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ON ASSOCIATIVE SPECTRA OF OPERATIONS

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Abstract. The distance of an operation from being associative can be "measured" by its associative spectrum, an appropriate sequence of positive integers. Associative spectra were introduced in a publication by B. Csákány and T. Waldhauser in 2000 for binary operations (see [1]). We generalize this concept to $2 \leq p$-ary operations, interpret associative spectra in terms of equational theories, and use this interpretation to find a characterization of fine spectra, to construct polynomial associative spectra, and to show that there are continuum many different spectra. Furthermore, an equivalent representation of bracketings is studied.

1. Introduction

B. Csákány and T. Waldhauser introduced associative spectra for binary operations in[1]. The main focus point in their paper was the spectrum of groupoids with two or three elements.

In this paper, we generalize first in Section 2 the definition of associative spectrum to $2 \leq p$-ary operations with the help of special unary terms, which will be called bracketings. Enumerations are used to distinguish between the variable symbols in a bracketing. Using these enumarations it is possible to define a reduct $\mathrm{Mod}_{\mathrm{Brack}} - \mathrm{Id}_{\mathrm{Brack}}$ of the well-known Galois-connection $\mathrm{Mod} - \mathrm{Id}$. The Galois-closed sets on the side of the identities are called fine spectra, which is a refinement of the notion of associative spectrum. Finally, some useful operations on bracketings are defined which are needed in the characterization of fine spectra. In Section 3 we give the characterization of fine spectra and a first application of it, a generalization of the generalized associative law. After that, insertion tuples are developed as an equivalent representation of bracketings in Section 4. With the help of these tuples the explicit formula of the generalized Catalan numbers is proven, where the generalized Catalan numbers count bracketings of a given length. In Section 5 three different polynomial spectra are presented which solve Problem 3 in [1]. The lattice of fine spectra is studied in Section 6. The covering relation, the atoms and the coatoms of this lattice are described. Furthermore, it is shown that there are continuum many different spectra. In Section 7 we look at some examples of finite groupoids (one of them has a polynomial spectrum from Section 5). It is shown that every finally associative spectrum appears as the fine spectrum of a finite groupoid. We conclude in Section 8 with the formulation of a few open problems.

2. Definitions and notation

The *algebra of p-ary bracketings* is defined as the term algebra

$$\mathbf{T}^{(\mathbf{p})} := \left(T_\omega(x), \omega^{\mathbf{T}^{(\mathbf{p})}}\right),$$

where p is a natural number greater or equal to 2, the alphabet is $\{x\}$ and the signature is $\{\omega\}$ with ω as a p-ary operation symbol.

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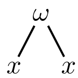
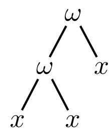
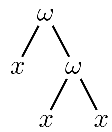
	$B_0^{(2)}$	$B_1^{(2)}$	$B_2^{(2)}$	
tree	x			
bracketing	x	ωxx	$\omega\omega xxx$	$\omega x\omega xx$
bracketing (infix)	x	(xx)	$((xx)x)$	$(x(xx))$
insertion tuple	$\emptyset$	(1)	$(1,1)$	$(1,2)$

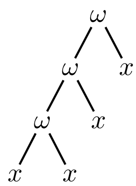
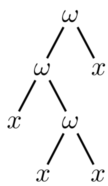
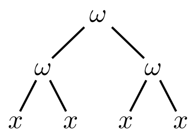
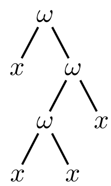
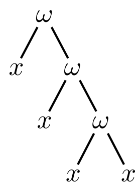
	$B_3^{(2)}$				
tree					
bracketing	$\omega\omega\omega xxxx$	$\omega\omega x\omega xxx$	$\omega\omega xx\omega xx$	$\omega x\omega\omega xxx$	$\omega x\omega x\omega xx$
bracketing (infix)	$(((xx)x)x)$	$((x(xx))x)$	$((xx)(xx))$	$(x((xx)x))$	$(x(x(xx)))$
insertion tuple	$(1,1,1)$	$(1,1,2)$	$(1,1,3)$	$(1,2,2)$	$(1,2,3)$

TABLE 1. Binary bracketings, their tree correspondences and their insertion tuples

We call the (unary) terms $t \in T_\omega(x)$ *p-ary bracketings* or simply bracketings if p is known. The *occurrence number* $|t|_\omega$ of a bracketing $t \in T_\omega(x)$ is defined as the number of occurrences of the operation symbol ω in t. The following trivial equalities hold:

$$|x|_\omega = 0 \quad \text{and} \quad \forall t_1, \dots, t_p \in T_\omega(x) : \; |\omega t_1 \dots t_p|_\omega = 1 + \sum_{k=1}^{p} |t_k|_\omega .$$

The *length* $|t|$ of a bracketing $t \in T_\omega(x)$ is defined as the number of occurrences of the variable symbol x in t. It is an easy observation to show that $|t| = (p-1) \cdot |t|_\omega + 1$ holds for all bracketings $t \in T_\omega(x)$. The length could be defined recursively too:

$$|x| = 1 \quad \text{and} \quad \forall t_1, \dots, t_p \in T_\omega(x) : \; |\omega t_1 \dots t_p| = \sum_{k=1}^{p} |t_k| .$$

Bracketings with occurrence number n can be viewed as trees with branching factor p, n inner nodes and $(p-1) \cdot n + 1$ leafs (because the symbols ω are the inner nodes and the symbols x are the leafs). We denote by

$$B_n^{(p)} := \{\, t \in T_\omega(x) \mid |t|_\omega = n \}$$

the set of all bracketings with occurrence number n. The *length function*, which transforms occurrence numbers into lengths, is given by

$$\begin{array}{rccc} \ell^{(p)} : & \mathbb{N} & \longrightarrow & \mathbb{N}^+ \\ & n & \longmapsto & (p-1) \cdot n + 1. \end{array}$$

For example, the first binary bracketings are given in Table 1 (insertion tuples are defined in Definition 4.2).

In the next step we want to distinguish between the variable symbols x in a bracketing. Therefore, we define the *enumerations* $\varepsilon_j : T_\omega(x) \longrightarrow T_\omega(X)$ by term induction as follows, where $\mathbf{T^{(p)}}(\mathbf{X}) = \left(T_\omega(X), \omega^{\mathbf{T^{(p)}}(\mathbf{X})}\right)$ is the term algebra over the alphabet $X = \{x_i \mid i \in \mathbb{N}^+\}$:

- $\forall j \in \mathbb{N}^+ : \varepsilon_j(x) = x_j$;
- $\forall t_1, \ldots, t_p \in T_\omega(x)\ \forall j \in \mathbb{N}^+ : \varepsilon_j(\omega t_1 \ldots t_p) = \omega \varepsilon_{j_1}(t_1) \ldots \varepsilon_{j_p}(t_p)$ with $j_m := j + \sum_{k=1}^{m-1} |t_k|$.

It is obvious that $\varepsilon_j(t)$ contains exactly the variable symbols $\left\{x_j, \ldots, x_{j+|t|-1}\right\}$. As an example we look again at some binary bracketings:

$$\varepsilon_j(\omega\omega xxx) = \omega\omega x_j x_{j+1} x_{j+2}\ ,\ \ \varepsilon_j(\omega x \omega xx) = \omega x_j \omega x_{j+1} x_{j+2}.$$

For a simpler notation we denote by

$$\Lambda^{(p)} := \{ (s,t) \in T_\omega(x) \times T_\omega(x) \mid |s|_\omega = |t|_\omega \}$$

the relation of all bracketings with the same occurrence number. It is an easy observation that $\Lambda^{(p)}$ is a congruence relation of $\mathbf{T^{(p)}}$. Further on, we will denote pairs $(s,t) \in \Lambda^{(p)}$ simply by $s \approx t$ and we will call them *identities*. From the example above we know that these identities can be interpreted via enumaration as generalized associativity conditions.

