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New unified Radon inversion formula

Árpád Kurusa

Abstract. We prove two unified Radon inversion formulas using elementary geometry and analysis.

1. Introduction

Let f be a real function on $\mathbb{R}^n$ and assume that it is integrable on each hyperplane. Let $\mathbb{P}^n$ denote the space of all hyperplanes in $\mathbb{R}^n$. The Radon transform Rf of f is defined by

$$Rf(\xi) = \int_\xi f(x)dx,$$

where dx is the natural measure on the hyperplane ξ. Each hyperplane $\xi \in \mathbb{P}^n$ can be written as $\xi = \{x \in \mathbb{R}^n \colon \langle x, \omega\rangle = p\}$, where $\omega \in S^{n-1}$ is a unit vector *and*$\langle ., .\rangle$*is* the usual inner product on $\mathbb{R}^n$. In what follows we identify the continuous functions ϕ on $\mathbb{P}^n$ with continuous functions ϕ on $S^{n-1} \times \mathbb{R}$ satisfying $\phi(\omega, p) = \phi(-\omega, -p)$.

We introduce also the dual transform R_t which maps a continuous function $\phi \in C^0(\mathbb{P}^n)$ to the function $R_t\phi \in C^0(\mathbb{R}^n)$ defined by

$$R_t\phi(x) = \int_{S^{n-}} \phi(\omega, \langle\omega, x\rangle)d\omega.$$

First Radon [10] and John [8] proved, that any C^∞ function f of compact support can be reconstructed from Rf. More precisely, if L denotes the Laplacian on $\mathbb{R}^n$ and $d\omega$ is the area element on S^{n-1} then

(1) $$f(x) = 2(2\pi)^{1-n}(-L)^{(n-1)/2} \int_{S^{n-1}} Rf(\omega, \langle\omega, x\rangle)d\omega \qquad \text{if } n \text{ is odd}$$

(2) $$f(x) = -(2\pi)^{-n}(-L)^{(n-2)/2} \int_{S^{n-1}} \int_{-\infty}^{\infty} \partial_2 Rf(\omega, p)\frac{dp}{\langle\omega, x\rangle - p}d\omega \qquad \text{if } n \text{ is even.}$$

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In formula (2), the Cauchy principal value is taken. Later these formulas were proved under many different assumptions [4,6,7,9,11]. These proofs are based on advanced potential analysis and the inversion formulas are different in the odd and even dimensional cases. Deans [3] gave a unified inversion formula which covered both cases but his formula was not so explicit as (1) and (2).

In this paper we prove two explicit unified inversion formulas, given in the next theorem, using elementary geometry and analysis rather than the potential theory employed by previous authors. In the following theorem, $S(\mathbb{R}^n)$ denotes the Schwartz-space of smooth rapidly decreasing functions on $\mathbb{R}^n$.

Theorem. *If $f \in S(\mathbb{R}^n)$, $2 \le n \in \mathbb{N}$ and*

$$\text{(1.1)}\quad h_A(\omega,p) = C \lim_{\varepsilon\to 0}\int_1^\infty (r^2-1)^{\frac{n-3}{2}} \times \\ \times \Big(\Big(\frac{d}{dr}\Big)^{n-1} Rf(\omega,p+r\varepsilon) + \Big(\frac{d}{dr}\Big)^{n-1} Rf(\omega,p-r\varepsilon)\Big)dr$$

$$\text{(1.2)}\quad h_B(\omega,p) = C \lim_{\varepsilon\to 0}\int_{|r|>\varepsilon} r^{n-2}\Big(\frac{d}{rdr}\Big)^{n-1}\Big(Rf(\omega,p-r)\Big)dr$$

where $C = (-1)^{n-1}\Gamma(n/2)\pi^{1/2}/\Gamma((n-1)/2)(2\pi)^n$, then $f = R_t h_A = R_t h_B$.

