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A Characterization of the Radon Transform's Range by a System of PDEs

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Abstract. Let g be a compactly supported function of d-planes in $\mathbb{R}^n$. We prove that then g is in the range of the Radon transform if and only if g satisfies an ultrahyperbolic system of PDEs. We parameterize the d-planes by $d+1$ points $x_0, x_1, \dots, x_d$ on them and get the PDE

$$\left(\frac{\partial^2}{\partial x_i^k \partial x_j^l} - \frac{\partial^2}{\partial x_i^l \partial x_j^k}\right) \frac{g(x_0, x_1, \dots, x_d)}{\mathrm{Vol}\{x_i - x_0\}_{i=1,d}} = 0,$$

where x_i^k denotes the $k-th$ coordinate of x_i. At the end we analyze in detail the case of $d = 1$.

1. Introduction

In this paper we consider the range of the (d, n) Radon transform R_d^n, which is defined by

$$R_d^n f(\xi) = \int_\xi f(x)dx,$$

where $f \in D(\mathbb{R}^n)$, ξ is an element of $G(d, n)$, the set of d dimensional hyperplanes in $\mathbb{R}^n$, $1 \le d \le n-2$ and dx is the surface measure on ξ.

There are many papers about the range of the Radon transform considered on several different spaces (*e.g.* [2],...,[12]), some of which ([3],[4],[8],[11]) give PDEs to characterize the range of the Radon transform.

The first one as far as I know in which a characterization by a system of PDEs was given is [8]. F. John characterized the range only in the case (1,3) and used in his proof many special properties of the 3 dimensional space and the Asgeirsson's lemma, but it was very geometrical compared with the others.

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These later extensions of John's result used strong tools of analysis like Fourier transforms etc. and a strong not too natural, by my opinion, definition of $S(G(d,n))$. On the other hand, as Richter also pointed out, Grinberg's and Gelfand's proof was incomplete. Other substantial difference between these papers is the parameterization's method of the d-planes. We will follow John's method parameterizing the d-planes by $d+1$ points on them. The other possible method, that was used by Richter, Grinberg and Gelfand is to parameterize as $y = Ax + c$, where A is an $(n-d)\times d$ matrix and $c \in \mathbb{R}^{n-d}$.

In the following we characterize the range of R_d^n with a system of PDEs on a relatively easy geometrical way deriving Helgason's moment condition directly from the differential equation.

At the end we analyze in detail the case of $(1,n)$ and give a short proof of John's main theorem which gives all the solutions of the ultrahyperbolic partial differential equation with four variables. This equation (5) plays part in the Yang-Mills theory [1, pp. 78-81].

2. Preliminaries

To any set of $d+1$ points $x_0, x_1, \ldots, x_d \in \mathbb{R}^n$, which are general position, one can associate a uniquely determined d-plane $\xi(x_0, x_1, \ldots, x_d) \in G(d,n)$ through them. Using this parameterization for $G(d,n)$ we can write

$$R_d^n f(x_0, x_1, \ldots, x_d) = \int_\xi f(x)dx.$$

It is worthwhile to note that $R_d^n f$ in this context is interpreted on a principal fibre bundle over $G(d,n)$ with the affine group of $\mathbb{R}^d$ as its structure group. The undermentioned function $R_d^n f/|\det U_x^{-1}|$ is definitely interpreted on this principal fibre bundle too. Let U_x be an automorphism of $\mathbb{R}^n$ transforming the system $\{x_i - x_0\}_{i=1,d}$ into an orthonormal system. Then a substitution gives

$$R_d^n f(x_0, x_1, \ldots, x_d) = \int_{\mathbb{R}^d} f\Big(x_0 + \sum_{i=1}^{d} \lambda_i (x_i - x_0)\Big) |\det U_x^{-1}|\ d\lambda,$$

where $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$. We note that $|\det U_x^{-1}|$ is the volume of the parallelepiped spanned by the vectors $\{x_i - x_0\}_{i=1,d}$. For example, in the case of $(1,n)$ $|\det U_x^{-1}| = |x_1 - x_0|$. We will use this remark frequently. Now we can state an easy lemma, which can be proven by applying the differentiation under the integral sign.