We call an algebra $\mathbf{A}$ to the signature $\{\omega\}$ a p-ary groupoid and denote it by $\mathbf{A} \in \mathrm{Alg}(\omega)$. Now we can define a reduct of the well-known Galois-connection $\mathrm{Mod} - \mathrm{Id}$. Let $\models_{\mathrm{Brack}} \subseteq \mathrm{Alg}(\omega) \times \Lambda^{(p)}$ be defined as

$$\mathbf{A} \models_{\mathrm{Brack}} s \approx t :\Longleftrightarrow \mathbf{A} \models (\varepsilon_1(s), \varepsilon_1(t)).$$

Because of the full invariance of $\mathrm{Id}\,\mathbf{A}$ it is obvious that

$$\mathbf{A} \models_{\mathrm{Brack}} s \approx t \Longrightarrow \forall j \in \mathbb{N}^+ :\ \mathbf{A} \models (\varepsilon_j(s), \varepsilon_j(t)).$$

The Galois-closed sets are given for any $\Sigma \subseteq \Lambda^{(p)}$ and $\mathcal{K} \subseteq \mathrm{Alg}(\omega)$ by

- $\mathrm{Mod}_{\mathrm{Brack}}\,\Sigma := \{\mathbf{A} \in \mathrm{Alg}(\omega) \mid \forall s \approx t \in \Sigma :\ \mathbf{A} \models_{\mathrm{Brack}} s \approx t\}$, which is of course a special variety that has additional properties (see open problems);
- $\mathrm{Id}_{\mathrm{Brack}}\,\mathcal{K} := \left\{ s \approx t \in \Lambda^{(p)} \,\middle|\, \forall \mathbf{A} \in \mathcal{K} :\ \mathbf{A} \models_{\mathrm{Brack}} s \approx t \right\}$, which is a reduct of the equational theory of $\mathcal{K}$.

We will further on denote any $\Sigma \subseteq \Lambda^{(p)}$ equivalently by the sequence

$$(\Sigma_n)_{n\in\mathbb{N}} := \left(\Sigma \cap (B_n^{(p)} \times B_n^{(p)})\right)_{n\in\mathbb{N}}.$$

For a p-ary groupoid $\mathbf{A}$ we define two different spectra:

- the *fine spectrum* of $\mathbf{A}$: $\sigma(\mathbf{A}) := \mathrm{Id}_{\mathrm{Brack}}\,\mathbf{A}$, or equivalently (see above) $(\sigma_n(\mathbf{A}))_{n\in\mathbb{N}}$ with $\sigma_n(\mathbf{A}) = \sigma(\mathbf{A}) \cap (B_n^{(p)} \times B_n^{(p)})$;
- the *associative spectrum* of $\mathbf{A}$: $(s_n(\mathbf{A}))_{n\in\mathbb{N}} := \left(\left|B_n^{(p)}/\sigma_n(\mathbf{A})\right|\right)_{n\in\mathbb{N}}$.

We say that $\mathbf{A}$ is *associative* iff $s_2(\mathbf{A}) = 1$. The following two observations are trivial.

Proposition 2.1. *If* $\mathbf{A} \in \mathrm{Alg}(\omega)$ *is a subgroupoid or a homomorphic image of* $\mathbf{B} \in \mathrm{Alg}(\omega)$*, then*

$$\sigma(\mathbf{A}) \supseteq \sigma(\mathbf{B}) \quad \textit{and} \quad \forall n \in \mathbb{N} :\ s_n(\mathbf{A}) \leq s_n(\mathbf{B}).$$

Proposition 2.2. *If* $\mathbf{A} \in \mathrm{Alg}(\omega)$ *and* $\mathbf{B} \in \mathrm{Alg}(\omega)$ *are isomorphic or antiisomorphic, then their spectra coincide:*

$$\sigma(\mathbf{A}) = \sigma(\mathbf{B}).$$

Finally, we define some useful operations for bracketings:

- $\gamma_i : T_\omega(x) \longrightarrow T_\omega(x)$ $(i = 1, \ldots, p)$ is defined as

$$\begin{array}{rccc} \gamma_i : & T_\omega(x) & \longrightarrow & T_\omega(x) \\ & t & \longmapsto & \omega t_1 \ldots t_p \end{array}$$

  with $t_i = t$ and $t_k = x$ for all $k \in \{1, \ldots, p\} \setminus \{i\}$. So $\gamma_i(t) = \omega x \ldots x\, t\, x \ldots x$ is the insertion of t at the i-th position in $\omega x \ldots x$.

- For the definition of β_i ($i \in \mathbb{N}^+$) we need some auxiliary functions α_i:
$$\begin{array}{rccl} \alpha_i: & X & \longrightarrow & T_\omega(x) \\ & x_k & \longmapsto & \begin{cases} \omega x \dots x \in B_1^{(p)}, & \text{if } k = i; \\ x, & \text{otherwise.} \end{cases} \end{array}$$
Denote (here and further on) by $\alpha_i^\# : T_\omega(X) \longrightarrow T_\omega(x)$ the unique homomorphism that continues α_i. Then β_i is defined as
$$\begin{array}{rccl} \beta_i: & T_\omega(x) & \longrightarrow & T_\omega(x) \\ & t & \longmapsto & \alpha_i^\# \left(\varepsilon_1 \left(t\right)\right). \end{array}$$
So $\beta_i(t)$ is the insertion of $\omega x \dots x$ at the i-th symbol x in t (if present).

It is easy to check that for any bracketing $t \in B_n^{(p)}$ the resulting bracketings $\gamma_i(t)$ ($i = 1, \dots, p$) and $\beta_i(t)$ ($i = 1, \dots, \ell^{(p)}(n)$) are in $B_{n+1}^{(p)}$.

To put these operators together we define for any positive natural number $n \in \mathbb{N}$ the *implication operator* δ_n as follows:
$$\begin{array}{rccl} \delta_n: & \operatorname{Eq} B_n^{(p)} & \longrightarrow & \operatorname{Eq} B_{n+1}^{(p)} \\ & \pi & \longmapsto & \left(\bigcup_{\xi \in \{\gamma_1, \dots, \gamma_p, \beta_1, \dots, \beta_{\ell^{(p)}(n)}\}} \{\xi(s) \approx \xi(t) \mid s \approx t \in \pi\}\right)^*, \end{array}$$
where $\operatorname{Eq} B_n^{(p)}$ denotes the set of equivalence relations on $B_n^{(p)}$ and τ^* denotes the transitive closure of τ.

3. Characterization of fine spectra

Our main goal is to characterize the Galois-closed sets $\operatorname{Id}_{\mathrm{Brack}} \mathcal{K}$ and the fine spectra of arbitrary groupoids.

First we need three preparatory lemmata. The first one shows a recursion formula for the operators β_i.

Lemma 3.1. For all $k \in \{1, \dots, p\}$ and for all $t_1, \dots, t_p \in T_\omega(x)$ with $i \in \{1, \dots, |t_k|\}$ we have:
$$\omega\, t_1 \dots t_{k-1}\, \beta_i(t_k)\; t_{k+1} \dots t_p = \beta_j(\omega t_1 \dots t_p),$$
where $j := i + \sum_{l=1}^{k-1} |t_l|$.

The proof is left to the reader; it is just a transformation of the insertion index. The next statement shows that all bracketings can be obtained with the operators β_i starting with x.

Lemma 3.2. For all $n \in \mathbb{N}$ we have
$$B_{n+1}^{(p)} = \left\{\beta_i(t) \;\middle|\; t \in B_n^{(p)}, i = 1, \dots, \ell^{(p)}(n)\right\}.$$

This follows directly from the previous lemma by induction on n.

Lemma 3.3. If $\Sigma \subseteq \Lambda^{(p)}$ is an equivalence relation that is closed under the implication operator, i.e. $\forall n \in \mathbb{N}: \ \delta_n(\Sigma_n) \subseteq \Sigma_{n+1}$, then
$$s \approx t \in \Sigma \Longrightarrow \alpha^\#(\varepsilon_1(s)) \approx \alpha^\#(\varepsilon_1(t)) \in \Sigma$$
holds for all $\alpha : X \longrightarrow T_\omega(x)$.

Proof. We choose an arbitrary but fixed identity $s \approx t \in \Sigma$. Then we apply induction on $n := \sum_{i=1}^{|s|} |\alpha(x_i)|_\omega$: For $n = 0$, $\alpha : X \longrightarrow T_\omega(x)$ must map each x_i on x ($i = 1, \dots, |s|$), thus
$$\alpha^\#(\varepsilon_1(s)) \approx \alpha^\#(\varepsilon_1(t)) = s \approx t \in \Sigma.$$
For the induction step from n to $n+1$, let $k \in \{1, \dots, |s|\}$ be a position where the occurrence number is greater than 0, i.e. $|\alpha(x_k)|_\omega > 0$. By the previous Lemma 3.2, we can find a bracketing t_k and a natural number $j \in \{1, \dots, |t_k|\}$ such that
$$\beta_j(t_k) = \alpha(x_k).$$

Now we can define a reduct of α:

$$\begin{array}{rccl} \tilde{\alpha}: & X & \longrightarrow & T_\omega(x) \\ & x_i & \longmapsto & \begin{cases} \alpha\left(x_i\right), & \text{if } i \neq k; \\ t_k, & \text{if } i = k. \end{cases} \end{array}$$

It is easy to see that $\sum_{i=1}^{|s|} \left|\tilde{\alpha}\left(x_i\right)\right|_\omega = n$ holds. With $l := j + \sum_{m=1}^{k-1} \left|\alpha\left(x_m\right)\right|$ it can be shown with the help of Lemma 3.1 that

$$\beta_l \circ \tilde{\alpha}^{\#} \circ \varepsilon_1 = \alpha^{\#} \circ \varepsilon_1$$

holds. Then, together with the induction hypothesis for $\tilde{\alpha}^{\#}$ and the prerequisite that Σ is closed under the implication operator, we get $\alpha^{\#}\left(\varepsilon_1(s)\right) \approx \alpha^{\#}\left(\varepsilon_1(t)\right) \in \Sigma$. $\square$

Now we are able to prove our main result, which is an analogon of the well-known characterization of equational theories.