It is well known, that the dual transform R_t has non-trivial kernel. So, for any function f, the above functions h_A and h_B are in the preimage of f at R_t, *i.e.* h_A, $h_B \in R_t^{-1} f$. For a clearer formulation, we introduce the operators $\Box$ and Ξ by

$$\Box f(\omega,p) = C \lim_{\varepsilon\to 0}\int_1^\infty (r^2-1)^{\frac{n-3}{2}}\Big(\Big(\frac{d}{dr}\Big)^{n-1} f(\omega,p+r\varepsilon) + \Big(\frac{d}{dr}\Big)^{n-1} f(\omega,p-r\varepsilon)\Big)dr$$

$$\Xi f(\omega,p) = C \lim_{\varepsilon\to 0}\int_{|r|>\varepsilon} r^{n-2}\Big(\frac{d}{rdr}\Big)^{n-1}\Big(f(\omega,p-r)\Big)dr$$

Then our inversion formulas appear in the form $f = R_t\Box Rf$ and $f = R_t \Xi Rf$. These formulas are very similar to the Radon formulas $f = cR_t\Lambda^{n-1}Rf$, where Λ is the Calderon-Zygmund operator in one dimension [12]. A straightforward but lengthy calculations on the Taylor expansion of $(r^2-1)^{(n-3)/2}$ show that $\Box = \Lambda^{n-1}$. Also $\Xi = \Lambda^{n-1}$ can be proved by partial integration (see (12) on *p.* 11 of [5]). We do not go into there detailes in this paper.

The dual transform notion R_t appears in the previously mentioned form in the literature [6]. Now we slightly modify this notion because this (equivalent!) version is more treatable in our considerations. To avoid the misunderstanding, this version is said to be boomerang transform and it is denoted by B [13]. The

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function-space $C^0B(\mathbb{R}^n\backslash 0)$ consists of such continuous real functions on $\mathbb{R}^n\backslash 0$ which can be extended into continuous functions also at the origin 0 along any line lying on 0. The boomerang transform $B\colon C^0B(\mathbb{R}^n\backslash 0) \to C^0(\mathbb{R}^n)$ is defined by

$$Bf(x) = \frac{1}{2}\int_{S^{n-1}} f(\omega\langle\omega, x\rangle)d\omega.$$

The simple connection between R_t and B can be described as follows. For a real function f on $\mathbb{R}^n$ let Pf be the function on $\mathbb{P}^n$ defined by $Pf\ (\xi) = f(x_\xi)$, where x_ξ is the orthogonal projection of the origin 0 on the hyperplane ξ. Then $Bf = R_t Pf$.

Also a useful geometric interpretation of the boomerang transform can be given as follows [13]. Let $f_\omega(t)$ be a continuous function defined on the line $l_\omega = \{t\omega\colon t \in \mathbb{R}\}$. Then the function $f_\omega^w \in C^0(\mathbb{R}^n)$, defined by $f_\omega^w(x) := f_\omega(\langle x, \omega\rangle)$, is a so called 'plane wave' with the axis ω. The function f_ω^w is constant along the hyperplanes which intersect the line l_ω orthogonally. Now take a function $f \in C^0B(\mathbb{R}^n\backslash 0)$ and for any $\omega \in S^{n-1}$ consider the function $f_\omega(t) := f(t\omega)$ on l_ω. Then the map $\omega \to f_\omega^w$ is a function-valued (plane wave-valued) function defined on S^{n-1}. The integral of this function is just Bf i.e.

$$Bf = \frac{1}{2}\int_{S^{n-1}} f_\omega^w d\omega.$$

Finally we sketch the main ideas of the paper. We start by the investigation of the radial function; this is the main point of our approach. First we show that the transform B is one to one on the space G_0 of smooth radial functions and prove three inversion formulas on this space G_0. (It is worth to note here that a different consideration of the boomerang transform on this space G_0 can be found also in [13].) In the next step, we prove inversion formulas for the radial functions which are defined around arbitrary point $P \in \mathbb{R}^n$. Using Dirac-sequences and convolution, we prove our general inversion formulas from these special ones.

2. Inversion formulas on radial functions

A function $f(x) \in C^0(\mathbb{R}^n)$ is said to be radial at $P \in \mathbb{R}^n$, if there exists a function $\bar{f}\colon \mathbb{R}_+ \to \mathbb{R}$ such that $f(x) = \bar{f}(|x - P|)$. If f is radial at 0 then

$$Bf(x) = |S^{n-2}|\int_0^{\pi/2} \cos^{n-2}(\alpha)\bar{f}(|x|\sin\alpha)d\alpha. \tag{2.1}$$

Lemma 2.1. *If h is a continuous radial function then*

$$Bh(x) = |S^{n-2}| \int_0^1 h(p|x|)(1-p^2)^{(n-3)/2} dp \tag{2.2}$$

and so if $f_i(x) = |x|^i$ $(i \in \mathbb{N})$ then

$$Bf_i(x) = f_i(x) \pi^{(n-1)/2} \frac{\Gamma((i+1)/2)}{\Gamma((n+i)/2)}. \tag{2.3}$$

The proof is a simple calculation which is left to the reader.

