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Support theorems for totally geodesic Radon transforms on constant curvature spaces

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Abstract. We prove a relation between the k-dimensional totally geodesic Radon transforms on the various constant curvature spaces using the geodesic correspondence between the spaces. Then we use this relation to obtain improved support theorems for these transforms.

0. Introduction

Let $\mathcal{M}^n$ be an n-dimensional simply connected Riemannian manifold of constant curvature κ. Normalizing the metric so that $\kappa = -1, 0$ or $+1$ we have to deal only with the hyperbolic space $\mathbb{H}^n$ $(\kappa = -1)$, the Euclidean space $\mathbb{R}^n$ $(\kappa = 0)$ and the sphere S^n $(\kappa = +1)$.

For a fixed k $(1 \le k \le n-1)$ let ξ be an arbitrary totally geodesic submanifold of $\mathcal{M}^n$ of dimension k. The k-dimensional totally geodesic Radon transform $\mathrm{R}f$ of $f \in \mathrm{L}^2(\mathcal{M}^n)$ is defined by

$$\mathrm{R}f(\xi) = \int_\xi f(x)\,dx, \tag{0.1}$$

where dx is the surface element on ξ induced by the metric of $\mathcal{M}^n$ [6].

Our first goal in this paper is to prove a link between these Radon transforms taken on different constant curvature spaces. The proof is based on the geodesic correspondence between the spaces [10]. The idea to use projection from constant curvature spaces to Euclidean spaces appeared also in [3,11].

The connection, established in Theorem 2.1 allows one to transpose results from one space to the other, hence it can be used to get inversion formulas, range characterizations [1] and so on. We shall use it to obtain support theorems.

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Roughly speaking a support theorem states that if the function f is in a suitable function space of $\mathcal{M}^n$ and the support of $\mathrm{R}f$, supp $\mathrm{R}f$, is bounded, then supp$f \subseteq P$ supp $\mathrm{R}f$, where P maps the set of the total geodesics into $\mathcal{M}^n$ so that the total geodesics correspond to their point closest to the origin. For more information about the applications of support theorems we refer to [4,5,7].

We consider the cases $k = n-1$ and $k \le n-2$ separately, because of their principal differences.

In the case $k = n-1$, Helgason's support theorem for the Euclidean space [5] via our Theorem 2.1 implies significant improvement on his support theorem for the hyperbolic space [5] replacing the rapid decrease by a decrease of order n. Transposing Helgason's support theorem to the sphere we find the support theorem for C^∞ functions vanishing with all their derivatives at the equator.

In the case $k \le n-2$, we use Schneider's result [12] to obtain support theorem on the sphere. After transposing this to the other spaces we find the support theorem on all the constant curvature spaces supposing only a finite order decrease on the functions, instead of the rapid decrease.

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1. Preliminaries

In this section we give the most necessary preliminaries and theorems upon which Section 2 is based. Most of the facts listed here are proved in [5,8,9,10,14].

The Riemannian metric on $\mathcal{M}^n$, one of the three spaces of constant curvature, is completely described by the 'size function' $\nu\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. The size function ν determines the radius $\nu(r)$ of the Euclidean sphere in $\mathbb{R}^n$ that is isometric to the geodesic sphere of radius r in $\mathcal{M}^n$ [14].

Other important function on the constant curvature spaces is the 'projector function' $\mu\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. This function generates a geodesic correspondence between the constant curvature spaces and the Euclidean space via the geodesic polar coordinatization [10]. Take a fixed point in $\mathcal{M}^n$, say O, and the origin in $\mathbb{R}^n$. Then the map, that makes the geodesic correspondence between these spaces is

$$\tilde{\mu}\colon \mathcal{M}^n \longrightarrow \mathbb{R}^n \qquad \Big(\mathrm{Exp}_O r\omega \mapsto \mu(r)\omega\Big), \tag{1.1}$$

where ω is a unit vector in $\mathrm{T}_O\mathcal{M}^n$, $r \in \mathbb{R}_+$ and $\mathbb{R}^n$ is identified with $\mathrm{T}_O\mathcal{M}^n$. Further on we shall use only the open half sphere $\mathbb{P}^n$ having the center point O rather than the whole sphere S^n because the odd functions have obviously zero Radon transform on the sphere [5].

