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The shadow picture problem for nonintersecting curves

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Abstract. We prove that two strictly convex bodies in the plane subtending the same angles at each of the points of two parallel straight lines and a big closed curve, must coincide.

1. Introduction

The problem of reconstructing a plane body from its shadow pictures, S-pictures, was raised in [5] (and for special cases in [3], [6]). The S-picture of a convex body $\mathcal{D}$ at a point $P \in \mathbb{R}^2$ is defined as the angle of the two supporting lines of $\mathcal{D}$ going through P. This angle is called the visual angle too. The question is: What set of S-pictures distinguishes any two convex bodies.

In [4] it is shown that the S-pictures taken from one curve only do not determine the convex bodies in general. However, roughly speaking, they do distinguish the polygons from each other [4]. Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{F}_1$ and $\mathcal{F}_2$ be closed convex domains with C^2 boundaries. Assume that the strictly convex bodies $\mathcal{F}_1$ and $\mathcal{F}_2$ subtend equal visual angles at each point of $\partial\mathcal{C}_1 \cup \partial\mathcal{C}_2$, and $\mathcal{F}_1 \cup \mathcal{F}_2$ are in the interior of $\mathcal{C}_1 \cap \mathcal{C}_2$. The author showed in [5] that if $\partial\mathcal{C}_1$ and $\partial\mathcal{C}_2$ *intersect each other in non-zero angles*, then $\mathcal{F}_1$ and $\mathcal{F}_2$ must coincide.

In this article we investigate the S-picture problem for curves that do not intersect each other. We prove uniqueness for S-pictures taken from two straight lines, or two hyperbolas. The method, we develop, can also be used for curves having asymptotes at infinity.

The basic idea of our method, coming from [2], is that the S-pictures generate such a measure on the set of straight lines that two domains having equal S-pictures

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have also equal volumes (perhaps infinite) with respect to this measure. For this measure, the finiteness of the volume of the difference of the two convex bodies is proved first. Then an infinite sequence of components of the difference of the bodies is constructed so, that the volumes of the components are the same (or almost the same). This implies that the volume of every component should be zero, hence the two bodies must coincide.

In the third section, we prove with elementary geometry that any two convex bodies can be distinguished by their S-pictures taken on any infinite set of curves.

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2. Preliminaries

First of all, we redefine the S-picture. Let $\mathcal{L}$ be the Grassman manifold of all the straight lines in the plane. Given a planar compact domain $\mathcal{D}$ with piecewise C^1 boundary, we define $\bar{\mathcal{D}}$ as a domain in $\mathcal{L}$ of all the straight lines intersecting $\mathcal{D}$ so that $\mathcal{D}$ has exactly two tangents parallel to l.

The S-picture function $S^{\mathcal{D}}$ of a domain $\mathcal{D}$ as above is defined then as

$$S^{\mathcal{D}}: \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad S^{\mathcal{D}}(P) = \frac{1}{2}\int_{S^1} \chi_{\bar{\mathcal{D}}}(l(\langle\omega, P\rangle, \omega))\,\mathrm{d}\omega,$$

where $\chi_{\bar{\mathcal{D}}}$ is the indicator function of $\bar{\mathcal{D}}$, $l(r,\omega)$ means the straight line through $r\omega$ that is perpendicular to ω, and $\langle .,. \rangle$ is the usual inner product. (The factor $1/2$ is needed because $l(r,\omega) = l(-r,-\omega)$). It is easy to see, that this definition is the same as the original one for convex bodies. For simplicity we shall write $l(r,\beta)$ instead of $l(r,\omega_\beta)$, where ω_β is the unit vector closing angle β with an appropriately fixed unit vector.

Let $g: \mathbb{R} \to \mathbb{R}^2$ be a C^1 curve parameterized by its arclength. Let $\xi(s,\alpha)$ denote the straight line $l(r,\omega_\beta)$ with $r = |g(s)|$ and making $\langle \omega_\beta, \dot{g}(s)\rangle = \sin\alpha$. From [7] the invariant measure on $\mathcal{L}$ is

$$\mathrm{d}\beta\,\mathrm{d}r = |\sin\alpha|\,\mathrm{d}\alpha\,\mathrm{d}s, \tag{1}$$

Let $\mathcal{D}$ be a strictly convex body with C^2 boundary so that the curve g is outside of $\mathcal{D}$. Let $a(s)$ and $b(s)$ be the lengths of the two tangents of $\mathcal{D}$ through

$g(s)$; $\alpha(s)$, $\beta(s)$ are the corresponding angles of these two tangents to $\dot g$, respectively. We proved in [5] that

$$\dot\nu = \frac{\sin\beta}{b} - \frac{\sin\alpha}{a}, \tag{2}$$

where $\nu(s) = S^{\mathcal{D}}(g(s))$.