Corollary 2.2. *If f is a continuous radial function, then*

(i) $f_{n-1}B(f_{n-1}Bf) = Q^{n-1}(f)(2\pi)^{n-1}$,

(ii) $f_{n-2}B(f_1Bf) = I^{n-1}(f)(2\pi)^{n-1}$,

(iii) $f_{n-1}Bf = Q^{(n-1)/2}(f)(2\pi)^{(n-1)/2}$ *if n odd, where*

$$Qf(x) = |x| \int_0^{|x|} \bar{f}(t)dt \quad \textit{and} \quad If(x) = \int_0^{|x|} \bar{f}(t)dt.$$

Proof. If $f = f_i$ then the formulas follow directly from Lemma 2.1. Since B, Q and I are linear operators, the formulas are valid for polynomials as well. As these integral operators are continuous with respect to the uniform convergence, the proof can be finished by the Weierstrass theorem.

■

Let G_P denote the space of C^∞ radial functions at the point $P \in \mathbb{R}^n$. The following theorem gives our inversion formulas for the radial functions $f \in G_0$.

Theorem 2.3. *The boomerang transform is an injection on G_0 onto G_0. If $f \in G_0$ then*

(i) $B^{-1}f = h_1 = (2\pi)^{1-n} \left(\frac{d}{drr}\right)^{n-1} \Big(f_{n-1}B(f_{n-1}f)\Big)$,

(ii) $B^{-1}f = h_2 = (2\pi)^{1-n} \left(\frac{d}{dr}\right)^{n-1} \Big(f_{n-2}B(f_1f)\Big)$,

(iii) $B^{-1}f = h_3 = (2\pi)^{(1-n)/2} \left(\frac{d}{drr}\right)^{(n-1)/2} (f_{n-1}f)$ *if n odd,*

where $\frac{d}{dr}$ is the radial differentiation.

Proof. Suppose that h is a continuous radial function and $Bh = 0$. Then by Corollary 2.2 we get $I^{n-1}(h) = 0$. Using differentiation $(n-1)$-times, we have $h = 0$, *i.e.* the boomerang transform B is one-to-one.

Since the three cases are very similar we deal only with the second one. $h_2 \in G_O$ follows immediately from

$$|x|^{n-2}B(f_1 f)(x) = |S^{n-2}| \int_0^{|x|} \overline{f}(p)p(|x|^2 - p^2)^{(n-3)/2}dp. \tag{2.4}$$

To see $f = Bh_2$, integrate (ii) $(n-1)$-times. Since h_2 is zero of order $n-1$ at the origin, we get

$$f_{n-2}B(f_1 f) = I^{n-1}(h_2)(2\pi)^{n-1}.$$

This implies $f = Bh_2$ by (ii) of Corollary 2.2.

■

The following statement easily follows from $h_\omega^w(x+y) = h_\omega(\langle x, \omega\rangle + \langle y, \omega\rangle)$ and from

$$Bh = \frac{1}{2}\int_{S^{n-1}} h_\omega^w d\omega. \tag{2.5}$$

Lemma 2.4. *If $f = Bh$, then*

$$f_y = B\Big(h\Big(x + x\frac{\langle x, y\rangle}{\langle x, x\rangle}\Big)\Big), \tag{2.6}$$

where $f_y(x) = f(x+y)$.

Notice that by this lemma and by Theorem 2.3, inversion formulas can be introduced for the radial functions at an arbitrary point P. Using radial Dirac-sequences and convolution, the procedure leads to the general inversion formulas. We follow this way in our proof. A sequence of functions $\{v_k\}$ is called delta-convergent if it tends to the Dirac distribution in the dual space of continuous bounded functions.

Proposition 2.5. *Let $f \in S(\mathbb{R}^n)$ and let $\{v_k\} \subset G_0$ be delta-convergent. If the sequence*

$$h_k(x) = \int_{\mathbb{R}} Rf(e_x, |x| - r)\overline{B^{-1}v_k}(|r|)dr, \qquad x \in \mathbb{R}^n \backslash 0 \tag{2.7}$$

where $e_x = x/|x|$, has limit function h, then $f = Bh$.