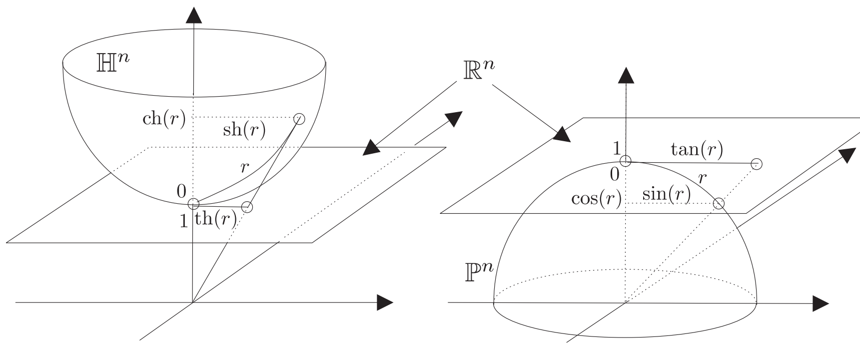

Figure 1. The geometric meaning of the function $\tilde{\mu}$ can be read off this figure, where the quadratic model of the hyperbolic space and the sphere in $\mathbb{R}^{n+1}$ [5] is 'projected' to the hyperplane $\mathbb{R}^n$, determined by the equation $x_{n+1} = 1$, by the straight lines through the origin of $\mathbb{R}^{n+1}$. For further details see the subject of the projective realization of constant curvature spaces in standard textbooks.

From the Figure 1 one can easily read off the entries of the table

$\mathcal{M}^n$	κ	ν	μ
$\mathbb{H}^n$	-1	$\sinh r$	$\tanh r$
$\mathbb{R}^n$	0	r	r
$\mathbb{P}^n$	$+1$	$\sin r$	$\tan r$

The point is that the geodesics in $\mathcal{M}^n$ are precisely the nonempty intersections of $\mathcal{M}^n$ with the two-planes through the origin of $\mathbb{R}^{n+1}$.

We shall need the following easy consequence of Schneider's result, and also Helgason's support theorem for the Euclidean space, that we recall after the proof.

Lemma 1.1. *Let $k \le n-2$, B be a spherical cap, and let the function $f \in \mathrm{C}(\mathrm{S}^n)$ be symmetric. If the integrals of f is zero over all the k-dimensional great subspheres ξ not intersecting the spherical cap B then f is zero outside B and its antipodal spherical cap.*

Proof. Let S^{k+1} be such an intersection of S^n with a $(k+2)$-dimensional subspace of $\mathbb{R}^{n+1}$ that does not intersect the spherical cap B. Then no k-dimensional great subsphere ξ of S^{k+1} intersects B, hence all the integrals of $f \in \mathrm{C}(\mathrm{S}^n)$ over the hypersubspheres of S^{k+1} vanish. Therefore $f \equiv 0$ on S^{k+1} by [12] that proves the lemma.

■

Theorem 1.2. (Theorem 2.6 in [5].) *Let $f \in \mathrm{C}(\mathbb{R}^n)$ satisfy the following conditions:*
(1) *For each integer $m > 0$, $|x|^m f(x)$ is bounded on $\mathbb{R}^n$.*
(2) *For each hyperplane ξ outside the unit ball, $|x| < 1$, $\mathrm{R}f(\xi) = 0$.*
Then $f(x) = 0$ for $|x| > 1$.

Note that counterexamples of Helgason show the condition (1) is necessary for the result.