3. The main results

Theorem 1. *$\mathcal{D}_1$ and $\mathcal{D}_2$ are strictly convex bodies, g_1 and g_2 are straight lines outside of $\mathcal{D}_1 \cup \mathcal{D}_2$ and $S^{\mathcal{D}_1} = S^{\mathcal{D}_2}$ on g_1 and g_2. If $\mathcal{D}_1$ and $\mathcal{D}_2$ have outer common tangent that intersects the straight lines, then $\mathcal{D}_1$ and $\mathcal{D}_2$ coincide.*

Proof. For intersecting straight lines this has already been proved in [5], hence we can assume $g_1 \parallel g_2$. We suppose further that the angle of g_1 to the fixed direction is zero, and therefore $\frac{1}{|\cos\beta|}\,\mathrm{d}\beta\,\mathrm{d}r = \mathrm{d}\alpha\,\mathrm{d}s$ by (1). From now on we use this measure on $\mathcal{L}$.

First we prove that $\mathcal{D}_1$ and $\mathcal{D}_2$ have infinitely many common tangents: Let t_0 be an assumed common tangent, and suppose t_0 intersects g_1 and g_2 at the points X_0 and Y_0, respectively. The visual angles of $\mathcal{D}_1$ and $\mathcal{D}_2$ are equal at Y_0, hence there exist an other common tangent t_1 through Y_0. The tangent t_1 intersects g_1 in X_1, where the visual angles of $\mathcal{D}_1$ and $\mathcal{D}_2$ are also equal. Therefore a third common tangent t_2 must go through X_1. t_2 intersects g_2 in Y_1 and so one can continue the procedure in the same way to get the sequence t_i of common tangents. This sequence of common tangents is certainly infinite, because the intersections $X_i = t_{2i} \cap g_1$ and $Y_i = t_{2i} \cap g_2$ make monotone sequences on g_1 and g_2, respectively. Further these sequences tend to infinity, because the visual angles of $\mathcal{D}_1$ and $\mathcal{D}_2$ at their respective limits would be zero otherwise.

The existence of the sequence t_i has as first consequence that $\mathcal{D}_1$ and $\mathcal{D}_2$ have two common tangents $\lim_{i\to\infty} t_{2i}$ and $\lim_{i\to\infty} t_{2i+1}$ parallel to g_1. Let their distances to g_1 be h and H, respectively, where $h < H$.

We prove next that any common tangent of $\mathcal{D}_1$ and $\mathcal{D}_2$ intersects the bodies in a point of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$. Let the distances of $\partial\mathcal{D}_1 \cap t_i$ and $\partial\mathcal{D}_2 \cap t_i$ from $t_i \cap g_1$ be d_i and $\bar d_i$, respectively. Let $e_i = \bar d_i - d_i$. Assuming $e_0 \geq 0$ one can easily see from (2) that $e_i \geq 0$ for all $i \in \mathbb{N}$

Equation (2) gives

$$\frac{\sin\alpha_{2i}}{d_{2i}+e_{2i}} - \frac{\sin\alpha_{2i-1}}{d_{2i-1}+e_{2i-1}} = \frac{\sin\alpha_{2i}}{d_{2i}} - \frac{\sin\alpha_{2i-1}}{d_{2i-1}}$$

for g_1 and

$$\frac{\sin\alpha_{2i}}{d_{2i}+e_{2i}+\frac{f}{\sin\alpha_{2i}}}-\frac{\sin\alpha_{2i+1}}{d_{2i+1}+e_{2i+1}+\frac{f}{\sin\alpha_{2i+1}}} = \frac{\sin\alpha_{2i}}{d_{2i}+\frac{f}{\sin\alpha_{2i}}}-\frac{\sin\alpha_{2i+1}}{d_{2i+1}+\frac{f}{\sin\alpha_{2i+1}}},$$

for g_2, where α_i is the angle between t_i and g_1, and f is the distance of g_1 and g_2. By an easy rearrangement we have