Proof. By the substitution $r = |x| - s$ and by the Fubini theorem we get

$$h_k(x) = \int_{\mathbb{R}} Rf(e_x, s)B^{-1}v_k(x - se_x)ds = \int_{\mathbb{R}^n} f(y)B^{-1}v_k\Big(x - x\frac{\langle x, y\rangle}{\langle x, x\rangle}\Big)dy. \tag{2.7}$$

From Lemma 2.4 we obtain

$$Bh_k(x) = \int_{\mathbb{R}^n} f(y)v_k(x - y)dy, \qquad x \in \mathbb{R}^n, \tag{2.9}$$

which proves the proposition completely.

■

3. Proof of the main Theorem

We need two technical lemmas. The first statement immediately follows by integrating in polar coordinates.

Lemma 3.1. *If $\{v_k\} \subset G_O$ is a delta-convergent sequence, then*

$$w_k\colon \mathbb{R} \to \mathbb{R} \quad (r \;\mapsto |r|^{n-1}\bar{v}_k(|r|)|S^{n-1}|/2) \tag{3.1}$$

is also delta-convergent.

Lemma 3.2. *If $\gamma \in C^\infty(\mathbb{R})$ and $f(r) = \gamma(r) - \gamma(-r)$, then $\frac{1}{r}\left(\frac{d}{drr}\right)^k f \in C^\infty$ and*

$$\lim_{r\to 0} \frac{1}{r}\Big(\frac{d}{drr}\Big)^k f(r) = f^{(2k+1)}(0)\frac{1}{(2k+1)!!}.$$

Proof. By induction we get

$$\Big(\frac{d}{drr}\Big)^k f(r) = \gamma_k(r) - \gamma_k(-r), \tag{3.2}$$

where $\gamma_k \in C^\infty$ and $\gamma_k(0) = 0$. This proves the first statement. The second assertion follows immediately using the Taylor expansion of γ.

■

Now we prove our main theorem. The function $Rf(e_x, |x| - r)$ is denoted by $\varphi(r)$. When the x-dependence is important, we denote it by $\varphi_x(r)$.

Proof of (1.1). By ii) of Theorem 2.3 and by Proposition 2.5, we have

$$(3.3)\qquad h_k(x) = (2\pi)^{1-n} \int_0^\infty (\varphi(r)+\varphi(-r)) \left(\frac{d}{dr}\right)^{n-1} (f_{n-2}B(f_1 v_k))\Big|_{re_x} dr.$$

By Theorem 2.3 there exists a function $U_k \in G_0$ such that $v_k = BU_k$ and so

$$(3.4)\qquad I^m U_k = (2\pi)^{1-n} \left(\frac{d}{dr}\right)^{n-1-m} (f_{n-2}B(f_1 v_k)).$$

Therefore by partial integration in (3.3) we get

$$(3.5)\qquad h_k(x) = (-2\pi)^{1-n} \int_0^\infty \left(\frac{d}{dr}\right)^{n-1} (\varphi(r)+\varphi(-r)) r^{n-2} B(f_1 v_k)(re_x) dr,$$

where the remainders vanish at 0 by (3.4) and at ∞ by $Rf \in S(S^{n-1}\times\mathbb{R})$ [7]. Use (2.4) and Lemma 3.1 furthermore reverse the order of integrations to see

$$(3.6)\qquad h_k(x) = (-2\pi)^{1-n} \frac{2|S^{n-2}|}{|S^{n-1}|} \int_0^\infty w_k(t) \int_t^\infty g(r) \frac{(r^2-t^2)^{(n-3)/2}}{t^{n-2}} dr dt,$$

where $g(r) = \left(\frac{d}{dr}\right)^{n-1}\varphi(r) + \left(\frac{-d}{dr}\right)^{n-1}\varphi(-r)$ and w_k comes from Lemma 3.1. Making use of the substitution $r = st$ in the inner integral results in

$$(3.7)\qquad h(x) = \lim_{k\to\infty} h_k(x) = (-2\pi)^{1-n} \frac{|S^{n-2}|}{|S^{n-1}|} \lim_{t\to 0} \int_t^\infty g(st)(s^2-1)^{(n-3)/2} ds,$$

which completes the proof.

