhd (R, R')$ and $\mathbf{M} \prec_w h^{-1}(R)$. Then $\mathbf{M} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$, where $\mathbf{a}_i \in h_{w_i}^{-1}(R_{w_i})$ for $i \in [n]$. Then $h_{w_i}(\mathbf{a}_i) \in R_{w_i}$ for all $i \in [n]$, so $h_w(\mathbf{M}) := (h_{w_1}(\mathbf{a}_1), \dots, h_{w_n}(\mathbf{a}_n)) \prec_w R$. Since $f \rhd (R, R')$, we have $f(h_w(\mathbf{M})) \in R'_s$. Thus $f_{(h,h')}(\mathbf{M}) = h'_s(f(h_w(\mathbf{M}))) \in h'_s(R'_s)$, and we conclude that $f_{(h,h')} \rhd (h^{-1}(R), h'(R'))$.

(ii) Again, the case of nullary relations is clear, so we assume that (T, T') has arity at least 1. Assume that $f_{(h,h')} \rhd (T, T')$ and $\mathbf{M} \prec_w h(T)$. Then $\mathbf{M} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$, where $\mathbf{a}_i \in h_{w_i}(T_{w_i})$ for $i \in [n]$. Then for each i there exists $\mathbf{b}_i \in T_{w_i}$ such that $\mathbf{a}_i = h_{w_i}(\mathbf{b}_i)$. Consequently, $(\mathbf{b}_1, \dots, \mathbf{b}_n) \prec_w T$. Since $f_{(h,h')} \rhd (T, T')$, we have $f_{(h,h')}(\mathbf{b}_1, \dots, \mathbf{b}_n) \in T'_s$. Since

$$\begin{aligned} f_{(h,h')}(\mathbf{b}_1, \dots, \mathbf{b}_n) &= h'_s(f(h_{w_1}(\mathbf{b}_1), \dots, h_{w_n}(\mathbf{b}_n))) \\ &= h'_s(f(\mathbf{a}_1, \dots, \mathbf{a}_n)) = h'_s(f(\mathbf{M})), \end{aligned}$$

we have $f(\mathbf{M}) \in {h'_s}^{-1}(T'_s)$, and we conclude that $f \rhd (h(T), h'^{-1}(T'))$.

(iii) The inclusion

$$\operatorname{mInv} F_{(h,h')} \subseteq \{(T, T') \in \mathcal{Q}_B \mid (h(T), h'^{-1}(T')) \in \operatorname{mInv} F\}$$

follows immediately from part (ii). In order to prove the converse inclusion, assume that $(T, T') \in \mathcal{Q}_B$ satisfies $(h(T), h'^{-1}(T')) \in \operatorname{mInv} F$. Then $(h^{-1}(h(T)), h'(h'^{-1}(T'))) \in \operatorname{mInv} F_{(h,h')}$ by part (i). Since $T \subseteq h^{-1}(h(T))$ and $T' \supseteq h'(h'^{-1}(T'))$ and since $\operatorname{mInv} F_{(h,h')}$ is closed under restrictions of antecedents and extensions of consequents, we have that $(T, T') \in \operatorname{mInv} F_{(h,h')}$. □

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