Theorem 3.4. *For any $\mathcal{K} \subseteq \mathrm{Alg}\,(\omega)$ and $\Sigma \subseteq \Lambda^{(p)}$ the following hold:*

(a) *If Σ is an equivalence relation that is closed under the implication operator then Σ is a congruence of $\mathbf{T}^{(\mathbf{p})}$.*
(b) *$\mathrm{Id}_{\mathrm{Brack}}\,\mathcal{K}$ is closed under the implication operator.*
(c) *If Σ is an equivalence relation that is closed under the implication operator then $\mathrm{Id}_{\mathrm{Brack}}\left\{\mathbf{T}^{(\mathbf{p})}/\Sigma\right\} = \Sigma$.*

Proof.

(a) It suffices to show that for all $i \in \{1, \ldots, p\}$, $s \approx t \in \Sigma$ and $t_1, \ldots, t_p \in T_\omega(x)$ we have

$$\omega t_1 \ldots t_{i-1}\, s\, t_{i+1} \ldots t_p \approx \omega t_1 \ldots t_{i-1}\, t\, t_{i+1} \ldots t_p \in \Sigma.$$

(The general case follows then by applying this rule repeatedly on each position). We know that Σ is closed under the implication operator, so:

$$\gamma_i(s) \approx \gamma_i(t) = \omega x \ldots x\, s\, x \ldots x \approx \omega x \ldots x\, t\, x \ldots x \in \Sigma.$$

According to Lemma 3.2 we have a sequence of natural numbers $i_1, \ldots, i_{|t_p|_\omega}$ such that

$$t_p = \beta_{i_{|t_p|_\omega}} \circ \cdots \circ \beta_{i_1}\left(x\right).$$

So it follows from Lemma 3.1 with $j_k := i_k + p - 2 + |s|$ that

$$\begin{aligned} \beta_{j_{|t_p|_\omega}} \circ \cdots \circ \beta_{j_1}\left(\gamma_i(s)\right) &= \omega x \ldots x\, s\, x \ldots x \left(\beta_{i_{|t_p|_\omega}} \circ \cdots \circ \beta_{i_1}\left(x\right)\right) \\ &= \omega x \ldots x\, s\, x \ldots x\, t_p \end{aligned}$$

holds. Similarly we have $\beta_{j_{|t_p|_\omega}} \circ \cdots \circ \beta_{j_1}\left(\gamma_i(t)\right) = \omega x \ldots x\, t\, x \ldots x\, t_p$ and since Σ is closed under the implication operator, we get

$$\omega x \ldots x\, s\, x \ldots x\, t_p \approx \omega x \ldots x\, t\, x \ldots x\, t_p \in \Sigma.$$

This construction step can be repeated, and finally we obtain

$$\omega t_1 \ldots t_{i-1}\, s\, t_{i+1} \ldots t_p \approx \omega t_1 \ldots t_{i-1}\, t\, t_{i+1} \ldots t_p \in \Sigma.$$

(b) This result follows obviously from the fact that $\mathrm{Id}\,\mathcal{K}$ is always a fully invariant congruence relation of $T_\omega(X)$ with the help of the following two equations:

$$\begin{aligned} \varepsilon_1\left(\gamma_i\left(t\right)\right) &= \omega x_1 \ldots x_{i-1}\, \varepsilon_i(t)\, x_{i+|t|} \ldots x_{p-1+|t|}, \\ \varepsilon_1\left(\beta_i\left(t\right)\right) &= \tilde{\alpha}_i^{\#}\left(\varepsilon_1(t)\right), \end{aligned}$$

where

$$\begin{array}{rccl} \tilde{\alpha}_i: & X & \longrightarrow & T_\omega(X) \\ & x_k & \longmapsto & \begin{cases} x_k, & \text{if } k < i; \\ \omega x_k \ldots x_{k+p-1}, & \text{if } k = i; \\ x_{k+p-1}, & \text{if } k > i. \end{cases} \end{array}$$

(c) We investigate two cases: "$\subseteq$": Let $s \approx t \in \mathrm{Id}_{\mathrm{Brack}}\left\{\mathbf{T}^{(\mathbf{p})}/\Sigma\right\}$. Then we have $\mathbf{T}^{(\mathbf{p})}/\Sigma \models \varepsilon_1(s) \approx \varepsilon_1(t)$. With the full invariance of $\mathrm{Id}\left\{\mathbf{T}^{(\mathbf{p})}/\Sigma\right\}$ and the function

$$\begin{array}{rccc} \alpha: & X & \longrightarrow & \mathbf{T}^{(\mathbf{p})}/\Sigma \\ & x_k & \longmapsto & [x] \end{array}$$

that maps each x_k to the equivalence class of x in $T_\omega(x)/\Sigma$ it follows that $[s] = \alpha^{\#}(\varepsilon_1(s)) = \alpha^{\#}(\varepsilon_1(t)) = [t]$. Therefore, $s \approx t$ is in Σ.

"$\supseteq$": Follows directly with Lemma 3.3.

□

The following corollary summarizes the last theorem.

Corollary 3.5. *For $\Sigma \subseteq \Lambda^{(p)}$ the following are equivalent:*

(a) Σ *is an equivalence relation and* Σ *is closed under the implication operator.*
(b) $\mathrm{Id}_{\mathrm{Brack}}\,\mathrm{Mod}_{\mathrm{Brack}}\,\Sigma = \Sigma$.
(c) *There exists a groupoid* $\mathbf{A}$ *such that* $\sigma(\mathbf{A}) = \Sigma$.

As a first application of our main result we show that the general associative law and a generalization of it hold.

Theorem 3.6. *For any p-ary groupoid $\mathbf{A}$ the following hold:*

(a) $s_0(\mathbf{A}) = s_1(\mathbf{A}) = 1$;
(b) $\mathbf{A}$ *is associative* $\Longleftrightarrow \forall n \in \mathbb{N}:\ s_n(\mathbf{A}) = 1$;
(c) $s_n(\mathbf{A}) = 1 \Longrightarrow \forall m \in \mathbb{N}, m \geq n:\ s_m(\mathbf{A}) = 1$ *for any* $n \geq 2$.

If the conclusion of (c) is fulfilled then we say that $\mathbf{A}$ *and the associative spectrum of* $\mathbf{A}$ *are finally associative.*

Proof.

(a) Absolutely clear because $B_0^{(p)} = \{x\}$ and $B_1^{(p)} = \{\omega x \dots x\}$.
(b) The direction "$\Longleftarrow$" is clear from the definition in Section 2. For the other direction we suppose that $\mathbf{A}$ is associative. For an arbitrary natural number $n \geq 2$ let $t \in B_n^{(p)}$ be the left associated bracketing $\omega \dots \omega x \dots x$, which means that all symbols ω occur before the x's. Assume we have another bracketing $s \in B_n^{(p)}$. We will prove that $\mathbf{A} \models_{\mathrm{Brack}} s \approx t$. Since s is not left associated, it has a subbracketing of the form $\omega(t_1 \dots t_k\, \omega(s_1 \dots s_p) \dots)$, where $k \geq 1$. Then we have by associativity

$$\mathbf{A} \models_{\mathrm{Brack}} \omega(t_1 \dots t_k\, \omega(s_1 \dots s_p) \dots) \approx \omega(\omega(t_1 \dots t_k\, s_1 \dots s_{p-k})\, s_{p-k+1} \dots s_p \dots).$$

This way one ω is moving to the left, and after finitely many such steps we reach t, i.e. $\mathbf{A} \models_{\mathrm{Brack}} s \approx t$.
(c) We show this fact via contradiction: Assume that there exists a groupoid $\mathbf{A} \in \mathrm{Alg}(\omega)$ with fine spectrum $\sigma(\mathbf{A})$ that has the property

$$\exists n \in \mathbb{N}, n \geq 2:\ s_n(\mathbf{A}) = 1 \ \text{ and } \ \exists m \in \mathbb{N}, m > n:\ s_m(\mathbf{A}) > 1.$$

Then define the sequence $(\Sigma_i)_{i \in \mathbb{N}}$ in $\Lambda^{(p)}$ as follows:

$$\Sigma_i := \begin{cases} B_i^{(p)} \times B_i^{(p)}, & \text{if } i < n; \\ \sigma_i(\mathbf{A}), & \text{otherwise.} \end{cases}$$

It is easy to see that the corresponding $\Sigma \subseteq \Lambda^{(p)}$ is an equivalence relation that is closed under the implication operator. Therefore, there exists a groupoid $\mathbf{B} \in \mathrm{Alg}(\omega)$ with $\sigma(\mathbf{B}) = \Sigma$. This is a contradiction to ((b)) because $s_2(\mathbf{B}) = 1$ but $s_m(\mathbf{B}) > 1$.