Lemma 2.1. *If $f \in D(\mathbb{R}^n)$ then*

$$(\partial_{i,k}\partial_{j,e} - \partial_{i,e}\partial_{j,k})\frac{R^n_d f(x_0, x_1, \dots, x_d)}{\mathrm{Vol}\{x_i - x_0\}_{i=1,d}} = 0,$$

where $0 \le i, j \le d$, $1 \le k, e \le n$ and $\partial_{i,k}$ denotes the differentiation with respect to the $k-th$ coordinate x_i^k of the $i-th$ point x_i.

Now we recall a definition and a statement of Helgason [6].

A function f on $G(d,n)$ is said to be in $D_H(G(d,n))$ if f is C^∞, has compact support and satisties the following condition: For each $k \in \mathbb{N}$ there exists a homogeneous kth-degree polynomial P_k on $\mathbb{R}^n$ such that for each d-dimensional subspace σ the polynomial

$$P_{\sigma,k}(u) = \int_{\sigma^\perp} f(x+\sigma)\langle x, u\rangle^k dx \qquad \text{for } u \in \sigma^\perp,$$

where $\sigma^\perp$ is the orthogonal complement of σ in $\mathbb{R}^n$ and dx is the surface measure on $\sigma^\perp$, coincides with the restriction $P_k|_{\sigma^\perp}$.

The crucial point of this condition is the independence of $P_{\sigma,k}(u)$ from σ if $u \in \sigma^\perp$, because it is obviously a homogeneous kth-degree polynomial.

Lemma 2.2. (Corollary 2.28. in [6]). *The (d,n) Radon transform is a bijection of $D(\mathbb{R}^n)$ onto $D_H(G(d,n))$.*

This statement characterize the range of our transforms and we will use it as starting point. The following result is a slight extension of Theorem 1.2. of [8] and can be proven by simple calculation.

Lemma 2.3. *Suppose that $v \in C^\infty(\mathbb{R}^{n(d+1)})$ such that it depends only on the d-planes of the $d+1$ points in $\mathbb{R}^n$, $\xi(x_0, x_1, \dots, x_d) \in G(d,n)$ and satisfies*

$$(\partial_{i,k}\partial_{j,e} - \partial_{i,e}\partial_{j,k})\frac{v(x_0, x_1, \dots, x_d)}{\mathrm{Vol}\{x_i - x_0\}_{i=1,d}} = 0, \tag{1}$$

where $0 \le i, j \le d$, $1 \le k, e \le n$. Then the function

$$w(x_0, x_1, \dots, x_d) = v(Ax_0, Ax_1, \dots, Ax_d)$$

also satisfies (1) and also depends only on $\xi(x_0, x_1, \dots, x_d) \in G(d,n)$ for any $A \in SO(n)$, the group of orthogonal automorphisms of determinant 1 of $\mathbb{R}^n$.

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3. The main result

Theorem 3.1. *Let $v \in D(G(d,n))$. Then there exists a function $f \in D(\mathbb{R}^n)$ such that $v = R^n_d f$ if and only if v satisfies the system of PDEs* (1)*.*

Proof. The necessity is proved by Lemma 2.1. Thus we only have to prove the sufficiency of the condition, i.e. that our condition implies the condition of Helgason in Lemma 2.2. From the definition we have for $u \in \sigma^\perp$

$$P_{\sigma,k}(u) = \int_{\sigma\perp} v(x, x+\sigma_1, x+\sigma_2, \dots, x+\sigma_d)\langle x,u\rangle^k dx,$$

where $\{\sigma_i\}_{i=1,d}$ is an orthonormal basis of σ. We have to show that $P_{\sigma,k}(u) = P_{\bar\sigma,k}(u)$ if $u \in \sigma^\perp \cap \bar\sigma^\perp$. First we simplify the statement to be proven.