2. Link between the Radon transforms

In this section we prove Theorem 2.1 that makes connection between the k-dimensional Radon transform on the Euclidean space and the k-dimensional totally geodesic Radon transform on the constant curvature spaces $\mathcal{M}^n$ ($1 \leq k \leq n-1$). Recently and independently this connection was discovered for the hyperbolic spaces in [1], where it is used to get range characterizations.

For convenience we use geodesic polar coordinatization for all the constant curvature spaces considered. For this reason, after fixing a point $O \in \mathcal{M}^n$ we write the function f in the form $f(\omega, r) = f(\mathrm{Exp}_O r\omega)$, where $\omega \in \mathrm{T}_O\mathcal{M}^n$ is a unit vector and r is the distance coordinate. For a k-dimensional totally geodesic submanifold ξ ($1 \leq k \leq n-1$) we shall use the notation $|\xi|$ to denote the distance of ξ from O.

Theorem 2.1. *If $f \in \mathrm{L}^2(\mathbb{R}^n)$ and $g(\tilde{\mu}^{-1}(x)) = f(x)(1+\kappa|x|^2)^{\frac{k+1}{2}}$ then*

$$(\mathrm{R}_k f)(\xi) = (1+\kappa|\xi|^2)^{-1/2}(\tilde{\mathrm{R}}_k g)(\tilde{\mu}^{-1}(\xi)), \tag{2.1}$$

where $1 \leq k \leq n-1$, R_k, resp. $\tilde{\mathrm{R}}_k$, denotes the k-dimensional Radon transform on $\mathbb{R}^n$, resp. $\mathcal{M}^n$, ξ is a k-dimensional hyperplane in $\mathbb{R}^n$, $x \in \mathbb{R}^n$, $\tilde{\mu}^{-1}$ is the inverse of $\tilde{\mu}$ and $\kappa \in \{-1, 0, +1\}$ is the curvature of $\mathcal{M}^n$. In the hyperbolic case we require $\mathrm{supp} f \subset \mathrm{B}^n$, B^n is the unit ball in $\mathbb{R}^n$.

Proof. Taking the $(k+1)$-dimensional totally geodesic submanifold in $\mathbb{R}^n$, resp. $\mathcal{M}^n$, spanned by O and ξ, resp. $\tilde{\mu}^{-1}(\xi)$, and restricting the Radon transform to this spanned submanifold one can assume that $k+1 = n$. We do this and omit the index k of R and $\tilde{\mathrm{R}}$.

According to Theorem 4.1 of [10] for $g \in \mathrm{L}^2(\mathcal{M}^n)$ we have

$$\tilde{\mathrm{R}}g(\bar{\omega}, h) = \int_{\mathrm{S}^{n-1}_{\bar{\omega},\cos\alpha_0(h)}} g\Big(\omega, \mu^{-1}\Big(\frac{\mu(h)}{\langle\omega,\bar{\omega}\rangle}\Big)\Big) \frac{\Big(\frac{\langle\omega,\bar{\omega}\rangle}{\mu^2(h)} + \kappa\Big)^{-n/2}}{\nu(h)}\, d\omega, \tag{2.2}$$

where we used the polar coordinates $(\bar{\omega}, h)$ of the point $\mathrm{Exp}_O(h\bar{\omega})$ of ξ nearest to the origin O. Further on we shall call $(\bar{\omega}, h)$ the polar coordinates of ξ. S^{n-1} is the unit sphere in $\mathrm{T}_O\mathcal{M}^n$ and $d\omega$ is the surface measure on it. For $t \in [-1, +1)$

$$\mathrm{S}^{n-1}_{\bar{\omega},t} = \{\omega \in \mathrm{S}^{n-1} : \langle \omega, \bar{\omega} \rangle > t\}, \tag{2.3}$$

