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Can you recognize the shape of a figure from its shadows?

János Kincses and Árpád Kurusa

Abstract. In connection with the Hammer's X-ray picture problem we discuss the following question: Given two convex compact sets inside a circle such that the sets subtend equal angles at each point of the circle, is it then true that the sets must coincide?

1. Introduction

In 1961 P.C. Hammer proposed the following question: How many X-ray pictures of a convex body must be taken to permit its exact reconstruction? There are, in fact, two different problems here, according as the pictures are taken from infinity, or from finite points. An X-ray picture of a convex body from a direction (from a point) is defined as a function which for any line parallel to the given direction (passing through the given point) gives the length of the segment in which the line intersects the body.

Both cases of Hammer's X-ray problem have nice solutions. In the parallel beam case R.J. Gardner and P. McMullen [3] proved that there are four universal directions such that if any two convex bodies have the same X-ray pictures from these directions then the bodies must coincide. The point source case was handled by K.J. Falconer [2] who proved that, except in certain awkward cases, if the line through the points P_1 and P_2 is known to intersect the interior of the body then the body is uniquely determined by the X-ray pictures taken from the two points.

These results show that the X-ray pictures contain a lot of information about the bodies. From this point of view it is natural to ask what we can say if we take simpler pictures of the body. The first named author proposed considering the shadow pictures. For a plane convex set the shadow picture from a direction (from a point) is defined as the length of the orthogonal projection of the set to

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the line perpendicular to the given direction (the angle which the set subtends at the given point). It is clear that in this case a finite set of directions or points is not enough to distinguish two sets. Moreover, in the parallel beam case the sets cannot be distinguished even if we know the shadow pictures from all directions, as the circle and the Reuleaux triangle show (there are also examples within the class of polygons). So the only interesting case is the point source case. In this paper we will discuss the following

Question. Given two compact convex sets inside a circle such that the sets subtend equal angles at each point of the circle. Is it true then that the sets must coincide?

For general convex sets the answer is negative because for each ellipse there is a circle containing it such that the ellipse subtend a right angle at each point of the circle. J.Green [4] characterized those concentric circles which can be distinguished from any other set. However, if we consider only the polygons then the answer is affirmative to the above question.

2. Results and counterexamples

For a planar compact convex set K we introduce the function

$$\alpha_K(X) = \text{the angle which the set } K \text{ subtends at the point } X.$$

First we investigate the simplest case, when both sets are "proper" segments (a segment is proper if it is not a point).

Lemma 2.1. *If S_1 and S_2 are segments inside the circle C, such that at each point of the circle the segments subtent equal angles, then S_1 and S_2 coincide.*

Proof. The segments must lie on the same line because the intersection points of the line of the segment S_i with the circle are the only zeros of the function $\alpha_{S_1}(X) = \alpha_{S_2}(X) = \alpha_S(X)$ $(X \in C)$. This common line divides the circle into two arcs C_1 and C_2. Choose one of them, say C_1, and consider the function $\alpha_S(X)$ on this arc.

If we consider the function $\alpha_S(X)$ on the whole plane then it is well known that the set $\{X\colon \alpha_S(X) = c\}$, $(c > 0)$ consists of two open circular arcs with endpoints the same as those of the segment S. This implies that the maximum of $\alpha_S(X)$ on C_1 is attained at the unique point M which has the property that the circular arc through M and the ends of the segment S_1 is tangent to C_1. Then

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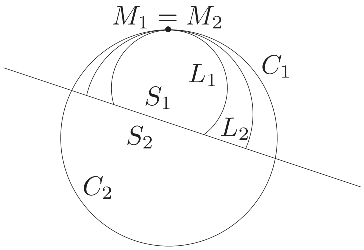

Figure 1.

M, being unique, must also be the corresponding point on C_1 for the segment S_2 (see Figure 1.). Since the length of the intersection of the common line with a circle internally tangent to C at M strictly increases with its radius, this shows that $S_1 = S_2$, as claimed.

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The next result will be the "local" version of Lemma 2.1. For this we need the exact form of the function $\alpha_S(X)$ for a segment S. With the notation of Figure 2. we have

$$\cos\alpha_S(X) = \frac{\overrightarrow{XA}\,.\,\overrightarrow{XB}}{|\overrightarrow{XA}|\cdot|\overrightarrow{XB}|}, \tag{1}$$

where the numerator of the fraction is the scalar product of the vectors , and the denominator is the product of their lengths.