Without any restriction of generality one can suppose that $|u| = 1$. Let $e_1, e_2, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$. Let $\varepsilon = \xi(0, e_1, e_2, \dots, e_d)$ and let $A \in SO(n)$ such that $\sigma_i = Ae_i$ and $u = Ae_n$. On substituting $x = Ay$ we get

$$P_{\sigma,k}(u) = P_{A\varepsilon,k}(e_n) = \int_{\varepsilon^\perp} v(Ay, A(y+e_1), \dots, A(y+e_d))\langle y, e_n\rangle^k dy,$$

hence by Lemma 2.3. it is enough to consider the case when $u = e_n$ and $\sigma_i = e_i$ to prove the statement below from which our theorem will follow easily.

(∗) If $\sigma_1, \sigma_2, \dots, \sigma_d$ is orthonormal system, $\bar\sigma_1 \perp \sigma_2, \dots, \sigma_d$, $|\bar\sigma_1| = 1$, $\sigma = \xi(0, \sigma_1, \sigma_2, \dots, \sigma_d)$ and $\bar\sigma = \xi(0, \bar\sigma_1, \sigma_2, \dots, \sigma_d)$ then $P_{\sigma,k}(e_n) = P_{\bar\sigma,k}(e_n)$.

Thus we are going to prove that if $E = \alpha_1 e_1 + \sum_{i=d+1}^{n-1} \alpha_i e_i$, where $\alpha_1^2 + \sum_{i=d+1}^{n-1} \alpha_i^2 = 1$, then $P_{\varepsilon,k}(e_n) = P_{\bar\varepsilon,k}(e_n)$, where $\bar\varepsilon = \xi(0, E, e_2, \dots, e_d)$. We will prove it by approximating E in a way that the $P_{\varepsilon_{\mathrm{appr}},k}(e_n)$ does not vary, where $\varepsilon_{\mathrm{appr}} = \xi(0, E_{\mathrm{appr}}, e_2, \dots, e_d)$.

As a first step, take $E_{\mathrm{appr}} = e_1 \sin\beta + e_i \cos\beta$, where $i > d$. Without any restriction of generality we can assume $i = d+1$. Let

$$\begin{aligned} P(\beta) &= P_{\varepsilon_{\mathrm{appr}},k}(e_n) \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} v(x, x+E_{\mathrm{appr}}, \dots, x+e_d)\lambda_n^k d\lambda_n d\lambda_{n-1} \dots d\lambda_{d+1}, \end{aligned}$$

where $x = (-e_1 \cos\beta + e_{d+1} \sin\beta)\lambda_{d+1} + \sum_{i=d+2}^{n} \lambda_i e_i$. Differentiating this with respect to β we obtain

$$P'(\beta) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Big(\lambda_{d+1} \sum_{j=0}^{d} (\sin\beta \partial_{j,1} v + \cos\beta \partial_{j,d+1} v) + \\ + \cos\beta \partial_{2,1} v - sin\beta \partial_{2,d+1} v\Big) \lambda_n^k d\lambda_n \ldots d\lambda_{d+1}.$$

To compute this integral, we need a simple fact. Since v is a function on $G(d, n)$ for every $\nu \in \mathbb{R}(\neq 0)$ we have

$$\frac{v(x_0, x_1, \ldots, x_d)}{\mathrm{Vol}\{x_i - x_0\}_{i=1,d}} = \nu \frac{v(y_0, y_1, \ldots, y_d)}{\mathrm{Vol}\{y_i - y_0\}_{i=1,d}},$$

where $y_k = x_k$ except the i-th for which $y_i = x_j + \nu(x_i - x_j)$. Then the differentiation of this equation with respect to ν at $\nu = 1$ gives

$$\frac{v(x_0, x_1, \ldots, x_d)}{\mathrm{Vol}\{x_i - x_0\}_{i=1,d}} + \sum_{m=1}^{n} (x_i^m - x_j^m) \partial_{i,m} \frac{v(x_0, x_1, \ldots, x_d)}{\mathrm{Vol}\{x_i - x_0\}_{i=1,d}} = 0. \tag{2}$$