where $\langle .,. \rangle$ is the standard scalar product in $\mathrm{T}_O\mathcal{M}^n$ identified with $\mathbb{R}^n$. Finally $\alpha_0(h)$ is the half of the angle the submanifold ξ subtends at the origin O. (Let γ be the unique geodesic through O perpendicular to ξ. For any point $P \in \xi$ let $\gamma(P)$ denote the geodesic joining P and O. The angle of γ and $\gamma(P)$ will tend to $\alpha_0(h)$ if P goes to a "farthest" point of ξ. Thus $\alpha_0(h) \equiv \frac{\pi}{2}$ for $\kappa = +1$ or $\kappa = 0$, and $\alpha_0(h) = \arccos \tanh h$ for $\kappa = -1$.) μ, ν and $\kappa \in \{-1, 0, +1\}$ are set according to $\mathcal{M}^n$, of course.

As it is well known, or take the equation (2.2) for the special entries $\kappa = 0$, $\mu(r) = r$ and $\nu(r) = r$, the Euclidean Radon transform R is

$$\mathrm{R}f(\bar{\omega}, h) = \int_{\mathrm{S}^{n-1}_{\bar{\omega},0}} f\Big(\omega, \frac{h}{\langle \omega, \bar{\omega} \rangle}\Big) \frac{h^{n-1}}{\langle \omega, \bar{\omega} \rangle^n} \, d\omega. \tag{2.4}$$

Let the polar coordinates of ξ be $(\bar{\omega}, h)$ in $\mathbb{R}^n$ and observe that then the polar coordinates of $\tilde{\mu}^{-1}(\xi)$ are $(\bar{\omega}, \mu^{-1}(h))$. Substituting into the equation (2.2) the function $g(\omega, r) = f(\omega, \mu(r))(1 + \kappa\mu^2(r))^{n/2}$, which is $g(\tilde{\mu}^{-1}(x)) = f(x)(1 + \kappa|x|^2)^{\frac{k+1}{2}}$ in polar coordinates, we obtain

$$\tilde{\mathrm{R}}g(\bar{\omega}, h) = \int_{\mathrm{S}^{n-1}_{\bar{\omega},\cos\alpha_0(h)}} f\Big(\omega, \frac{h}{\langle \omega, \bar{\omega} \rangle}\Big) \frac{\Big(1 + \kappa \frac{h^2}{\langle \omega, \bar{\omega} \rangle}\Big)^{n/2}}{\Big(\frac{\langle \omega, \bar{\omega} \rangle}{h^2} + \kappa\Big)^{n/2}} \frac{1}{\nu(\mu^{-1}(h))} \, d\omega. \tag{2.5}$$

Since $\frac{\mu(r)}{\nu(r)} = \sqrt{1 + \kappa\mu^2(r)}$ this equation immediately gives (take $r = \mu^{-1}(h)$) the formula of the theorem. (Note that $\alpha_0(h)$ can be less than $\frac{\pi}{2}$ only in the hyperbolic case, but then the support of f is appropriately restricted to this angle.)

∎

We note here, that by Beltrami's theorem, the only Riemannian manifolds that have geodesic correspondences between each other are the spaces of constant curvature, therefore similar connections between Radon transforms on more general spaces are unlikely.

3. The support theorems

In principle, in this section we only transpose the known results Lemma 1.1 and Theorem 1.2 from their spaces to other spaces via our Theorem 2.1.

Theorem 3.1. *Let $k \le n-2$ and assume $g \in \mathrm{C}(\mathcal{M}^n)$ satisfies the following conditions:*

(i) *The function $g(\omega, r)\nu^{k+1}(r)$ tends to a finite number $G(\omega)$ for each $\omega \in \mathrm{S}^{n-1} \subset \mathrm{T}_O\mathcal{M}^n$ as $\left\{\begin{matrix} r \to \pi/2 & if & \kappa = +1 \\ r \to \infty & if & \kappa = 0, -1 \end{matrix}\right\}$ and $G(\omega) = G(-\omega)$.*

(ii) *$\tilde{\mathrm{R}}_k g(\xi) = 0$ for each k-dimensional totally geodesic ξ with $|\xi| > 1$.*

Then $g(\omega, h) = 0$ for $h > 1$.