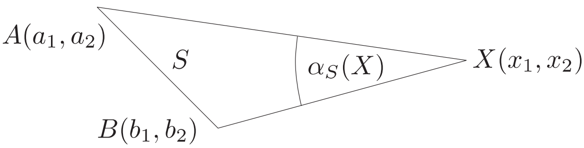

Figure 2.

Lemma 2.2. *If S_1 and S_2 are segments inside the circle C, and there is an arc C' of C such that the segments subtend equal angles at each point of this arc, then S_1 and S_2 coincide.*

Proof. Without loss of generality we may suppose that C is the unit circle. We put the analytic expression $X(t) = (x_1(t), x_2(t)) = (\cos t, \sin t)$, $(0 \le t \le 2\pi)$ of the circle C into the formula of $\cos\alpha_{S_1}(X)$ and $\cos\alpha_{S_2}(X)$ $(i = 1, 2)$

$$\cos\alpha_{S_i}(X(t)) = \frac{(a_1^i - \cos t)(b_1^i - \cos t) + (a_2^i - \sin t)(b_2^i - \sin t)}{\sqrt{(a_1^i - \cos t)^2 + (a_2^i - \sin t)^2}\sqrt{(b_1^i - \cos t)^2 + (b_2^i - \sin t)^2}}$$

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We obtain two analytic functions which are equal on an open interval. Therefore $\cos\alpha_{S_1}(X(t)) = \cos\alpha_{S_2}(X(t))$ on the whole interval $(0 \le t < 2\pi)$, which implies that $\alpha_{S_1}(X(t)) = \alpha_{S_2}(X(t))$ for $0 \le t < 2\pi$. Using Lemma 2.1, we conclude that $S_1 = S_2$.

■

Now we can answer our question positively for polygons.

Theorem 2.3. *If P_1 and P_2 are convex polygons inside the circle C, such that the polygons subtend equal angles at each point of the circle, then P_1 and P_2 coincide.*

Proof. The angle functions in each of the open arcs cut out on C by the edges of P_1 and P_2 uniquely determine, by Lemma 2.2, certain diagonals of P_1 resp. P_2 (see Figure 3).

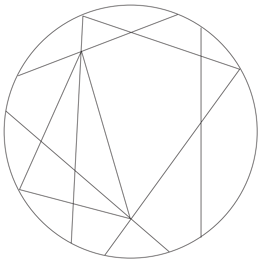

Figure 3.

It is easily seen that P_1 resp. P_2 is the convex hull of these diagonals (note that each vertex of P_1 resp. P_2 is the end-point of some such diagonal). Therefore P_1 and P_2 coincide.

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Using the standard approximation procedure, one might expect that this result can be extended to general convex sets. Surprisingly this is not true, as the following examples show.

The first example is well known in elementary geometry: For any ellipse there is a circle C containing the ellipse such that the ellipse subtends a right angle at each point of the circle. This means that the ellipse cannot be distinguished from the circle concentric to C with radius $\frac{1}{\sqrt{2}}$ times the radius of C.

J.W.Green [4] characterized those angles α for which there is a non-circular convex set which subtends the angle α at each point of the circle. He proved that these angles are exactly those which can be written as $(1 - \frac{m}{n})\pi$, where m is odd, with m and n relatively prime.

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It is very natural to consider here the question of which convex figures are "typical": those which can be distinguished from any other convex figure or those which can not? Define a convex body K to be *distinguishable* from a curve C in which it lies if K is determined by the angles which it subtends at points of C. Now we have the following

Conjecture. The set of compact convex sets in a circle C which are not distinguishable from C is of first Baire category with respect to the Haussdorff metric.

This conjecture would be implied by the stronger statement that any polygon is distinguishable.

Question 1. Is it true that polygons are distinguishable (in the family of all convex bodies)?

3. Lines and other curves

Now let us survey the general features of the proof of the previous section from the viewpoint of replaceing the circle by other curves. The above arguments yield a framework to prove such results.

It is easy to see that Lemma 2.1 remains true if we replace the circle by any convex closed curve. In Lemma 2.2 we strongly used the fact that the circle is an analytic curve, but this property is sufficient. Combining these lemmas we obtain the following generalization of Theorem 2.3:

Theorem 3.1. *Let C be a closed convex curve which is analytic. Then convex polygons are distinguishable from C (in the family of convex polygons).*

This theorem covers for example the case of ellipses. However, our method can give further results. Consider for example the line. First we prove the analog of Lemma 2.1.