$$e_{2i}\frac{\sin\alpha_{2i}}{d_{2i}(d_{2i}+e_{2i})}=e_{2i-1}\frac{\sin\alpha_{2i-1}}{d_{2i-1}(d_{2i-1}+e_{2i-1})}$$

and

$$e_{2i+1}\frac{\sin\alpha_{2i+1}}{(d_{2i+1}+\frac{f}{\sin\alpha_{2i+1}})(d_{2i+1}+e_{2i+1}+\frac{f}{\sin\alpha_{2i+1}})} = e_{2i}\frac{\sin\alpha_{2i}}{(d_{2i}+\frac{f}{\sin\alpha_{2i}})(d_{2i}+e_{2i}+\frac{f}{\sin\alpha_{2i}})}.$$

Therefore

$$\frac{e_{2i+1}}{e_{2i-1}}=\frac{\sin\alpha_{2i-1}}{\sin\alpha_{2i+1}}\frac{(d_{2i+1}+\frac{f}{\sin\alpha_{2i+1}})(d_{2i+1}+e_{2i+1}+\frac{f}{\sin\alpha_{2i+1}})}{d_{2i-1}(d_{2i-1}+e_{2i-1})}\times \times\frac{d_{2i}(d_{2i}+e_{2i})}{(d_{2i}+\frac{f}{\sin\alpha_{2i}})(d_{2i}+e_{2i}+\frac{f}{\sin\alpha_{2i}})}.$$

To find the limit of the right hand side as $i\to\infty$ first we observe that the sequence e_{2i-1} is bounded and

$$H=\lim_{i\to\infty} d_{2i}\sin\alpha_{2i} \quad\text{and}\quad h=\lim_{i\to\infty} d_{2i+1}\sin\alpha_{2i+1}. \tag{3}$$

It is easy to see, that

$$\tan\alpha_{2i+1}\sim\frac{H+f}{|Y_i|},\qquad \tan\alpha_{2i-1}\sim\frac{H}{|X_i|}\qquad\text{and}\qquad \frac{|Y_i|}{|X_i|}\sim\frac{h+f}{h} \tag{4}$$

where $\sim$ denotes asymptotic equivalence. Therefore

$$\lim_{i\to\infty}\frac{\sin\alpha_{2i-1}}{\sin\alpha_{2i+1}}=\lim_{i\to\infty}\frac{\tan\alpha_{2i-1}}{\tan\alpha_{2i+1}}=\frac{H}{H+f}\frac{h+f}{h}.$$

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Taking the above limits into our expression for e_{2i+1}/e_{2i-1} we obtain

$$\lim_{i\to\infty}\frac{e_{2i+1}}{e_{2i-1}}=\frac{H}{H+f}\frac{h+f}{h}>1.$$

Since e_i is obviously bounded, this gives $e_0=0$, i.e. each common tangent intersects $\mathcal{D}_1$ and $\mathcal{D}_2$ in a point of $\partial\mathcal{D}_1\cap\partial\mathcal{D}_2$. Consequently, there is a common tangent at each point P of $\partial\mathcal{D}_1\cap\partial\mathcal{D}_2$, because otherwise a common tangent with different points of intersection would there exist.

In sum, we have proved that $\mathcal{D}_1\cap\mathcal{D}_2$ has nonempty interior, and therefore $\operatorname{Int}\mathcal{D}_1\setminus\mathcal{D}_2$ (resp. $\operatorname{Int}\mathcal{D}_2\setminus\mathcal{D}_1$) is the union of connected components, that are bounded by one arc of $\partial\mathcal{D}_1$ and by an other arc of $\partial\mathcal{D}_2$. These arcs intersect each other in two points, where they have common tangents. It follows from these, that $\operatorname{Int}(\bar{\mathcal{D}}_1\triangle\bar{\mathcal{D}}_2)$ consists of components in $\mathcal{L}$, too.

Let $\Phi_j l$ denote the straight line through $l\cap g_j$ which makes the angle $S^{\mathcal{D}_1}(l\cap g_j)=S^{\mathcal{D}_2}(l\cap g_j)$ with l in the appropriate direction. Let $\bar{\mathcal{R}}$ be a component of $\operatorname{Int}(\bar{\mathcal{D}}_1\triangle\bar{\mathcal{D}}_2)$ and set $\Psi_j\bar{\mathcal{R}}=\{\Phi_j l : l\in\bar{\mathcal{R}}\}$. Since $\Phi_j l$ cuts a component of $\operatorname{Int}(\mathcal{D}_1\triangle\mathcal{D}_2)$ exactly when l does, $\Psi_j\bar{\mathcal{R}}$ is a component of $\operatorname{Int}(\bar{\mathcal{D}}_2\triangle\bar{\mathcal{D}}_1)$.