Applying this in our situation for $j = 1$ and $j = 0$ our integral becomes

$$P'(\beta) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\cos\beta \partial_{2,1} v - \sin\beta \partial_{2,d+1} v) \lambda_n^k d\lambda_n \ldots d\lambda_{d+1}.$$

Then by partial integration with respect to λ_n we have

$$(k+1) P'(\beta) \\ = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{d}{d\lambda_n} (\cos\beta \partial_{2,1} v - \sin\beta \partial_{2,d+1} v) \lambda_n^{k+1} d\lambda_n \ldots d\lambda_{d+1}.$$

But $\frac{d}{d\lambda_n} = \sum_{j=0}^{d} \partial_{j,n}$ in our case therefore (1) gives that

$$(k+1) P'(\beta) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{-d}{d\lambda_{d+1}} \partial_{2,n} v \ \lambda_n^{k+1} d\lambda_n \ldots d\lambda_{d+1}.$$

Since $n > d + 1$ the integration with respect to λ_{d+1} shows $P'(\beta) \equiv 0$, *i.e.* $P_{\varepsilon_{\mathrm{appr}},k}(e_n) = P_{\varepsilon,k}(e_n)$. One can easily see that with at most $n - 1$ steps of this kind we can reach the general E, therefore $(*)$ is proved.

Now let $\sigma_1, \sigma_2, \ldots, \sigma_d$ and $\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_d$ be orthonormal systems and $\sigma = \xi(0, \sigma_1, \sigma_2, \ldots, \sigma_d)$, $\bar{\sigma} = \xi(0, \bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_d)$. We prove $P_{\sigma,k}(e_n) = P_{\bar{\sigma},k}(e_n)$ step by step showing a sequence of the d dimensional subspaces σ^r such that $P_{\sigma,k}(e_n) = P_{\sigma^1,k}(e_n) = \cdots = P_{\sigma^m,k}(e_n) = P_{\bar\sigma,k}(e_n)$, where all these equations are true by $(*)$.

If $\bar\sigma_i$ is at most for one index j not perpendicular to σ_j it can be substituted into the place of σ_j by $(*)$ on such a way that $P_{\sigma,k}(e_n)$ does not change *i.e.* $P_{\sigma,k}(e_n) = P_{\sigma^1,k}(e_n)$. Let us continue this replacements with the last obtained σ^{r-1} as far as possible. This procedure can stop at the σ^r if and only if there are at least two elements of $\{\sigma_i^r\}_{i=1,d}$, which are not perpendicular to $\bar\sigma_i$.

In this case let σ_j^r and σ_k^r be two vectors not perpendicular to $\bar\sigma_i$. Transforming these two vectors as

$$\sigma_j^{r+1} = \sigma_j^r \cos\alpha + \sigma_k^r \sin\alpha \quad \sigma_k^{r+1} = \sigma_j^r \sin\alpha - \sigma_k^r \cos\alpha,$$

where $\tan\alpha = \langle\bar\sigma_i, \sigma_k^r\rangle / \langle\bar\sigma_i, \sigma_j^r\rangle$, and leaving $\sigma_s^{r+1} = \sigma_s^r$ for other indexes s we obtain the $\sigma^{r+1} = \xi(0, \sigma_1^{r+1}, \sigma_2^{r+1}, \dots, \sigma_d^{r+1})$ subspace, which satisfies obviously $P_{\sigma^{r+1},k}(e_n) = P_{\sigma^r,k}(e_n)$ and has more vectors perpendicular to $\bar\sigma_i$. Continuing this procedure as far as possible we will get a subspace the orthonormal spanning system of which will become treatable by the previous method. As a result of this line of reasoning finally we shall obtain the desired sequence, which proves the theorem.