Lemma 3.2. *If S_1 and S_2 are segments on the same side of the line l, such that the segments subtend equal angles at each point of the line, then S_1 and S_2 coincide.*

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Proof. We distinguish two cases.

First case: The segments are on the same line l'.

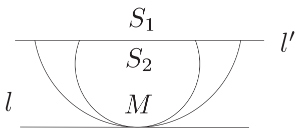

Figure 4.a

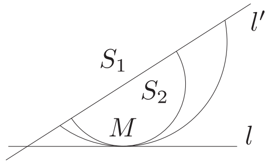

Figure 4.b

This case can be proved in the same way as Lemma 2.1. The function $\alpha_S(X) = \alpha_{S_1}(X) = \alpha_{S_2}(X)$ $(X \in l)$ attains its global maximum at a unique point M on the line l, if l' is parallel to l. If l and l' are not parallel then there are exactly two local maxima places, one for each of the halflines of l determined by the intersection point of l and l' (see Figures 4). The point M is characterized by the property that the circle through M and the endpoints of the segment S_i is tangent to l. But this implies that one of the segments contains the other which is possible only if $S_1 = S_2$ because the value of the maximum is the same for both segments.

Second case: The segments are on different lines. Each of the functions $\alpha_{S_1}(X)$, $\alpha_{S_2}(X)$ $(X \in l)$ can have at most one zero only so either both segments are parallel to l or the lines of the segments intersect each other in a point on l. Label the two segments AB and CD, so that both segments are on the same side of the line AC.

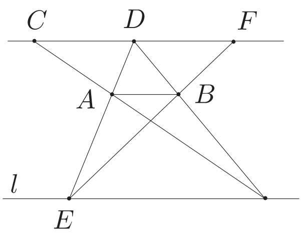

Figure 5.a

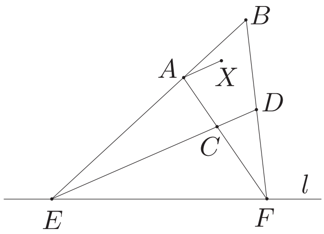

Figure 5.b

Case A: The segments are parallel to l. The lines CA and DB must intersect each other in a point on l. With the notations of Figure 5.a we get, using the properties of similitudes, that $CD = DF$. This means that ED is a median and

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also a bisector of the angle CEF, which implies that ED is perpendicular to CF. Similarly, CB is perpendicular to CF, but this is impossible.

Case B: The lines of the segments intersect each other in a point on l (see Figure 5.b). If the line AC is parallel to l then BD is also parallel to l. In this case the segments AC and BD are parallel to l and subtend equal angles at each point of the line l. By the Case A this is impossible. If the line AC intersects l in a point F then the line BD must contain F. We may suppose that the line AC separates the point F and the segment BD. Using the notations of Figure 6. we get, with the notation $\alpha_S(x,0) = \alpha_S(x)$, that

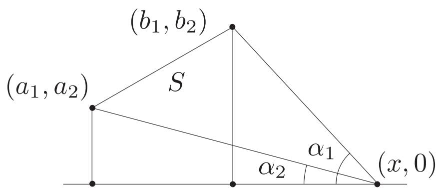

Figure 6.

$$
\begin{aligned}
x\sin\alpha_S(x) &= x\sin(\alpha_1(x) - \alpha_2(x)) \\
&= x(\sin\alpha_1(x)\cos\alpha_2(x) - \cos\alpha_1(x)\sin\alpha_2(x)) \\
&= \frac{x\cdot b_2}{\sqrt{(x-b_1)^2+b_2^2}}\cos\alpha_2(x) - \frac{x\cdot a_2}{\sqrt{(x-a_1)^2+a_2^2}}\cos\alpha_1(x).
\end{aligned} \tag{2}
$$

where (a_1, a_2) and (b_1, b_2) are the coordinates of the vertices of S. This gives that

$$
\lim_{x\to\infty} x\sin\alpha_S(x) = b_2 - a_2, \tag{3}
$$

where $b_2 - a_2$ geometrically means the 'height' of the segment S. Using formula (3) we get that the height of the segments AB and CD are equal. Now translate the segment CD so that the image of the point C is A. The new position of D is X (see Figure 5.b). The point X must be in the quadrangle $ABCD$ because $EAF_\angle < ECF_\angle$ and $EDF_\angle < ECF_\angle$. At the same time X must be on the line going through B and parallel to l, because the segments AX and AB have the same height. This means that $X = B$, that is AB and CD are parallel. But this is impossible, since they have a common point E.