Proof. If $\kappa = +1$ consider the function $h\colon \mathrm{S}^n \longrightarrow \mathbb{R}$ defined on the open upper half of S^n by $g(\omega, r)$, on the open bottom half of S^n by the reflection of the upper half with respect to the center of S^n, and on the equator by $G(\omega)$. Then h is a symmetric continuous function on S^n because of (i) and of $\nu(\pi/2) = 1$, hence Lemma 1.1 gives the assertion.

If $\kappa = 0$ consider the function $h\colon \mathbb{P}^n \longrightarrow \mathbb{R}$ defined by $h(\tilde{\mu}^{-1}(x)) = g(x)(1 + |x|^2)^{\frac{k+1}{2}}$ where $\tilde{\mu}$ corresponds to $\mathbb{P}^n$. By Theorem 2.1, then $(\mathrm{R}_k g)(\xi)\sqrt{1 + |\xi|^2} = (\tilde{\mathrm{R}}_k h)(\tilde{\mu}^{-1}(\xi))$ derives. Thus to get the statement of the theorem we only have to verify if the function h satisfies our theorem's condition (i). Using its definition in polarcoordinates, $h(\omega, r) = g(\omega, \tan r)(1 + \tan^2 r)^{\frac{k+1}{2}}$, this is done by

$$\lim_{r \to \frac{\pi}{2}} h(\omega, r)\sin^{k+1} r = \lim_{r \to \frac{\pi}{2}} g(\omega, \tan r)(1 + \tan^2 r)^{\frac{k+1}{2}}$$
$$= \lim_{t \to \infty} g(\omega, t)(1 + t^2)^{\frac{k+1}{2}} = \lim_{t \to \infty} g(\omega, t)t^{k+1} = G(\omega),$$

where the last equation comes just from the condition (i) on g.

If $\kappa = -1$ consider the function $h\colon \mathrm{B}^n \longrightarrow \mathbb{R}$ defined by $g(\tilde{\mu}^{-1}(x)) = h(x)(1 - |x|^2)^{\frac{k+1}{2}}$. Then

$$h(\omega, \tanh r) = \frac{g(\omega, r)}{(1 - \tanh^2 r)^{\frac{k+1}{2}}}$$

in polar coordinates and Theorem 2.1 gives $(\mathrm{R}_k h)(\xi)\sqrt{1 - |\xi|^2} = (\tilde{\mathrm{R}}_k g)(\tilde{\mu}^{-1}(\xi))$. Thus the statement of the theorem will follow after the verification of our theorem's condition (i) on h. To do this an easy observation is that

$$\lim_{t \to 1} h(\omega, t) = \lim_{r \to \infty} \frac{g(\omega, r)}{(1 - \tanh^2 r)^{\frac{k+1}{2}}}$$
$$= \lim_{r \to \infty} g(\omega, r)\cosh^{k+1} r = \lim_{r \to \infty} g(\omega, r)\sinh^{k+1} r = G(\omega).$$

Thus h can be extended to a function h_* onto the closed unit ball so that it is uniformly continuous. Then h_* extends continuously onto $\mathbb{R}^n$ by Tietze extension theorem so that the extended function h_{**} has compact support. This function, h_{**}, obviously satisfies the condition (i) of our theorem that completes the proof. ■

Now we turn to the Radon transforms integrating on the 1-codimensional total geodesic submanifolds.

