

A characterization of the radon transform and its dual on Euclidean space

ÁRPÁD KURUSA

Abstract. In this paper we present a characterization of the Radon transform without any restriction on its range. We also consider the boomerang transform B [5], which is essentially the dual of the Radon transform.

A. Hertle [3] gave a characterization of the Radon transform by investigating its behavior under rotations, dilations and translations and making some restrictions on its range. We present an alternative characterization that needs no restriction on the range.

By definition

$$R(f)_{(\omega,p)} = \int_{\langle x,\omega \rangle=p} f(x)dx, \quad B(h)_{(x)} = \int_{S_x^{n-1}} h(\langle \omega, x \rangle) d\omega,$$

where $\omega \in S^{n-1}$, $0 \leq p \in \mathbb{R}$, dx and $d\omega$ is the surface measure, h and f are good enough functions on \mathbb{R}^n (for example continuous functions with compact support) and

$$S_x^{n-1} = \{\omega \in S^{n-1} : \langle \omega, x \rangle \geq 0\}.$$

One can easily verify (see [4]), that the boomerang transform has the following properties:

(1) $B(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 B(f_1) + \alpha_2 B(f_2)$

(2) $B(f \circ U) = B(f) \circ U$

(3) $B(f(\delta x))_{(y)} = B(f)_{(\delta y)}$

(4) $B(f)_{(y+z)} = B\left(f\left(x \frac{\langle x, x+z \rangle}{\langle x, x \rangle}\right)\right)_{(y)}$

(5) If $f_k \rightarrow f$ locally uniformly on \mathbb{R}^n then $B(f_k)_{(x)} \rightarrow B(f)_{(x)}$ for any $x \in \mathbb{R}^n$, where $\alpha_1, \alpha_2 \in \mathbb{R}$, f, f_1, f_2, f_k are functions on \mathbb{R}^n , U is an orthogonal transformation, $0 < \delta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$.

Theorem 1. *If a nonzero function-transformation β has properties (1) — (5) then $\beta = \frac{\beta(1)}{B(1)}B$ on $C(\mathbb{R}^n \setminus \{0\})$.*

Proof. Let $f_i(x) = |x|^i$. The following simple equations prove, that $\beta(f_i) = f_i \cdot c'_i$, where the constants c'_i depends only on $i \in \mathbb{N}$.

$$\begin{aligned}\beta(f_i)_{(\delta x)} &= \beta(f_i(\delta y))_{(x)} = \delta^i \beta(f_i)_{(x)} \\ \beta(f_i)_{(Ux)} &= \beta(f_i(Uy))_{(x)} = \beta(f_i)_{(x)}\end{aligned}$$

This implies that if i is even

$$\begin{aligned}c'_i(t + |x|)^i &= \beta(f_i)_{(x+(tx/|x|))} = \beta(f_i(y + te_y \langle e_y, e_x \rangle))_{(x)} = \\ &= \sum_{j=0}^i \binom{i}{j} t^{i-j} \beta(|y|^j \langle e_x, e_y \rangle^{i-j})_{(x)},\end{aligned}$$

where $e_x = \frac{x}{|x|}$ and $t \in \mathbb{R}$. Since this is an equality of two polynomials, we have obtained for even i that

$$(*) \quad c'_i f_j(x) = \beta(|y|^j \langle e_x, e_y \rangle^{i-j})_{(x)}$$

Now this equation and the condition (4) imply

$$\beta(\langle e, e_y \rangle^i)_{(e_0)} = \beta(\langle e, e_y \rangle^i)_{(e+e_0)} = c'_i,$$

where e, e_0 are unit vectors. Let $c_i = B(f_i)/f_i$. The integration of our last equation over $S_{e_0}^{n-1}$ gives

$$c'_i B(1) = \int_{S_{e_0}^{n-1}} \beta(\langle e, e_y \rangle^i)_{(e_0)} de = \beta\left(\int_{S_{e_0}^{n-1}} \langle e, e_y \rangle^i de\right)_{(e_0)} = \beta(c_i)_{(e_0)} = c_i \beta(1).$$

This way, we have derived the following equation for any even i :

$$B(f_i) = \frac{\beta(1)}{B(1)} B(f_i).$$

Furthermore if i is even

$$\begin{aligned}\beta(f_i(x+y)) &= \beta(\langle x+y, x+y \rangle^{i/2}) \\ &= \sum_{k,l,m} \binom{i/2}{k,l,m} 2^m |x|^{2l+m} \beta(|y|^{2k+m} \langle e_x, e_y \rangle^m),\end{aligned}$$

where $k+l+m = i/2$ and $k, l, m \in \mathbb{N}$. Taking (*) into account this gives rise to (by calculating $B(f_i(x+y))$)

$$B(f_i(x+y)) = \frac{\beta(1)}{B(1)} B(f_i(x+y)).$$

Since the translates of radial polynomials are locally uniformly dense in $C(\mathbb{R}^n)$ this completes the proof. ■