□

4. Equivalent representation of bracketings

In this section we are going to define an equivalent representation of bracketings. First let us look at the number of bracketings with a given occurrence number.

Definition 4.1. The *generalized* Catalan *numbers* $C_n^{(p)}$ $(n \geq 0, p \geq 1)$ are defined by the following recursion:

- $C_0^{(p)} := 1$;
- $C_n^{(p)} := \sum_{i_1,\dots,i_p \in \mathbb{N}, \sum_{k=1}^{p} i_k = n-1} \left(\prod_{l=1}^{p} C_{i_l}^{(p)} \right)$.

With respect to the definition of the bracketings as terms of $\mathbf{T^{(p)}}$ and the properties of the occurrence number stated in Section 2, $\left| B_n^{(p)} \right| = C_n^{(p)}$ follows for all $n \in \mathbb{N}$. We know from [3] that

$$C_n^{(p)} = \frac{1}{(p-1) \cdot n + 1} \cdot \binom{p \cdot n}{n}$$

holds for all $n \in \mathbb{N}$. We will prove this in a more general form in Theorem 4.6.

With the same idea as above and the fact that $\sigma(\mathbf{A})$ is a congruence relation in $\mathbf{T^{(p)}}$ we see that for any groupoid $\mathbf{A}$:

$$\forall n \in \mathbb{N}^+ : \ s_n(\mathbf{A}) \leq \sum_{i_1,\dots,i_p \in \mathbb{N}, \sum_{k=1}^{p} i_k = n-1} \left(\prod_{l=1}^{p} s_{i_l}(\mathbf{A}) \right).$$

Now we are going to define an equivalent representation for bracketings which will be useful to prove the explicit formula of the generalized Catalan numbers.

Definition 4.2. The *insertion tuple* of a bracketing t is the tuple $\mathrm{IT}(t) \in \mathbb{N}^{|t|_\omega}$ whose i-th entry is one plus the number of x's preceding the i-th symbol ω in (the prefix notation of) t. It can be also defined recursively as follows.

- $\mathrm{IT}(x) := \emptyset \in \mathbb{N}^0$;
- $\mathrm{IT}(\omega t_1 \dots t_p) := \left(1, \mathbf{v}^1, \dots, \mathbf{v}^p\right)$ is the consecutive sequence of 1 and the $\mathbf{v}^i$ where $\mathbf{v}^i$ is the insertion tuple of t_i with an additional shift that is added to each entry of the tuple:

$$\mathbf{v}^i := \mathrm{IT}(t_i) + \sum_{k=1}^{i-1} |t_k| \in \mathbb{N}^{|t_i|_\omega}.$$

The shift is exactly the sum of the lengths of the previous bracketings just as in Lemma 3.1.

The insertion tuples of the first binary bracketings have been presented in Section 2. We introduce the following notation, which we will need later. For any $k \in \mathbb{N}^+$ and $n \in \mathbb{N}$ let

$$M_{n,k}^{(p)} := \{ \mathbf{u} \in \mathbb{N}^n \ \mid 1 \leq u_i \leq (p-1) \cdot (i-1) + k \text{ and } u_i \leq u_j \ \ (1 \leq i \leq j \leq n) \}.$$

The insertion tuples can be characterized as follows.

Proposition 4.3. *For $t \in B_n^{(p)}$ the following hold:*

(a) $\mathrm{IT}(t) \in \mathbb{N}^n$.

(b) *If we know the insertion tuple $\mathbf{u} := \mathrm{IT}(t)$ of t, then we also know the insertion tuple $\mathbf{v} := \mathrm{IT}(\beta_i(t))$ of $\beta_i(t)$ (for $i = 1, \dots, \ell^{(p)}(n)$):*

$$v_q = \begin{cases} u_q, & \text{if } q \in \{1, \dots, l\}; \\ i, & \text{if } q = l+1; \\ u_{q-1} + p - 1, & \text{if } q \in \{l+2, \dots, n+1\}, \end{cases}$$

where $l := \max \{0\} \cup \{ q \in \{1, \dots, n\} \mid u_q \leq i \}$.

(c) $\mathrm{IT}[B_n^{(p)}] = M_{n,1}^{(p)}$

(d) IT *is an injective map.*

(e) *The name insertion tuple is justified, because with* $\mathbf{u} := \mathrm{IT}\,(t)$ *and the two definitions* $t_0 := x$, $t_i := \beta_{u_i}\,(t_{i-1})$ *(for* $i \in \{1, \ldots, n\}$*) we have* $t = t_n$.

Proof.

(a) This is clear from the definition.

(b) Remember that $\beta_i(t)$ is the insertion of $\omega x \ldots x$ at the i-th symbol x in t and that $u_q - 1$ equals the number of x's preceding the q-th symbol ω. The statement becomes clear if we observe that l is the position of the last symbol ω having less than i many x's before it.

(c) We investigate two cases: "$\subseteq$": Let $t \in B_n^{(p)}$ be an arbitrary bracketing and let $\mathbf{u} := \mathrm{IT}(t)$. It is clear from the definition that $1 \leq u_i$ and $\mathbf{u}$ is monotone. For the upper bound let us consider the i-th occurrence of the symbol ω in t. If we delete all symbols from this ω to the end of the bracketing then we obtain the prefix of a bracketing with occurrence number $i-1$ with at least one x missing. Therefore, the number of x's preceding this ω is at most $\ell^{(p)}(i-1)-1$. Hence, $u_i \leq \ell^{(p)}(i-1)$.

"$\supseteq$": We use induction on n to show this. The case $n = 0$ is clear, because both sides are $\{\emptyset\}$. For the induction step from n to $n+1$ let $\mathbf{u} \in M_{n+1,1}^{(p)}$. Then $(u_q)_{q=1,\ldots,n} \in M_{n,1}^{(p)}$. By induction hypothesis we have a bracketing $t \in B_n^{(p)}$ such that $\mathrm{IT}\,(t) = (u_q)_{q=1,\ldots,n}$. With the help of ((b)) and the monotonicity of $\mathrm{IT}\,(t)$ we see that $\mathrm{IT}\,\left(\beta_{u_{n+1}}\,(t)\right) = \mathbf{u}$.

(d) The entries of the insertion tuples tell us the positions of the symbols ω.

(e) It can be shown by induction on i with the help of ((b)) that

$$\mathrm{IT}(t_i) = (u_q)_{q=1,\ldots,i} \quad (i = 1, \ldots, n)$$

holds. So we have $\mathrm{IT}(t_n) = \mathbf{u} = \mathrm{IT}(t)$. Then by injectivity $t = t_n$ follows.

□

In [2] a bijection is given between bracketings and certain lattice paths called p-good paths. We invite the reader to find a bijection between p-good paths and insertion tuples. The sets $M_{n,k}^{(p)}$ generalize insertion tuples, and the corresponding paths generalize the p-good paths by shifting the bounding line $k-1$ steps up. Therefore, Theorem 4.6 can be considered as a generalization of Theorem 0.4 from [2].

Lemma 4.4. For any $k \in \mathbb{N}^+$ and $n \in \mathbb{N}$ we have

(a) $|M_{0,k}^{(p)}| = 1$;

(b) $|M_{n+1,k}^{(p)}| = \sum_{l=0}^{k-1} |M_{n,p+l}^{(p)}|$.