∎

Now comes the "local" version. The proof is analogous to the proof of Lemma 2.2. since the line is also analytic with the parameterization $(t, 0)$.

Lemma 3.3. *If S_1 and S_2 are segments on the same side of the line l, and there is an interval on the line such that the segments subtend equal angles at each point of the interval, then S_1 and S_2 coincide.*

The transition from segments to polygons can be handled in the same way as in the case of circle.

Theorem 3.4. *If P_1 and P_2 are convex polygons on the same side of the line l, such that the polygons subtend equal angles at each point of the line, then P_1 and P_2 coincide.*

We can also prove the analog of Theorem 2.3 for polygons.

Theorem 3.5. *If P_1 and P_2 are convex polygons inside the convex polygon P, such that the polygons P_1 and P_2 subtend equal angles at each point of the polygon P, then P_1 and P_2 coincide.*

Proof. The sides of P_1 and P_2 and vertices of P divide the boundary of P into finitely many line segments. For any of these segments there is a diagonal of P_1 and P_2 resp. such that from each point of the segment these diagonals can be seen from P_1 and P_2 resp. and for each vertex of P_1 (P_2) there is a "visible" diagonal of P_1 (P_2) which contains the given vertex. According to Lemma 3.3 these diagonals coincide, which gives that $P_1 = P_2$.

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Theorems 3.4 and 3.5 shows that we can distinguish polygons from the points of convex analytic curves or polygons.

Question 2. Let C be an arbitrary convex closed curve. P_1 and P_2 are convex polygons inside C, such that the polygons P_1 and P_2 subtend equal angles at each point of C. Is it true that P_1 and P_2 must coincide?

In the case of the circle we have seen the existence of convex figures that can not be distinguished from the circle by their visual angles. The situation is principally different in the case of the straight line because it goes to infinity, but in this case we can also construct a non-circular convex domain subtending the same angle as a circle at each point of the straight line. We achieve this by perturbing the support function of the circle.

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Example. Let the straight line be the x-axis and let K be the unit circle centered at the point $(0,2)$. There are two tangents to K from the point $(x,0)$. The angles of them to the x-axis can be easily calculated as the sum of the angle $(2,0)(x,0)(0,0)\angle$ and the visual angle of the circle at the point $(x,0)$. Thus we obtain

$$\alpha_l(x) = \arctan\frac{-2}{x} + \arctan\frac{1}{\sqrt{x^2+3}} + \begin{cases} \pi, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

as the angle of the 'left hand side tangent' and

$$\alpha_r(x) = \arctan\frac{-2}{x} - \arctan\frac{1}{\sqrt{x^2+3}} + \begin{cases} \pi, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

as the angle of the 'right hand side tangent'. Perturbing these angle-functions with $\varphi(x) = \arctan\frac{\varepsilon}{\sqrt{x^2+3}}$, where ε is a small constant that will be fixed later, we get the new domain K_ε which is bounded by the envelope curves of the straight lines through $(x,0)$ with angle-functions $\beta_l(x) = \alpha_l(x) - \varphi(x)$ and $\beta_r(x) = \alpha_r(x) - \varphi(x)$. (See Figure 7.)

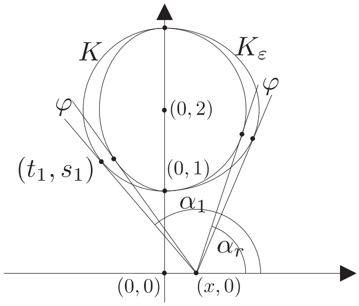

Figure 7.

First we determine the boundary curves of K_ε and then consider their convexity. For brevity we deal only with the straight line with angle-function β_l. Suppose the corresponding envelope curve is parametrized by $(t_l(x), s_l(x))$ for $x \in (-\infty, \infty)$. Calculating the angle of its tangent in different ways, we obtain the equations

$$\tan\beta_l(x) = \frac{s_l(x)}{t_l(x) - x} = \frac{ds_l}{dt_l}\left(= \frac{ds_l/dx}{dt_l/dx}\right).$$

Differentiating the first equation and using for substitution the second one, we obtain the solutions as

$$s_l(x) = \frac{F^2(x)}{\dot{F}(x)} \qquad \text{and} \qquad t_l(x) = x + \frac{F(x)}{\dot{F}(x)},$$

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where $F(x) = \tan\beta_l(x)$. Note that to divide by $\dot{F}(x) = \frac{\dot{\beta}_l(x)}{\cos^2\beta_l(x)}$ is legal because $\dot{\beta}_l > 0$ for small ε since

$$\dot{\beta}_l(x) = \frac{2\sqrt{x^2+3}+x}{(x^2+4)\sqrt{x^2+3}} - \frac{\varepsilon x}{(x^2+3+\varepsilon^2)\sqrt{x^2+3}},$$

from the definition of β_l.