Let $g_j(s_1)$ and $g_j(s_2)$ be the intersections of g_j with the tangents at the endpoints of $\mathcal{R}$. Further, let M and N be the volume of $\bar{\mathcal{R}}$ and $\Psi_j\bar{\mathcal{R}}$, respectively. Then

$$\begin{aligned}M=\int_{\bar{\mathcal{R}}}\frac{\mathrm{d}\beta\,\mathrm{d}r}{|\cos\beta|}&=\int_{\bar{\mathcal{R}}}\mathrm{d}\alpha\,\mathrm{d}s=\int_{s_1}^{s_2}S^{\bar{\mathcal{R}}}(g_j(s))\,\mathrm{d}s\\&=\int_{s_1}^{s_2}\int\chi_{\bar{\mathcal{R}}}(\xi(s,\alpha))\,\mathrm{d}\alpha\,\mathrm{d}s=\int_{s_1}^{s_2}\int\chi_{\Psi_j\bar{\mathcal{R}}}(\Phi_j\xi(s,\alpha))\,\mathrm{d}\alpha\,\mathrm{d}s\\&=\int_{s_1}^{s_2}\int\chi_{\Psi_j\bar{\mathcal{R}}}(\xi(s,\alpha))\,\mathrm{d}\alpha\,\mathrm{d}s=\int_{s_1}^{s_2}S^{\Psi_j\bar{\mathcal{R}}}(g_j(s))\,\mathrm{d}s\\&=\int_{\Psi_j\bar{\mathcal{R}}}\mathrm{d}\alpha\,\mathrm{d}s=\int_{\Psi_j\bar{\mathcal{R}}}\frac{\mathrm{d}\beta\,\mathrm{d}r}{|\cos\beta|}=N,\end{aligned}$$

hence $\bar{\mathcal{R}}$ and $\Psi_j\bar{\mathcal{R}}$ have the same volume.

To see that the volume of $\bar{\mathcal{D}}_1\triangle\bar{\mathcal{D}}_2$ is finite, one needs to consider only the straight lines close to the common tangents parallel to g_1, because the measure $\frac{\mathrm{d}\beta\,\mathrm{d}r}{|\cos\beta|}$ has singularity only at $\beta=\pm\pi/2$. Let us choose the common tangent $l(h,\pi/2)$ and calculate the volume of $\operatorname{Int}\bar{\mathcal{D}}_1\setminus\bar{\mathcal{D}}_2$ in a small $\varepsilon>0$ neighborhood of $l(h,\pi/2)$. We have

$$\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon}\int\frac{\mathrm{d}\beta\,\mathrm{d}r}{|\cos\beta|}=\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon}\int\mathrm{d}r\frac{\mathrm{d}\beta}{|\cos\beta|}=\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon}\varrho(\beta)\frac{\mathrm{d}\beta}{|\cos\beta|}=\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon}\sigma(\beta)\,\mathrm{d}\beta,$$

where $\varrho(\beta)$ is the distance of the two tangent of $\operatorname{Int}\mathcal{D}_1 \setminus \mathcal{D}_2$ perpendicular to ω_β. $\sigma(\beta)$ is the length of the segment these two tangents cut out from $l(h, \pi/2)$. Since $l(h, \pi/2)$ meets $\mathcal{D}_1$ and $\mathcal{D}_2$ in the same point, $\sigma(\beta) \to 0$ as $\beta \to \pi/2$, hence the volume is finite by the last integral.

Since $\bigcup_{k\in\mathbb{N}}(\Psi_1\Psi_2)^k\bar{\mathcal{R}}$ has finite volume and $(\Psi_1\Psi_2)^k\bar{\mathcal{R}}$ is infinite sequence of disjoint sets having constant volume, we conclude that $\bar{\mathcal{R}}$ and in the same way any other component should be empty. This completes the proof.

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Using S-pictures taken from a third curve, we can skip from the condition on the common tangent.