Proof. (a) is trivial. For (b) we partition the set $M_{n+1,k}^{(p)}$ into the disjoint subsets

$$S_l := \left\{ \mathbf{u} \in M_{n+1,k}^{(p)} \;\middle|\; u_1 = l \right\} \quad (l = 1, \ldots, k).$$

It is easy to verify that the map

$$\begin{array}{rccc} \varphi_l : & S_l & \longrightarrow & M_{n,k+p-l}^{(p)} \\ & \mathbf{u} & \longmapsto & (u_q - (l-1))_{q=2,\ldots,n+1} \end{array}$$

is a bijection for $l = 1, \ldots, k$. Therefore

$$\left|M_{n+1,k}^{(p)}\right| = \sum_{l=1}^{k} \left|M_{n,k+p-l}^{(p)}\right| = \sum_{l=0}^{k-1} \left|M_{n,p+l}^{(p)}\right|.$$

□

The following lemma can be shown by a straightforward induction on k.

Lemma 4.5. For any $k \in \mathbb{N}^+$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} &\sum_{m=0}^{k-1} \frac{p+m}{(p-1)\cdot(n+1)+m+1} \cdot \prod_{l=m+1}^{n+m} ((p-1)\cdot(n+1)+l) \\ = \quad & \frac{1}{n+1} \cdot \frac{k}{(p-1)\cdot(n+1)+k} \cdot \prod_{l=k}^{n+k} ((p-1)\cdot(n+1)+l)\,. \end{aligned}$$

Theorem 4.6. *For any $k \in \mathbb{N}^+$ and $n \in \mathbb{N}$ we have*

$$\begin{aligned} \left| M_{n,k}^{(p)} \right| &= \frac{1}{n!} \cdot \frac{k}{(p-1)\cdot n+k} \cdot \prod_{l=k}^{n+k-1} ((p-1)\cdot n+l) \\ &= \frac{k}{(p-1)\cdot n+k} \cdot \binom{p\cdot n+k-1}{n}. \end{aligned}$$

Proof. It is a routine induction proof using the recursion formula from Lemma 4.4 and the previous Lemma 4.5. $\square$

5. Polynomial spectra

In this section we give three different examples of polynomial spectra which solve problem 3 in [1].

Example 5.1. Let k be a fixed natural number. We define an equivalence relation $\Sigma \subseteq \Lambda^{(2)}$ as follows. For a bracketing $s = \omega t_1 t_2 \in B_n^{(2)}$ we call t_1 the *left factor of* s and denote it by $\mathrm{left}\,(s)$, and we put $\mathrm{left}\,(x) = x$ (cf. [1]). For $s \approx t \in B_n^{(2)} \times B_n^{(2)}$ let

$$s \approx t \in \Sigma :\Longleftrightarrow \left|\mathrm{left}^i\,(s)\right| = \left|\mathrm{left}^i\,(t)\right| \text{ for } i = 1,\dots,k.$$

The set Σ is closed under the implication operator, thus it appears as the fine spectrum of some groupoid $\mathbf{A}$. The corresponding associative spectrum is a polynomial of degree k:

$$s_n\,(\mathbf{A}) = \binom{n-1}{k} + \binom{n-1}{k-1} + \dots + \binom{n-1}{1} + \binom{n-1}{0}.$$

It is straightforward to check that $\delta_n\,(\Sigma_n) \subseteq \Sigma_{n+1}$ holds for all $n \in \mathbb{N}$. Let $s \in B_n^{(2)}$ be an arbitrary bracketing, and let us abbreviate $\left|\mathrm{left}^i\,(s)\right|$ by l_i. Clearly we have

$$1 \le l_k \le l_{k-1} \le \dots \le l_2 \le l_1 \le n,$$

where the inequalities are strict, except maybe for a couple of repeated 1s at the beginning. We have to count how many such k-tuples exist. If the number of 1s at the beginning is i, then we have to choose $k-i$ different numbers from the set $\{2,\dots,n\}$, hence the number of possibilities is $\binom{n-1}{k-i}$. Thus we have

$$\left|B_n^{(2)}/\Sigma_n\right| = \binom{n-1}{k} + \binom{n-1}{k-1} + \dots + \binom{n-1}{1} + \binom{n-1}{0},$$

which is indeed a polynomial of degree k.

Example 5.2. Let $k \in \mathbb{N}^+$ be an integer and define the relation $\Sigma \subseteq \Lambda^{(p)}$ as follows: For an identity $s \approx t \in B_n^{(p)} \times B_n^{(p)}$ denote by $\mathbf{u} := \mathrm{IT}(s)$ and $\mathbf{v} := \mathrm{IT}(t)$ the insertion tuples of the bracketings. We define Σ by

$$s \approx t \in \Sigma :\Longleftrightarrow \begin{cases} s = t, & \text{if } n < k; \\ \forall i \in \{n-k+1,\dots,n\} : \; u_i = v_i, & \text{if } n \ge k. \end{cases}$$

It holds that Σ is an equivalence relation that is closed under the implication operator. Therefore, there exists a groupoid $\mathbf{A}$ such that $\sigma\,(\mathbf{A}) = \Sigma$. The associative spectrum of $\mathbf{A}$ is

$$s_n\,(\mathbf{A}) = \begin{cases} C_n^{(p)}, & \text{if } n < k; \\ \dfrac{(p-1)\cdot(n-k)+1}{(p-1)\cdot n+1} \cdot \dbinom{(p-1)\cdot n+k}{k}, & \text{if } n \ge k. \end{cases}$$

Remember that $s \approx t \in \Sigma$ means that the last k entries of the insertion tuples of s and t are equal, or equivalently that the last k symbols ω are in the same place in s and t. Therefore, it is easy to verify that $\delta_n(\Sigma_n) \subseteq \Sigma_{n+1}$ holds for all $n \in \mathbb{N}$.

To know the associative spectrum we have to count the insertion tuples with different last k entries. From Proposition 4.3 we know that $\mathrm{IT}[B_n^{(p)}] = M_{n,1}^{(p)}$. If we look at the last $k \leq n$ entries we see that they form exactly the set $M_{k,(p-1)\cdot(n-k)+1}^{(p)}$. So the formula can be obtained from Theorem 4.6.

Example 5.3. The binary groupoid $\mathbf{G} := (\mathbb{Z}_6[Y], \oplus)$ (where $\mathbb{Z}_6[Y]$ is the polynomial ring over $\mathbb{Z}_6$ in the variable Y) with the binary operation

$$\begin{array}{rccc} \oplus : & \mathbb{Z}_6[Y]^2 & \longrightarrow & \mathbb{Z}_6[Y] \\ & (X_1, X_2) & \longmapsto & 3Y \cdot X_1 + 2Y \cdot X_2 \end{array}$$

has the associative spectrum

$$\forall n \in \mathbb{N}, n \geq 2 : \ s_n(\mathbf{G}) = \frac{n^2+n-2}{2}.$$

Instead of $\mathbb{Z}_6[Y]$ another ring can be chosen which has zero divisors (in this case $3Y$ and $2Y$) whose powers are all different.

In [1] the notion of left and right depth sequence were defined. Here, we only need two special cases: for $s \in B_n^{(2)}$ let $d_l(s)$ denote the left depth of the first variable of s, and let $d_r(s)$ denote the right depth of the last variable of s. On the binary tree corresponding to s one can see $d_l(s)$ as the length of the path connecting the root and the leftmost leaf and $d_r(s)$ as the length of the path connecting the root and the rightmost leaf. From this interpretation it is clear that for all $t_1, t_2 \in T_\omega(x)$:

$$d_l(\omega t_1 t_2) = d_l(t_1) + 1 \quad \text{and} \quad d_r(\omega t_1 t_2) = d_r(t_2) + 1. \tag{1}$$

Later it will be useful to compute these numbers from the insertion tuple $\mathbf{u} := \mathrm{IT}(s)$:

$$\begin{aligned} d_l(s) &= \left| \left\{ q \in \{1, \ldots, n\} \;\middle|\; u_q = 1 \right\} \right|, \\ d_r(s) &= \left| \left\{ q \in \{1, \ldots, n\} \;\middle|\; u_q = \ell^{(2)}(q-1) \right\} \right|. \end{aligned}$$

It is a routine induction using (1) to check that for $n > 0$

$$(\varepsilon_1(s))^{\mathbf{G}}(X_1, \ldots, X_{n+1}) = (3Y)^{d_l(s)} \cdot X_1 + (2Y)^{d_r(s)} \cdot X_{n+1}$$

holds. Remember that the length function is $\ell^{(2)}(n) = n + 1$ and that $\varepsilon_1(s)$ contains exactly the variable symbols $x_1, \ldots, x_{\ell^{(2)}(n)}$ such that $\varepsilon_1(s)$ can be interpreted as a $(n+1)$-ary term.