To see the convexity of the envelope curve (t_l, s_l), for small ε, we must prove that

$$0 < \frac{d^2 t_l}{ds_l^2} = \frac{d}{ds_l}\left(\frac{1}{F(x)}\right) = \frac{-\dot{F}(x)}{F^2(x)}\frac{dx}{ds_l}.$$

Since $\dot{F} = \frac{\dot{\beta}_l(x)}{\cos^2\beta_l(x)} > 0$, this is equivalent to

$$0 > \frac{ds_l}{dx} = \left(\frac{F^2(x)}{\dot{F}(x)\cos\beta_l(x)}\right)^2 (2\dot{\beta}_l^2(x)\cot\beta_l(x) - \ddot{\beta}_l(x)).$$

Thus it is enough to show that $0 > 2\dot{\beta}_l^2(x)\cot\beta_l(x) - \ddot{\beta}_l(x)$. Our first observation is that this is true for $\varepsilon = 0$, because then $\beta_l \equiv \alpha_l$ and the circle is convex. But $2\dot{\beta}_l^2(x)\cot\beta_l(x) - \ddot{\beta}_l(x)$ depends continuously on ε, therefore for any finite interval ε can be chosen so small, that the function remains negative on the given interval. Thus we need to observe the required inequality only at infinity.

Let us estimate the order of the functions $\cot\beta_l$, $\dot{\beta}_l$ and $\ddot{\beta}_l$ in terms of powers of $\frac{1}{x}$. This is to divide the nominator's polynomial with the denominator's polynomial so that the highest power in the nominator decreases. We use the easy equation $|x|\sqrt{x^2+3} = x^2 + \frac{3}{2} + O(x^{-2})$, where for the ususal order - function $\lim xO(x^{-1}) =$ *some constant* at infinity. Suppose that $|\varepsilon| < \varepsilon_0$, where ε_0 is small enough. Using the definition of β_l we have

$$\begin{aligned}\cot\beta_l(x) &= \frac{x(x^2+3-\varepsilon) + 2(1+\varepsilon)\sqrt{x^2+3}}{(1+\varepsilon)x\sqrt{x^2+3} - 2(x^2+3-\varepsilon)} \\ &= x\frac{1}{(1+\varepsilon)\operatorname{sgn} x - 2} + \frac{1}{x}\frac{(1+\varepsilon)(2(1+\varepsilon) - (\frac{5}{2}+\varepsilon)\operatorname{sgn} x)}{((1+\varepsilon)\operatorname{sgn} x - 2)^2} + O(x^{-3}),\end{aligned}$$

where $O(x^{-3})$ does not depend on ε, but may depend on ε_0. Similarly

$$\begin{aligned}\dot{\beta}_l(x) &= \frac{(x^2+3+\varepsilon^2)(2\sqrt{x^2+3} - x) - \varepsilon x(x^2+4)}{(x^2+4)\sqrt{x^2+3}(x^2+3+\varepsilon^2)} \\ &= \frac{1}{x^2}(2 - (1+\varepsilon)\operatorname{sgn} x) + \frac{1}{x^4}((5.5 + 4.5\varepsilon + \varepsilon^3)\operatorname{sgn} x - 8) + O(x^{-6})\end{aligned}$$

and

$$\ddot{\beta}_l(x) = \frac{-4x}{(x^2+4)^2} + \frac{(x^2-4)\sqrt{x^2+3} + (x^2+4)\frac{x^2}{\sqrt{x^2+3}}}{(x^2+3)(x^2+4)^2} + \\ + \varepsilon \frac{(x^2-3-\varepsilon^2)\sqrt{x^2+3} + (x^2+3+\varepsilon^2)\frac{x^2}{\sqrt{x^2+3}}}{(x^2+3)(x^2+3+\varepsilon^2)^2} \\ = \frac{1}{x^3} 2((1+\varepsilon)\operatorname{sgn} x - 2) + \frac{1}{x^5}(35 + 3\varepsilon - (25 + 21\varepsilon + 4\varepsilon^3)\operatorname{sgn} x) + O(x^{-7}),$$

where $O(x^{-6})$ and $O(x^{-7})$ are independent of ε ($|\varepsilon| < \varepsilon_0$!). These mean that