Theorem 2. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be strictly convex bodies and let C be a compact domain so that $\mathcal{D}_1 \cup \mathcal{D}_2 \subset \operatorname{Int}\mathcal{C}$. Let g_1 and g_2 be straight lines not intersecting $\mathcal{D}_1 \cup \mathcal{D}_2$. If the visual angles of $\mathcal{D}_1$ and $\mathcal{D}_2$ are equal at each point of g_1, g_2 and $\partial\mathcal{C}$ then $\mathcal{D}_1 \equiv \mathcal{D}_2$.*

Proof. Since neither of $\mathcal{D}_1$ and $\mathcal{D}_2$ can contain the other, they have a common outer tangent t. This cuts $\partial\mathcal{C}$, and therefore they have an other common tangent through $t \cap \partial\mathcal{C}$ that is not parallel to t. Now one uses Theorem 1 to conclude the statement.

■

In the following theorem we substitute the straight lines g_1 and g_2 with hyperbolas. A hyperbola h divides the plane into tree parts, two of which are convex. We denote the union of these two convex parts by $C(h)$. The proof of the next theorem is very similar to the previous one, therefore we are going into details only where nontrivial differences occur.

Theorem 3. *Let the asymptotes of the hyperbola h_1 be parallel to the asymptotes of the hyperbola h_2. Suppose that h_1 does not intersect h_2 and the strictly convex bodies $\mathcal{D}_1$ and $\mathcal{D}_2$ are in $\operatorname{Int} C(h_1) \cap \operatorname{Int} C(h_2)$. If the visual angles of $\mathcal{D}_1$ and $\mathcal{D}_2$ are equal at each point of h_1 and h_2 then $\mathcal{D}_1 \equiv \mathcal{D}_2$.*

Proof. For easier calculation we assume that the asymptotes of h_1 and h_2 are not the same. As the reader will see, a slight modification of our proof can handle that case too ($h_1 \not\equiv h_2$ is necessary of course). We choose the asymptote closest to $\mathcal{D}_1 \cup \mathcal{D}_2$ to be the x-axis and parameterize the hyperbolas by arclength. We shall follow the steps of the proof of Theorem 1 using the same notations where possible.

Obviously any common tangent of $\mathcal{D}_1$ and $\mathcal{D}_2$ intersects h_1 and h_2, and so, one can construct an infinite sequence of common tangents, as in the proof of Theorem 1.

To show that each common tangent intersects $\mathcal{D}_1$ and $\mathcal{D}_2$ in a point of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$, we can apply (2) with a calculation very similar to that used in the proof of Theorem 1.

Therefore we have the same situation, i.e. $\operatorname{Int}\mathcal{D}_1 \cap \operatorname{Int}\mathcal{D}_2$ is not empty and $\operatorname{Int}(\mathcal{D}_1 \triangle \mathcal{D}_2)$ is the union of its components. These components are bounded by one arc of $\partial\mathcal{D}_1$ and by an other arc of $\partial\mathcal{D}_2$. These arcs have only two common points where $\mathcal{D}_1$ and $\mathcal{D}_2$ have common tangents.

We define the mappings Φ_1, Φ_2, Ψ_1 and Ψ_2 just as in the proof of Theorem 1. Again, the sequence $(\Psi_1\Psi_2)^k\bar{\mathcal{R}}$ of components is infinite for any component $\bar{\mathcal{R}}$ of $\operatorname{Int}\bar{\mathcal{D}}_1 \setminus \bar{\mathcal{D}}_2$.

In the present case it is no longer true that a component and its images by Ψ_1 and Ψ_2 have the same volume. It is true, however, that if the component $\bar{\mathcal{R}}$ is not empty, then the volume of $\bigcup_{j=1}^{\infty}(\Psi_2\Psi_1)^j\bar{\mathcal{R}}$ is infinite w.r.t. the measure $\frac{\mathrm{d}\beta\,\mathrm{d}r}{|\cos\beta|}$.

To show this, let us first observe that all the components $(\Psi_2\Psi_1)^j\bar{\mathcal{R}}$ are disjoint. Let $\bar{\mathcal{R}}_j = (\Psi_2\Psi_1)^j\bar{\mathcal{R}}$ and the volume of $\bar{\mathcal{R}}_j$ be M_j. We shall estimate the ratio M_{j+1}/M_j when $j \to \infty$.