■

Proof of (1.2). By (i) of Theorem 2.3 and by Proposition 2.5 we have

$$(3.8)\qquad h_k(x) = (2\pi)^{1-n} \int_0^\infty (\varphi(r)+\varphi(-r)) \left(\frac{d}{drr}\right)^{n-1} (f_{n-1}B(f_{n-1}v_k))\Big|_{re_x} dr.$$

As in the previous proof, use integration by parts to get

$$(3.9)\quad h_k(x) = (-2\pi)^{1-n} \int_0^\infty \left(\frac{d}{rdr}\right)^{n-1} (\varphi(r)+\varphi(-r)) r^{n-1} B(f_{n-1}v_k)(re_x) dr.$$

The function $g(r) = \left(\frac{d}{rdr}\right)^{n-1}(\varphi(r)+\varphi(-r))r^{n-1}$ is of class C^∞ by Lemma 3.2. Therefore use Lemma 3.1, Lemma 2.1 and the Fubini theorem to get

$$(3.10)\qquad h_k(x) = (-2\pi)^{1-n} \frac{2|S^{n-2}|}{|S^{n-1}|} \int_0^\infty w_k(s) \int_s^\infty \frac{g(r)}{r} (1-s^2/r^2)^{(n-3)/2} dr ds,$$

where w_k comes from Lemma 3.1. Thus we have

$$h(x) = \lim_{k\to\infty} h_k(x) = (-2\pi)^{1-n}\frac{|S^{n-2}|}{|S^{n-1}|}\lim_{s\to 0}\int_s^\infty \frac{g(r)}{r}(1-s^2/r^2)^{(n-3)/2}dr. \tag{3.11}$$

To obtain the theorem, we have to prove that

$$0 = \lim_{s\to 0}\int_s^\infty \frac{g(r)}{r}(1-(1-s^2/r^2)^{(n-3)/2})dr. \tag{3.12}$$

For this purpose break up the integral into two parts as $[s, 2s]$ and $(2s, \infty)$ and transform the first part into an integral on [1,2] to see that it tends to zero. The other integral on $(2s, \infty)$ converges to zero simply by the Lebesgue dominated convergence theorem. This completes the proof.

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It should be mentioned that the odd dimensional inversion formula (1) can be proved easily using (iii) of Theorem 2.3, Proposition 2.5 and Lemma 3.1.

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References

[1] A. M. Cormack, Representation of a function by its line integrals with some radiological applications I.,II., *J. Appl. Phys.*, **34** (1963), 2722–2727; 35(1964), 2908–2913.

[2] A. M. Cormack and E. T. Quinto, A Radon transform on spheres through the origin in $\mathbf{R}^n$ and applications to the Darboux equation, *Trans. AMS*, **260** (1980), 575–581.

[3] S. R. Deans, A unified Radon inversion formula, *J. Math. Phys.*, **19** (1978), 2346–2349.

[4] I. M. Gelfand and G. E. Shilov, *Generalized Functions* vol. **1.**, Academic Press, Inc., 1964.

[5] I. M. Gelfand, M. I. Graev and N. Ya. Vilenkin, *Generalized Functions* vol. **5.**, Academic Press, Inc., 1966.

[6] S. Helgason, *Groups and geometric analysis*, Academic Press, Inc., 1984.

[7] S. Helgason, The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds, *Acta Math.*, **113** (1965), 153–180.

[8] F. John, *Plane waves and spherical means*, Springer-Verlag, Reprint, (1981).

[9] D. Ludwig, The Radon transform on Euclidean space, *Comm. Pure Appl. Math.*, **69** (1966), 49–81.

[10] J. Radon, Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten, *Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Nat. kl.*, **69** (1917), 262–277.
[11] D. C. Solmon, The $X - ray$ transform, *J. Math. Anal. Appl.*, **56** (1976), 61–83.
[12] D. C. Solmon, Asymptotic formulas for the dual Radon transform and applications, *Math. Z.*, **195** (1987), 321–343.
[13] Z. I. Szabó, Hilbert's fourth problem I., *Adv. in Math.*, **59** (1986), 185–301.

Á. Kurusa, Bolyai Institute, Aradi vértanúk tere 1., $H - 6720$ Szeged, Hungary; *e-mail*: kurusa@math.u-szeged.hu