■

4. The $(1, n)$ case

From now on we concentrate on the special case of $(1, n)$ Radon transform, when we integrate over the lines. We parameterize the lines $G(1, n)$ by Plücker coordinates.

Let $\xi = (\xi_1, \dots, \xi_n)$ and $\eta = (\eta_1, \dots, \eta_n)$ be two different points of the line g. The Plücker coordinates of g are

$$p_{i,k} = \det \begin{bmatrix} \xi_i & \xi_k \\ \eta_i & \eta_k \end{bmatrix} \qquad \text{and} \qquad q_j = \det \begin{bmatrix} \xi_j & 1 \\ \eta_j & 1 \end{bmatrix},$$

where $1 \le i < k \le n$ and $1 \le j \le n$. As it is well known ratios of the coordinates $p_{i,n}$ and q_j are unchanged under replacing ξ and η by two different points of g and determine the line uniquely. Below for brevity we simply write p_i for $p_{i,n}$.

By using the Plücker coordinates now we define a bijection between functions on $G(1, n)$ and the functions on $\mathbb{R}^{2n-2}$. To any function v on $G(1, n)$ let us associate a function u on $\mathbb{R}^{2n-2}$ by the equation

$$v(\xi, \eta) = \left(\sum_{i=1}^{n} \left(\frac{q_i}{q_n} \right)^2 \right)^{1/2} u\left(\frac{x_1}{q_n}, \frac{x_2}{q_n}, \frac{x_3}{q_n}, \dots, \frac{x_{2n-2}}{q_n} \right), \tag{3}$$

where $x_i = \sum_{j=1}^{n-1} (\lambda_{i,j} p_j + \delta_{i,j} q_j)$ and $\lambda_{i,j}, \delta_{i,j} \in \mathbb{R}$ $(1 \le i \le 2n-2)$. The function v can be recovered from u if and only if the matrices

$$\Lambda = [\lambda_{i,j}]_{\substack{i=1,2n-2\\ j=1,n-1}} \qquad \text{and} \qquad \Delta = [\delta_{i,j}]_{\substack{i=1,2n-2\\ j=1,n-1}}$$

are chosen on the way that the $(2n-2) \times (2n-2)$ matrix $\Gamma = [\Lambda, \Delta]$ is invertible.

Since $|\xi - \eta| = \left(\sum_{i=1}^{n} q_i^2\right)^{1/2}$ derivation of $v(\xi, \eta)/|\xi - \eta|$ with respect to ξ_k and η_e, equations (1) and (3) result in

$$\begin{aligned}(4)\qquad &\sum_{i=1}^{2n-2} (\lambda_{i,e}\delta_{i,k} + \lambda_{i,k}\delta_{i,e})\partial_i^2 u + \\ &\quad + \sum_{1 \le i < j}^{2n-2} (\lambda_{i,e}\delta_{j,k} - \lambda_{i,k}\delta_{j,e} + \lambda_{j,e}\delta_{i,k} - \lambda_{j,k}\delta_{i,e})\partial_i\partial_j u = 0.\end{aligned}$$

This means that relation (3) defines an equivalence between the systems of PDEs (1) and (4) if $k, e < n$.

Lemma 4.1. *Relation* (3) *gives a bijection between* $C^2(G(1,n))$ *and* $C^2(\mathbb{R}^{2n-2})$. *For a pair of functions* $u \in C^2(\mathbb{R}^{2n-2})$ *and* $v \in C^2(G(1,n))$ *related by* (3) *the systems of PDEs* (4) *and* (1) *are equivalent.*

Proof. The only non trivial thing to prove here is that if u satisfies (4) and v is defined by (3) then v satisfies (1) not only for $k, e < n$ but also for $e = n$. From (2) we know that

$$\frac{v(\xi,\eta)}{|\xi - \eta|} + \sum_{e=1}^{n} (\xi_e - \eta_e) \frac{\partial}{\partial \xi_e} \frac{v(\xi,\eta)}{|\xi - \eta|} = 0.$$