Let the angle between $\dot{h}_i(s)$ and the x-axis be $\gamma_i(s)$. Then $\beta = \alpha_i - \gamma_i - \frac{\pi}{2}$, i.e. $\cos\beta = \sin(\alpha_i - \gamma_i)$, where (s_i, α_i) is the parameter of the straight line $\xi_i(s_i, \alpha_i)$ on h_i. By (1) we have $\mathrm{d}\beta\,\mathrm{d}r = \sin\alpha_i\,\mathrm{d}\alpha_i\,\mathrm{d}s_i$, hence $\frac{\mathrm{d}\beta\,\mathrm{d}r}{|\cos\beta|} = \frac{\sin\alpha_1}{\sin(\alpha_1-\gamma_1)}\,\mathrm{d}\alpha_1\,\mathrm{d}s_1$ that implies

$$(5)\qquad M_{j+1} = \int_{\bar{\mathcal{R}}_{j+1}} \frac{\sin\alpha_2}{\sin(\alpha_2-\gamma_2)}\,\mathrm{d}\alpha_2\,\mathrm{d}s_2 = \int_{\Psi_1\bar{\mathcal{R}}_j} \frac{\sin(\alpha_2-\nu_2)}{\sin(\alpha_2-\nu_2-\gamma_2)}\,\mathrm{d}\alpha_2\,\mathrm{d}s_2,$$

where $\nu_i(s) = S^{\mathcal{D}_1}(h_i(s)) = S^{\mathcal{D}_2}(h_i(s))$ is the visual angle at $h_i(s)$ and the second equality follows from the definition $\Psi_2\Psi_1\bar{\mathcal{R}} = \{\Phi_2 l : l \in \Psi_1\bar{\mathcal{R}}\}$. By (1) $\mathrm{d}\alpha_2\,\mathrm{d}s_2 = \frac{\sin\alpha_1}{\sin\alpha_2}\,\mathrm{d}\alpha_1\,\mathrm{d}s_1$ for $\xi_1(s_1,\alpha_1) = \xi_2(s_2,\alpha_2)$ therefore

$$M_{j+1} = \int_{\Psi_1\bar{\mathcal{R}}_j} \frac{\sin(\alpha_2-\nu_2)\sin\alpha_1}{\sin(\alpha_2-\nu_2-\gamma_2)\sin\alpha_2}\,\mathrm{d}\alpha_1\,\mathrm{d}s_1,$$

Substituting $\Phi_1 l$ again, by $\Psi_1\bar{\mathcal{R}} = \{\Phi_1 l : l \in \bar{\mathcal{R}}\}$ we arrive to

$$M_{j+1} = \int_{\bar{\mathcal{R}}_j} \frac{\sin(\alpha_2-\nu_2)\sin(\alpha_1+\nu_1)}{\sin(\alpha_2-\nu_2-\gamma_2)\sin\alpha_2}\,\mathrm{d}\alpha_1\,\mathrm{d}s_1,$$

where α_2, ν_2 and γ_2 are the h_2-parameters of the straight line $\xi_1(s_1, \alpha_1+\nu_1)$. Substituting $\alpha_2 = \alpha_1 + \nu_1 - \gamma_1 + \gamma_2$ and combining the result with

$$M_j = \int_{\bar{\mathcal{R}}_j} \frac{\sin\alpha_1}{\sin(\alpha_1-\gamma_1)}\,\mathrm{d}\alpha_1\,\mathrm{d}s_1$$

one gets

$$M_{j+1} - M_j = \int_{\bar{\mathcal{R}}_j} \mu(s_1, \alpha_1) \frac{\sin \alpha_1}{\sin(\alpha_1 - \gamma_1)} \, \mathrm{d}\alpha_1 \, \mathrm{d}s_1, \tag{6}$$

where

$$\mu(s_1, \alpha_1) = \frac{\sin(\alpha_1 + \nu_1 - \nu_2 - \gamma_1 + \gamma_2) \sin(\alpha_1 + \nu_1) \sin(\alpha_1 - \gamma_1)}{\sin(\alpha_1 + \nu_1 - \nu_2 - \gamma_1) \sin(\alpha_1 + \nu_1 - \gamma_1 + \gamma_2) \sin \alpha_1} - 1.$$

