Romanov's theorem in higher dimensions

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Dedicated to Prof. K. Tandori's 70-th and to Prof. L. Leindler' 60-th birthday

Abstract. It is proved that functions are determined by their integrals over rotation invariant families of hypersurfaces.

1. Introduction

Investigating the problem of determining the vector-function b from the known solution $u\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the Cauchy differential equation

$$(1) \qquad \frac{\partial^2}{\partial t^2} u(x,t) = \triangle u + 4\pi\delta(x-z,t), \qquad u|_{t<0} \equiv 0,$$

where

$$\triangle u = \sum_{i,j=1}^{n} a_{i,j}\partial_i\partial_j u + \sum_{i=1}^{n} b_i \partial_i u + cu,$$

$A = \{a_{i,j}\}_{i,j=1}^n$ positive definite, $A \in \mathrm{C}^3$, $b \in \mathrm{C}^2$ and $c \in \mathrm{C}$, Romanov [7] arrived to the following integral geometric problem.

There is a rotation invariant family of curves in the plane and the integrals of a function are given over each curve of this family. Determine the function!

Romanov solved this problem in [7]. In his solution a system of analytic conditions are given for the family of curves. Later on several other authors considered similar problems [1], [2], [4], [5], [6] but the generalization of this problem to higher dimensions seems to be missing.

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We determine a function by its integrals given over the hypersurfaces of a rotation invariant family of hypersurfaces in $\mathbb{R}^n$ $(n \in \mathbb{N})$.

We deal also with the original problem: we avoid the difficulties tighten between Romanov's analytic conditions and their geometric meaning by geometrically constructing the family of curves. This allows us to calculate explicitly with spherical harmonics. This method leads also for the higher dimensional generalization.

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2. Curves in the plane

Let Ω_α be a set of curves for each $\alpha \in [0, 2\pi)$, called spread, so that Ω_α is Ω_0 rotated around the origin by angle α. Ω_0 consists of curves c_r, where $r \in (0, \infty)$. We work with the family $\Omega = \bigcup_{\alpha=0}^{2\pi} \Omega_\alpha$ of curves.

We suppose that each curve c_r is closed, goes through the origin and has exactly two interions with each of the circle of radius $\varrho \in (0, r)$ centered to the origin. We assume further, that $|P_r| = r$ for the point $P_r \in c_r$ farthest from the origin and that c_r is symmetric with respect to the straight line through P_r and the origin. Without loss of generality, we can also suppose, that the points P_r $(r \in (0, \infty))$ are on the x-axis.

For easier handling, we regard Ω as a two-parameter family of curves $c_{r,\alpha}$. $c_{r,\alpha}$ is c_r rotated around the origin by angle α.

The conditions on the curves imply that in polar coordinates c_r can be parameterized by $c_r(\varrho) = (\varrho, \varphi_r(\varrho))$ in the positive (upper) half plane and by $c_r(\varrho) = (\varrho, -\varphi_r(\varrho))$ in the negative (downward) half plane so that $|c_r(\varrho)| = \varrho \in [0, r]$ and $\varphi_r(\varrho)$ is the angle of $c_r(\varrho)$ to the x-axis.

Our key result follows. Let $\mathbb{T}^2 = \{(r, \rho) \in \mathbb{R}^2 : 0 \le \rho \le r\}$ and $\bar{\mathbb{T}}^2 = \{(r, \rho) \in \mathbb{R}^2 : 0 \le r \le \rho\}$.

Theorem 1. *Let* $f \in \mathrm{L}^2(\mathbb{R}^2)$ *be zero in a neighborhood of the origin,* $\frac{\varphi_r(\varrho)}{\sqrt{r-\varrho}} \in \mathrm{C}^2(\mathbb{T}^2)$. *If the integral of* f *over the curves* $c_{r,\alpha}$ *is zero for* $0 < r < \mu$, *then* $f(x) = 0$ *for* $|x| < \mu$.

Proof. There must exist a $\nu > 0$ so that $f(x) = 0$ for $|x| < \nu$. Therefore the statement is not obvious only in the case when $\nu < \mu$.