On differentiating with respect to η_k it follows that

$$\left(\sum_{e=1}^{n} (\xi_e - \eta_e) \frac{\partial^2}{\partial \eta_k \partial \xi_e} - \frac{\partial}{\partial \xi_k} + \frac{\partial}{\partial \eta_k} \right) \frac{v(\xi,\eta)}{|\xi - \eta|} = 0.$$

We add to this equation the one obtained by replacing ξ and η and get in this manner

$$\sum_{e=1}^{n} (\xi_e - \eta_e) \left(\frac{\partial^2}{\partial \eta_k \partial \xi_e} - \frac{\partial^2}{\partial \xi_k \partial \eta_e} \right) \frac{v(\xi,\eta)}{|\xi - \eta|} = 0.$$

Since v satisfies (1) for all $1 \le k, e \le n-1$ finally we obtain the desired formula for $e = n$.

■

Now we can state one of the main theorems of [8] as a simple consequence of our results.

Theorem 4.2. *The function $u\colon\mathbb{R}^4 \to \mathbb{R}$ is a C_c^∞ solution of the ultrahyperbolic PDE*

$$(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)u = 0 \tag{5}$$

if and only if there exists a C_c^∞ function $f\colon\mathbb{R}^3 \to \mathbb{R}$ such that

$$R_1^3 f(\xi,\eta) = \left(\sum_{i=1}^{3}\left(\frac{q_i}{q_3}\right)^2\right)^{1/2} u\left(\frac{p_1+q_2}{q_3}, \frac{-p_2+q_1}{q_3}, \frac{p_1-q_2}{q_3}, \frac{-p_1-q_1}{q_3}\right),$$

where p_i and q_i $(1 \le i \le 3)$ are the Plücker coordinates of the straightline through ξ and η.

Proof. In (2) let us choose the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

as Γ. Then the relationship between v and u defined by (3) is one-to-one and (4) gives only one equation for u namely (5). By Lemma 4.1. and Theorem 3.1. this gives our theorem.

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To find such a nice formula for higher dimension it would be necessary that the PDE (4) does not depend on k and e. This condition gives a lot of linear equations for the elements of the matrix Γ from which one can conclude by counting these equations that for $n \ge 4$ to find matrix Γ that gives only one PDE in (4) is impossible.

In Theorem 3.1. we essentially characterized the range of the (d,n) Radon transform R_d^n by the system of PDEs (1). The fact that the function $v(\xi,\eta)$ depends only on the straightline through ξ and η plays a role of an additional condition there. The following corollary shows how this unessential condition can be removed by making certain transformation on the range of R_1^n. At the same time this result may prove useful in solving partial differential equations.

Corollary 4.3. *The function $u\colon \mathbb{R}^{2(n-1)} \to \mathbb{R}$ is a C_c^∞ solution of the system of ultrahyperbolic PDEs*

$$\left(\frac{\partial^2}{\partial x_k \partial y_e} - \frac{\partial^2}{\partial y_k \partial x_e}\right) u = 0 (1 \le e, k \le n-1), \tag{6}$$

where $u = u(x,y)$, $x = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $y = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$, if and only if there exists a C_c^∞ function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$R_1^n f(\xi,\eta) = \left(\sum_{i=1}^n \left(\frac{q_i}{q_n}\right)^2\right)^{1/2} u\left(\frac{p_1+q_1}{q_n}, \ldots, \frac{p_{n-1}+q_{n-1}}{q_n}, \frac{p_1-q_1}{q_n}, \ldots, \frac{p_{n-1}-q_{n-1}}{q_n}\right),$$

where p_i and q_i $(1 \le i \le n-1)$ are the Plücker coordinates of the straightline through ξ and η.

Proof. In (2) let us choose the following matrix

$$\begin{bmatrix} U & U \\ U & -U \end{bmatrix}$$

as Γ, where U is the unit $(n-1) \times (n-1)$ matrix. One can conclude the proof as in the previous theorem.

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