Theorem 3.2. *Let $g \in \mathrm{C}(\mathcal{M}^n)$ satisfy the following conditions ($k = n-1$):*
(i^-) *If $\kappa = -1$, the function $g(\omega, r)\sinh^n(r)$ tends to a finite number for each $\omega \in \mathrm{S}^{n-1} \subset \mathrm{T}_O\mathbb{H}^n$ as $r \to \infty$.*
(i^+) *If $\kappa = 0, +1$, the function $g(\omega, r)\mu^m(r)$ is bounded for each integer $m > 0$.*
(ii) *$\tilde{\mathrm{R}}g(\xi) = 0$ for each $(n-1)$-dimensional totally geodesic ξ if $|\xi| > 1$.*
Then $g(\omega, h) = 0$ for $h > 1$.

Proof. Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by $g(\tilde{\mu}^{-1}(x)) = f(x)(1 + \kappa|x|^2)^{n/2}$, or in polar coordinates $g(\omega, r) = f(\omega, \mu(r))(1 + \kappa\mu^2(r))^{n/2}$. Then by the continuity of μ^{-1} and g the function f is continuous. Note that in the hyperbolic case this continuity is meant only in the open unit ball $\mathrm{B}^n \subset \mathbb{R}^n$.

If $\kappa = -1$ the proof goes in the same way as in the previous theorem, so we leave it to the reader.

If $\kappa = 0, 1$ the condition (i^+) guarantees directly the first condition of Theorem 1.2 . To see this one has to observe only $\kappa = 1$. Then $f(\omega, \tan r)\tan^m r = g(\omega, r)\cos^n r \tan^m(r)$, and this with condition (i^+) gives the boundedness of $f(\omega, t)t^m$ for each integer $m > 0$.

By our Theorem 2.1 $(\mathrm{R}f)(\xi)\sqrt{1 + \kappa|\xi|^2} = (\tilde{\mathrm{R}}g)(\tilde{\mu}^{-1}(\xi))$, hence the function f satisfies also the second condition of Theorem 1.2 for $\mu(1)$. If namely $|\xi| > \mu(1)$ then $|\tilde{\mu}^{-1}(\xi)| > 1$ since the function μ^{-1} is strictly increasing. Therefore, for such an $(n-1)$-dimensional hyperplane ξ, $|\xi| > \mu(1)$, we have $\tilde{\mathrm{R}}g(\tilde{\mu}^{-1}(\xi)) = 0$ by condition (ii) and so $\mathrm{R}f(\xi) = 0$ too.

Now, Theorem 1.2 implies that $f(x)$ must be zero for $|x| > \mu^{-1}(1)$, hence $g(\omega, r) = 0$ follows for $|r| > 1$. This proves the theorem. ■

Note that Theorem 3.1 is just Lemma 1.1 in the case of the sphere ($\kappa = +1$), and Theorem 3.2 is equivalent to Theorem 1.2 in the case of the Euclidean space ($\kappa = 0$).

For the hyperbolic space, our Theorem 2.2 gives a considerable improved version of Helgason's similar theorem [4,5,6,7] requiring only a decrease of order n

if $k = n-1$. I do not know any other result for $k \leq n-2$. It is interesting to note, that there is no difference between the cases $k = n-1$ and $k \leq n-2$!

For the sphere, our condition in the case of $k = n-1$ is equivalent to the function being C^∞ at the equator, formed by the points having $\pi/2$ distance from the point O, and all of its derivatives being zero there. This condition can not be omitted, because the decay condition in Theorem 1.2 is known to be necessary. Theorem 3.2 therefore shows that the condition imposed in [6,7] that g should vanish on the equator along with its odd order derivatives (since g is symmetric) is not sufficient for the conclusion. As I know support theorems on the sphere had been proved before only with the assumption that the function vanishes in some belt around the equator [9,11].

In the most classical case, on the Euclidean space, Theorem 3.1 gives the support theorem for $k \leq n-2$ requesting the function to decrease only of order $k+1$, that seems to have connection with Solmon's Theorem 7.7 in [13]. This relation will be detailed in a forthcoming paper of Berenstein-Casadio Tarabusi-Kurusa.

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