Since all the indicated angles tend to zero as $s_1 \to \infty$, we have

$$\begin{aligned} \lim_{s_1 \to \infty} s_1 \mu(s_1, \alpha_1) &= \lim_{s_1 \to \infty} s_1 \left(\frac{(\alpha_1 + \nu_1 - \nu_2 - \gamma_1 + \gamma_2)(\alpha_1 + \nu_1)(\alpha_1 - \gamma_1)}{(\alpha_1 + \nu_1 - \nu_2 - \gamma_1)(\alpha_1 + \nu_1 - \gamma_1 + \gamma_2)\alpha_1} - 1 \right) \\ &= \lim_{s_1 \to \infty} s_1 \frac{\alpha_1 \nu_2 \gamma_2 - \nu_2(\alpha_1 + \nu_1 - \nu_2 - \gamma_1 + \gamma_2)\gamma_1}{(\alpha_1 + \nu_1 - \nu_2 - \gamma_1)(\alpha_1 + \nu_1 - \gamma_1 + \gamma_2)\alpha_1}. \end{aligned}$$

Elementary calculation gives $\lim_{s_1 \to \infty} s_1^2 \sin \gamma_i = \mathit{constant}$. By this and (4) the terms α_1, ν_1 and ν_2 are asymptotically equivalent to c/s_1 for appropriate constants c. Therefore $\lim_{s_1 \to \infty} s_1 \mu(s_1, \alpha_1) = k$ for some constant k.

The component $\bar{\mathcal{R}}_j$ has two common tangents of $\mathcal{D}_1$ and $\mathcal{D}_2$. They intersect h_1 in two points, say, in $h_1(\sigma_j)$ and $h_1(\bar{\sigma}_j)$ so that $\sigma_j < s_1 < \bar{\sigma}_j$ for all $\xi_1(s_1, \alpha_1) \in \bar{\mathcal{R}}_j$. Therefore the integral in (6) can be estimated as

$$|M_{j+1} - M_j| \leq \frac{|k| + \varepsilon}{\sigma_j} M_j$$

for j big enough and $\varepsilon > 0$ small enough. In more suitable equivalent form this means

$$1 - \frac{|k| + \varepsilon}{\sigma_j} \leq \frac{M_{j+1}}{M_j} \leq 1 + \frac{|k| + \varepsilon}{\sigma_j}.$$

We shall show that σ_j grows exponentially with $j \to \infty$. This proves $\lim_{j \to \infty} M_j = \mathit{constant} > 0$ that gives $\sum_{j=1}^{\infty} M_j = \infty$ as we stated.

Let the common tangent $t \in \Psi_1 \bar{\mathcal{R}}_j$ through $h_1(\sigma_j)$ cut h_2 in $h_2(\tau_j)$. Then the common tangent $\Phi_2 t$ is in $\bar{\mathcal{R}}_{j+1}$ and therefore cuts h_1 in $h_1(\sigma_{j+1})$. Let f be the distance of the two asymptotes parallel to the x-axis. Let h and H denote the distances from the x-axis of the two common tangents parallel to the x-axis, so that $h < H$. Obviously,

$$\lim_{j \to \infty} \frac{\tau_j}{\sigma_j} = \frac{h + f}{h} \qquad \text{and} \qquad \lim_{j \to \infty} \frac{\sigma_{j+1}}{\tau_j} = \frac{H}{H + f},$$

hence

$$\lim_{j\to\infty}\frac{\sigma_{j+1}}{\sigma_j}=\frac{H}{H+f}\frac{h+f}{h}>1,$$

which was to be shown.

To complete the proof we only have to note that the volume of $\bar{\mathcal{D}}_1\triangle\bar{\mathcal{D}}_2$ is finite by the same reason as in the case of the straight lines.

■

The above method can be used for a large class of curves. These curves should have two important properties: asymptotes in the infinity and something that ensures that the sequence of common tangents gets onto the asymptotic part. We state this observation in the following theorem.

Theorem 4. *Let h_1 and h_2 be convex curves so that*

1) h_i $(i=1,2)$ *has an asymptotic straight line* g_i,
2) g_1 *is parallel to* g_2 *and the intersection of the convex envelopes of* h_1 *and* h_2 *is not empty,*
3) *the angle* $\gamma_i(s)$ *between* g_i *and* $\dot{h}_i(s)$ *satisfies* $\lim_{s\to\infty} s\gamma_i(s)=0$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be strictly convex bodies so that $\mathcal{D}_1\cup\mathcal{D}_2$ is in the interior of the intersection of the convex envelopes of h_1 and h_2. If the visual angles of $\mathcal{D}_1$ and $\mathcal{D}_2$ are equal at each point of h_1 and h_2 then $\mathcal{D}_1\equiv\mathcal{D}_2$.