Theorem 2. *If a nonzero function-transformation ϱ has the following properties then $\varrho = cR$, for some $c \in \mathbb{R}$ on $C_c(\mathbb{R}^n)$, the compactly supported continuous functions.*

$$(1') \quad \varrho(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \varrho(f_1) + \alpha_2 \varrho(f_2)$$

$$(2') \quad \varrho(f \circ U) = \varrho(f) \circ U$$

$$(3') \quad \varrho(f(\delta x))_{(\omega, r)} = \delta^{1-n} \varrho(f)_{(\omega, \delta r)}$$

$$(4') \quad \varrho(f(x+y))_{(\omega, r)} = \varrho(f)_{(\omega, r + \langle \omega, y \rangle)}$$

$$(5') \quad \text{If } f_k \rightarrow f \text{ locally uniformly on } \mathbb{R}^n \text{ then } \varrho(f_k) \rightarrow \varrho(f) \text{ (with common support)}$$

The notations are the same as in Theorem 1.

Proof. Let $\hat{f} = B(\varrho(f))$. Then we have

$$(a) \quad (\alpha_1 f_1 + \alpha_2 f_2)^\wedge = \alpha_1 \hat{f}_1 + \alpha_2 \hat{f}_2$$

$$(b) \quad (f \circ U)^\wedge = \hat{f} \circ U$$

$$(c) \quad (f(\delta x))^\wedge_{(y)} = \delta^{1-n} \hat{f}(\delta y)$$

$$(d) \quad (f(x+y))^\wedge_{(z)} = \hat{f}(z+y)$$

$$(e) \quad \text{If } f_k \rightarrow f \text{ locally uniformly on } \mathbb{R}^n \text{ then } \hat{f}_k \rightarrow \hat{f} \text{ (with common support)}$$

It is well known [1], that if a transform satisfies (a), (d) and (e) then there is a function g on \mathbb{R}^n such that

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(y)g(x-y) \, dy.$$

A short calculation with (b) and (c) gives that

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(y) \frac{d}{|x-y|} \, dy,$$

where d is a suitable constant. This implies by [2, pp.104] that $\hat{f}(x) = cB(R(f))$ for some nonzero constant c . Thus we have for any function f that

$$B(\varrho(f) - cR(f)) = 0.$$

From Theorem 1.3 of [4] we know the invertibility of the boomerang transform on the set of radial continuous functions. So if we prove that the continuity of f implies the continuity of $\varrho(f)$ then we shall obtain that ϱ is equal to cR on the continuous radial functions. Since the translates of radial continuous functions are locally uniformly dense in $C_c(\mathbb{R}^n)$ this will imply the theorem.

Thus to finish the proof we only have to prove the continuity of $\varrho(f)$ for any continuous radial function f . For any $\varepsilon > 0$ we have

$$\begin{aligned} \varrho(f)_{(r+\varepsilon)} - \varrho(f)_{(r)} &= \varrho \left(\left(\frac{r+\varepsilon}{r} \right)^{n-1} f \left(\frac{r+\varepsilon}{r} x \right) \right)_{(r)} - \varrho(f)_{(r)} \\ &= \varrho \left(\left(\frac{r+\varepsilon}{r} \right)^{n-1} f \left(\frac{r+\varepsilon}{r} x \right) - f(x) \right)_{(r)} \end{aligned}$$

If ε tends to zero then

$$\left(\frac{r+\varepsilon}{r} \right)^{n-1} f \left(\frac{r+\varepsilon}{r} x \right) - f(x) \rightarrow 0$$

uniformly on $\text{supp}(f)$. This just means that $\varrho(f)$ is a continuous function. ■

Theorem 1 (resp. Theorem 2) is valid for any other function space in which $C(\mathbb{R}^n \setminus \{0\})$ (resp. $C_c(\mathbb{R}^n)$) is dense.

The author would like to thank L. Fehér for making valuable suggestions on the form of this article.

References

- [1] W. F. DONOGHUE, *Distributions and Fourier Transforms*, Academic Press, Inc., 1969.
- [2] S. HELGASON, *Groups and geometric analysis* (Pure and applied mathematics), Academic Press, Inc., 1984.
- [3] A. HERTLE, A characterization of Fourier and Radon transforms on Euclidean space, *Trans. Amer. Math. Soc.*, **273** (1982), 595–608.
- [4] Á. KURUSA, New Radon inversion formulas, *Acta Math. Hung.*, **60(1-2)** (1992), 283–290.
- [5] Z. I. SZABÓ, Hilbert's Fourth Problem I., *Adv. in Math.*, **59** (1986), 185–301.

Á. KURUSA, Bolyai Institute, Aradi vértanúk tere 1., 6720 Szeged, Hungary; *e-mail*: kurusa@math.u-szeged.hu