The arclength measure on c_r is $\sqrt{1 + \varrho^2 \dot{\varphi}_r^2(\varrho)}$. Therefore the integral $F(r, \alpha)$ of the function f over the curve $c_{r,\alpha}$ takes the form

$$F(r,\alpha) = \int_\nu^r \left(f(\varrho, \alpha + \varphi_r(\varrho)) + f(\varrho, \alpha - \varphi_r(\varrho))\right) \sqrt{1 + \varrho^2 \dot{\varphi}_r^2(\varrho)} \, \mathrm{d}\varrho \tag{1}$$

in polar coordinates. The Fourier expansions of f and F are

$$f(\varrho,\beta)=\sum_{k=-\infty}^{\infty} f_k(\varrho)e^{ik\beta} \quad\text{and}\quad F(r,\alpha)=\sum_{k=-\infty}^{\infty} F_k(r)e^{ik\alpha},$$

where i is the imaginary unit. Substituting these into (1) we obtain

$$F_k(r)=2\int_\nu^r f_k(\varrho)\cos(k\varphi_r(\varrho))\sqrt{1+\varrho^2\dot\varphi_r^2(\varrho)}\,\mathrm{d}\varrho. \tag{2}$$

To solve this integral equation, let us consider its kernel

$$\bar K(r,\varrho)=\cos(k\varphi_r(\varrho))\sqrt{1+\varrho^2\dot\varphi_r^2(\varrho)}.$$

It is proved in [4], that

$$\lim_{\varepsilon\to 0}\dot\varphi_r(r-\varepsilon)\sqrt{\varepsilon}=\frac{-1}{\sqrt{2r(r\kappa_r-1)}}, \tag{3}$$

and

$$\lim_{\varepsilon\to 0}\varphi_r(r-\varepsilon)/\sqrt{\varepsilon}=\sqrt{\frac{2}{r(r\kappa_r-1)}}, \tag{3'}$$

where κ_r is the curvature of c_r at P_r. (It is bigger than $1/r$ because c_r cuts every circle of radius less than r.) Let $K(r,\varrho)=\bar K(r,\varrho)\sqrt{r-\varrho}$. Then $K(r,r)=\sqrt{\frac{r}{2(r\kappa_r-1)}}\neq 0$ and $K\in \mathrm{C}^1([\nu,\mu]^2)$ by the differentiability condition on φ_r. Therefore, Theorem B in [6] implies the uniqueness of the solution of our equation

$$F_k(r)=2\int_\nu^r f_k(\varrho)\frac{K(r,\varrho)}{\sqrt{r-\varrho}}\,\mathrm{d}\varrho. \tag{2'}$$

Since $f_k=0$ is a solution to this equation, our theorem is proved.

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The uniqueness can be proved with the above method also to an other class of curves that is the family of the above curves inverted to a circle centered to the origin. For this family of curves the function space on which our proof works will change to the compactly supported L^2 functions. Let us denote the inverted curves by $\bar c_r$. The function $\bar\varphi_r$ can be defined according to the definition of φ_r.

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Theorem 2. *Let $f \in L^2_0(\mathbb{R}^2)$ and $\frac{\bar\varphi_r(\varrho)}{\sqrt{\varrho-r}} \in C^2(\bar{\mathbb{T}}^2)$. If the integral of f over the curves $\bar c_{r,\alpha}$ is zero for $\mu < r$, then $f(x) = 0$ for $\mu < |x|$.*

3. Higher dimensions

In this section we are going to prove the higher dimensional equivalent of Theorem 1. From now on the dimension $n \geq 3$.

Let $\tilde\omega \in S^{n-1}$, S^{n-1} is the unit sphere, and let $\Omega_{\tilde\omega}$ be a set of hypersurfaces that are fixed by all the rotations that fix the origin and $\tilde\omega$. Let Ω_ω be a set of hypersurfaces for each $\omega \in S^{n-1}$, so that Ω_ω is $\Omega_{\tilde\omega}$ rotated by any of the rotations that fixes the origin and takes $\tilde\omega$ to ω. $\Omega_{\tilde\omega}$ consists of hypersurfaces s_r, where $r \in (0,\infty)$, so that for the point P_r of s_r farthest from the origin $|P_r| = r$. Note that s_r is fixed by all the rotations that fixes the origin and $\tilde\omega$, hence $P_r = r\tilde\omega$. We work with the family $\Omega = \bigcup_{\omega\in S^{n-1}} \Omega_\omega$ of hypersurfaces.

Let the curve c_r be given by a plane cut of s_r through the axis OP_r. We parameterize it by $(\varrho, \varphi_r(\varrho))$ in polar coordinates, just like in Theorem 1.

We suppose that each curve c_r is closed, goes through the origin and has exactly two intersections with each of the spheres of radius $\varrho \in (0,r)$ centered to the origin. Without loss of generality, we can also assume, that the points P_r ($r \in (0,\infty)$) are on the first coordinate axis.