From this it follows that the fine spectrum of $\mathbf{G}$ can be characterized as:

$$s \approx t \in \sigma(\mathbf{G}) \iff d_l(s) = d_l(t) \text{ and } d_r(s) = d_r(t).$$

To obtain the formula for the associative spectrum we have to count all possibilities for $d_l(s)$ and $d_r(s)$. For $n \geq 2$ we have the following restrictions:

$$d_l(s) \geq 1,\ d_r(s) \geq 1 \text{ and } 3 \leq d_l(s) + d_l(r) \leq n + 1.$$

This is pretty clear using the insertion tuple because the first entry $u_1 = 1$ counts for both depths, the second entry u_2 can either be 1 or $2 = \ell^{(2)}(1)$ and the other entries u_k $(k = 3, \ldots, n)$ can be 1, $\ell^{(2)}(k-1)$ or something in between. It is also clear that all such possibilities can occur. So we have

$$\left(\sum_{l=1}^{n} n + 1 - l \right) - 1 = \left(\sum_{k=1}^{n} k \right) - 1 = \frac{n^2+n-2}{2}$$

possibilities.

6. The lattice of fine spectra

The Galois-closed sets $\mathrm{Id}_{\mathrm{Brack}}\,\mathcal{K}$ ($\mathcal{K} \subseteq \mathrm{Alg}\,(\omega)$) form a complete lattice as a closure system in $\mathcal{P}\left(\Lambda^{(p)}\right)$. We will denote this complete lattice by $\mathbf{FS} = (FS, \wedge, \vee)$ (FS stands for fine spectra because we know from Corollary 3.5 that the elements of this lattice are exactly all fine spectra). To unterstand associative spectra it would be very useful to unterstand this lattice. As a beginning we will look at the covering relation $\prec$.

Proposition 6.1. *For any $\sigma(\mathbf{A}), \sigma(\mathbf{B}) \in FS$ the following holds:*

$$\sigma(\mathbf{A}) \prec \sigma(\mathbf{B}) \iff \exists! n \in \mathbb{N}: \ \sigma_n(\mathbf{A}) \neq \sigma_n(\mathbf{B}), \text{ and for this } n \text{ we have } \sigma_n(\mathbf{A}) \prec \sigma_n(\mathbf{B}) \text{ in the lattice } \mathrm{Eq}\big(B_n^{(p)}\big).$$

Proof. It is clear that the condition on the right is sufficient. For the necessity let us assume that $\sigma(\mathbf{A}) \prec \sigma(\mathbf{B})$. If $\sigma(\mathbf{A})$ and $\sigma(\mathbf{B})$ differ at least at two positions, say $\sigma_n(\mathbf{A}) \neq \sigma_n(\mathbf{B})$ and $\sigma_m(\mathbf{A}) \neq \sigma_m(\mathbf{B})$ for some $n < m \in \mathbb{N}$, then

$$\Sigma_k := \begin{cases} \sigma_k(\mathbf{A}), & \text{if } k \leq n; \\ \sigma_k(\mathbf{B}), & \text{if } k > n \end{cases}$$

defines a fine spectrum that is strictly between $\sigma(\mathbf{A})$ and $\sigma(\mathbf{B})$ contradicting that $\sigma(\mathbf{A}) \prec \sigma(\mathbf{B})$. If $\sigma(\mathbf{A})$ and $\sigma(\mathbf{B})$ differ only at one position, say $\sigma_n(\mathbf{A}) \neq \sigma_n(\mathbf{B})$ and $\sigma_n(\mathbf{A}) \not\prec \sigma_n(\mathbf{B})$ in the lattice $\mathrm{Eq}\big(B_n^{(p)}\big)$, then

$$\Sigma_k := \begin{cases} \sigma_k(\mathbf{A}), & \text{if } k \neq n; \\ \pi, & \text{if } k = n \end{cases}$$

defines a fine spectrum that is strictly between $\sigma(\mathbf{A})$ and $\sigma(\mathbf{B})$ if $\pi \in \mathrm{Eq}\big(B_n^{(p)}\big)$ is an equivalence relation such that $\sigma_n(\mathbf{A}) < \pi < \sigma_n(\mathbf{B})$. □

As a consequence we the obtain the following characterization of the atoms and coatoms of $\mathbf{FS}$:

Corollary 6.2. *There are no atoms in* $\mathbf{FS}$. *For any $\sigma(\mathbf{A}) \in FS$ we have:*

$$\sigma(\mathbf{A}) \text{ is a coatom in } \mathbf{FS} \iff \forall n \in \mathbb{N} \setminus \{2\}: \ s_n(\mathbf{A}) = 1 \ \text{ and } \ s_2(\mathbf{A}) = 2.$$

Proof. The description for the coatoms follows from the above proposition. For the atoms we assume that $\sigma(\mathbf{A}) \in FS$ is an atom in $\mathbf{FS}$. From the previous proposition we see that $\sigma_n(\mathbf{A})$ is the equality relation on $B_n^{(p)}$ for all but one $n \in \mathbb{N}$. It is clear that such a $\sigma(\mathbf{A})$ cannot be closed under the implication operator. □

From the previous corollary we know that the number of coatoms is exactly the number of possibilities to group p elements into two classes (because $|B_2^{(p)}| = p$). So we get:

Corollary 6.3. *There are exactly $2^{p-1} - 1$ coatoms in* $\mathbf{FS}$.

We prove that the cardinality of the set of sequences of natural numbers that arise as associative spectra is continuum. Clearly, it cannot be more, so it suffices to construct continuously many different spectra, and it suffices to do it in the binary case. First we need a definition: if $\omega xx = (xx)$ is a subbracketing of $s \in B_n^{(2)}$, then we say that (xx) is a *pair of eggs* in s. (Actually the two x's are the eggs, see [1].)

Lemma 6.4. Let τ_n be the equivalence relation on $B_n^{(2)}$, where the bracketings with at least 3 pairs of eggs form one class, and all the other bracketings are singletons. Then $\tau_n \supsetneq \delta_{n-1}(\tau_{n-1})$ for all $n \geq 5$.

Proof. The operators γ_i, β_i do not decrease the number of eggs, hence $\delta_{n-1}(\tau_{n-1}) \subseteq \tau_n$ for all $n \geq 1$. For every $n \geq 5$ one can find a bracketing with occurrence number n, which cannot be obtained by these operators from any bracketing (with occurrence number $n-1$) with at least three pairs of eggs. For example,

$$t = (\ldots(((xx)(xx))x)x\ldots x)(xx)$$

is such a bracketing. Thus t is a singleton in $\delta_{n-1}(\tau_{n-1})$, but not a singleton in τ_n. This shows that $\delta_{n-1}(\tau_{n-1}) \neq \tau_n$ if $n \geq 5$.

□

Theorem 6.5. *There exist $2^{\aleph_0}$ different associative spectra.*

Proof. Let S be the set of 0–1 sequences whose first five entries are 0. For every $\mathbf{a} = \{a_n\}_{n=0}^{\infty} \in S$ we construct a sequence of equivalence relations $\sigma_n^{\mathbf{a}} \subseteq B_n^{(2)} \times B_n^{(2)}$ recursively:

$$\sigma_n^{\mathbf{a}} = \begin{cases} \delta_{n-1}\left(\sigma_{n-1}^{\mathbf{a}}\right), & \text{if } a_n = 0; \\ \tau_n, & \text{if } a_n = 1. \end{cases}$$

Note that we do not have to define the "initial value" $\sigma_0^{\mathbf{a}}$ since $B_0^{(2)}$ is a one-element set. Observe also that $\sigma_n^{\mathbf{a}}$ is the equality relation on $B_n^{(2)}$ for $n \leq 4$ for every $\mathbf{a} \in S$.

First we claim that $\sigma_n^{\mathbf{a}} \subseteq \tau_n$ for every $\mathbf{a} \in S$. This is clear for $n = 0$ (and also for $n = 1, 2, 3, 4$), and then we can proceed by induction. Suppose that $\sigma_{n-1}^{\mathbf{a}} \subseteq \tau_{n-1}$. If $a_n = 1$, then $\sigma_n^{\mathbf{a}} = \tau_n$; if $a_n = 0$, then $\sigma_n^{\mathbf{a}} = \delta_{n-1}\left(\sigma_{n-1}^{\mathbf{a}}\right) \subseteq \delta_{n-1}(\tau_{n-1}) \subsetneq \tau_n$ by the previous lemma, and by the monotonicity of δ_{n-1}.