$$2\dot{\beta}_l^2(x) \cot \beta_l(x) - \ddot{\beta}_l(x) = \frac{1}{x^5} f(\varepsilon) + O(x^{-7}),$$

where f is a continuous function of ε. By the above formulas we have $f(0) = 1 - 2\operatorname{sgn} x$. Since the sign of $2\dot{\beta}_l^2(x) \cot \beta_l(x) - \ddot{\beta}_l(x)$ near infinity is the same as the sign of $\frac{f(\varepsilon)}{x^5}$ this proves that $2\dot{\beta}_l^2(x) \cot \beta_l(x) - \ddot{\beta}_l(x) < 0$ for $\varepsilon < \varepsilon_0$ and $|x| > x_0$ if ε_0 is small enough and x_0 is big enough. Thus the perturbed figure K_ε is convex for small ε as was to be proved.

4. Further investigations, generalizations

The above considerations show that the shadow pictures taken from the points of a line or a circle are not enough to distinguish any two convex figures. How many pictures should be taken? If we are greedy then we try the ring.

Theorem 4.1. *Let R be a circular ring determined by the concentric circles $C_1 \subset C_2$. If the compact convex sets K_1 and K_2 are inside C_1, such that the sets K_1 and K_2 subtend equal angles at each point of the ring R, then K_1 and K_2 coincide.*

Proof. If $K_1 \neq K_2$ then neither of them contains the other, so they have a common tangent. Let e be a common tangent and denote by A and B the intersection points of e with C_1 and C_2 (see Figure 8.).

From the equality of the visual angles we see that there must be another common tangent through B. Let C be that intersection point of this new tangent and the circle C_1 which is closer to B. Using again the equality of the visual angles we have that through each point of the segment BC there is a common tangent different from BC. This gives that if e is a common tangent then for any line l enclosing a sufficiently small angle with e there is a common tangent parallel to l.

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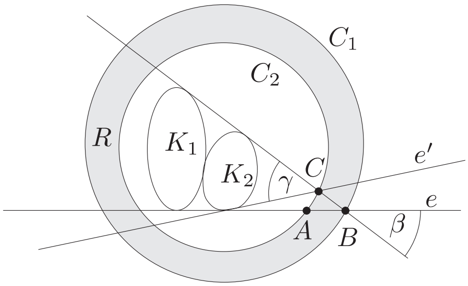

Figure 8.

This means that the set of angles for which there is a common tangent enclosing this angle with a fixed reference ray is open. But it is obviously closed and nonempty which implies that for each line there is a common tangent parallel to it, that is $K_1 = K_2$.

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Nietsche [8] answered a conjecture of Klamkin which can be reformulated the following way.

Theorem 4.2. *Let $C_1 \subset C_2$ be two concentric circles. If the convex figure K is inside the circle C_1, and the set K subtends an angle β_1 at each point of C_1, and the set K subtends anangle β_2 at each point of C_2, then K must be a circle concentric to C_1.*

This fact and some other considerations strengthen our feeling that the answer is the case of two concentric circles.

Question 3. Let $C_1 \subset C_2$ two concentric circles. If the compact convex sets K_1 and K_2 are inside C_1, such that the sets K_1 and K_2 subtend equal angles at each point of C_1 and C_2, then is it true that K_1 and K_2 coincide.?

Recently some new results became known in connection with this problem. In [5],[6] Kurusa proved, that K_1 and K_2 can be distinguished by their visual angles using either two arbitrary but intersecting curves or two “parallel” hyperbolas instead of the circles.

In higher dimensions there are two different generalizations. We may define the shadow picture as the supporting cone of the body from a point and we may ask:

Question 4. If K_1 and K_2 are compact convex bodies inside the sphere S, and for each point of S the supporting cones of K_1 and K_2 from this point are congruent, then is it true that $K_1 = K_2$?

Matsuura [7] proved that the answer is yes if one of the bodies is a ball and Bianchi and Gruber [1] proved that if one of the bodies is an ellipsoid then the other body must also be an ellipsoid. The general case is open.

The other natural definition of shadow picture is the spherical measure of the supporting cone.

Question 5. If K_1 and K_2 are compact convex bodies inside the sphere S, and for each point of S the spherical measure of the supporting cones of K_1 and K_2 from this point are equal, then is it true that $K_1 = K_2$?

There is only one result about this question. T. Ódor informed us, that he could generalize Nietsche's theorem in the obvious way.

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