The conditions on the hypersurfaces s_r imply that in polar coordinates the curve c_r can be parameterized by $c_r(\varrho) = (\varrho, \varphi_r(\varrho))$ in the positive (upper) half plane and by $c_r(\varrho) = (\varrho, -\varphi_r(\varrho))$ in the negative (downward) half plane so that $|c_r(\varrho)| = \varrho$ and φ_r is the angle of $c_r(\varrho)$ to the first axis.

For easier handling, we regard Ω as a two-parameter family of hypersurfaces $s_{r,\omega}$, where $s_{r,\omega}$ is s_r rotated by a rotation that fixes the origin and takes $\tilde\omega$ to ω.

The main result of this section is

Theorem 3. *Let $f \in L^2(\mathbb{R}^n)$ be zero in a neighborhood of the origin, $\frac{\varphi_r(\varrho)}{\sqrt{r-\varrho}} \in C^{[\frac{n+2}{2}]}(\mathbb{T}^2)$. If the integral of f over the hypersurfaces $s_{r,\omega}$ is zero for $0 < r < \mu$, then $f(x) = 0$ for $|x| < \mu$.*

Proof. There must exist a $\nu > 0$ so that $f(x) = 0$ for $|x| < \nu$. Therefore the statement is not obvious only in the case when $\nu < \mu$.

The surface measure on s_r is $(\varrho \sin\varphi_r(\varrho))^{n-2}\sqrt{1+\varrho^2\dot\varphi_r^2(\varrho)}\,d\omega\,d\varrho$, where $d\omega$ is the $(n-2)$-dimensional measure on S^{n-2}. Therefore the integral $F(r,\bar\omega)$ of the function f over the hypersurface $s_{r,\bar\omega}$ with respect to the surface measure takes the form

$$F(r,\bar\omega)=\int_\nu^r\int_{\mathrm{S}^{n-2}\perp\bar\omega} f(\varrho,\omega\sin\varphi_r(\varrho)+\bar\omega\cos\varphi_r(\varrho))(\varrho\sin\varphi_r(\varrho))^{n-2}\times \times\sqrt{1+\varrho^2\dot\varphi_r^2(\varrho)}\,\mathrm{d}\omega\,\mathrm{d}\varrho \tag{4}$$

in polar coordinates, where $\bar\omega$ is a unit vector and $\mathrm{S}^{n-2}\perp\bar\omega$ denotes the $(n-2)$-dimensional unit sphere in $\mathbb{R}^n$ perpendicular to $\bar\omega$. Using the Dirac delta distribution δ we get

$$F(r,\bar\omega)=\int_\nu^r\int_{\mathrm{S}^{n-1}} f(\varrho,\omega)\delta(\langle\omega,\bar\omega\rangle-\cos\varphi_r(\varrho))\varrho^{n-2}\sin\varphi_r(\varrho)\sqrt{1+\varrho^2\dot\varphi_r^2(\varrho)}\,\mathrm{d}\omega\,\mathrm{d}\varrho, \tag{5}$$

where $\langle.,.\rangle$ denotes the usual inner product on $\mathbb{R}^n$.

For easier calculation, let $f(r,\omega)=h(r)Y_{k,m}(\omega)$ for a spherical harmonic function $Y_{k,m}$. Then the equation (5) becomes

$$F(r,\bar\omega)=\int_\nu^r h(\varrho)\varrho^{n-2}\sin\varphi_r(\varrho)\sqrt{1+\varrho^2\dot\varphi_r^2(\varrho)}\times \times\int_{\mathrm{S}^{n-1}} Y_{k,m}(\omega)\delta(\langle\omega,\bar\omega\rangle-\cos\varphi_r(\varrho))\,\mathrm{d}\omega\,\mathrm{d}\varrho, \tag{6}$$

Using the Funk–Hecke theorem [8] for the Dirac delta δ we see

$$Y_{k,m}(\bar\omega)\frac{|\mathrm{S}^{n-2}|}{C_m^\lambda(1)}C_m^\lambda(t)(1-t^2)^{\lambda-\frac12}=\int_{\mathrm{S}^{n-1}}\delta(\langle\omega,\bar\omega\rangle-t)Y_{k,m}(\omega)\,\mathrm{d}\omega,$$

where $\lambda=\frac{n-2}{2}$, $|\mathrm{S}^{n-2}|$ is the surface volume of S^{n-2} and C_m^λ is the Gegenbauer polynomial of the first kind. Therefore

$$F(r,\bar\omega)=Y_{k,m}(\bar\omega)\frac{|\mathrm{S}^{n-2}|}{C_m^\lambda(1)}\int_\nu^r h(\varrho)\varrho^{n-2}\sin^{n-2}\varphi_r(\varrho)C_m^\lambda(\cos\varphi_r(\varrho))\sqrt{1+\varrho^2\dot\varphi_r^2(\varrho)}\,\mathrm{d}\varrho. \tag{7}$$