Now we can verify that $\sigma^{\mathbf{a}}$ is a fine spectrum: if $a_n = 0$, then $\sigma_n^{\mathbf{a}} = \delta_{n-1}\left(\sigma_{n-1}^{\mathbf{a}}\right)$; if $a_n = 1$, then $\sigma_n^{\mathbf{a}} = \tau_n \supsetneq \delta_{n-1}(\tau_{n-1}) \supseteq \delta_{n-1}\left(\sigma_{n-1}^{\mathbf{a}}\right)$, hence Corollary 3.5 applies.

We need to check yet that different elements of S give different associative spectra. Let $\mathbf{a} \neq \mathbf{b} \in S$, and suppose that $a_i \neq b_i$, say $a_i = 0$ and $b_i = 1$. Then we have $\sigma_i^{\mathbf{a}} = \delta_{i-1}\left(\sigma_{i-1}^{\mathbf{a}}\right) \subseteq \delta_{i-1}(\tau_{i-1}) \subsetneq \tau_i = \sigma_i^{\mathbf{b}}$. We have proved that $\sigma_i^{\mathbf{a}} \subsetneq \sigma_i^{\mathbf{b}}$, and this means that not only the two fine spectra, but the corresponding spectra are also different: $\left|B_i^{(2)}/\sigma_i^{\mathbf{a}}\right| > \left|B_i^{(2)}/\sigma_i^{\mathbf{b}}\right|$. □

Remark 6.6. From the previous proof we see that

$$\forall \mathbf{a}, \mathbf{b} \in S: \ \sigma^{\mathbf{a}} \subseteq \sigma^{\mathbf{b}} \Longleftrightarrow \mathbf{a} \leq \mathbf{b},$$

which means that S embeds into $\mathbf{FS}$ as a poset (not as a lattice!). Since S is isomorphic to $\mathcal{P}(\mathbb{N})$, we have $\mathcal{P}(\mathbb{N})$ as a subposet in $\mathbf{FS}$. On the other hand clearly $\mathbf{FS}$ embeds into $\mathcal{P}\left(\Lambda^{(p)}\right) \cong \mathcal{P}(\mathbb{N})$. Therefore, $\mathbf{FS}$ and $\mathcal{P}(\mathbb{N})$ are *equimorphic*. This shows for example, that there is a chain and an antichain of continuum cardinality in $\mathbf{FS}$.

7. Spectra of finite groupoids

There are only countably many finite groupoids, hence Theorem 6.5 shows that there are spectra which can be realized only on infinite groupoids. It would be interesting to see, under what conditions a (fine) spectrum is realizable on a finite groupoid. One obvious necessary condition: the spectrum has to be recursive (computable by a Turing machine). If there exists $N \in \mathbb{N}$ such that $\sigma_n = \delta_{n-1}(\sigma_{n-1})$ holds for all $n > N$, then the sequence σ_n is recursive. We conjecture that this condition is sufficient in order to realize a fine spectrum on a finite groupoid (cf. Proposition 7.3). The condition is not necessary, as the following example shows.

Example 7.1. We construct a finite groupoid with the "three-egg spectrum" τ_n. First let us consider the groupoid $\mathbf{A}$ given by the following multiplication table.

	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	2

One can prove by induction that for any bracketing s the maximal value of the corresponding term function $s^{\mathbf{A}}$ is $\max(3 - e, 0)$ where e is the number of pairs of eggs in s. This maximal value is attained for example at $s^{\mathbf{A}}(3, \ldots, 3)$. This shows that $\sigma_n(\mathbf{A}) \supseteq \tau_n$, since bracketings with at least three pairs of eggs induce constant 0 term functions. (Actually one can verify that $\mathbf{A} \models s_1 \approx s_2$ iff either both s_1 and s_2 contain at least three pairs of eggs, or both contain at most two pairs of eggs and these are at the same positions in s_1 and s_2.)

In order to isolate bracketings with at most two eggs, we blow up the nonzero elements of **A** using the Sheffer operation on the two-element set. We present this operation with somewhat unusual notation:

	$\widehat{\square}$	$\widetilde{\square}$
$\widehat{\square}$	$\widetilde{\square}$	$\widehat{\square}$
$\widetilde{\square}$	$\widehat{\square}$	$\widehat{\square}$

We replace each nonzero element of **A** with two elements: one wearing a hat, the other one wearing a tilde, and we define the multiplication such that the numbers get multiplied as in **A**, and headgears get multiplied according to the Sheffer operation. We obtain the following seven-element groupoid $\widehat{\mathbf{A}}$:

	0	$\widehat{1}$	$\widetilde{1}$	$\widehat{2}$	$\widetilde{2}$	$\widehat{3}$	$\widetilde{3}$
0	0	0	0	0	0	0	0
$\widehat{1}$	0	0	0	0	0	$\widetilde{1}$	$\widehat{1}$
$\widetilde{1}$	0	0	0	0	0	$\widehat{1}$	$\widehat{1}$
$\widehat{2}$	0	0	0	$\widetilde{1}$	$\widehat{1}$	$\widetilde{2}$	$\widehat{2}$
$\widetilde{2}$	0	0	0	$\widehat{1}$	$\widehat{1}$	$\widehat{2}$	$\widehat{2}$
$\widehat{3}$	0	$\widetilde{1}$	$\widehat{1}$	$\widetilde{2}$	$\widehat{2}$	$\widetilde{2}$	$\widehat{2}$
$\widetilde{3}$	0	$\widehat{1}$	$\widehat{1}$	$\widehat{2}$	$\widehat{2}$	$\widehat{2}$	$\widehat{2}$

We did not blow up 0, hence bracketings with at least three pairs of eggs still induce constant term functions, and thus we have $\sigma_n\big(\widehat{\mathbf{A}}\big) \supseteq \tau_n$. On the other hand, if s contains at most two pairs of eggs, then $s^{\mathbf{A}}(x_1, \ldots, x_n) \neq 0$ if $x_1, \ldots, x_n \in \big\{\widehat{3}, \widetilde{3}\big\}$. This means that substituting $\widehat{3}$s and $\widetilde{3}$s into $s^{\mathbf{A}}$ we can recover all information about hats and tildes, that is we can determine the term function corresponding to s over the Sheffer operation. This operation is Catalan, hence from the term function we can recover the bracketing. Consequently s is a singleton in $\sigma_n(\widehat{\mathbf{A}})$, hence $\sigma_n(\widehat{\mathbf{A}}) = \tau_n$.

In Section 5 we gave examples for polynomial spectra using Corollary 3.5. The groupoids that we obtained this way were infinite groupoids of the form $\mathbf{T}^{(\mathbf{P})}/\Sigma$, but below we will construct a finite groupoid with a polynomial spectrum.

Example 7.2. Let us define a binary operation on the set $A = \{0, 1, \ldots, k+1\}$ by

$$x \cdot y = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } x \neq 0 = y; \\ \min(x+1, k+1), & \text{if } x \neq 0 \neq y. \end{cases}$$

The associative spectrum of the groupoid $\mathbf{A} = (A; \cdot)$ is a polynomial of degree k.

Indeed, let Σ be the equivalence relation defined in Example 5.1. We prove that the fine spectrum of **A** is $\sigma_n(\mathbf{A}) = \Sigma_n$. To avoid notational difficulties we prove it only for $k = 3$; it will be clear from the proof how the construction works for arbitrary k. To have a better view of the operation, let us write out the multiplication table.

	0	1	2	3	4
0	0	0	0	0	0
1	1	2	2	2	2
2	1	3	3	3	3
3	1	4	4	4	4
4	1	4	4	4	4

First we prove that $\sigma_n(\mathbf{A}) \subseteq \Sigma_n$ for all $n \in \mathbb{N}$. It suffices to show that for any $t \in B_n^{(2)}$, the values of l_1, l_2, l_3 can be read off from the term function $t^{\mathbf{A}}$ corresponding to t. Taking a look at the multiplication table, we see immediately that $x \cdot y = 0$ iff $x = 0$. Therefore a product of arbitrarily many elements (with arbitrarily inserted parentheses) equals 0 iff the first (i.e. leftmost) element is 0. We record this fact with the following (hopefully intuitive) notation, where $*$ symbolises an arbitrary nonzero element:

$$(0 \cdots) = 0, \quad (* \cdots) \neq 0. \tag{2}$$

From this observation and from the idempotence of the element 4 we infer

$$\begin{aligned}(4 \cdots 4) \cdot (4 \cdots 0) &= 4 \cdot * = 4,\\ (4 \cdots 4) \cdot (0 \cdots 0) &= 4 \cdot 0 = 1.\end{aligned}$$

This means that l_1 can be computed from the values of $t^{\mathbf{A}}$:

$$l_1 = \max \Big\{ i \;\Big|\; t^{\mathbf{A}} (\underbrace{4, \ldots, 4}_{i}, 0, \ldots 0) = 1 \Big\}.$$

Knowing the value of l_1, we can find l_2 using the following observations:

$$\begin{aligned}((4 \cdots 4) \cdot (4 \cdots 0)) \cdot (4 \cdots 4) &= (4 \cdot *) \cdot 4 = 4 \cdot 4 = 4,\\ ((4 \cdots 4) \cdot (0 \cdots 0)) \cdot (4 \cdots 4) &= (4 \cdot 0) \cdot 4 = 1 \cdot 4 = 2.\end{aligned}$$

Hence l_2 can be recovered from $t^{\mathbf{A}}$ as

$$l_2 = \max \Big\{ i \;\Big|\; t^{\mathbf{A}} \big(\overbrace{\underbrace{4, \ldots, 4}_{i}, 0, \ldots 0}^{l_1}, 4, \ldots, 4\big) = 2 \Big\}.$$

Note that if $l_1 = 1$, then we cannot make such substitutions, but in this case clearly $l_2 = l_3 = 1$.