4. Infinitely many arbitrary curves

Although the results above and in [5] cover many cases, we still do not know any general result for S-pictures taken from two concentric circles, the first case considered in the literature. We offer the following theorem as a first step in this direction.

Theorem 5. *Let $\mathcal{C}_i$ $(i\in\mathbb{N})$, $\mathcal{D}_1$ and $\mathcal{D}_2$ be convex bodies so that $\mathcal{D}_1\cup\mathcal{D}_2\subset\operatorname{Int}\mathcal{C}_0$ and $\mathcal{C}_i\subset\operatorname{Int}\mathcal{C}_{i+1}$ for $(i\in\mathbb{N})$. If the S-pictures of $\mathcal{D}_1$ and $\mathcal{D}_2$ are equal at all points of each $\partial\mathcal{C}_i$, then $\mathcal{D}_1\equiv\mathcal{D}_2$.*

Proof. Let t_0 be a common supporting line of $\mathcal{D}_1$ and $\mathcal{D}_2$. It intersects each $\partial\mathcal{C}_i$ in two points, P_i and Q_i. Let $T_1=t_0\cap\mathcal{D}_1$ and $T_2=t_0\cap\mathcal{D}_2$. T_1 and T_2 divide the straight line t_0 into parts. On one of the two infinite parts are the intersection points P_i and on the other one are the intersection points Q_i $(i=0,1,2,\dots)$.

Let P be a limit point of $\{P_i\}$, which may be the infinity. At each point P_i there must be an other common supporting line of $\mathcal{D}_1$ and $\mathcal{D}_2$, say t_i. Since P is a limit point, the sequence t_i must have a limit straight line t^P through P which is a common supporting lines. Since t^P is a limit of common supporting line, it must meet the bodies in a point of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$.

The point P must also be a limit point of the intersection points $t^P \cap \partial\mathcal{C}_i$. Through these intersection points there must be other common supporting lines t_i^P. Obviously t_0 is a limit straight line of these common supporting lines t_i^P, hence T_1 should coincide with T_2, i.e. every common supporting line intersects the two bodies in $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$.

Also every point of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$ is an intersection point of a common supporting line, because otherwise a common supporting line bridging the point of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$ would exist, which intersects $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ in different points.

Let $\tilde{P}_i \in \tilde{\mathcal{C}}_i$ be a monotone subsequence of P_i tending to P. Then the sequence $\tilde{\mathcal{C}}_i$ is also monotone with respect to the inclusion relation, i.e. either $\tilde{\mathcal{C}}_i \subset \tilde{\mathcal{C}}_{i+1}$ for each i or $\tilde{\mathcal{C}}_{i+1} \subset \tilde{\mathcal{C}}_i$ for each i. Let $\tilde{Q}_i \in (t_0 \cap \partial\tilde{\mathcal{C}}_i)$ be different from $\tilde{P}_i$. The sequence $\tilde{Q}_i$ is monotone, because $\tilde{\mathcal{C}}_i$ is monotone. Moreover, if I denotes the intersection of t_0 and $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$, then the distance $d(\tilde{P}_i, I)$ is decreasing or increasing together with $d(\tilde{Q}_i, I)$. Let Q be the limit of $\tilde{Q}_i$, which may be the infinity. As in the case of P we obtain a new common supporting line through Q, say t^Q. Q is a limit point of the intersection points $t^Q \cap \partial\tilde{\mathcal{C}}_i$, which implies a sequence of common supporting lines t_i^Q through these points tending to t_0. Obviously, the intersections of t_i^Q and t_j^P with $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$ are on the same side of t_0 and their distances to I tends to zero.

Therefore a common supporting line is a limit line of convergent sequences of common supporting lines from both clockwise and anti-clockwise directions. The normals of the common supporting lines then constitutes a closed subset $\mathcal{N}$ of the unit circle S^1, so that any point of $\mathcal{N}$ has convergent sequence in $\mathcal{N}$ from both clockwise and anti-clockwise directions. Let ω be a unit vector. Since $\mathcal{N}$ is closed, there is a point $\alpha \in \mathcal{N}$ closest to ω. Both arcs of $\omega\alpha$ in S^1 contain sequences in $\mathcal{N}$ convergent to α, therefore $\omega = \alpha$, hence $\mathcal{N} = S^1$.

So, $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same set of supporting lines, hence they coincide.

■

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