For general function f this means

$$F_{k,m}(r)=\frac{|\mathrm{S}^{n-2}|}{C_m^\lambda(1)}\int_\nu^r f_{k,m}(\varrho)\varrho^{n-2}\sin^{n-2}\varphi_r(\varrho)C_m^\lambda(\cos\varphi_r(\varrho))\sqrt{1+\varrho^2\dot\varphi_r^2(\varrho)}\,\mathrm{d}\varrho, \tag{8}$$

where $F_{k,m}$ is the coefficient of $Y_{k,m}$ in the spherical harmonic expansion of F.

As in the two dimensional case, we are now interested in the behavior of the kernel

$$\bar K(r,\varrho)=\frac{|\mathrm{S}^{n-2}|}{C_m^\lambda(1)}\varrho^{n-2}\sin^{n-2}\varphi_r(\varrho)C_m^\lambda(\cos\varphi_r(\varrho))\sqrt{1+\varrho^2\dot\varphi_r^2(\varrho)}$$

of the integral equation (8). Let $K(r,\varrho)=\bar K(r,\varrho)\sqrt{r-\varrho}^{\,n-3}$. Then, our integral equation is

$$F_{k,m}(r)=\int_\nu^r f_{k,m}(\varrho)\frac{K(r,\varrho)}{\sqrt{r-\varrho}^{\,n-3}}\,\mathrm{d}\varrho, \tag{8'}$$

where $K(r,r)=\frac{|\mathrm{S}^{n-2}|}{2}\sqrt{\frac{2r}{r\kappa_r-1}}^{\,n-1}\neq 0$ by (3) and (3′), and $K\in \mathrm{C}^{[n/2]}([\nu,\mu]^2)$ by the differentiability condition on φ_r.

The integral equations of this type are proved to have unique solutions [6, pp. 515 (3.9); 7, pp.41] , hence the solution $f_{k,m}=0$ is unique and our theorem is proved.

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As in the previous section, we can apply the method presented above to the family of the above defined hypersurfaces inverted to a sphere centered to the origin. For this family of hypersurfaces the function space on which our proof works is the compactly supported L^2 functions. Let the inverted hypersurfaces denoted by $\bar s_t$. The function $\bar\varphi_r$ can now be defined analogously to φ_r.

Theorem 2. *Let $f\in \mathrm{L}^2_0(\mathbb{R}^2)$ and $\frac{\bar\varphi_r(\varrho)}{\sqrt{\varrho-r}^{\,n-3}}\in \mathrm{C}^2(\bar{\mathbb{T}}^2)$. If the integral of f over the hypersurfaces $\bar s_{r,\omega}$ is zero for $\mu<r$, then $f(x)=0$ for $\mu<|x|$.*

4. Generalizations

Our theorems can be generalized considerable by the following observations. First of all, we could use measures different from the arclength resp. surface measure on the curves resp. hypersurfaces. We should only impose the rotation invariance and some differentiability condition on the measures.

Next, and more importantly, we can state the same result for curves resp. hypersurfaces that satisfy the conditions of the theorems only near the point P_r by the following reasoning.

Let $\varepsilon>0$ and assume the curves c_r satisfy all the conditions of Theorem 1 (the same can be done for Theorem 2, 3, and 4 of course), in the ring $\frac{r}{1+\varepsilon}<|X|\le r$. If $f\in \mathrm{L}^2(\mathbb{R}^2)$ is zero for $|x|<\mu$ and its integral $F(r,\alpha)$ is zero for $r<\mu(1+\varepsilon)$ then the nonzero part of the integration for $F(r,\alpha)$ uses at most only the part of c_r lying in the ring $\frac{r}{1+\varepsilon}<|X|\le r$, hence Theorem 1 gives that f is zero for $|X|<\mu(1+\varepsilon)$. If $F(r,\alpha)$ is zero for $r<\nu$ then we have to use this step $\log_{1+\varepsilon}\nu/\mu$-times to get the result of Theorem 1. Note that this "onion peeling" trick allows us to consider very general curves. Among others, an interesting example is the family of ellipsoids having one of their focuses in the origin. This generalizes Romanov's results in [7, Sect.1]. Theorem 1 with the above generalization applies to these curves, hence the support theorem is valid for the ellipsoids.

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