Similarly, l_3 can be obtained, since we have

$$\begin{aligned}(((4 \cdots 4) \cdot (4 \cdots 0)) \cdot (4 \cdots 4)) \cdot (4 \cdots 4) &= ((4 \cdot *) \cdot 4) \cdot 4 = (4 \cdot 4) \cdot 4 = 4 \cdot 4 = 4,\\ (((4 \cdots 4) \cdot (0 \cdots 0)) \cdot (4 \cdots 4)) \cdot (4 \cdots 4) &= ((4 \cdot 0) \cdot 4) \cdot 4 = (1 \cdot 4) \cdot 4 = 2 \cdot 4 = 3,\end{aligned}$$

and therefore in case $l_2 > 1$ we have

$$l_3 = \max \Big\{ i \;\Big|\; t^{\mathbf{A}} \big(\overbrace{\underbrace{4, \ldots, 4}_{i}, 0, \ldots 0}^{l_2}, 4, \ldots, 4\big) = 3 \Big\}.$$

Now we prove the inclusion $\sigma_n (\mathbf{A}) \supseteq \Sigma_n$, i.e. the fact that the numbers l_1, l_2, l_3 determine the term function $t^{\mathbf{A}}$. First we observe that $\mathbf{A}$ satisfies the identity $x (yz) \approx xy$, and from this we conclude by induction that

$$\mathbf{A} \models \varepsilon_1 (t) \approx \varepsilon_1 (\mathrm{left} (t)) \cdot x_{l_1+1}.$$

Applying this identity to the left factor of t we obtain

$$\mathbf{A} \models \varepsilon_1 (t) \approx \big(\varepsilon_1 \big(\mathrm{left}^2 (t)\big) \cdot x_{l_2+1}\big) \cdot x_{l_1+1}.$$

Let us repeat this procedure until the left factor becomes the single variable x_1. Suppose this happens after s steps, i.e. $1 = l_s < l_{s-1} < \cdots < l_2 < l_1$. Then we have

$$\mathbf{A} \models \varepsilon_1 (t) \approx \big(\big(\cdots \big((x_1 \cdot x_{l_s+1}) \cdot x_{l_{s-1}+1}\big) \cdots \big) \cdot x_{l_2+1}\big) \cdot x_{l_1+1}.$$

This already shows that $t^{\mathbf{A}}$ is determined by the numbers $l_1, l_2, \ldots, l_s$. We have to show that actually the first three of these numbers are sufficient. If $s \leq 3$ then we have nothing to prove, and if $s \geq 4$, then using (2) and the multiplication table we get the following formula for $t^{\mathbf{A}}$:

$$\begin{aligned} t^{\mathbf{A}} (x_1, \cdots, x_n) &= \big(\big(\cdots \big((x_1 \cdot x_{l_s+1}) \cdot x_{l_{s-1}+1}\big) \ldots \big) \cdot x_{l_2+1}\big) \cdot x_{l_1+1} \\ &= \begin{cases} 0, & \text{if } x_1 = 0; \\ 1, & \text{if } x_1 \neq 0 = x_{l_1+1}; \\ 2, & \text{if } x_1, x_{l_1+1} \neq 0 = x_{l_2+1}; \\ 3, & \text{if } x_1, x_{l_1+1}, x_{l_2+1} \neq 0 = x_{l_3+1}; \\ 4, & \text{if } x_1, x_{l_1+1}, x_{l_2+1}, x_{l_3+1} \neq 0. \end{cases} \end{aligned}$$

The next proposition shows that every finally associative spectrum appears as the fine spectrum of a finite groupoid.

Proposition 7.3. *Let* $\mathbf{A} \in \mathrm{Alg}(\omega)$ *be a* p*-ary groupoid with*

$$\exists n \in \mathbb{N}, n \geq 2 : \; s_n(\mathbf{A}) = 1.$$

Then there exists a finite groupoid $\mathbf{B} \in \mathrm{Alg}(\omega)$ *with* $\sigma(\mathbf{B}) = \sigma(\mathbf{A})$.

Proof. Let $\mathbf{A} \in \mathrm{Alg}(\omega)$ be a groupoid with the above property and denote by $\Sigma := \sigma(\mathbf{A})$ the fine spectrum of $\mathbf{A}$. We know from Theorem 3.6 that

$$\forall m \in \mathbb{N}, m \geq n : \; \Sigma_m = B_m^{(p)} \times B_m^{(p)}$$

holds. And by Theorem 3.4 we know that $\sigma\left(\mathbf{T}^{(\mathbf{p})}/\Sigma\right) = \Sigma$ holds. Define $\mathbf{B} = \left(B, \omega^{\mathbf{B}}\right)$ as

$$B := \left\{ [t]_\Sigma \;\middle|\; t \in B_k^{(p)}, k < n \right\} \cup \{*\}$$

with the operation

$$\begin{array}{cccc} \omega^{\mathbf{B}}: & B^p & \longrightarrow & B \\ & \left([t_1]_\Sigma, \ldots, [t_p]_\Sigma\right) & \longmapsto & \begin{cases} [\omega t_1 \ldots t_p]_\Sigma & \text{if } \left|\omega t_1 \ldots t_p\right|_\omega < n \\ * & \text{otherwise} \end{cases} \end{array}$$

and $\omega^{\mathbf{B}}(b_1, \ldots, b_p) := *$ if one of the arguments is $*$.

We have to show that $\sigma(\mathbf{B}) = \sigma\left(\mathbf{T}^{(\mathbf{p})}/\Sigma\right)$ holds. This is pretty clear because $\mathbf{B}$ is nearly the same as $\mathbf{T}^{(\mathbf{p})}/\Sigma$. The only difference is that the equivalence classes containing all bracketings of one size $m \geq n$ are equalized to $*$. □

8. Open problems

In conclusion, we formulate a few problems:

1. Another idea to unterstand the lattice **FS** is to translate constructions for groupoids into constructions in **FS** and vice versa. A very simple example of this is the direct product $\prod$ and the meet $\bigwedge$. Let $\mathbf{A_i} \in \mathrm{Alg}(\omega)$ for $i \in I$ (arbitrary index set). Then we have

$$\sigma\left(\prod_{i \in I} \mathbf{A_i}\right) = \bigwedge_{i \in I} \sigma(\mathbf{A_i}).$$

   Are there other correspondences between certain constructions, e.g. the join $\bigvee$ in **FS**?
2. We have studied the Galois-closed sets $\mathrm{Id}_{\mathrm{Brack}}\,\mathcal{K}$ for any $\mathcal{K} \subseteq \mathrm{Alg}(\omega)$. What is the analogon of a variety, i.e. what are the Galois-closed sets $\mathrm{Mod}_{\mathrm{Brack}}\,\Sigma$ on the groupoid side?
3. What additional properties have fine spectra of finite algebras? Prove or disprove that the following condition is sufficient in order to realize a fine spectrum σ on a finite groupoid:

$$\exists N \in \mathbb{N}, \; \forall n \in \mathbb{N}, n > N : \; \sigma_n = \delta_{n-1}(\sigma_{n-1}).$$

References

[1] Csákány, B. and Waldhauser, T., *Associative Spectra of Binary Operations.* Mult.-Valued Log. **5** (2000), pp. 175 - 200

[2] Hilton, P. and Pedersen, J., *Catalan Numbers, Their Generalization, and Their Uses.* Math. Intelligencer **13** (1991), no. 2, pp. 64 - 75

[3] Klarner, D. A., *Correspondences Between Plane Trees and Binary Sequences.* J. Combinatorial Theory **9** (1970), pp. 401 - 411

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